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# The Lane-Emden Function and Nonlinear Eigenvalues Problems 

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#### Abstract

RÉSUMÉ. - Nous considérons un problème aux valeurs propres, semilinéaire elliptique, sur une boule de $\mathbb{R}^{n}$ et montrons que ces valeurs et fonctions propres peuvent s'obtenir à partir de la fonction de Lane-Emden.


Abstract. - We consider a semilinear elliptic eigenvalues problem on a ball of $\mathbb{R}^{n}$ and show that all the eigenfunctions and eigenvalues, can be obtained from the Lane-Emden function.

## 1. Introduction

We consider the problem

$$
\left(P_{\lambda}^{\alpha}\right)\left\{\begin{array}{l}
\Delta u+\lambda(1+u)^{\alpha}=0, \text { in } B_{1} \\
u>0, \text { in } B_{1} \\
u=0, \text { on } \partial B_{1}
\end{array}\right.
$$

where $B_{1}$ is the unit ball of $\mathbb{R}^{n}, n \geqslant 3, \lambda>0$ and $\alpha>1$.
This problem arises in many physical models like the nonlinear heat generation and the theory of gravitational equilibrium of polytropic stars (cf. [2] and [11]). It is well known (cf. [2], [10], [12]) that there exists a critical constant $\lambda^{*}(\alpha)$, such that $\left(P_{\lambda}^{\alpha}\right)$ admits, at least, one solution if $0<$ $\lambda<\lambda^{*}(\alpha)$ and no solution if $\lambda>\lambda^{*}(\alpha)$. We deal here with these critical constants and the corresponding eigenfunctions.

[^0]Let $\phi$ be the Lane-Emden function(cf. [1], [5], [6], [15]) in the $n$-dimensional space and $r_{0}$ the first "zero" of $\phi$, we show that

$$
\lambda^{*}(\alpha)=\max _{r \in\left[0, r_{0}[ \right.} r^{2} \phi^{\alpha-1}(r)
$$

We use this formula to compute $\lambda^{*}(\alpha)$, when $\alpha$ is the Critical Sobolev Exponent. We also extend, to the subcritical case, an estimate of $\lambda^{*}(\alpha)$ given in [10] and show qualitative properties of the eigenfunctions.
In the Appendix, we show how to approximate $\phi$, so one can use numerical approaches (Maple or Matlab) to get estimates of $\lambda^{*}(\alpha)$.

## 2. Scalings of the Lane-Emden function as solutions

When $0<\lambda \leqslant \lambda^{*}(\alpha)$, it is known that any regular solution of $\left(P_{\lambda}^{\alpha}\right)$ is radial and the minimal one is stable and analytical (cf.[8], [12]).

Proposition 2.1. - Let $u$ be a regular solution of $\left(P_{\lambda}^{\alpha}\right)$, then

$$
u(r)=(1+u(0)) \phi\left(\sqrt{\lambda}(1+u(0))^{\frac{\alpha-1}{2}} r\right)-1, \forall r \in[0,1]
$$

where $\phi$ is the Lane-Emden function, in the $n$-dimensional space.
Proof. - The Lane-Emden function(cf. [1], [5], [6], [15]) is the solution of

$$
(L-E)\left\{\begin{array}{l}
\phi "(r)+\frac{n-1}{r} \phi^{\prime}(r)+\phi(r)|\phi(r)|^{\alpha-1}=0 \\
\phi(0)=1, \phi^{\prime}(0)=0
\end{array}\right.
$$

The proof of the proposition is quite immediate.

## 3. The Subcritical Case

Let us consider the problem $\left(P_{\lambda}^{\alpha}\right)$, with $1<\alpha<\frac{n+2}{n-2}$. Let $\phi$ be the Lane-Emden function.

Proposition 3.1. - There exists $r_{0}>0$, such that $\phi\left(r_{0}\right)=0, \phi(r)>0$, $\forall r \in\left[0, r_{0}[\right.$ and

$$
\lambda^{*}(\alpha)=\max _{\rho \in\left[0, r_{0}\right]} \rho^{2} \phi^{\alpha-1}(\rho)
$$

We also have

$$
\lambda^{*}(\alpha) \geqslant \frac{2}{(\alpha-1)^{2}}(\alpha(n-2)-n), \text { if } \frac{n}{n-2}<\alpha<\frac{n+2}{n-2} .
$$

Proof. - As $\phi(0)>0$, we infer that $\phi>0$, on a maximal interval $\left[0, r_{0}[\right.$. The problem

$$
\left\{\begin{array}{l}
\Delta u+u^{\alpha}=0, \text { in } \mathbb{R}^{n} \\
u>0, \text { in } \mathbb{R}^{n}
\end{array}\right.
$$

does not admit a solution (cf.[4]), so we infer that $r_{0}<\infty$ and $\phi\left(r_{0}\right)=0$.
Let us put

$$
\psi_{\rho}(r)=\frac{\phi(\rho r)-\phi(\rho)}{\phi(\rho)}, \forall r \in[0,1]
$$

with $0<\rho<r_{0}$, then $\psi_{\rho}$ is a solution of $\left(P_{\lambda}^{\alpha}\right)$, with $\lambda=\rho^{2} \phi^{\alpha-1}(\rho)$. We infer that

$$
\max _{\rho \in\left[0, r_{0}\right]} \rho^{2} \phi^{\alpha-1}(\rho) \leqslant \lambda^{*}(\alpha) .
$$

Let us suppose that

$$
\max _{\rho \in\left[0, r_{0}\right]} \rho^{2} \phi^{\alpha-1}(\rho)<\lambda^{*}(\alpha)
$$

if $u_{\lambda^{*}(\alpha)}$ is the unique solution of $\left(P_{\lambda^{*}(\alpha)}^{\alpha}\right)(c f .[10])$, one can use Proposition 1 to show that
$u_{\lambda^{*}(\alpha)}(r)=\left(1+u_{\lambda^{*}(\alpha)}(0)\right)\left(\phi\left(\left(\lambda^{*}(\alpha)\right)^{\frac{1}{2}}\left(1+u_{\lambda^{*}(\alpha)}(0)\right)^{\frac{\alpha-1}{2}} r\right)-\frac{1}{1+u_{\lambda^{*}(\alpha)}(0)}\right)$.
Let us put $\rho_{\lambda^{*}(\alpha)}=\left(\lambda^{*}(\alpha)\right)^{\frac{1}{2}}\left(1+u_{\lambda^{*}(\alpha)}(0)\right)^{\frac{\alpha-1}{2}}$. As $u_{\lambda^{*}(\alpha)} \geqslant 0$, we infer that $\rho_{\lambda^{*}(\alpha)}<r_{0}$. As $u_{\lambda^{*}(\alpha)}(1)=0$, we infer that

$$
\frac{1}{1+u_{\lambda^{*}(\alpha)}(0)}=\phi\left(\left(\lambda^{*}(\alpha)\right)^{\frac{1}{2}}\left(1+u_{\lambda^{*}(\alpha)}(0)\right)^{\frac{\alpha-1}{2}}\right) .
$$

So we get

$$
u_{\lambda^{*}(\alpha)}(r)=\frac{\phi\left(\rho_{\lambda^{*}(\alpha)} r\right)-\phi\left(\rho_{\lambda^{*}(\alpha)}\right)}{\phi\left(\rho_{\lambda^{*}(\alpha)}\right)} \text { and } \lambda^{*}(\alpha)=\left(\rho_{\lambda^{*}(\alpha)}\right)^{2} \phi^{\alpha-1}\left(\rho_{\lambda^{*}(\alpha)}\right) .
$$

The last equality leads to a contradiction.
To prove the last statement, we use the fact that the maximum here is achieved at a unique $r_{\alpha}$ (see the next lemma). So we get

$$
\begin{gathered}
\phi^{\prime}\left(r_{\alpha}\right)=-\frac{2}{(\alpha-1) r_{\alpha}} \phi\left(r_{\alpha}\right), \text { and } \\
\phi^{\alpha-3}\left(r_{\alpha}\right)\left(2 \phi^{2}\left(r_{\alpha}\right)+4 r_{\alpha}(\alpha-1) \phi\left(r_{\alpha}\right) \phi^{\prime}\left(r_{\alpha}\right)+(\alpha-1) r_{\alpha}^{2}\left((\alpha-2)\left(\phi^{\prime}\left(r_{\alpha}\right)\right)^{2}+\phi\left(r_{\alpha}\right) \phi^{\prime \prime}\left(r_{\alpha}\right)\right)\right) \leqslant 0
\end{gathered}
$$

We first replace $\phi^{\prime \prime}\left(r_{\alpha}\right)$ by its value from $(L-E)$ and then $\phi^{\prime}\left(r_{\alpha}\right)$, from the previous equality, to get

$$
\phi^{\alpha-1}\left(r_{\alpha}\right)\left(-(\alpha-1) \lambda^{*}(\alpha)+2(n-4)+4 \frac{\alpha-2}{\alpha-1}\right) \leqslant 0
$$

Simplifying, one gets the estimate.
Remark 3.2. - The last statement in Proposition 2 is also true for $\alpha \geqslant$ $\frac{n+2}{n-2}$, with the same proof, provided that $\sup _{r \in \mathbb{R}_{+}} r^{2} \phi^{\alpha-1}(r)$ is attained (see the next Proposition 6); this has been proved in [10], using sophisticated arguments.

Lemma 3.3. - Let us put $g(r)=r^{2} \phi^{\alpha-1}(r), r \in\left[0, r_{0}\right]$, there exists $\left.\rho_{0} \in\right] 0, r_{0}\left[\right.$ such that $g$ is increasing on $\left[0, \rho_{0}\right]$ and decreasing on $\left[\rho_{0}, r_{0}\right]$.

Proof. - Let $\rho$ be an arbitrary positive constant with $\rho<r_{0}$, then, as we have already mentioned $\psi_{\rho}$ is a solution of $\left(P_{\gamma}^{\alpha}\right)$, where $\gamma=g(\rho)$. As $g^{\prime}(r)=r \phi^{\alpha-2}(r)\left(2 \phi(r)+(\alpha-1) r \phi^{\prime}(r)\right)$, we infer that $g$ is increasing on a maximal interval $I_{0} \subset\left[0, r_{0}\right]$ with $0 \in I_{0}$.

Using Proposition 2, there exists $\left.\rho_{0} \in\right] 0, r_{0}\left[\right.$, such that $g\left(\rho_{0}\right)=$ $\max _{r \in\left[0, r_{0}\right]} g(r)=\lambda^{*}(\alpha)$. This $\rho_{0}$ is unique, otherwise, if there exists $\lambda \in\left[0, r_{0}\right]$, such that $g(\lambda)=\max _{r \in\left[0, r_{0}\right]} g(r)=\lambda^{*}(\alpha)$, then $\psi_{\rho_{0}}$ and $\psi_{\lambda}$ are both solutions of the problem $\left(P_{\lambda^{*}(\alpha)}^{\alpha}\right)$. As $\phi$ is decreasing on [0, $\left.r_{0}\right]$, we infer that $\psi_{\rho_{0}}(0)=\frac{1-\phi\left(\rho_{0}\right)}{\phi\left(\rho_{0}\right)} \neq \frac{1-\phi(\lambda)}{\phi(\lambda)}=\psi_{\lambda}(0)$. So we get two different solutions of the problem $\left(P_{\lambda^{*}(\alpha)}^{\alpha}\right)$. This leads to a contradiction (cf. [10]). As $g\left(r_{0}\right)=0$, we infer that $I_{0} \neq\left[0, r_{0}\right]$. Let us put $\delta=\sup I_{0}$. The function $g$ can't be constant on a nontrivial interval $J \subset\left[\delta, r_{0}\right]$, for if $g(r)=c$ in $J$, then for every $\lambda \in J, \psi_{\lambda}$ is a solution of $\left(P_{c}^{\alpha}\right)$. As $\psi_{\lambda_{1}}(0) \neq \psi_{\lambda_{2}}(0)$, if $\lambda_{1}, \lambda_{2} \in J$ and $\lambda_{1} \neq \lambda_{2}$, we infer that the problem $\left(P_{c}^{\alpha}\right)$ admits an infinity of solutions. This leads again to a contradiction (cf. [10]).

So if $g$ is not decreasing on $\left[\delta, r_{0}\right]$, then there exists $\beta_{1}$ and $\beta_{2}$ with $r_{0}>\beta_{2}>\beta_{1}>\delta$, such that $g$ is decreasing on $\left[\delta, \beta_{1}\right]$ and increasing on $\left[\beta_{1}, \beta_{2}\right]$. Let us put $c_{0}=\min \left(g(\delta), g\left(\beta_{2}\right)\right)$, then $c_{0}>g\left(\beta_{1}\right)$. Let us choose $c \in$ $] g\left(\beta_{1}\right), c_{0}[$, so the problem $g(t)=c$ admits at least three different solutions $\left.\lambda_{i} \in\right] 0, \beta_{2}\left[, 1 \leqslant i \leqslant 3\right.$. As $\psi_{\lambda_{i}}(0) \neq \psi_{\lambda_{j}}(0)$, if $i \neq j, 1 \leqslant i, j \leqslant 3$, we obtain three solutions for the problem $\left(P_{c}^{\alpha}\right)$. So we get a contradiction.

We conclude that $g$ is increasing on $[0, \delta]$, decreasing on $\left[\delta, r_{0}\right]$ and $\delta=\rho_{0}$.
Proposition 3.4. - If $\lambda=\lambda^{*}(\alpha)$, there exists a unique $\left.\rho_{\lambda^{*}(\alpha)} \in\right] 0, r_{0}[$, such that
$\lambda^{*}(\alpha)=\left(\rho_{\lambda^{*}(\alpha)}\right)^{2} \phi^{\alpha-1}\left(\rho_{\lambda^{*}(\alpha)}\right)$ and the unique solution $u_{\lambda^{*}(\alpha)}$ of $\left(P_{\lambda^{*}(\alpha)}^{\alpha}\right)$ is

$$
u_{\lambda^{*}(\alpha)}(r)=\frac{\phi\left(\rho_{\lambda^{*}(\alpha)} r\right)-\phi\left(\rho_{\lambda^{*}(\alpha)}\right)}{\phi\left(\rho_{\lambda^{*}(\alpha)}\right)}=\psi_{\rho_{\lambda^{*}(\alpha)}}(r), \forall r \in[0,1]
$$

When $0<\lambda<\lambda^{*}(\alpha)$, there exist exactly two constants $r_{\lambda}$ and $\rho_{\lambda}$, such that $0<r_{\lambda}<\rho_{\lambda^{*}(\alpha)}<\rho_{\lambda}<r_{0}, \lambda=r_{\lambda}^{2} \phi^{\alpha-1}\left(r_{\lambda}\right)=\rho_{\lambda}^{2} \phi^{\alpha-1}\left(\rho_{\lambda}\right)$ and the only two solutions of $\left(P_{\lambda}^{\alpha}\right)$ are

$$
u_{\lambda}=\psi_{r_{\lambda}}, v_{\lambda}=\psi_{\rho_{\lambda}}
$$

the minimal one(cf.[2]) is $u_{\lambda}, \lim _{\lambda \rightarrow 0} u_{\lambda}=0$ in $C^{0}\left(\overline{B_{1}}\right)$ and $\lim _{\lambda \rightarrow 0} v_{\lambda}(r)=\infty, \forall r \in[0,1[$.

Proof. - Using Proposition 2 and Lemma 1, one infers that the only solution of $\left(P_{\lambda^{*}(\alpha)}^{\alpha}\right)$ is $\psi_{\rho_{0}}$. We put $\rho_{\lambda^{*}(\alpha)}=\rho_{0}$. If $0<\lambda<\lambda^{*}(\alpha)$, using the lemma again, we infer that $g(t)=\lambda$ admits exactly two solutions $r_{\lambda}$ and $\rho_{\lambda}$, with $0<r_{\lambda}<\rho_{\lambda^{*}(\alpha)}<\rho_{\lambda}<r_{0}$. Let us put $u_{\lambda}=\psi_{r_{\lambda}}$ and $v_{\lambda}=\psi_{\rho_{\lambda}}$, $u_{\lambda}(0) \neq v_{\lambda}(0)$. These two functions $u_{\lambda}$ and $v_{\lambda}$ are solutions of the the problem $\left(P_{\lambda}^{\alpha}\right)$, which admits only two ones (cf. [10]).

As $\phi$ is decreasing on $\left[0, r_{0}\right]$, one can verify that $u_{\lambda}(0)<v_{\lambda}(0)$, so we infer that the minimal solution (cf.[2]) is $u_{\lambda}$.

As $\lambda=r_{\lambda}^{2} \phi^{\alpha-1}\left(r_{\lambda}\right)=\rho_{\lambda}^{2} \phi^{\alpha-1}\left(\rho_{\lambda}\right), 0<r_{\lambda}<\rho_{\lambda^{*}(\alpha)}<\rho_{\lambda}<r_{0}$, we get $\lim _{\lambda \rightarrow 0} r_{\lambda}=0, \lim _{\lambda \rightarrow 0} \rho_{\lambda}=r_{0}, \lim _{\lambda \rightarrow 0} u_{\lambda}(r)=\lim _{r_{\lambda} \rightarrow 0} \frac{\phi\left(r_{\lambda} r\right)}{\phi\left(r_{\lambda}\right)}-1=$ 0 , and $\lim _{\lambda \rightarrow 0} v_{\lambda}(r)=\lim _{\rho_{\lambda} \rightarrow r_{0}} \frac{\phi\left(\rho_{\lambda} r\right)-\phi\left(\rho_{\lambda}\right)}{\phi\left(\rho_{\lambda}\right)}=\phi\left(r_{0} r\right)\left(\lim _{\rho_{\lambda} \rightarrow r_{0}} \frac{1}{\phi\left(\rho_{\lambda}\right)}\right)=$ $\infty, \forall r \in[0,1[$.

## 4. The Critical Sobolev Exponent Case

In this section, we suppose that $\alpha=\frac{n+2}{n-2}$ and $n \geqslant 3$.
Let us consider the following problem

$$
\left(P^{\alpha}\right)\left\{\begin{array}{l}
\Delta u+u^{\alpha}=0, \text { in } \mathbb{R}^{n} \\
u>0, \text { in } \mathbb{R}^{n} .
\end{array}\right.
$$

Remark 4.1. - Every radially symmetrical solution of $\left(P^{\alpha}\right)$ verifies $\lim _{r \rightarrow \infty} u(r)=0$ (cf. [9]).

Following the method of Pohozaev in [14], the problem

$$
\left(Q^{\alpha}\right)\left\{\begin{array}{l}
u "(r)+\frac{n-1}{r} u^{\prime}(r)+u^{\alpha}(r)=0, \forall r>0 \\
u>0, u(0)=1, u^{\prime}(0)=0
\end{array}\right.
$$

admits a solution $\phi$.

Lemma 4.2. - Let $u$ be a radially symmetrical regular solution of $\left(P^{\alpha}\right)$, then

$$
u(r)=u(0) \phi\left(u(0)^{\frac{\alpha-1}{2}} r\right)
$$

Proof. - This proof is immediate.
Lemma 4.3. - Let us put $g(r)=r^{2} \phi^{\alpha-1}(r), r \in \mathbb{R}_{+}$, then there exists $r_{0}>0$, such that $g$ is increasing on $\left[0, r_{0}\right]$, decreasing on $\left[r_{0}, \infty[\right.$, with $\lim _{r \rightarrow \infty} g(r)=0$.

Proof. - As we have already mentioned, $g$ is increasing near 0. Let us assume that $g$ is nondecreasing on $[0, \infty[$, then we have two possibilities

$$
\lim _{r \rightarrow \infty} g(r)=\infty \text { or } \lim _{r \rightarrow \infty} g(r)=c, 0<c<\infty
$$

For every $\rho>0, \psi_{\rho}$ is a solution of $\left(P_{\gamma}^{\alpha}\right)$, with $\gamma=\rho^{2} \phi^{\alpha-1}(\rho)=g(\rho)$. We infer (cf. [2], [10]) that $g(r) \leqslant \lambda^{*}(\alpha), \quad \forall r>0$, so the first limit becomes impossible.

In the second case, we have two subcases: $c$ is achieved or not.
If $c$ is not achieved, then $\forall l$ such that $0<l<c$, there exists $r_{l}>0$ such that $g\left(r_{l}\right)=l$. One can verify that $\forall 0<l<c$, the problem $\left(P_{l}^{\alpha}\right)$ admits the solution $\psi_{r_{l}}$, so we infer that $c \leqslant \lambda^{*}(\alpha)$. Let $u$ be a radially symmetrical solution (cf. [2], [10] and [3]) of $\left(P_{c}^{\alpha}\right)$. As in the proof of Proposition 2, one can verify that

$$
u=\psi_{\rho}, \rho=\sqrt{c}(1+u(0))^{\frac{\alpha-1}{2}} \text { and } \frac{1}{1+u(0)}=\phi(\rho) .
$$

As $c=\rho^{2} \phi^{\alpha-1}(\rho)=g(\rho)$, we get a contradiction.
Let us suppose that $c$ is achieved, as $g$ is assumed to be nondecreasing, there exists $r_{0}$ such that $g(r)=c, \forall r \geqslant r_{0}$. Let us choose, an arbitrary constant $\rho>0$ such that $\rho \geqslant r_{0}$. The function $\psi_{\rho}$ is a solution of the problem $\left(P_{\gamma}^{\alpha}\right)$, where $\gamma=\rho^{2} \phi^{\alpha-1}(\rho)=g(\rho)=c, \forall \rho \geqslant r_{0}$. This means that this problem, with such a $\gamma$, admits an infinity of solutions $\psi_{\rho}$; this leads to a contradiction (cf. [2], [10]). So $g$ is not nondecreasing on [0, $\infty[$. As $g$ can't be constant on a nontrivial interval, we deduce that there exists positive constants $r_{1}$ and $r_{2}$, such that $r_{1}<r_{2}$, with $g$ is increasing on [ $0, r_{1}$ ] and decreasing on a maximal interval $\left[r_{1}, r_{2}\right.$. Let us suppose that $g$ increases again on $\left[r_{2}, r_{3}\right]$, with $r_{2}<r_{3}$. If $\left.\gamma \in\right] g\left(r_{2}\right), \min \left(g\left(r_{1}\right), g\left(r_{3}\right)\right)$ [, then $g(r)=\gamma$ admits, at least, three roots, so the problem $\left(P_{\gamma}^{\alpha}\right)$ admits, at least, three solutions; this gives again a contradiction (cf. [10]).

Finally, we get the existence of $r_{0}>0$, such that $g$ is increasing on [ $0, r_{0}$ ] and decreasing on $\left[r_{0}, \infty\left[\right.\right.$. As $g>0$, we infer that $\lim _{r \rightarrow \infty} g(r)=c_{0} \geqslant 0$. If $c_{0}>0$, then for every $\left.c \in\right] 0, c_{0}\left[\right.$, there exists a unique $\rho_{c} \in \mathbb{R}_{+}$, verifying $g\left(\rho_{c}\right)=c$. As $c<\lambda^{*}(\alpha)$, the problem $\left(P_{c}^{\alpha}\right)$ admits exactly two solutions (cf. [10]). One of these two solutions is $\psi_{\rho_{c}}$. Let $u_{c}$ be the other one, then, using Proposition 2 again, we get

$$
u_{c}(r)=\psi_{\gamma}, \gamma=c^{\frac{1}{2}}\left(1+u_{c}(0)\right)^{\frac{\alpha-1}{2}}=c^{\frac{1}{2}} \phi^{\frac{1-\alpha}{2}}\left(c^{\frac{1}{2}}\left(1+u_{c}(0)\right)^{\frac{\alpha-1}{2}}\right)
$$

So we infer that $c=g(\gamma)$. As the two solutions are different, $\rho_{c} \neq \gamma$ and $\gamma$ is another root of $g(r)=c$. This gives a contradiction and proves that necessarily $c=0$. This ends the proof of the lemma.

Proposition 4.4. - Let us assume $\alpha=\frac{n+2}{n-2}, n \geqslant 3$, then

$$
\lambda^{*}(\alpha)=\max _{r \in] 0, \infty[ } g(r)
$$

Proof. - Let $\gamma=g(\rho)=\rho^{2} \phi^{\alpha-1}(\rho), \rho \in \mathbb{R}_{+}^{*}$, we have seen that $\psi_{\rho}$ is a solution of $\left(P_{\gamma}^{\alpha}\right)$. So we infer that $g(\rho) \leqslant \lambda^{*}(\alpha), \forall \rho \in \mathbb{R}_{+}$.

Let us suppose that

$$
\max _{r \in] 0, \infty[ } g(r)<\lambda^{*}(\alpha)
$$

and let $u$ be the unique solution (cf. [10]) of $\left(P_{\lambda^{*}(\alpha)}^{\alpha}\right)$. As in the proof of Proposition 2, we get that $u=\psi_{\rho}$ and $\lambda^{*}(\alpha)=g(\rho)$. This gives a contradiction.

Proposition 4.5. - We have $\lambda^{*}(\alpha)=\frac{n(n-2)}{4}$. There exists a unique $r_{\lambda^{*}(\alpha)}=\sqrt{n(n-2)}$, such that $\lambda^{*}(\alpha)=r_{\lambda^{*}(\alpha)}^{2} \phi^{\alpha-1}\left(r_{\lambda^{*}(\alpha)}\right)$ and a unique solution of $\left(P_{\lambda^{*}(\alpha)}^{\alpha}\right)$

$$
u_{\lambda^{*}(\alpha)}=\psi_{r_{\lambda^{*}(\alpha)}} .
$$

If $0<\lambda<\lambda^{*}(\alpha)$, there exist exactly two constants
$r_{\lambda}=\frac{\sqrt{1-\frac{2 \lambda}{n(n-2)}-\sqrt{1-\frac{4 \lambda}{n(n-2)}}}}{(n(n-2))^{-1} \sqrt{2 \lambda}}$ and $\rho_{\lambda}=\frac{\sqrt{1-\frac{2 \lambda}{n(n-2)}+\sqrt{1-\frac{4 \lambda}{n(n-2)}}}}{(n(n-2))^{-1} \sqrt{2 \lambda}}$
such that $0<r_{\lambda}<r_{\lambda^{*}(\alpha)}<\rho_{\lambda}, \lambda=g\left(r_{\lambda}\right)=g\left(\rho_{\lambda}\right)$ and the only two solutions of $\left(P_{\lambda}^{\alpha}\right)$ are

$$
u_{\lambda}=\psi_{r_{\lambda}} \text { and } v_{\lambda}=\psi_{\rho_{\lambda}},
$$

the minimal one (cf. [2]) is $u_{\lambda} ; \lim _{\lambda \rightarrow 0} u_{\lambda}=0$, in $C^{0}\left(\overline{B_{1}}\right)$ and $\left.\left.\lim _{\lambda \rightarrow 0} v_{\lambda}(r)=r^{2-n}-1, \forall r \in\right] 0,1\right]$.

Proof. - One can use Lemma 3 to get the existence (and the uniqueness) of $r_{\lambda^{*}(\alpha)}=r_{0}, r_{\lambda}$ and $\rho_{\lambda}$. It is then easy to verify that $\psi_{r_{\lambda^{*}(\alpha)}}$ is a solution of $\left(P_{\lambda^{*}(\alpha)}^{\alpha}\right), u_{\lambda}=\psi_{r_{\lambda}}$ and $v_{\lambda}=\psi_{\rho_{\lambda}}$ are solutions of $\left(P_{\lambda}^{\alpha}\right)$. The problem $\left(P_{\lambda}^{\alpha}\right)$ admits only two solutions (cf. [10]), as $\phi$ is decreasing on $\mathbb{R}_{+}^{*}$, one can verify that $u_{\lambda}(0)<v_{\lambda}(0)$, so $u_{\lambda} \neq v_{\lambda}$. We conclude that $u_{\lambda}$ and $v_{\lambda}$ are the only solutions of $\left(P_{\lambda}^{\alpha}\right)$ and the minimal one (cf. [2]) is $u_{\lambda}$.

Let us compute the constants $r_{\lambda^{*}(\alpha)}, r_{\lambda}$ and $\rho_{\lambda}$.
It is well known (cf. [13]) that, if $\alpha=\frac{n+2}{n-2}$, the problem $\left(Q^{\alpha}\right)$ admits the continuum of spherically symmetrical "instantons"

$$
u_{\gamma}(r)=\gamma^{\frac{n-2}{2}}(n(n-2))^{\frac{n-2}{4}}\left(\gamma^{2}+r^{2}\right)^{\frac{2-n}{2}}, \gamma>0
$$

Let us fix $\gamma>0$, so $u_{\gamma}(0)=\gamma^{\frac{2-n}{2}}(n(n-2))^{\frac{n-2}{4}}$. Using Lemma 2, we get the expression of the Lane-Emden function

$$
\phi(r)=\frac{1}{u_{\gamma}(0)} u_{\gamma}\left(u_{\gamma}(0)^{\frac{-2}{n-2}} r\right)=\left(1+\frac{r^{2}}{n(n-2)}\right)^{\frac{2-n}{2}} .
$$

As $\alpha-1=\frac{n+2}{n-2}-1=\frac{4}{n-2}$, we infer that

$$
g(r)=r^{2} \phi^{\alpha-1}(r)=r^{2}\left(1+\frac{r^{2}}{n(n-2)}\right)^{-2}
$$

Using Proposition 4, a direct calculation gives

$$
\begin{gathered}
\lambda^{*}(\alpha)=\max _{r>0} r^{2}\left(1+\frac{r^{2}}{n(n-2)}\right)^{-2} \\
=r^{2}\left(1+\frac{r^{2}}{n(n-2)}\right)_{\left.\right|_{r=r_{\lambda}(\alpha)}=\sqrt{n(n-2)}}^{-2}=\frac{n(n-2)}{4} .
\end{gathered}
$$

In [7], the previous constant has been computed, using the Pohozaev Identity. If $0<\lambda<\lambda^{*}(\alpha)$, the equation $g(r)=\lambda$ admits two positive roots

$$
r_{\lambda}=\frac{\sqrt{1-\frac{2 \lambda}{n(n-2)}-\sqrt{1-\frac{4 \lambda}{n(n-2)}}}}{(n(n-2))^{-1} \sqrt{2 \lambda}} \text { and } \rho_{\lambda}=\frac{\sqrt{1-\frac{2 \lambda}{n(n-2)}+\sqrt{1-\frac{4 \lambda}{n(n-2)}}}}{(n(n-2))^{-1} \sqrt{2 \lambda}} .
$$

This gives us $u_{\lambda}=\psi_{r_{\lambda}}$ and $v_{\lambda}=\psi_{\rho_{\lambda}}$; as $r_{\lambda}<\rho_{\lambda}$, we get $u_{\lambda}(0)<v_{\lambda}(0)$, so $u_{\lambda}$ is the minimal solution.

As $\lambda=r_{\lambda}^{2} \phi^{\alpha-1}\left(r_{\lambda}\right)=\rho_{\lambda}^{2} \phi^{\alpha-1}\left(\rho_{\lambda}\right), 0<r_{\lambda}<r_{\lambda^{*}(\alpha)}<\rho_{\lambda}<\infty$, one can verify that $\lim _{\lambda \rightarrow 0} r_{\lambda}=0, \lim _{\lambda \rightarrow 0} \rho_{\lambda}=\infty, \lim _{\lambda \rightarrow 0} u_{\lambda}=0$, in $C^{0}\left(\overline{B_{1}}\right)$ and $\left.\left.\lim _{\lambda \rightarrow 0} v_{\lambda}(0)=\lim _{\rho_{\lambda} \rightarrow \infty} \frac{\phi\left(\rho_{\lambda} r\right)}{\phi\left(\rho_{\lambda}\right)}-1=r^{2-n}-1, \forall r \in\right] 0,1\right]$.

## 5. The Supercritical Case

We consider here the case $\alpha>\frac{n+2}{n-2}, n \geqslant 3$. Let us put

$$
f(\alpha)=\frac{4 \alpha}{\alpha-1}+4 \sqrt{\frac{\alpha}{\alpha-1}}, \forall \alpha>1 .
$$

Let's first detail a condition, $f(\alpha)>n-2$, used in [10].
Lemma 5.1. - If $\left(3 \leqslant n \leqslant 10\right.$ and $\left.\alpha>\frac{n+2}{n-2}\right)$
or $\left(n>10\right.$ and $\left.\frac{n+2}{n-2}<\alpha<\frac{n-2 \sqrt{n-1}}{n-2 \sqrt{n-1}-4}\right)$,
then $f(\alpha)>n-2$. If $n>10$ and $\frac{n-2 \sqrt{n-1}}{n-2 \sqrt{n-1}-4} \leqslant \alpha$, then $f(\alpha) \leqslant n-2$.
Proof. - Let us put $p(t)=4 t^{2}+4 t$ and $u=\sqrt{\frac{\alpha}{\alpha-1}}$, so we get $f(\alpha)=$ $p(u)$. The only positive root of $p(t)=n-2$, is $t_{0}=\frac{\sqrt{n-1}-1}{2}$ and the equation $u=\frac{\sqrt{n-1}-1}{2}$ has the only solution $\alpha_{0}=\frac{n-2 \sqrt{n-1}}{n-2 \sqrt{n-1}-4}$. But $\alpha_{0}>0$, if and only if $n>10$.

For every $\alpha>\frac{n+2}{n-2}$, we have $\alpha>1$ so we get $\sqrt{\frac{\alpha}{\alpha-1}}>1>\frac{\sqrt{n-1}-1}{2}$, if $3 \leqslant n \leqslant 10$. We infer that $f(\alpha)>n-2$, if $3 \leqslant n \leqslant 10$.

If $n>10$, we have $\alpha_{0}>\frac{n+2}{n-2}>1$, one can verify that if $\frac{n+2}{n-2}<\alpha<\alpha_{0}$, then $f(\alpha)>n-2$ and $f(\alpha) \leqslant n-2$, if $\alpha \geqslant \alpha_{0}$.

Proposition 5.2. - Let us put $\lambda_{s}=\frac{2}{(\alpha-1)^{2}}(\alpha(n-2)-n)$. If $\left(3 \leqslant n \leqslant 10\right.$ and $\left.\frac{n+2}{n-2}<\alpha\right)$ or $\left(n>10\right.$ and $\left.\frac{n+2}{n-2}<\alpha<\frac{n-2 \sqrt{n-1}}{n-2 \sqrt{n-1}-4}\right)$ then

$$
\lambda^{*}(\alpha)=\max _{\mathbb{R}_{+}^{*}} g(r), \lambda^{*}(\alpha)>\lambda_{s} \text { and } \phi(r) \sim \lambda_{s}^{\frac{1}{\alpha-1}} r^{\frac{2}{1-\alpha}} \text {, as } r \rightarrow \infty
$$

If $\left(\rho_{i}\right)$ is an increasing sequence of positive reals, such that $\left(\psi_{\rho_{i}}\right)$ are solutions of $\left(P_{\lambda_{s}}^{\alpha}\right)$ and $\lim _{i \rightarrow \infty} \rho_{i}=\infty$, then $\lim _{i \rightarrow \infty} \psi_{\rho_{i}}(r)=\lambda_{s}^{\frac{1}{\alpha-1}}\left(r^{\frac{2}{1-\alpha}}-\right.$ 1), $\forall r \in] 0,1]$.

If $n>10$ and $\frac{n-2 \sqrt{n-1}}{n-2 \sqrt{n-1}-4} \leqslant \alpha$ then

$$
\lambda^{*}(\alpha)=\sup _{\mathbb{R}_{+}^{*}} g(r)=\lambda_{s} \text { and } \phi(r) \sim \lambda_{s}^{\frac{1}{\alpha-1}} r^{\frac{2}{1-\alpha}} \text {, as } r \rightarrow \infty \text {. }
$$

If $\left(\lambda_{i}\right)$ is an increasing positive sequence such that $\lim _{i \rightarrow \infty} \lambda_{i}=\lambda_{s}$ and $\forall i, w_{i}$ is the unique solution of $\left(P_{\lambda_{i}}^{\alpha}\right)$, then
$\left.\left.\lim _{i \rightarrow \infty} w_{i}(r)=\lambda_{s}^{\frac{1}{\alpha-1}}\left(r^{\frac{2}{1-\alpha}}-1\right), \forall r \in\right] 0,1\right]$.
Proof. - As in the proof of Proposition 4, one can verify that $\lambda^{*}(\alpha)=$ $\sup _{\mathbb{R}_{+}^{*}} g(r)$, where $g(r)=r^{2} \phi^{\alpha-1}(r)$.

If $\left(3 \leqslant n \leqslant 10\right.$ and $\left.\frac{n+2}{n-2}<\alpha\right)$ or $\left(n>10\right.$ and $\left.\frac{n+2}{n-2}<\alpha<\frac{n-2 \sqrt{n-1}}{n-2 \sqrt{n-1}-4}\right)$, using Lemma 4 , we get $f(\alpha)>n-2$. So we can use Theorem 1 in [10] to infer that $\lambda^{*}(\alpha)>\lambda_{s},\left(P_{\lambda^{*}(\alpha)}^{\alpha}\right)$ admits a unique solution and $\left(P_{\lambda_{s}}^{\alpha}\right)$ admits an infinity of solutions. Using the unique solution $u_{\lambda^{*}(\alpha)}$ of $\left(P_{\lambda^{*}(\alpha)}^{\alpha}\right)$, one can deduce from Proposition 1 that $u_{\lambda^{*}(\alpha)}=\psi_{\rho}$, where $\rho \in \mathbb{R}_{+}^{*}$ and $g(\rho)=$ $\lambda^{*}(\alpha)$. We conclude that the supremum is achieved and $\lambda^{*}(\alpha)=\max _{\mathbb{R}_{+}^{*}} g(r)$.

Let us suppose that

$$
a=\liminf _{r \rightarrow \infty} g(r)<A=\limsup _{r \rightarrow \infty} g(r)
$$

For every $\lambda \in] a, A\left[\right.$, the equation $g(r)=\lambda$ admits a sequence of roots $\left(r_{i}\right)$, with $\lim _{i \rightarrow \infty} r_{i}=\infty$. As for every $i, \psi_{r_{i}}$ is a solution of $\left(P_{\lambda}^{\alpha}\right)$, we get an infinity of solutions for this problem; but an infinity of solutions exists only when $\lambda=\lambda_{s}$ (cf. [10]). We get a contradiction and infer that

$$
a=A=\lambda_{s}=\lim _{r \rightarrow \infty} g(r), \text { so } \phi(r) \sim \lambda_{s}^{\frac{1}{\alpha-1}} r^{\frac{2}{1-\alpha}} \text {, as } r \rightarrow \infty
$$

If $\left(\rho_{i}\right)$ is an increasing sequence of positive constants, such that $\left(\psi_{\rho_{i}}\right)$ are solutions of $\left(P_{\lambda_{s}}^{\alpha}\right)$ and $\lim _{i \rightarrow \infty} \rho_{i}=\infty$, then one can use the previous asymptotic behavior of $\phi$ to get $\left.\left.\lim _{i \rightarrow \infty} \psi_{\rho_{i}}(r)=\lambda_{s}^{\frac{1}{\alpha-1}}\left(r^{\frac{2}{1-\alpha}}-1\right), \forall r \in\right] 0,1\right]$.

If $n>10$ and $\frac{n-2 \sqrt{n-1}}{n-2 \sqrt{n-1}-4} \leqslant \alpha$, we get from Lemma 4 that $f(\alpha) \leqslant n-2$. Using [10] again, we infer that $\lambda^{*}(\alpha)=\lambda_{s},\left(P_{\lambda}^{\alpha}\right)$ admits a unique solution for every $\lambda \in] 0, \lambda^{*}(\alpha)[$. As the function $g$ is increasing near $r=0$, we infer that $g$ is increasing on $\mathbb{R}_{+}^{*}$. For, on one hand, if $g$ decreases on a nontrivial open interval $I \subset \mathbb{R}_{+}^{*}$, then the equation $g(r)=\lambda$ admits at least two roots $r_{1}<r_{2}$, if $\left.\lambda \in\right] \min _{I} g(r), \max _{I} g(r)\left[\right.$. As $\psi_{r_{1}}$ and $\psi_{r_{2}}$ are solutions of $\left(P_{\lambda}^{\alpha}\right)$,
with $\psi_{r_{1}}(0) \neq \psi_{r_{2}}(0)$, this violates the uniqueness result of [10]. On another hand, the function $g$ can't be constant on a nontrivial interval, otherwise we get an infinity of solutions for some $\lambda$. One can then see that

$$
\lim _{r \rightarrow \infty} g(r)=\sup _{\mathbb{R}_{+}^{*}} g(r)=\lambda^{*}(\alpha) ; \lambda^{*}(\alpha)=\lambda_{s}(c f .[10])
$$

So $\phi(r) \sim \lambda_{s}^{\frac{1}{\alpha-1}} r^{\frac{2}{1-\alpha}}$, as $r \rightarrow \infty$.
Using this asymptotic behavior, one can show the last statement of the proposition.

Let us put

$$
\left(Q_{\lambda}^{\alpha}\right)\left\{\begin{array}{l}
\Delta u+\lambda(1+u)^{\alpha}=0, \text { in } B_{r_{0}} \\
u>0, \text { in } B_{r_{0}} \\
u=0, \text { on } \partial B_{r_{0}}
\end{array}\right.
$$

where $B_{r_{0}}=\left\{x \in \mathbb{R}^{n},\|x\|<r_{0}\right\}$. For every solution $u$ of $\left(Q_{\lambda}^{\alpha}\right)$, we put $v(r)=u\left(r_{0} r\right)$ for every $r \in[0,1]$. Let $\lambda_{r_{0}}^{*}(\alpha)$, be the maximal eigenvalue of $\left(Q_{\lambda}^{\alpha}\right)$.

Lemma 5.3. - A function $u$ is a solution of $\left(Q_{\lambda}^{\alpha}\right)$, if and only if $v$ is a solution of $\left(P_{r_{0}^{2} \lambda}^{\alpha}\right)$. In particular, we get $\lambda_{r_{0}}^{*}(\alpha)=r_{0}^{2} \lambda^{*}(\alpha)$.

Proof. - The proof is easy.
Remark 5.4. - According to the previous lemma, the results obtained here for $\left(P_{\lambda}^{\alpha}\right)$ (on the unit ball $B_{1}$ ), can be easily stated for $\left(Q_{\lambda}^{\alpha}\right)$ (on any ball $B_{r_{0}}$ ).

## 6. Appendix

Let $S_{k}^{i}$ be the set of all the $(k-i)$-selections of $\{1, \ldots, i\}$ and $s(j)$ the multiplicity of the element $j, 1 \leqslant j \leqslant i$. If $u$ is a analytical solution of $\left(P_{\lambda}^{\alpha}\right)$, with $u(r)=\sum_{k=0}^{\infty} a_{k} r^{k}$ near $r=0, r_{0}$ the convergence radius of this series, then

Proposition 6.1. -

$$
\begin{gathered}
\forall k \geqslant 0, a_{2 k+1}=0, \quad a_{2}=\frac{\lambda}{n-2}\left(1+a_{0}\right)^{\alpha}\left(\frac{1}{n}-\frac{1}{2}\right) \\
\quad \text { and } \forall k>1, \quad a_{2 k}=\frac{\lambda}{n-2}\left(\frac{1}{2 k+n-2}-\frac{1}{2 k}\right) \times \\
\sum_{i=1}^{k-1}\left(1+a_{0}\right)^{\alpha-i} \frac{1}{i!} \Pi_{p=0}^{i-1}(\alpha-p) \Sigma_{s \in S_{k-1}^{i}} \Pi_{j=1}^{i} a_{2(1+s(j))} .
\end{gathered}
$$

Proof. - Let us choose $0<r \leqslant \rho<r_{0}$, by standard integrations, we get

$$
\begin{gathered}
u(r)-u(\rho)=\frac{\lambda}{n-2} \times \\
\left(\left(r^{2-n}-\rho^{2-n}\right) \int_{0}^{r} t^{n-1}(1+u(t))^{\alpha} d t+\int_{r}^{\rho}\left(t-\rho^{2-n} t^{n-1}\right)(1+u(t))^{\alpha} d t\right) .
\end{gathered}
$$

Let us point out that

$$
\begin{gathered}
(1+u(r))^{\alpha}=(1+u(0)-u(0)+u(r))^{\alpha} \\
=(1+u(0))^{\alpha}\left(1+\frac{u(r)-u(0)}{1+u(0)}\right)^{\alpha}=\left(1+a_{0}\right)^{\alpha}\left(1+\Sigma_{i=1}^{\infty} \frac{a_{i}}{1+a_{0}} r^{i}\right)^{\alpha}, u(0)=a_{0}
\end{gathered}
$$

By the Maximum Principle, we have $\forall r \in] 0,1[, 0<u(r)<u(0)$, so we get

$$
\left|\frac{u(0)-u(r)}{1+u(0)}\right|<1, \forall r \in[0,1]
$$

we infer that

$$
(1+u(r))^{\alpha}=\left(1+a_{0}\right)^{\alpha}\left(1+\Sigma_{j=1}^{\infty} \frac{\alpha(\alpha-1) \ldots(\alpha-j+1)}{j!}\left(\Sigma_{i=1}^{\infty} \frac{a_{i}}{1+a_{0}} r^{i}\right)^{j}\right) .
$$

All these series are uniformly convergent on $[0, \rho]$. If we put $(1+u(r))^{\alpha}=\Sigma_{j=0}^{\infty} c_{j} r^{j}$, we get

$$
\begin{aligned}
u(r)= & \frac{\lambda}{n-2}\left(\left(r^{2-n}-\rho^{2-n}\right) \int_{0}^{r} t^{n-1} \Sigma_{j=0}^{\infty} c_{j} t^{j} d t+\int_{r}^{\rho}\left(t-\rho^{2-n} t^{n-1}\right) \Sigma_{j=0}^{\infty} c_{j} t^{j} d t\right) \\
= & \frac{\lambda}{n-2}\left(\Sigma_{j=0}^{\infty} c_{j} \frac{r^{2+j}}{j+n}-\Sigma_{j=0}^{\infty} c_{j} \frac{\rho^{2-n} r^{j+n}}{j+n}+\Sigma_{j=0}^{\infty} c_{j} \frac{\rho^{j+2}}{j+2}-\Sigma_{j=0}^{\infty} c_{j} \frac{\rho^{j+2}}{j+n}\right) \\
& +\frac{\lambda}{n-2}\left(-\Sigma_{j=0}^{\infty} c_{j} \frac{r^{j+2}}{j+2}+\Sigma_{j=0}^{\infty} c_{j} \frac{\rho^{2-n} r^{j+n}}{j+n}\right) \\
= & \frac{\lambda}{n-2}\left(\Sigma_{j=2}^{\infty} c_{j-2} \frac{r^{j}}{j+n-2}+\Sigma_{j=0}^{\infty} c_{j} \frac{\rho^{j+2}}{j+2}-\Sigma_{j=0}^{\infty} c_{j} \frac{\rho^{j+2}}{j+n}-\Sigma_{j=2}^{\infty} c_{j-2} \frac{r^{j}}{j}\right) .
\end{aligned}
$$

We finally obtain
(2) $u(r)=\frac{\lambda}{n-2}\left(\Sigma_{j=2}^{\infty} c_{j-2}\left(\frac{1}{j+n-2}-\frac{1}{j}\right) r^{j}+\Sigma_{j=0}^{\infty} c_{j} \rho^{j+2}\left(\frac{1}{j+2}-\frac{1}{j+n}\right)\right)$.

Using the previous identity, we obtain

$$
a_{1}=0, \quad \forall k>1, a_{k}=\frac{\lambda}{n-2}\left(\frac{1}{k+n-2}-\frac{1}{k}\right) c_{k-2}
$$

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Using (1), we get

$$
c_{0}=\left(1+a_{0}\right)^{\alpha}, c_{1}=\alpha\left(1+a_{0}\right)^{\alpha-1} a_{1}=0
$$

and

$$
\begin{aligned}
\forall k>1, & c_{k}=\left(1+a_{0}\right)^{\alpha} \Sigma_{j=1}^{k} \frac{1}{j!} \Pi_{p=0}^{j-1}(\alpha-p) \frac{1}{\left(1+a_{0}\right)^{j}} \Sigma_{s \in S_{k}^{j}} \Pi_{i=1}^{j} a_{1+s(i)} \\
& =\Sigma_{j=1}^{k} \frac{1}{j!} \Pi_{p=0}^{j-1}(\alpha-p)\left(1+a_{0}\right)^{\alpha-j} \Sigma_{s \in S_{k}^{j}} \Pi_{i=1}^{j} a_{1+s(i)}
\end{aligned}
$$

Using the previous relation and the fact that $a_{1}=0$, one can verify (by induction) that $a_{2 k+1}=0, \forall k>0$. We then obtain from (2) and the expression of $c_{k}$

$$
\begin{gathered}
a_{2 k}=\frac{\lambda}{n-2}\left(\frac{1}{2 k+n-2}-\frac{1}{2 k}\right) c_{2 k-2} \\
=\frac{\lambda}{n-2}\left(\frac{1}{2 k+n-2}-\frac{1}{2 k}\right) \Sigma_{j=1}^{k-1} \frac{1}{j!} \Pi_{p=0}^{j-1}(\alpha-p)\left(1+a_{0}\right)^{\alpha-j} \Sigma_{s \in S_{k-1}^{j}} \Pi_{i=1}^{j} a_{2(1+s(i))} . \\
\forall j \in[1, k-1], \operatorname{Card}\left(S_{k-1}^{j}\right)=C_{k-2}^{j-1} .
\end{gathered}
$$

Let us put

$$
\begin{gathered}
d_{2}=\frac{1}{2 n} \text { and } \forall k>1, \\
d_{2 k}=\frac{1}{(2 k+n-2)(2 k)} \Sigma_{i=1}^{k-1} \frac{1}{i!} \Pi_{p=0}^{i-1}(\alpha-p) \Sigma_{s \in S_{k-1}^{i}} \Pi_{j=1}^{i} d_{2(1+s(j))},
\end{gathered}
$$

then
LEMMA 6.2. $-a_{2 k}=(-1)^{k} \lambda^{k}\left(1+a_{0}\right)^{k(\alpha-1)+1} d_{2 k}, \quad \forall k>1$.
Proof. -

$$
\begin{gathered}
a_{4}=\frac{\alpha \lambda^{2}}{(n-2)^{2}}=\left(1+a_{0}\right)^{2 \alpha-1}\left(\frac{1}{n+2}-\frac{1}{4}\right)\left(\frac{1}{n}-\frac{1}{2}\right) \\
=\lambda^{2}\left(1+a_{0}\right)^{2 \alpha-1} \frac{1}{4(n+2)} \frac{\alpha}{2 n}=\lambda^{2}\left(1+a_{0}\right)^{2(\alpha-1)+1} \frac{1}{4(n+2)} \frac{\alpha}{2 n} . \\
d_{4}=\frac{1}{4(n+2)} \Sigma_{i=1}^{1} \frac{1}{i!} \Pi_{p=0}^{i-1}(\alpha-p) \Sigma_{s \in S_{1}^{i}} \Pi_{j=1}^{i} d_{2(1+s(j))} \\
=\frac{\alpha}{4(n+2)} d_{2}=\frac{1}{4(n+2)} \frac{\alpha}{2 n}
\end{gathered}
$$

so we infer that the formula is true for $k=2$. Let us suppose it true for every $j$, such that $2 \leqslant j \leqslant k$. From Proposition 7, we have

$$
\begin{aligned}
& =\frac{\lambda}{n-2}\left(\frac{1}{2 k+n}-\frac{1}{2(k+1)}\right) \Sigma_{j=1}^{k} \frac{1}{j!} \Pi_{p=0}^{j-1}(\alpha-p)\left(1+a_{0}\right)^{\alpha-j} \Sigma_{s \in S_{k}^{j}} \Pi_{i=1}^{j} a_{2(1+s(i))} \\
& =\frac{-\lambda}{(2(k+1)+n-2)(2(k+1))} \Sigma_{j=1}^{k} \frac{1}{j!} \Pi_{p=0}^{j-1}(\alpha-p)\left(1+a_{0}\right)^{\alpha-j} \Sigma_{s \in S_{k}^{j}} \Pi_{i=1}^{j} a_{2(1+s(i))} . \\
& \quad \forall j \in[1, k], \forall s \in S_{k}^{j}, \text { if } i \in[1, j], \text { then } 1 \leqslant 1+s(i) \leqslant k
\end{aligned}
$$

so one can use the hypothesis to get $\forall i \in[1, j]$,

$$
a_{2(1+s(i))}=(-1)^{1+s(i)} \lambda^{1+s(i)}\left(1+a_{0}\right)^{(s(i)+1)(\alpha-1)+1} d_{2(1+s(i))}
$$

We then obtain

$$
\begin{gathered}
\Pi_{i=1}^{j} a_{2(1+s(i))} \\
=(-1)^{\Sigma_{i=1}^{j}(1+s(i))} \lambda^{\Sigma_{i=1}^{j}(1+s(i))}\left(1+a_{0}\right)^{\Sigma_{i=1}^{j}\{(\alpha-1)(s(i)+1)+1\}} \Pi_{i=1}^{j} d_{2(1+s(i))} \\
=(-1)^{j+\Sigma_{i=1}^{j} s(i)} \lambda^{j+\Sigma_{i=1}^{j} s(i)}\left(1+a_{0}\right)^{\alpha j+(\alpha-1) \Sigma_{i=1}^{j} s(i)} \Pi_{i=1}^{j} d_{2(1+s(i))} .
\end{gathered}
$$

But for every $s \in S_{k}^{j}$, we have $\Sigma_{i=1}^{j} s(i)=k-j$.
We infer that

$$
\begin{gathered}
\Pi_{i=1}^{j} a_{2(1+s(i))}=(-1)^{k} \lambda^{k}\left(1+a_{0}\right)^{\alpha j+(\alpha-1)(k-j)} \Pi_{i=1}^{j} d_{2(1+s(i))} \\
=(-1)^{k} \lambda^{k}\left(1+a_{0}\right)^{(\alpha-1) k+j} \Pi_{i=1}^{j} d_{2(1+s(i))}
\end{gathered}
$$

Substituting in the expression of $a_{2(k+1)}$, we obtain

$$
\begin{aligned}
& a_{2(k+1)}=(-1)^{k+1} \lambda^{k+1}\left(1+a_{0}\right)^{k(\alpha-1)+\alpha} \frac{1}{(2(k+1)+n-2)(2(k+1))} \times \\
& \sum_{j=1}^{k} \frac{1}{j!} \Pi_{p=0}^{j-1}(\alpha-p) \Sigma_{s \in S_{k}^{j}} \Pi_{i=1}^{j} d_{2(1+s(i))} \\
& =(-1)^{k+1} \lambda^{k+1}\left(1+a_{0}\right)^{(k+1)(\alpha-1)+1} \frac{1}{(2(k+1)+n-2)(2(k+1))} \times \\
& \Sigma_{j=1}^{k} \frac{1}{j!} \Pi_{p=0}^{j-1}(\alpha-p) \Sigma_{s \in S_{k}^{j}} \Pi_{i=1}^{j} d_{2(1+s(i))} . \\
& =(-1)^{k+1} \lambda^{k+1}\left(1+a_{0}\right)^{(k+1)(\alpha-1)+1} d_{2(k+1)} .
\end{aligned}
$$

Let us compute the first terms of the Lane-Emden function, $\phi(r)=\sum_{i=0}^{\infty} a_{2 i} r^{2 i}$, near $r=0$, where $a_{0}=1$, and $a_{2 i}=(-1)^{i} 2^{i(\alpha-1)+1} d_{2 i}, \forall i>1$.

$$
\begin{gathered}
d_{0}=1 ; d_{2}=\frac{1}{2 n} ; d_{4}=\frac{1}{4(n+2))} \alpha d_{2}=\frac{\alpha}{(2 n)(4(n+2))} ; \\
d_{6}=\frac{1}{6(n+4)}\left(\alpha d_{4}+\frac{1}{2} \alpha(\alpha-1) d_{2}^{2}\right)=\frac{1}{6(n+4)}\left\{\frac{\alpha^{2}}{(2 n)(4(n+2))}+\frac{\alpha(\alpha-1)}{2(2 n)^{2}}\right\} ; \\
d_{8}=\frac{1}{8(n+6)}\left(\alpha d_{6}+\alpha(\alpha-1) d_{4} d_{2}+\frac{\alpha(\alpha-1)(\alpha-2)}{6} d_{2}^{3}\right) \\
=\frac{1}{8(n+6)}\left\{\frac{\alpha^{3}}{(2 n)(4(n+2))(6(n+4))}+\frac{\alpha^{2}(\alpha-1)}{2(2 n)^{2}(6(n+4))}+\frac{\alpha^{2}(\alpha-1)}{(2 n)^{2}(4(n+2))}\right. \\
\left.+\frac{\alpha(\alpha-1)(\alpha-2)}{6(2 n)^{3}}\right\} ; \\
d_{10}=\frac{1}{10(n+8)}\left\{\alpha d_{8}+\frac{\alpha(\alpha-1)}{2}\left(2 d_{2} d_{6}+d_{4}^{2}\right)+3 \frac{\alpha(\alpha-1)(\alpha-2)}{6} d_{2}^{2} d_{4}+\right. \\
\left.=\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{24} d_{2}^{4}\right\} \\
+\frac{1}{10(n+8)}\left\{\frac{\alpha^{4}}{(2 n)(4(n+2))(6(n+4))(8(n+6))}+\frac{\alpha^{3}(\alpha-1)}{2(2 n)^{2}(6(n+4))(8(n+6))}\right. \\
+\frac{\alpha^{3}(\alpha-1)}{(2 n)^{2}(4(n+2))(8(n+6))}+\frac{\alpha^{2}(\alpha-1)(\alpha-2)}{6(2 n)^{3}(8(n+6))}+\frac{\alpha^{3}(\alpha-1)}{(2 n)^{2}(4(n+2))(6(n+4))} \\
+\frac{\alpha^{2}(\alpha-1)^{2}}{2(2 n)^{3}(6(n+4))}+\frac{\alpha^{3}(\alpha-1)}{2(2 n)^{2}(4(n+2))^{2}}+\frac{\alpha^{2}(\alpha-1)(\alpha-2)}{2(2 n)^{3}(4(n+2))} \\
\left.+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{24(2 n)^{4}}\right\} .
\end{gathered}
$$

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