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A remark on the action of the mapping class group on the unit tangent bundle

J. Souto[1]

Abstract. — We prove that the standard action of the mapping class group $\mathrm{Map}(\Sigma)$ of a surface Σ of sufficiently large genus on the unit tangent bundle $T^1\Sigma$ is not homotopic to any smooth action.

Résumé. — On montre que l'action standard du groupe modulaire $\mathrm{Map}(\Sigma)$ d'une surface Σ de genre assez grand sur le fibré unitaire tangent $T^1\Sigma$ n'est pas homotopique à une action lisse.

From now on let Σ be a closed orientable surface of genus at least 12 and consider its unit tangent bundle $\pi : T^1\Sigma \to \Sigma$. The kernel of the homomorphism $\pi_* : \pi_1(T^1\Sigma) \to \pi_1(\Sigma)$ is characteristic and hence π_* induces a homomorphism

$$\mathrm{Out}(\pi_1(T^1\Sigma)) \to \mathrm{Out}(\pi_1(\Sigma))$$

between the corresponding groups of outer automorphisms. In particular, any continuous action $G \curvearrowright T^1\Sigma$ of a group on the unit tangent bundle induces a homomorphism $G \to \mathrm{Out}(\pi_1(\Sigma))$.

By the Baer-Dehn-Nielsen theorem [5], $\mathrm{Out}(\pi_1(\Sigma))$ is isomorphic to the mapping class group $\mathrm{Map}(\Sigma)$ of Σ, i.e. to the group of isotopy classes of self-diffeomorphisms. In [13], Morita proved that there is no smooth action $\mathrm{Map}(\Sigma) \curvearrowright \Sigma$ inducing the isomorphism $\mathrm{Map}(\Sigma) \simeq \mathrm{Out}(\pi_1(\Sigma))$. This result

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was extended by Markovic [10] who proved that there is also no such action by homeomorphisms.

On the other hand, $\mathrm{Map}(\Sigma)$ acts on the (total space of the) unit tangent bundle $T^1\Sigma$ in such a way that the induced homomorphism

$$\mathrm{Map}(\Sigma) \to \mathrm{Out}(\pi_1(T^1\Sigma)) \to \mathrm{Out}(\pi_1(\Sigma))$$

agrees with the isomorphism $\mathrm{Map}(\Sigma) \simeq \mathrm{Out}(\pi_1(\Sigma))$. This standard action $\mathrm{Map}(\Sigma) \curvearrowright T^1\Sigma$ is only Hölder, but we will deduce below from results due to Deroin-Kleptsyn-Navas [4] and Sullivan [18] that it is conjugated to an action by Lipschitz homeomorphisms. It follows from Theorem 0.1 below that the standard action is not conjugated, and even not homotopic, to a smooth action. Through out the whole paper, smooth means C^∞; only at the end of the paper, we will discuss briefly the degree of smoothness needed in our argument.

Theorem 0.1. — *Suppose that Σ is a closed orientable surface of genus $g \geqslant 12$. Then there is no smooth action of* $\mathrm{Map}(\Sigma)$ *on* $T^1\Sigma$ *which induces the isomorphism* $\mathrm{Map}(\Sigma) \to \mathrm{Out}(\pi_1(\Sigma))$.

Observe that Theorem 0.1 gives, for the surfaces under consideration, a different albeit quite inefficient proof of Morita's non-lifting theorem. We use the adjective *inefficient* because while Morita's original argument [13] applies for finite index subgroups as well, we will need here to work with the full mapping class group. More concretely, orientation reversing elements in the mapping class group will play a central role.

We sketch the proof of Theorem 0.1. Seeking a contradiction, suppose that there is smooth action $\mathrm{Map}(\Sigma) \curvearrowright T^1\Sigma$ inducing the Baer-Dehn-Nielsen isomorphism. We will use a topological trick to show that a certain subgroup G of $\mathrm{Map}(\Sigma)$, isomorphic to the mapping class group of a surface with at least genus 6, stabilizes a smooth circle $\mathbb{S}^1 \subset T^1\Sigma$. It follows from the work of Parwani [15] that the so-obtained action $G \curvearrowright \mathbb{S}^1$ is trivial. A result of Thurston [19] and the fact that $H_1(G;\mathbb{Z}) = 0$ imply that G is in the kernel of the action $\mathrm{Map}(\Sigma) \curvearrowright T^1\Sigma$, contradicting the assumption that this action induces the isomorphism $\mathrm{Map}(\Sigma) \simeq \mathrm{Out}(\pi_1(\Sigma))$.

The proof of Theorem 0.1 is slightly simpler if the genus of the surface is even and through out most of this paper we will assume that this is the case. The modifications needed to prove Theorem 0.1 for surfaces of odd genus will be discussed after the proof in the even genus case has been completed.

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1.

In this section we recall the construction of an action by homeomorphisms $\mathrm{Map}(\Sigma) \curvearrowright T^1\Sigma$ which induces the Baer-Dehn-Nielsen isomorphism $\mathrm{Map}(\Sigma) \simeq \mathrm{Out}(\pi_1(\Sigma))$. Mostly, the material in this section is well-known; we include it here for the sake of completeness. The interested reader can find in Casson-Bleiler [3] all the needed facts on hyperbolic geometry.

Let Σ be a closed hyperbolic surface and identify its universal cover with the hyperbolic plane $\mathbb{H}^2$. Let $\partial\mathbb{H}^2 \simeq \mathbb{S}^1$ be the circle at infinity of $\mathbb{H}^2$ and consider the space of (distinct) triples

$$\{(a_1, a_2, a_3) \in \partial\mathbb{H}^2 \times \partial\mathbb{H}^2 \times \partial\mathbb{H}^2 | a_i \neq a_j \ \forall\ i \neq j\}$$

The group with 2 elements acts on the space of triples via the fixed-point free involution

$$(a_1, a_2, a_3) \mapsto (a_2, a_1, a_3)$$

The quotient Θ_3 of the space of triples under this involution is diffeomorphic to the unit tangent bundle $T^1\mathbb{H}^2$ via the map (figure 1) which associates to (a_1, a_2, a_3) the unique unit tangent vector v normal to the geodesic in $\mathbb{H}^2$ with endpoints a_1, a_2 and pointing to a_3. Here, pointing to a_3 means that $a_3 = \lim_{t\to\infty} \exp(tv)$ where $\exp(\cdot)$ is the geodesic exponential map of $\mathbb{H}^2$.

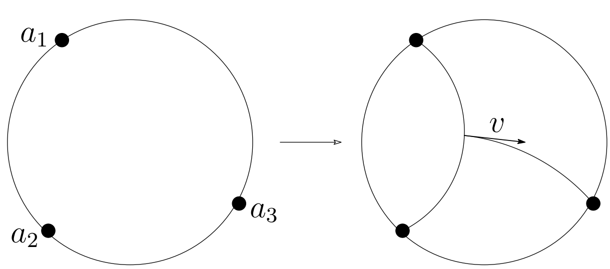

Figure 1. — The diffeomorphism between Θ_3 and $T^1\Sigma$

The action by deck-transformations $\pi_1(\Sigma) \curvearrowright \mathbb{H}^2$ extends to an action on the circle at infinity and hence on an action $\pi_1(\Sigma) \curvearrowright \Theta_3$. At the same time, the action $\pi_1(\Sigma) \curvearrowright \mathbb{H}^2$ induces, via the differential, an action on $T^1\mathbb{H}^2$ in such a way that $T^1\Sigma = \pi_1(\Sigma)\backslash T^1\mathbb{H}^2$. It follows directly from the construction that the diffeomorphism $\Theta_3 \to T^1\mathbb{H}^2$ conjugates both actions of $\pi_1(\Sigma)$. In particular, $T^1\Sigma$ is diffeomorphic to $\pi_1(\Sigma)\backslash\Theta_3$.

Suppose now that $\phi : \Sigma \to \Sigma$ is a homeomorphism and let $\tilde{\phi} : \mathbb{H}^2 \to \mathbb{H}^2$ be a lift. It is well-known that the map $\tilde{\phi}$ extends continuously to a homeomorphism

$$\partial\tilde{\phi} : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$$

Moreover, if $\psi : \Sigma \to \Sigma$ is homotopic to ϕ and $\tilde{\psi} : \mathbb{H}^2 \to \mathbb{H}^2$ is any lift of ψ, then the boundary extensions $\partial\tilde{\phi}$ and $\partial\tilde{\psi}$ differ by the boundary extension of a deck-transformation of the cover $\mathbb{H}^2 \to \Sigma$.

More precisely, the subgroup $\mathcal{G} \subset \mathrm{Homeo}(\partial_\infty\mathbb{H}^2)$ formed by all the boundary extensions of all possible lifts of self-homeomorphisms of Σ fits in the following exact sequence:

$$1 \to \pi_1(\Sigma) \to \mathcal{G} \to \mathrm{Map}(\Sigma) \to 1 \tag{1.1}$$

Here, the normal subgroup $\pi_1(\Sigma)$ corresponds to the boundary extensions of deck-transformations. It follows that the action $\mathcal{G} \curvearrowright \partial\mathbb{H}^2$ induces an action

$$\mathrm{Map}(\Sigma) \simeq \mathcal{G}/\pi_1(S) \curvearrowright \pi_1(S)\backslash\Theta_3 \simeq T^1\Sigma$$

The exact sequence (1.1) induces a homomorphism from $\mathrm{Map}(\Sigma)$ to $\mathrm{Out}(\pi_1(\Sigma))$; this is the isomorphism between these two groups given by the Baer-Dehn-Nielsen theorem. It follows that the action

$$\mathrm{Map}(\Sigma) \curvearrowright T^1\Sigma \tag{1.2}$$

induces the Baer-Dehn-Nielsen isomorphism $\mathrm{Map}(\Sigma) \simeq \mathrm{Out}(\pi_1(\Sigma))$, as desired.

Before moving on observe that, up to conjugacy in $\mathrm{Homeo}(T^1\Sigma)$, the action (1.2) does not depend on the a hyperbolic metric on Σ. However, for any choice of metric, the standard action (1.2) is not better than Hölder [9]. We prove now that it is conjugated to a Lipschitz action:

Proposition 1.1. — *The standard action (1.2) is conjugated to an action by Lipschitz homeomorphisms.*

The key tool in the proof of Proposition 1.1 is the following result due to Deroin, Kleptsyn and Navas [4, Theorem D]: *Every countable subgroup* *of* Homeo($\mathbb{S}^1$) *is topologically conjugated to a group of Lipschtiz homeomorphisms.* In [4], the Deroin-Kleptsyn-Navas theorem is only stated for orientation preserving homeomorphisms of $\mathbb{S}^1$; the proof for subgroups of Homeo($\mathbb{S}^1$) remains the same.

Proof of Proposition 1.1. — By the Deroin-Kleptsyn-Navas theorem, there is a homeomorphism of $\partial\mathbb{H}^2 \simeq \mathbb{S}^1$ conjugating the action $\mathcal{G} \curvearrowright \partial\mathbb{H}^2$ to a Lipschitz action. This action induces a Lipschitz action on the space of triples and hence Θ_3. The quotient M of Θ_3 under the restriction of this Lipschitz action to the subgroup $\pi_1(\Sigma)$ is a Lipschitz 3-manifold on which the mapping class group Map(Σ) acts by Lipschitz homeomorphisms. The map $\Theta_3 \to \Theta_3$ induced by the conjugating homeomorphism of $\mathbb{S}^1$ induces a homeomorphism $M \to T^1\Sigma$. Transporting the Lipschitz structure of M and the action $\mathrm{Map}(\Sigma) \curvearrowright M$ we obtain a Lipschitz action of Map(Σ) on T^1M for some Lipschtiz structure on $T^1\Sigma$. However, a theorem due to Sullivan [18] asserts that every 3-manifold admits a unique Lipschitz structure up to homeomorphism close to the identity. In other words, we can conjugate the so constructed action $\mathrm{Map}(\Sigma) \curvearrowright T^1\Sigma$ by a homeomorphism close to the identity to obtain an action $\mathrm{Map}(\Sigma) \curvearrowright T^1\Sigma$ which is Lipschitz with respect to the standard smooth structure of $T^1\Sigma$. By construction, this action is conjugated to the standard action (1.2). $\square$

2.

Suppose that Σ has even genus and let $\sigma : \Sigma \to \Sigma$ be a smooth orientation reversing involution on Σ fixing a single curve (figure 2).

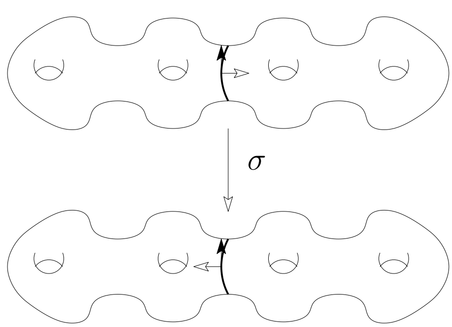

Figure 2. — The involution σ

In this section we prove:

Proposition 2.1. — *Suppose that $F : T^1\Sigma \to T^1\Sigma$ is a smooth involution inducing the same element as σ in $\mathrm{Out}(\pi_1(\Sigma))$. Then the fixed point set (F) of F consists of one or two smooth disjoint circles.*

Beginning with the proof of Proposition 2.1 we consider $\pi : T^1\Sigma \to \Sigma$ as a circle bundle; this is a very particular, and particularly nice, type of Seifert manifold. See [17] for classical facts on Seifert manifolds.

The center of the group $\pi_1(T^1\Sigma)$ is the cyclic subgroup $\mathcal{Z}$ represented by the fiber. Since the involution F has to preserve the center of $\pi_1(T^1\Sigma)$, we deduce that the image under F of the fibers of $\pi : T^1\Sigma \to \Sigma$ are freely homotopic to fibers. In the terminology of Meeks-Scott [11], this means that F *preserves the fibration up to homotopy.* The key tool in the proof of Proposition 2.1 is the following result due to Meeks and Scott [11, Theorem 2.2]:

Theorem (Meeks-Scott). — *Let M be a compact, $\mathbb{R}P^2$-irreducible Seifert fiber space with infinite fundamental group. If G is a finite group acting on M which preserves the given Seifert fibration up to homotopy, then M possesses a G-invariant Seifert fibration homotopic to the original fibration.*

Recall that a 3-manifold is $\mathbb{R}P^2$-irreducible if every embedded 2-dimensional sphere bounds a ball and if it does not contain any two sided real projective plane $\mathbb{R}P^2$. There are only two Seifert manifolds which are not $\mathbb{R}P^2$-irreducible, namely $\mathbb{S}^2 \times \mathbb{S}^1$ and the connected sum of 2 copies of $\mathbb{R}P^3$. In particular, $T^1\Sigma$ is $\mathbb{R}P^2$-irreducible and hence the Meeks-Scott theorem applies.

Since a Seifert manifold with hyperbolic base orbifold has a unique Seifert fibered structure up to isotopy [20, Lemma 3.5], there is some diffeomorphism $f_1 : T^1\Sigma \to T^1\Sigma$ isotopic to the identity such that $F_1 = f_1 \circ F \circ f_1^{-1}$ maps fibers of the bundle $\pi : T^1\Sigma \to \Sigma$ to fibers. So far, we have only used that F has finite order. We are now going to use the remaining assumptions.

The diffeomorphism F_1 induces a diffeomorphism $\hat{F}_1 : \Sigma \to \Sigma$ mapping $x \in \Sigma$ to the base point of the fiber $F_1(T^1_x\Sigma)$. Observe that $\hat{F}_1$ is an involution. The assumption that F induces the same element as σ in $\mathrm{Out}(\pi_1(\Sigma))$, together with the Baer-Dehn-Nielsen theorem, imply that $\hat{F}_1$ and σ represent the same element in the mapping class group of Σ. Since $\hat{F}_1$ and σ

have both finite order and are isotopic to each other, they are conjugated by some $\hat{f}_2 : \Sigma \to \Sigma$ isotopic to the identity:

$$\sigma = \hat{f}_2 \circ \hat{F}_1 \circ \hat{f}_2^{-1}$$

(See for example case 3 in the proof of Theorem 1.2 in [1] for a discussion of this fact.) Let $f_2 : T^1\Sigma \to T^1\Sigma$ be any diffeomorphism mapping fibers to fibers and inducing $\hat{f}_2$. For instance, such a f_2 can be constructed choosing a smooth connection on $\pi : T^1\Sigma \to \Sigma$ and lifting horizontally an isotopy between the identity of Σ and $\hat{f}_2$.

Consider now the smooth involution $F_2 = f_2 \circ F_1 \circ f_2^{-1}$ of $T^1\Sigma$ and observe that for every $x \in \Sigma$ we have $T^1_{\sigma(x)}\Sigma = F_2(T^1_x(\Sigma))$. It follows that the fixed-point set of F_2 is contained in the pre-image under π of the unique curve point-wise fixed by σ

$$\mathrm{Fix}(F_2) \subset \pi^{-1}(\mathrm{Fix}(\sigma))$$

The orientation of Σ induces an orientation on the fibers of $T^1\Sigma$. Since $\pi : T^1\Sigma \to \Sigma$ is a circle bundle with Euler-number $e(T^1\Sigma) = \chi(\Sigma) = 2-2g \neq 0$ (see [12]), every homeomorphism of $T^1\Sigma$ is orientation preserving; compare with [14, p.189]. In particular, $F_2 : T^1\Sigma \to T^1\Sigma$ is orientation preserving. Since the induced map on the base $\hat{F}_2 = \sigma$ reverses the orientation, it follows that F_2 has also to reverse the orientation of the fibers.

In particular, the restriction of F_2 to the torus $\pi^{-1}(\mathrm{Fix}(\sigma))$ is an orientation reversing involution which maps every fiber of

$$\pi^{-1}(\mathrm{Fix}(\sigma)) \to \mathrm{Fix}(\sigma) \tag{2.3}$$

to itself. Hence, the restriction of F_2 to any of the fibers of (2.3) is an orientation reversing involution, which therefore has exactly two fixed points on each fiber. Being a the fix set of a smooth element of finite order, $\mathrm{Fix}(F_2)$ is a submanifold. It follows now that $\mathrm{Fix}(F_2) \subset \pi^{-1}(\mathrm{Fix}(\sigma))$ consists of one or two smooth curves in $T^1\Sigma$. Since F and F_2 are conjugated, the claim of Proposition 2.1 follows.

3.

Before moving any further we need a little bit more of notation. The (full) mapping class group $\mathrm{Map}(X)$ of a compact surface X with boundary ∂X is the group of isotopy classes of homeomorphisms (or equivalently, diffeomorphisms) of X. Here we do not assume that isotopies fix point-wise the boundary of X. We denote by $\mathrm{Map}_+(X)$ the subgroup of the full mapping class group consisting of those mapping classes represented by orientation preserving diffeomorphisms of X which fix as a set every boundary component of X. In the course of the proof of Theorem 0.1 we will need several times the following vanishing theorem for the homology of $\mathrm{Map}_+(X)$.

THEOREM (Powell [16], Korkmaz [8]). — *If X is a compact surface with possibly non-empty boundary and at least genus 3 then*

$$H_1(\mathrm{Map}_+(X), \mathbb{Z}) = 0$$

We deduce now Theorem 1 from Proposition 2.1 and from previous results due to Parwani [15], Thurston [19] and an argument taken from Franks-Handel [7].

THEOREM 1. — *Suppose that Σ is a closed orientable surface of genus $g \geqslant 12$. Then there is no smooth action of $\mathrm{Map}(\Sigma)$ on $T^1\Sigma$ which induces the isomorphism $\mathrm{Map}(\Sigma) \to \mathrm{Out}(\pi_1(\Sigma))$.*

We assume for the time being that g is even. The case that g is odd will be discussed in the end of this section.

As above, let σ be an orientation reversing involution of Σ fixing exactly one curve. The quotient $\Sigma/\langle\sigma\rangle$ is a surface Z with at least genus 6 and a boundary component. We identify Z with the closure in Σ of one of the two connected components of $\Sigma \setminus \mathrm{Fix}(\sigma)$. Every homeomorphism $f : Z \to Z$ induces a homeomorphism

$$\hat{f} : \Sigma \to \Sigma$$

by $\hat{f}(x) = f(x)$ for $x \in Z$ and $\hat{f}(x) = \sigma(f(\sigma(x)))$ for $x \notin Z$. The map $f \to \hat{f}$ induces a homomorphism

$$\iota : \mathrm{Map}(Z) \to \mathrm{Map}(\Sigma) \tag{3.4}$$

We denote by G the image of $\mathrm{Map}_+(Z)$ under the *doubling homomorphism* (3.4). Observe that by construction the image of (3.4) centralizes σ. Notice also that the image $\iota(f) \in \mathrm{Map}(\Sigma)$ of $f \in \mathrm{Map}(Z)$ preserves the π_1-injective surface Z; since the induced mapping class of Z is equal to the original f we deduce that (3.4) is injective. In other words we have:

LEMMA 3.1. — *The doubling homomorphism (3.4) is injective and its image is contained in the centralizer of σ.*

Seeking a contradiction to the claim of Theorem 0.1, suppose that there is a smooth action $\mathrm{Map}(\Sigma) \curvearrowright T^1\Sigma$ inducing the Baer-Dehn-Nielsen isomorphism. In particular, the associated homomorphism

$$\Phi : \mathrm{Map}(\Sigma) \to \mathrm{Diff}(T^1\Sigma), \quad \gamma \mapsto \Phi_\gamma$$

is injective. Since $G = \iota(\mathrm{Map}_+(Z))$ centralizes σ we have $\Phi_\gamma \circ \Phi_\sigma = \Phi_\sigma \circ \Phi_\gamma$ and hence

$$\Phi_\gamma(\mathrm{Fix}(\Phi_\sigma)) = \mathrm{Fix}(\Phi_\sigma)$$

for all $\gamma \in G$. By Proposition 2.1, $\mathrm{Fix}(\Phi_\sigma)$ consists of one or two smooth circles. Taking into account that G is isomorphic to $\mathrm{Map}_+(Z)$ we have that $H_1(G;\mathbb{Z}) = 0$ by the homology vanishing theorem above. It follows that for all $\gamma \in G$ the diffeomorphism Φ_γ preserves each one of the connected components of $\mathrm{Fix}(\Phi_\sigma)$.

Let from now on S be a connected component of $\mathrm{Fix}(\Phi_\sigma)$ and recall that S is a smooth circle. So far, we have found out that the smooth action $\mathrm{Map}(\Sigma) \curvearrowright T^1\Sigma$ induces a smooth action $G \curvearrowright S$. The following theorem due to Parwani [15, Theorem 1.1] implies that the action $G \curvearrowright S$ is trivial.

Theorem (Parwani). — *Let Z be a connected surface with finitely many punctures, finitely many boundary components and genus at least 6. Then any C^1 action of $\mathrm{Map}_+(Z)$ on the circle is trivial.*

Fix from now on a point $x \in S$ and a basis v_1, v_2, v_3 of the tangent space $T_x(T^1\Sigma)$ such that v_1 is tangent to S; using this basis, identify $T_x(T^1\Sigma)$ with $\mathbb{R}^3$. Since x is fixed by every element of G we obtain a representation

$$G \to \mathrm{GL}_3\mathbb{R}, \quad \gamma \mapsto D(\Phi_\gamma)_x \tag{3.5}$$

We claim:

Lemma 3.2. — *The representation (3.5) is trivial.*

Proof. — To begin with we observe that for all $\gamma \in G$ we have $D(\Phi_\gamma)_x v_1 = v_1$. In particular, the matrix $D(\Phi_\gamma)_x$ has the following form

$$D(\Phi_\gamma)_x = \begin{pmatrix} 1 & b_\gamma \\ 0 & A_\gamma \end{pmatrix}$$

with $A_\gamma \in \mathrm{GL}_2\mathbb{R}$ and $b_\gamma \in \mathbb{R}^2$. The map

$$G \to \mathrm{GL}_2\mathbb{R}, \quad \gamma \mapsto A_\gamma$$

is a group homomorphism. We claim first that this representation is trivial. To begin with, the homomorphism

$$G \to \mathbb{R}^* = \mathbb{R} \setminus \{0\},\ \gamma \mapsto \det(A_\gamma)$$

must be trivial because $H_1(G;\mathbb{Z}) = 0$. In particular, $A_\gamma \in \mathrm{SL}_2\mathbb{R}$ for all γ. In oder to prove that a matrix $A \in \mathrm{SL}_2\mathbb{R}$ is the identity it suffices to show that it acts trivially on the circle $\mathbb{S}^1 = (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_+$. Composing the representation $\gamma \mapsto A_\gamma$ with the action $\mathrm{SL}_2\mathbb{R} \curvearrowright \mathbb{S}^1$ we obtain a smooth action of G on the circle $\mathbb{S}^1$. By Parwani's theorem, this action is trivial. This proves that $A_\gamma = \mathrm{Id}_2$ for all $\gamma \in G$.

Summing up, we have that for all $\gamma \in G$ the matrix $D(\Phi_\gamma)_x$ has the following form

$$D(\Phi_\gamma)_x = \begin{pmatrix} 1 & b_\gamma \\ 0 & \mathrm{Id}_2 \end{pmatrix}$$

with $b_\gamma \in \mathbb{R}^2$. The map $\gamma \mapsto b_\gamma$ is a group homomorphism with image in $\mathbb{R}^2$. Using again that $H_1(G;\mathbb{Z}) = 0$ we deduce that this group homomorphism is trivial, meaning that $b_\gamma = (0,0)$ for all γ. We have proved that $D(\Phi_\gamma)_x = \mathrm{Id}_3$ for all $\gamma \in G$ as claimed. $\square$

We can now conclude the proof of Theorem 0.1 using the following result due to Thurston [19, Theorem 3]:

Theorem (Thurston). — *Let G be a finitely generated group acting on a connected manifold with a global fixed point x. If the action is C^1 and Dg_x is the identity for all $g \in G$, then either there is a nontrivial homomorphism of G into $\mathbb{R}$ or G acts trivially.*

It follows from Lemma 3.2 that the action $G \curvearrowright T^1\Sigma$ satisfies the assumptions of Thurston's theorem. In particular, using again the assumption that $H_1(G;\mathbb{Z}) = 0$, we deduce that the action $G \curvearrowright T^1\Sigma$ must be trivial. This implies that for each $\gamma \in G$ the element in $\mathrm{Out}(\pi_1(T^1\Sigma))$ induced by Φ_γ is trivial as well. Hence, the element of $\mathrm{Out}(\pi_1(\Sigma)) = \mathrm{Out}(\pi_1(T^1\Sigma)/\mathcal{Z})$ induced by Φ_γ is also trivial. By assumption, γ and Φ_γ induce the same element of $\mathrm{Out}(\pi_1(\Sigma))$. It follows now from the Baer-Dehn-Nielsen theorem that every $\gamma \in G$ is trivial in $\mathrm{Map}(\Sigma)$. This contradiction to Lemma 3.1 concludes the proof of Theorem 0.1 if Σ has even genus.

We discuss now briefly the proof of Theorem 0.1 for surface of odd genus. To begin with we let $\sigma : \Sigma \to \Sigma$ be an orientation reversing involution with two fixed curves γ_1, γ_2 and identify the quotient $Z = \Sigma/\sigma$ with a connected component of $\Sigma \setminus (\gamma_1 \cup \gamma_2)$.

Suppose that $\mathrm{Map}(\Sigma) \curvearrowright T^1\Sigma$ is a smooth action inducing the Baer-Dehn-Nielsen isomorphisms and, with the same notation as above, let Φ_σ be the diffeomorphism of $T^1\Sigma$ corresponding to σ. Using the same argument as in Proposition 2.1 we obtain for every one of the curves γ_1, γ_2 one or two fixed curves of Φ_σ. Moreover, the fixed curves corresponding to γ_1 (resp. γ_2) have the property that they project to curves in Σ homotopic to a power of γ_1 (resp. γ_2).

As above we denote by G the image of the group $\mathrm{Map}_+(Z)$ into $\mathrm{Map}(\Sigma)$ under the doubling homomorphism. Again, the image of G in $\mathrm{Diff}(T^1\Sigma)$ under the homomorphism

$$\Phi : \mathrm{Map}(\Sigma) \to \mathrm{Diff}(T^1\Sigma)$$

centralizes Φ_σ. Hence, G acts on the fixed point set of Φ_σ. Noting that the elements in $\mathrm{Map}_+(Z)$ fix the homotopy class of γ_1, we deduce that G acts on the fixed curve, or the union of the two fixed curves, of F corresponding to γ_1. Now the argument proceeds word-by-word as above.

This concludes the proof of Theorem 0.1.

Before moving on, observe that in the course of the final step of the proof of Theorem 0.1 we have actually shown:

Proposition 3.3. — *Suppose that Z is a connected surface with at least genus* 6 *and that M is a connected 3-manifold. Any smooth action*

$$\mathrm{Map}_+(Z) \curvearrowright M$$

which preserves a smooth circle in M is trivial.

Notice that the assumption that the action is C^1 is necessary. An example of a non-trivial continuous action preserving a curve can be, for example, given as follows. Start with the action $\mathrm{Map}_+(\Sigma) \curvearrowright T^1\Sigma$ provided in section 1 and, with the same notation as in the proof of Theorem 0.1, restrict it to the subgroup $\iota(\mathrm{Map}_+(Z)) \simeq \mathrm{Map}_+(Z)$ where ι is the homomorphism (3.4).

We conclude with a few mussings which may be of interest to the reader:

• In the proof of Theorem 0.1, it plays a crucial role that we considered the whole mapping class group; not even the group of orientation preserving mapping classes would have sufficed. In fact, we used the involution σ to ensure that some subgroup $G \subset \mathrm{Map}(\Sigma)$ isomorphic to the mapping class group $\mathrm{Map}(Z)$ of the surface $Z = \Sigma/\langle\sigma\rangle$ acts on a smooth circle $S \subset T^1\Sigma$.

• Another reason why the proof of Theorem 0.1 does not apply to finite index subgroup of the mapping class group is that we used several times that $H_1(G;\mathbb{Z})=0$. It is conjectured that for a sufficiently large surface Z, the group $\mathrm{Map}_+(Z)$ does not contain finite index subgroups Γ with $H^1(\Gamma;\mathbb{Z})\neq 0$.

• Theorem 0.1 asserts that certain smooth actions do not exist; we discuss now briefly the degree of smoothness needed in the proof. In the final steps of the proof, it would have sufficed to have a C^1-action; for instance, in the statement of Proposition 3.3 we could replace smooth by C^1 leaving the proof unchanged. However, smoothness was also used in the proof of Proposition 2.1. The key step of the proof was the Meeks-Scott theorem whose proof makes use of the theory of minimal surfaces for some invariant metric on $T^1\Sigma$. The needed facts on minimal surfaces do not need the metric to be smooth; probably C^2 suffices. However, if the involution F is only C^1, it is not clear why should there be any F-invariant metric which is better than C^0.

• The statement of Theorem 0.1 can be easily extended to cover surfaces of genus $6, 8$ and 10 as well. The reason for this is that, instead of using Parwani's theorem one could refer to the following result of Farb-Franks [6, Theorem 1.2]: *If Z is a compact surface of genus at least 3 and at most a puncture, then every C^2-action $\mathrm{Map}_+(Z)\curvearrowright\mathbb{S}^1$ is trivial.*

• Observe that there is a different argument to prove Proposition 2.1, namely the classification of 3-dimensional orbifolds: consider the orbifold $(T^1\Sigma)/F$ and use that it is geometrizable to prove that it is homeomorphic to $(T^1\Sigma)/d\sigma$ where $d\sigma: T^1\Sigma\to T^1\Sigma$ is the differential of the original involution $\sigma:\Sigma\to\Sigma$. Perhaps using this approach one can prove Proposition 2.1 for F only C^1; as mentioned above, this would show that Theorem 0.1 remains true replacing smooth by C^1. However, the author of this note is not even sure that under this assumption the quotient $(T^1\Sigma)/F$ is an orbifold; observe for instance that the uniqueness and existence theorem for geodesics does not hold for C^0-metrics. When considering this problem, one should keep in mind that Bing [2] constructed continuous involutions of the sphere $\mathbb{S}^3$ whose fixed point set is the Alexander sphere and which thus are not conjugated to the standard involution.

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