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On log K-stability for asymptotically log Fano varieties

Kento Fujita[1]

Résumé. — La notion de variété asymptotiquement log Fano a été proposée par Cheltsov et Rubinstein. Dans ce travail on montre que, si une variété asymptotiquement log Fano (X, D) vérifie que D est irréductible et $-K_X - D$ est big, alors X n'admet pas de métrique Kähler-Einstein conique d'angle $2\pi\beta$ sur D, quelque soit l'angle rationnel positif β suffisamment petit. Ce résultat donne une réponse positive à une conjecture de Cheltsov et Rubinstein.

Abstract. — The notion of asymptotically log Fano varieties was given by Cheltsov and Rubinstein. We show that, if an asymptotically log Fano variety (X, D) satisfies that D is irreducible and $-K_X - D$ is big, then X does not admit Kähler-Einstein edge metrics with angle $2\pi\beta$ along D for any sufficiently small positive rational number β. This gives an affirmative answer to a conjecture of Cheltsov and Rubinstein.

Contents

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Article proposé par Vincent Guedj.

1. Introduction

The purpose of this article is to give a simple necessary criterion for log K-stability of $((X, D), -K_X - (1-\beta)D)$ with cone angle $2\pi\beta$ in the sense of [10], where X is projective log terminal and D is a reduced Weil divisor with $-K_X - (1-\beta)D$ ample. The motivation comes from a recent preprint of Cheltsov and Rubinstein [4], who treated the case that the dimension of X is equal to two. In this article, we show the following result.

Theorem 1.1. — *Let X be a normal projective variety which is log terminal, D be a nonzero reduced Weil divisor on X which is $\mathbb{Q}$-Cartier, and $0 \leqslant \beta \leqslant 1$ be a rational number. Assume that the pair $(X, (1-\beta)D)$ is dlt, $-K_X - (1-\beta)D$ is ample, and $((X, D), -K_X - (1-\beta)D)$ is log K-stable (resp. log K-semistable) with cone angle $2\pi\beta$. Then we have $\eta_\beta(D) > 0$ (resp. $\geqslant 0$), where*

$$\eta_\beta(D) := \beta \cdot \mathrm{vol}_X(-K_X - (1-\beta)D) - \int_0^\infty \mathrm{vol}_X(-K_X - (1-\beta+x)D)dx.$$

Note that vol_X is the volume function (see [8]).

Theorem 1.1 immediately gives the following theorem.

Theorem 1.2. — *Let X be a smooth projective variety and D be a nonzero reduced simple normal crossing divisor on X. Assume that $-K_X - (1-\beta)D$ is ample for any $0 < \beta \ll 1$ and the divisor $-K_X - D$ is big. Then $((X, D), -K_X - (1-\beta)D)$ is not log K-semistable with cone angle $2\pi\beta$ for any $0 < \beta \ll 1$ with $\beta \in \mathbb{Q}$. In particular, X does not admit Kähler-Einstein edge metrics with angle $2\pi\beta$ along D for any $0 < \beta \ll 1$ with $\beta \in \mathbb{Q}$.*

Theorem 1.2 gives an affirmative answer for a conjecture of Cheltsov and Rubinstein for asymptotically log Fano varieties [3] with irreducible boundaries in any dimension. Although the following corollary is a special case of Theorem 1.2, we state the assertion for the readers' convenience.

Corollary 1.3 (see [3, Conjecture 1.11 (i)]). — *Let (X, D) be an asymptotically log Fano variety with D irreducible, that is, X is a smooth projective variety and D is a smooth irreducible divisor on X such that $-K_X - (1-\beta)D$ is ample for any $0 < \beta \ll 1$. If the divisor $-K_X - D$ is big, then X does not admit Kähler-Einstein edge metrics with angle $2\pi\beta$ along D for any $0 < \beta \ll 1$ with $\beta \in \mathbb{Q}$.*

Remark 1.4. —

(1) In [4, Theorem 1.6] (see also [4, Conjecture 1.5]), Cheltsov and Rubinstein proved Corollary 1.3 in dimension two by using a construction of flops on the deformation to the normal cone.

(2) We always assume $\beta \in \mathbb{Q}$ for a technical reason. Indeed, if β is irrational, then the $\mathbb{C}$-algebra in Remark 3.3 is not finitely generated.

The strategy for the proof of Theorem 1.1 is essentially same as the strategy in [5]. We consider a kind of "log-version" of divisorial stability along D in the sense of [5]. We construct a specific log semi test configuration from certain section ring (see Remark 3.3) and calculate its log Donaldson-Futaki invariant explicitly by using the theory of "geography of models" (see Theorem 3.4).

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A *variety* stands for a reduced, irreducible, separated and of finite type scheme over the complex number field $\mathbb{C}$. For the theory of minimal model program, we refer the readers to [7]. For any Weil divisor E on a normal variety X, the *divisorial sheaf* on X is denoted by $\mathcal{O}_X(E)$. More precisely, the section $\Gamma(U, \mathcal{O}_X(E))$ on any open subscheme $U \subset X$ is given by the following:

$$\{f \in \mathbb{C}(X) \,|\, \mathrm{div}(f)|_U + E|_U \geqslant 0\},$$

where $\mathbb{C}(X)$ is the function field of X.

For varieties X_1 and X_2, let $p_i \colon X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection morphisms.

2. Log K-stability

We recall the definition of log K-stability.

Definition 2.1 (see [10]). — *Let X be an n-dimensional normal projective variety, L be an ample line bundle on X, and D be a reduced Weil divisor on X.*

(1) *A coherent ideal sheaf* $\mathcal{I} \subset \mathcal{O}_{X\times\mathbb{A}^1_t}$ *is said to be a* flag ideal *if* $\mathcal{I}$ *is of the form*

$$\mathcal{I} = I_M + I_{M-1}t^1 + \cdots + I_1 t^{M-1} + (t^M) \subset \mathcal{O}_{X\times\mathbb{A}^1_t},$$

where $I_M \subset \cdots \subset I_1 \subset \mathcal{O}_X$ *is a sequence of coherent ideal sheaves of* X*.*

(2) *Let* $m \in \mathbb{Z}_{>0}$*, and let* $\mathcal{I} \subset \mathcal{O}_{X\times\mathbb{A}^1}$ *be a flag ideal. A* log semi test configuration $((\mathcal{X},\mathcal{D}),\mathcal{M})/\mathbb{A}^1$ of $((X,D),L^{\otimes m})$ obtained by $\mathcal{I}$ *is given from the following data:*

- $\Pi\colon \mathcal{X} \to X\times\mathbb{A}^1$ *is the blowing up along* $\mathcal{I}$*,* $\mathcal{D}\subset\mathcal{X}$ *is given by the blowing up of* $D\times\mathbb{A}^1$ *along* $\mathcal{I}|_{D\times\mathbb{A}^1}$*, that is,* $\mathcal{D}$ *is the strict transform of* $D\times\mathbb{A}^1$ *on* $\mathcal{X}$*, and* $E\subset\mathcal{X}$ *is the Cartier divisor defined by* $\mathcal{O}_{\mathcal{X}}(-E) = \mathcal{I}\cdot\mathcal{O}_{\mathcal{X}}$*,*
- $\mathcal{M}$ *is the line bundle on* $\mathcal{X}$ *defined by* $\mathcal{M} := \Pi^*p_1^*L^{\otimes m}\otimes\mathcal{O}_{\mathcal{X}}(-E)$*,*

such that we require the following:

- $\mathcal{I}$ *is not of the form* (t^M)*, and*
- $\mathcal{M}$ *is semiample over* $\mathbb{A}^1$*.*

(3) *Assume that* $((\mathcal{X},\mathcal{D}),\mathcal{M})/\mathbb{A}^1$ *is a log semi test configuration of* $((X,D),L^{\otimes m})$ *obtained by* $\mathcal{I}$*. Then the multiplicative group* $\mathbb{G}_m$ *naturally acts on* $(\mathcal{X},\mathcal{M})$ *and* $(\mathcal{D},\mathcal{M}|_{\mathcal{D}})$*. For* $k\in\mathbb{Z}_{>0}$*, let* $w(k)$ *be the total weight of* $\mathbb{G}_m$*-action on* $H^0(\mathcal{X}_0,\mathcal{M}^{\otimes k}|_{\mathcal{X}_0})$ *and* $\tilde{w}(k)$ *be the total weight of* $\mathbb{G}_m$*-action on* $H^0(\mathcal{D}_0,\mathcal{M}^{\otimes k}|_{\mathcal{D}_0})$*, where* $\mathcal{X}_0\subset\mathcal{X}$ *and* $\mathcal{D}_0\subset\mathcal{D}$ *are the scheme-theoretic fibers at* $0\in\mathbb{A}^1$*, respectively. It is known that, for* $k\gg 0$*,* $w(k)$ *(resp.* $\tilde{w}(k)$*) is a polynomial function of degree at most* $n+1$ *(resp.* n*). For* $k\gg0$*, we set*

$$\begin{aligned}
\chi(X,L^{\otimes mk}) &= a_0k^n + a_1k^{n-1} + O(k^{n-2}),\\
\chi(D,L|_D^{\otimes mk}) &= \tilde{a}_0k^{n-1} + O(k^{n-2}),\\
w(k) &= b_0k^{n+1} + b_1k^n + O(k^{n-1}),\\
\tilde{w}(k) &= \tilde{b}_0k^n + +O(k^{n-1}).
\end{aligned}$$

For any $\beta\in[0,1]$*, we set the* log Donaldson-Futaki invariant $\mathrm{DF}_\beta((\mathcal{X},\mathcal{D}),\mathcal{M})$ with cone angle $2\pi\beta$ *as*

$$\mathrm{DF}_\beta((\mathcal{X},\mathcal{D}),\mathcal{M}) := 2(b_0a_1 - b_1a_0) + (1-\beta)(a_0\tilde{b}_0 - b_0\tilde{a}_0).$$

(4) *Let* $\beta\in[0,1]$*.* $((X,D),L)$ *is said to be* log K-stable *(resp.* log K-semistable*) with cone angle* $2\pi\beta$ *if* $\mathrm{DF}_\beta((\mathcal{X},\mathcal{D}),\mathcal{M}) > 0$ *(resp.* $\geqslant 0$*)*

holds for any $m \in \mathbb{Z}_{>0}$, for any flag ideal $\mathcal{I}$, and for any log semi test configuration $((\mathcal{X}, \mathcal{D}), \mathcal{M})/\mathbb{A}^1$ of $((X, D), L^{\otimes m})$ obtained by $\mathcal{I}$. For an ample $\mathbb{Q}$-divisor A on X, $((X, D), A)$ is said to be log K-stable *(resp.* log K-semistable*)* with cone angle $2\pi\beta$ *if $((X, D), \mathcal{O}_X(aA))$ is so for some $a \in \mathbb{Z}_{>0}$ with aA Cartier (this definition does not depend on the choice of a).*

The following theorem is important.

Theorem 2.2 (see [2, 4, 10]). — *Let X be a smooth projective variety, D be a reduced simple normal crossing divisor on X, and let $\beta \in [0, 1] \cap \mathbb{Q}$. Assume that $-K_X - (1 - \beta)D$ is ample and X admits Kähler-Einstein edge metrics with angle $2\pi\beta$ along D. Then $((X, D), -K_X - (1 - \beta)D)$ is log K-semistable with cone angle $2\pi\beta$.*

3. Constructing log semi test configurations

In this section, from a pair (X, D), we construct a specific log semi test configuration via D. The construction is essentially in the same way as in [5, §3]. We fix the following condition:

Assumption 3.1. — Let X be an n-dimensional normal projective variety which is log terminal, D is a nonzero reduced Weil divisor on X which is $\mathbb{Q}$-Cartier, and $\beta \in [0, 1] \cap \mathbb{Q}$. Assume that the pair $(X, (1 - \beta)D)$ is dlt and $-K_X - (1 - \beta)D$ is ample.

Definition 3.2. — *Under Assumption 3.1, we set*

$$\begin{aligned} \tau(D) &:= \sup\{\tau \in \mathbb{R}_{>0} \mid -K_X - \tau D \text{ big}\}, \\ \tau_\beta(D) &:= \sup\{\tau \in \mathbb{R}_{>0} \mid -K_X - (1 - \beta + \tau)D \text{ big}\}. \end{aligned}$$

It is obvious that $\tau_\beta(D) = \tau(D) - (1 - \beta)$. We remark that $\tau(D) > 1$ holds if and only if the divisor $-K_X - D$ is big.

Remark 3.3. — By [1, Corollary 1.1.9], the $\mathbb{C}$-algebra

$$\bigoplus_{\substack{k \in \mathbb{Z}_{\geqslant 0} \\ j \in \mathbb{Z}_{\geqslant 0}}} H^0(X, \mathcal{O}_X(\lfloor k(-K_X - (1 - \beta)D) - jD \rfloor))$$

is finitely generated, where $\lfloor k(-K_X - (1 - \beta)D) - jD \rfloor$ is the biggest $\mathbb{Z}$-divisor which is contained by $k(-K_X - (1 - \beta)D) - jD$. We note that $H^0(X, \mathcal{O}_X(\lfloor k(-K_X - (1 - \beta)D) - jD \rfloor)) = 0$ if $j > k\tau_\beta(D)$. Thus, there exists $r \in \mathbb{Z}_{>0}$ such that

- $L_\beta := r(-K_X - (1-\beta)D)$ is Cartier, and
- the $\mathbb{C}$-algebra
$$\bigoplus_{\substack{k\in\mathbb{Z}_{\geqslant 0}\\ j\in[0,kr\tau_\beta(D)]\cap\mathbb{Z}}} H^0(X, \mathcal{O}_X(kL_\beta - jD))$$
is generated by
$$\bigoplus_{j\in[0,r\tau_\beta(D)]\cap\mathbb{Z}} H^0(X, \mathcal{O}_X(L_\beta - jD)).$$

From now on, we fix such r (and L_β).

Theorem 3.4 ([6, Theorem 4.2]). — *Under Assumption 3.1, there exist*

- *a sequence of rational numbers*
$$0 = \tau_0 < \tau_1 < \cdots < \tau_m = \tau_\beta(D),$$
- *normal projective varieties $X_1, \ldots, X_m$ such that $X_1 = X$, and*
- *mutually distinct birational contraction maps $\phi_i \colon X \dashrightarrow X_i$ with $\phi_1 = \mathrm{id}_X$ $(1 \leqslant i \leqslant m)$*

such that the following hold:

- *for any $x \in [\tau_{i-1}, \tau_i]$, ϕ_i is a semiample model (see [6, Definition 2.3]) of $-K_X - (1-\beta+x)D$, and*
- *if $x \in (\tau_{i-1}, \tau_i)$, then ϕ_i is the ample model (see [6, Definition 2.3]) of $-K_X - (1-\beta+x)D$.*

Proof. — By [1, Corollary 1.4.3], there exists a projective birational morphism $\sigma \colon \tilde{X} \to X$ such that σ is an isomorphism in codimension one and $\tilde{X}$ is $\mathbb{Q}$-factorial. Let $\tilde{D}$ be the strict transform of D on $\tilde{X}$. A semiample model (resp. the ample model) of $-K_{\tilde{X}} - (1-\beta+x)\tilde{D}$ is a semiample model (resp. the ample model) of $-K_X - (1-\beta+x)D$. Moreover, the $\mathbb{C}$-algebra
$$\bigoplus_{\substack{k\in\mathbb{Z}_{\geqslant 0}\\ j\in\mathbb{Z}_{\geqslant 0}}} H^0(X, \mathcal{O}_X(\lfloor k(-K_X - (1-\beta)D) - jD\rfloor))$$
is equal to the $\mathbb{C}$-algebra
$$\bigoplus_{\substack{k\in\mathbb{Z}_{\geqslant 0}\\ j\in\mathbb{Z}_{\geqslant 0}}} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\lfloor k(-K_{\tilde{X}} - (1-\beta)\tilde{D}) - j\tilde{D}\rfloor)).$$
Thus we can apply [6, Theorem 4.2]. □

We construct a log semi test configuration of $((X,D),\mathcal{O}_X(L_\beta))$ from D. For any $j\in[0,r\tau_\beta(D)]\cap\mathbb{Z}$, we set

$$I_j := \mathrm{Image}(H^0\left(X,\mathcal{O}_X(L_\beta - jD)\right)\otimes_{\mathbb{C}}\mathcal{O}_X(-L_\beta)\to\mathcal{O}_X),$$

where the homomorphism is the evaluation. Note that, for any $j\in[0,r\tau_\beta(D)]\cap\mathbb{Z}$, $I_j\subset\mathcal{O}_X(-jD)$ and

$$0\subset I_{r\tau_\beta(D)}\subset\cdots\subset I_1\subset I_0=\mathcal{O}_X$$

hold. For $k\in\mathbb{Z}_{>0}$ and $j\in[0,kr\tau_\beta(D)]\cap\mathbb{Z}$, we define

$$J_{(k,j)} := \sum_{\substack{j_1+\cdots+j_k=j,\\ j_1,\dots,j_k\in[0,r\tau_\beta(D)]\cap\mathbb{Z}}} I_{j_1}\cdots I_{j_k}\subset\mathcal{O}_X.$$

Lemma 3.5 (see [5, Lemma 3.3]). — *The $J_{(k,j)}\subset\mathcal{O}_X$ is equal to*

$$\mathrm{Image}(H^0\left(X,\mathcal{O}_X(kL_\beta - jD)\right)\otimes_{\mathbb{C}}\mathcal{O}_X(-kL_\beta)\to\mathcal{O}_X).$$

In particular, we have

$$H^0\left(X,\mathcal{O}_X(kL_\beta - jD)\right) = H^0\left(X,\mathcal{O}_X(kL_\beta)\cdot J_{(k,j)}\right).$$

Proof. — Set

$$V_{k,j} := H^0(X,\mathcal{O}_X(kL_\beta - jD))$$

for simplicity. We remark that, by the choice of $r\in\mathbb{Z}_{>0}$, the homomorphism

$$\bigoplus_{\substack{j_1+\cdots+j_k=j,\\ j_1,\dots,j_k\in[0,r\tau_\beta(D)]\cap\mathbb{Z}}} V_{1,j_1}\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}}V_{1,j_k}\to V_{k,j}$$

is surjective. For any $1\leqslant i\leqslant k$, the ideal sheaf I_{j_i} is nothing but

$$\mathrm{Image}(V_{1,j_i}\otimes_{\mathbb{C}}\mathcal{O}_X(-L_\beta)\to\mathcal{O}_X).$$

Thus the assertion follows. □

Set the flag ideal $\mathcal{I}$ such that

$$\mathcal{I} := I_{r\tau_\beta(D)} + I_{r\tau_\beta(D)-1}t^1+\cdots+I_1t^{r\tau_\beta(D)-1}+(t^{r\tau_\beta(D)})\subset\mathcal{O}_{X\times\mathbb{A}^1_t}.$$

For any $k\in\mathbb{Z}_{>0}$, we have

$$\mathcal{I}^k = J_{(k,kr\tau_\beta(D))}+J_{(k,kr\tau_\beta(D)-1)}t^1+\cdots+J_{(k,1)}t^{kr\tau_\beta(D)-1}+(t^{kr\tau_\beta(D)})$$

by the construction of $J_{(k,j)}$. Let $\Pi\colon \mathcal{X}\to X\times\mathbb{A}^1$ be the blowing up along $\mathcal{I}$ and let $E\subset\mathcal{X}$ be the Cartier divisor given by the equation $\mathcal{O}_{\mathcal{X}}(-E)=\mathcal{I}\cdot\mathcal{O}_{\mathcal{X}}$. Set $\mathcal{L}_\beta:=\Pi^*p_1^*\mathcal{O}_X(L_\beta)\otimes\mathcal{O}_{\mathcal{X}}(-E)$. Let $\mathcal{D}\to D\times\mathbb{A}^1$ be the blowing up along $\mathcal{I}|_{D\times\mathbb{A}^1}$. We note that $\mathcal{I}|_{D\times\mathbb{A}^1}=(t^{r\tau_\beta(D)})\subset\mathcal{O}_{D\times\mathbb{A}^1}$ since $I_j\subset\mathcal{O}_X(-jD)\subset\mathcal{O}_X(-D)$ for any $j>0$. In particular, $\mathcal{D}\simeq D\times\mathbb{A}^1$ holds.

Lemma 3.6 (see [5, Lemma 3.4]). — *$((\mathcal{X},\mathcal{D}),\mathcal{L}_\beta)/\mathbb{A}^1$ is a log semi test configuration of $((X,D),L_\beta)$.*

Proof. — Set $\alpha:=p_2\circ\Pi\colon\mathcal{X}\to\mathbb{A}^1$. It is enough to check that $\mathcal{L}_\beta$ is α-semiample. By Lemma 3.5, the homomorphism

$$H^0(X,\mathcal{O}_X(kL_\beta)\cdot J_{(k,j)})\otimes_{\mathbb{C}}\mathcal{O}_X\to\mathcal{O}_X(kL_\beta)\cdot J_{(k,j)}$$

is surjective for any $k\in\mathbb{Z}_{>0}$ and $j\in[0,kr\tau_\beta(D)]\cap\mathbb{Z}$. Thus

$$H^0\left(X\times\mathbb{A}^1,p_1^*\mathcal{O}_X(kL_\beta)\cdot\mathcal{I}^k\right)\otimes_{\mathbb{C}[t]}\mathcal{O}_{X\times\mathbb{A}^1}\to p_1^*\mathcal{O}_X(kL_\beta)\cdot\mathcal{I}^k$$

is surjective for any $k\in\mathbb{Z}_{>0}$. From [8, Lemma 5.4.24], we have

$$\begin{aligned}\alpha^*\alpha_*\mathcal{L}_\beta^{\otimes k} &\simeq \Pi^*(p_2)^*(p_2)_*(p_1^*\mathcal{O}_X(kL_\beta)\cdot\mathcal{I}^k)\\ &= \Pi^*\left(H^0\left(X\times\mathbb{A}^1,p_1^*\mathcal{O}_X(kL_\beta)\cdot\mathcal{I}^k\right)\otimes_{\mathbb{C}[t]}\mathcal{O}_{X\times\mathbb{A}^1}\right)\\ &\twoheadrightarrow \Pi^*\left(p_1^*\mathcal{O}_X(kL_\beta)\cdot\mathcal{I}^k\right)\\ &\twoheadrightarrow \Pi^*p_1^*\mathcal{O}_X(kL_\beta)\otimes\mathcal{O}_{\mathcal{X}}(-kE)=\mathcal{L}_\beta^{\otimes k}\end{aligned}$$

for $k\gg 0$. □

Definition 3.7. — *We say the $((\mathcal{X},\mathcal{D}),\mathcal{L}_\beta)/\mathbb{A}^1$ is the* basic log semi test configuration of $((X,D),\mathcal{O}_X(L_\beta))$ via D.

Now we calculate the log Donaldson-Futaki invariant of the basic log semi test configuration $((\mathcal{X},\mathcal{D}),\mathcal{L}_\beta)/\mathbb{A}^1$ of $((X,D),\mathcal{O}_X(L_\beta))$ via D. By the asymptotic Riemann-Roch theorem, we have

$$a_0=\frac{(L_\beta^{\cdot n})}{n!},\quad a_1=\frac{(L_\beta^{\cdot n-1}\cdot -K_X)}{2\cdot(n-1)!},\quad \tilde{a}_0=\frac{(L_\beta^{\cdot n-1}\cdot D)}{(n-1)!}.$$

(We follow the notation in Definition 2.1.) By [9, §3],

$$\begin{aligned}w(k) &= -\dim\left(\frac{H^0(X\times\mathbb{A}^1,p_1^*\mathcal{O}_X(kL_\beta))}{H^0(X\times\mathbb{A}^1,p_1^*\mathcal{O}_X(kL_\beta)\cdot\mathcal{I}^k)}\right)\\ &= -kr\tau_\beta(D)\cdot h^0(X,\mathcal{O}_X(kL_\beta))+v(k),\end{aligned}$$

where

$$v(k) := \sum_{j=1}^{kr\tau_\beta(D)} h^0(X, \mathcal{O}_X(kL_\beta - jD)).$$

By the same argument,

$$\begin{aligned}\tilde{w}(k) &= -\dim\left(\frac{H^0(D\times\mathbb{A}^1, p_1^*\mathcal{O}_X(kL_\beta)|_D)}{H^0(D\times\mathbb{A}^1, p_1^*\mathcal{O}_X(kL_\beta)|_D\cdot(t^{kr\tau_\beta(D)}))}\right)\\ &= -kr\tau_\beta(D)\cdot h^0(D, \mathcal{O}_X(kL_\beta)|_D).\end{aligned}$$

Thus

$$\tilde{b}_0 = -r\tau_\beta(D)\frac{(L_\beta^{\cdot n-1}\cdot D)}{(n-1)!}.$$

We set $v(k) = v_0k^{n+1} + v_1k^n + O(k^{n-1})$. We calculate the values v_0 and v_1. Let $L_{\beta,i}$ and D_i be the divisors on X_i which are the push-forwards of L_β and D, respectively. For $k \gg 0$ sufficiently divisible, by [6, Remark 2.4 (i)] and [5, Proposition 4.1], $v(k)$ is equal to

$$\begin{aligned}&\sum_{i=1}^{m}\sum_{j=kr\tau_{i-1}+1}^{kr\tau_i} h^0\left(X_i, \mathcal{O}_{X_i}\left(kL_{\beta,i} - jD_i\right)\right)\\ = &\sum_{i=1}^{m}\Big(\frac{(kr)^{n+1}}{n!}\int_{\tau_{i-1}}^{\tau_i}\left(((1/r)L_{\beta,i} - xD_i)^{\cdot n}\right)dx\\ &-\frac{(kr)^n}{2\cdot(n-1)!}\int_{\tau_{i-1}}^{\tau_i}\Big(((1/r)L_{\beta,i} - xD_i)^{\cdot n-1}\cdot(K_{X_i}+D_i)\Big)dx\Big)\\ &+O(k^{n-1}).\end{aligned}$$

This implies that

$$\begin{aligned}v_0 &= \frac{r^{n+1}}{n!}\sum_{i=1}^{m}\int_{\tau_{i-1}}^{\tau_i}\left(((1/r)L_{\beta,i} - xD_i)^{\cdot n}\right)dx,\\ v_1 &= \frac{-r^n}{2\cdot(n-1)!}\sum_{i=1}^{m}\int_{\tau_{i-1}}^{\tau_i}\Big(((1/r)L_{\beta,i} - xD_i)^{\cdot n-1}\cdot(K_{X_i}+D_i)\Big)dx.\end{aligned}$$

Thus we have

$$\begin{aligned}&\mathrm{DF}_\beta((\mathcal{X},\mathcal{D}),\mathcal{L}_\beta)\\ = &\ 2(v_0a_1 - v_1a_0) + (1-\beta)(a_0\tilde{b}_0 - (v_0 - r\tau_\beta(D)a_0)\tilde{a}_0)\\ = &\ \frac{n\cdot r^n(L_\beta^{\cdot n})}{(n!)^2}\sum_{i=1}^{m}\int_{\tau_{i-1}}^{\tau_i}(\beta - x)\left((-K_{X_i} - (1-\beta+x)D_i)^{\cdot n-1}\cdot D_i\right)dx.\end{aligned}$$

Lemma 3.8 (cf. [5, Theorem 5.2]). — *We have*

$$\eta_\beta(D) = n\sum_{i=1}^{m}\int_{\tau_{i-1}}^{\tau_i}(\beta - x)\left((-K_{X_i} - (1-\beta+x)D_i)^{\cdot n-1}\cdot D_i\right)dx.$$

Proof. — By [6, Remark 2.4 (i)], we have

$$\mathrm{vol}_X(-K_X - (1-\beta+x)D) = ((-K_{X_i} - (1-\beta+x)D_i)^{\cdot n})$$

for any $x \in [\tau_{i-1}, \tau_i]$. From partial integration, we have

$$\begin{aligned} & n\sum_{i=1}^{m}\int_{\tau_{i-1}}^{\tau_i}(\beta - x)\left((-K_{X_i} - (1-\beta+x)D_i)^{\cdot n-1}\cdot D_i\right)dx \\ = \ & \sum_{i=1}^{m}\Big([(x-\beta)\mathrm{vol}_X(-K_X - (1-\beta+x)D)]_{\tau_{i-1}}^{\tau_i} \\ & -\int_{\tau_{i-1}}^{\tau_i}\mathrm{vol}_X(-K_X - (1-\beta+x)D)dx\Big) = \eta_\beta(D). \end{aligned}$$

We remark that $\mathrm{vol}_X(-K_X - (1-\beta+x)D) = 0$ if $x \geqslant \tau_\beta(D)$. □

Proof of Theorem 1.1. — Follows immediately from Lemma 3.8. □

Remark 3.9. — If $\beta = 1$, then the value $\eta_1(D)$ is noting but the value $\eta(D)$ in [5, Definition 1.1].

Corollary 3.10. — *Let X be a normal projective variety which is log terminal, D is a nonzero reduced Weil divisor on X which is $\mathbb{Q}$-Cartier. Assume that $(X, (1-\beta)D)$ is klt and $-K_X - (1-\beta)D$ is ample for any $0 < \beta \ll 1$. Moreover, we assume that $-K_X - D$ is big. Then for any $0 < \beta \ll 1$ rational number, $((X, D), -K_X - (1-\beta)D)$ is not log K-semistable with cone angle $2\pi\beta$.*

Proof. — We have $\eta_\beta(D) = \eta_+(\beta) - \eta_-$, where

$$\begin{aligned} \eta_+(\beta) &:= \beta\cdot\mathrm{vol}_X(-K_X - (1-\beta)D) - \int_{1-\beta}^{1}\mathrm{vol}_X(-K_X - xD)dx, \\ \eta_- &:= \int_1^\infty \mathrm{vol}_X(-K_X - xD)dx. \end{aligned}$$

If $-K_X - D$ is big, then $\eta_- > 0$. On the other hand, $\lim_{\beta\to 0}\eta_+(\beta)$ is equal to zero. Indeed, we know that

$$\eta_+(\beta) = \int_{1-\beta}^{1}\left(\mathrm{vol}_X\left(-K_X - (1-\beta)D\right) - \mathrm{vol}_X(-K_X - xD)\right)dx \geqslant 0$$

and

$$\eta_+(\beta) \leqslant \int_{1-\beta}^{1} \mathrm{vol}_X(-K_X)dx = \beta \cdot \mathrm{vol}_X(-K_X).$$

Thus the assertion follows from Theorem 1.1. □

Proof of Theorem 1.2. — Follows from Theorem 2.2 and Corollary 3.10. □

4. Examples

We see some examples.

Example 4.1. — Let X be an n-dimensional Fano manifold, and let D be a smooth divisor on X with $-K_X \sim_{\mathbb{Q}} lD$ for some $l \in [1, n+1] \cap \mathbb{Q}$. Then $-K_X - (1-\beta)D$ is ample for any $\beta \in (0,1]$. In this case, we have

$$\eta_\beta(D) = \frac{n}{n+1}\mathrm{vol}_X(-K_X - (1-\beta)D)\left(\beta - \frac{l-1}{n}\right).$$

If $\beta < (l-1)/n$, then $\eta_\beta(D) < 0$ holds.

Remark 4.2. — In Example 4.1, if $D \sim -K_X$ (i.e., $l = 1$), then $\eta_\beta(D) > 0$ for any $\beta \in (0,1]$. Thus our argument does not give any destabilizing information in this case. Compare with [4, §6].

Example 4.3. — Let $Y := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, C be a section of the $\mathbb{P}^1$-bundle $Y/\mathbb{P}^1$ with $(C^{\cdot 2}) = 1$, $\pi\colon X \to Y$ be the blowing up along $p \in Y$ with $p \in C$, E be the exceptional divisor of π, and set $D := \pi_*^{-1}C$. Then $-K_X - (1-\beta)D$ is ample for any $\beta \in (0,1]$. (The pair (X, D) is nothing but [3, (I8B.1)].) In this case, $\tau_1 = \beta$, $X_2 = Y$, $\tau_2 = \tau_\beta(D) = 1 + \beta$ and

$$\mathrm{vol}_X(-K_X - xD) = \begin{cases} -4x + 7 & \text{if } x \in [0,1], \\ x^2 - 6x + 8 & \text{if } x \in [1,2]. \end{cases}$$

Thus $\eta_\beta(D) = 2(\beta^2 - 2/3)$. For example, if we set $\beta := 1/2$, then $\eta_{1/2}(D) < 0$ holds. In this case ($\beta = 1/2$), we can check that $r := 2$ satisfies the condition in Remark 3.3 and the corresponding flag ideal $\mathcal{I}$ is of the form

$$\mathcal{I} = \mathcal{O}_X(-3D - 2E) + \mathcal{O}_X(-2D - E)t^1 + \mathcal{O}_X(-D)t^2 + (t^3).$$

We remark that X does not admit Kähler-Einstein edge metrics with angle $2\pi\beta$ along D for any $\beta \in (0,1] \setminus \{\sqrt{3} - 1\}$ by [4, Example 2.8].

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