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Algorithmic aspects of branched coverings (*)

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Abstract. — This is a survey, and a condensed version, of a series of articles on the algorithmic study of Thurston maps. We describe branched coverings of the sphere in terms of group-theoretical objects called bisets, and develop a theory of decompositions of bisets.

We introduce a canonical "Levy" decomposition of an arbitrary Thurston map into homeomorphisms, metrically-expanding maps and maps doubly covered by torus endomorphisms. The homeomorphisms decompose themselves into finite-order and pseudo-Anosov maps, and the expanding maps decompose themselves into rational maps.

As an outcome, we prove that it is decidable when two Thurston maps are equivalent. We also show that the decompositions above are computable, both in theory and in practice.

Résumé. — Ce texte est un survol, et une version condensée, d'une série d'articles étudiant algorithmiquement les applications de Thurston. Nous décrivons les revêtements ramifiés de la sphère en termes d'objets de la théorie des groupes appelés « bi-ensembles », et développons une théorie de leur décomposition.

Nous introduisons une décomposition canonique « de Levy » d'une application de Thurston quelconque en homéomorphismes, applications métriquement dilatantes et applications doublement revêtues par un endomorphisme du tore. Les homéomorphismes se décomposent eux-mêmes en applications d'ordre fini et pseudo-Anosov, et les applications dilatantes se décomposent elles-mêmes en applications rationelles.

Comme conséquence, nous prouvons qu'il est algorithmiquement décidable si deux applications de Thurston sont combinatoirement équivalentes. Nous montrons aussi que les décompositions décrites ci-dessus sont calculables, en théorie et en pratique.

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Article proposé par Jean-Pierre Otal.

Contents

0. Introduction

Nielsen [27], and later Thurston [39], have achieved an impressive classification of surface self-homeomorphisms. Given a compact surface X and a self-homeomorphism $f\colon X \circlearrowleft$, there exists a canonical set $\mathscr{C}$ of curves, invariant up to isotopy by f, that separate X into simpler surfaces, and such that the induced first return maps on these pieces are isotopic to either finite-order or *pseudo-Anosov* maps.

These induced maps all preserve a geometric structure: finite-order transformations are hyperbolic isometries (so preserve a complex structure), while pseudo-Anosov maps preserve a pair of transverse foliations, expanding one and contracting the other.

This classification result may also be viewed as a bridge between topology and group theory: f naturally acts by automorphisms on the fundamental group G of X; the collection $\mathscr{C}$ of curves determines a splitting of G as amalgamated free product over cyclic subgroups, and the induced automorphisms of the pieces are either finite-order or irreducible in a strong sense ("iwip": irreducible with irreducible powers). Thus the decomposition of X as an amalgam over circles naturally parallels a decomposition of G as an amalgam over cyclic subgroups.

Our aim is to do the same for branched self-coverings of compact surfaces, namely maps $f\colon X \circlearrowleft$ that are coverings away from a finite set of *branch points*, where they admit local models of the form $z \mapsto z^d$ in complex charts.

If f has degree > 1, the only surfaces to consider are the sphere and the torus, by the Riemann–Hurwitz formula. Examples of branched self-coverings of the sphere include rational maps in $\mathbb{C}(z)$ and their compositions with homeomorphisms; branched self-coverings of the torus admit no branch points, so are genuine coverings and can all be represented on the torus $\mathbb{R}^2/\mathbb{Z}^2$ (after punctures are filled in) as $z \mapsto Mz + b$ for some $M \in \mathbb{Z}^{2\times 2}$ and some $b \in \mathbb{R}^2$. These maps may descend to the sphere: if $2b \in \mathbb{Z}^2$ and $\wp\colon \mathbb{R}^2/\mathbb{Z}^2 \to S^2$ is a branched covering satisfying $\wp(-p) = \wp(p)$ then $\wp \circ f \circ \wp^{-1}$ is a branched self-covering of S^2.

An extra ingredient, besides topology and group theory, becomes available if f has degree > 1: it may happen that X admits a metric that is expanded by f. Even better, X may admit a complex structure that is preserved by f, in which case there exists a conformal metric that is expanded by f.

0.1. Overview of results

We consider self-branched coverings $f\colon (S^2, A) \circlearrowleft$, with A a finite subset of S^2 containing $f(A)$ and the critical values of f; such maps are called *marked Thurston maps*. For example, A could be the *post-critical set* $P_f = \bigcup_{n>0} f^n(\text{critical points})$.

Since Thurston's fundamental work, it is customary to consider such maps f up to *combinatorial equivalence*: two maps f_0, f_1 are equivalent if they can be deformed smoothly into one another along a path $f_t\colon (S^2, A_t) \circlearrowleft$ of marked Thurston maps.

This notion is a combination of two stricter notions: *conjugacy* by a homeomorphism $(S^2, A_0) \to (S^2, A_1)$ and *isotopy rel* A, namely along a path of Thurston maps with constant marked set A. The *centralizer* of a Thurston map $f\colon (S^2, A) \circlearrowleft$ is the group of pure mapping classes $g \in \mathbf{Mod}(S^2, A)$ such that $g^{-1}fg$ is isotopic to f.

Our work makes essential use of a fundamental invariant introduced by Nekrashevych, the *biset* of a branched covering. Choose a basepoint $* \in S^2\backslash A$ and write $G = \pi_1(S^2\backslash A, *)$. Then the biset of a branched self-covering $f\colon (S^2, A) \circlearrowleft$ is a set $B(f)$ equipped with two commuting actions of G, whose (appropriately defined) isomorphism class is a complete invariant of f up to combinatorial equivalence. Since G is a free group, calculations in $B(f)$ are easy to perform.

The theory is slightly complicated by a family of branched self-coverings $f\colon (S^2, A) \circlearrowleft$ that come from a self-covering of the torus $\tilde f\colon T^2 \circlearrowleft$ via a degree-2 branched covering $\wp\colon T^2 \to S^2$; i.e. $\wp \circ \tilde{f} = f \circ \wp$. Suppose we have $\tilde{f}(z) = Mz + b$ on the model $\mathbb{R}^2/\mathbb{Z}^2$ of T^2, and assume that the eigenvalues of M are real but not rational. We call the map f *irrational doubly covered by a torus endomorphism*. It may furthermore be *expanding*, if all eigenvalues of M have norm > 1.

A *Levy obstruction* is a cycle of curves $\gamma_0, \gamma_1, \dots, \gamma_n = \gamma_0$ on $S^2 \backslash A$ with f mapping γ_i to γ_{i+1} by degree 1, up to isotopy. It is clearly an obstruction to the existence of an f-expanding metric, and we show that it is the only one:

Theorem A. — *Suppose $f\colon (S^2, A) \circlearrowleft$ is a Thurston map with degree at least 2 such that f admits no Levy obstruction. Then either f is isotopic to a map expanding a metric on $S^2 \backslash A$, or f is isotopic to the quotient by the involution $z \mapsto -z$ of an affine map on $\mathbb{R}^2/\mathbb{Z}^2$ whose eigenvalues are different from ± 1.*

By a *decomposition* of a map $f\colon (S^2, A) \circlearrowleft$ we mean a decomposition of $S^2 \backslash A$ into punctured spheres along an f-invariant multicurve; the *pieces* of the decomposition are the return maps of f on the sub-spheres.

We first show that every Thurston map $f\colon (S^2, A) \circlearrowleft$ may canonically be decomposed into pieces that, up to isotopy, (1) are homeomorphisms, or (2) expand a metric on $S^2 \backslash A$, or (3) are non-expanding irrational doubly covered by torus endomorphisms; see Figure 0.1.

The second case (expanding a metric) is equivalent to (2′) a topological property of f (it does not admit Levy cycles), and to (2″) a group-theoretical property (the biset of f is contracting).

According to the classical Nielsen–Thurston theory, homeomorphisms in case (1) may canonically be further decomposed, again up to isotopy, into maps of finite order and pseudo-Anosov homeomorphisms, namely homeomorphisms that preserve a pair of transverse foliations on $S^2 \backslash A$.

The decomposition theory of Pilgrim [28] lets us decompose expanding maps that are not doubly covered by irrational torus endomorphisms into pieces that preserve a complex structure, namely are rational maps. Therefore (2) implies another group-theoretical property (4): the biset of f decomposes as an amalgam over cyclic bisets (i.e. transitive bisets over cyclic groups), with rational pieces, and an algebraic-geometry property (5): there exists a complex stable curve (an algebraic variety $\mathbb{X}$ consisting of complex spheres with marked points, arranged as a cactoid) and a rational map $\mathbb{X} \dashrightarrow \mathbb{X}$ that becomes isotopic to f when the nodes of $\mathbb{X}$ are resolved; see Theorem D.

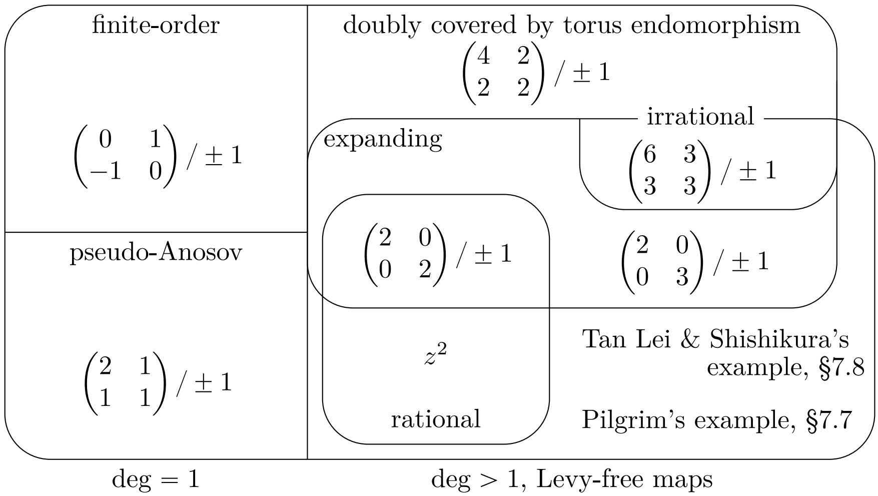


Figure 0.1. Geometric maps

We show how decision problems — isotopy rel A and computation of centralizers — can be promoted from pieces in a decomposition to the global map. We also show that the points in $A\backslash P_f$ can be encoded in group-theoretical language, as finite sequences of biset elements. Finally, isotopy and centralizers can be computed for each of the pieces: finite-order, pseudo-Anosov, rational maps and maps doubly covered by torus endomorphisms. Our main result follows:

Theorem B. — *It is decidable whether or not two Thurston maps are combinatorially equivalent.*

Furthermore, the centralizer of a Thurston map f (i.e. the set of homeomorphisms that commute with f up to isotopy) is effectively computable.

This extends a series of partial results: Bonnot, Braverman and Yampolsky show in [11] that equivalence to a rational map is decidable, and Selinger and Yampolsky show in [33, Main Theorem III] that it is decidable whether g is equivalent to f provided that all return maps in the canonical decomposition of f are rational maps with hyperbolic orbifolds.

Even though the first examples of branched self-coverings of the sphere of degree > 1 are rational maps, they benefit greatly from sometimes being considered as topological maps. This is because surgery may be performed on topological maps: one may decompose them into smaller pieces, glue two maps together ("mating", see §IV.3), etc. A fundamental theorem of

Thurston asserts that, once a topological condition is satisfied ("no annular obstruction"), the topological map can be isotoped back into a unique rational map. What we are proposing, in this research, is to express the topology in a group-theoretical language in which fundamental questions become decidable and effectively computable, see §7.

Note that, in contrast to non-invertible branched self-coverings, the algorithmic theory of **Mod** is fairly advanced, and in particular the Nielsen–Thurston decomposition is known to be efficiently computable, starting with train tracks as shown by Bestvina and Handel [10], and actually in polynomial-time as recently announced by Margalit, Strenner and Yurttas.

0.2. Structure of the papers

In the first article [3], we develop the general machinery of bisets and decompositions of bisets. The main definitions are graphs of groups and graphs of bisets, and the main result is a van Kampen-like theorem: given a correspondence $Y \leftarrow Z \rightarrow X$ and appropriately compatible covers of X, Y, Z, a graph of bisets is obtained by restricting the correspondence to the sets in the cover; and the van Kampen theorem expresses the biset of the correspondence as the "fundamental biset" of the graph of bisets, just as the fundamental group of a space is the fundamental group of its graph of groups.

In the second article [4], we specialize to punctured spheres, or more generally orbispace structures on spheres, which we treat as groups of the form

$$G = \langle \gamma_1, \dots, \gamma_n \mid \gamma_1^{e_1} = \dots = \gamma_n^{e_n} = \gamma_1 \cdots \gamma_n = 1 \rangle$$

with all $e_i \in \{2, 3, \dots, \infty\}$. Curves on $S^2 \backslash A$ are treated as conjugacy classes in G, and multicurves as collections of conjugacy classes. We use detailed structure about pure mapping class groups to explain how conjugacy and centralizer problems for pieces of a decomposition can be promoted to the original Thurston map.

In the third article [5], we study the effect of erasing punctures (periodic cycles or marked preimages of post-critical points) from a Thurston map. We show how these erased points can be encoded by a finite subset of the biset called a *portrait*. We use this language to study more carefully the maps that are doubly covered by torus endomorphisms and characterize them via elementary group theory.

In the fourth article [6], we prove the first decomposition theorem, of an arbitrary Thurston map into homeomorphisms, expanding maps and maps doubly covered by torus endomorphisms. We also give the characterization of expanding maps as Levy-free maps and as maps with contracting biset. In particular, the above decomposition is along a minimal Levy multicurve such that all pieces are Levy-free or homeomorphisms.

In the fifth article [7], we describe algorithms, and their implementations, that

- convert the Poirier description of a complex polynomial by its external angles into a biset;
- convert the Hubbard tree description of a polynomial into a biset;
- convert a polynomial biset into its Poirier description by external angles;
- convert a floating-point approximation of a rational map into a biset;
- convert a sphere biset into a complex rational map with algebraic coefficients, or produce an invariant multicurve that testifies to the inexistence of a rational map.

The first three algorithms are entirely symbolic, while the last two require floating-point calculations as well as manipulations of triangulations on the sphere. All these algorithms have been implemented in the software package Img [1] within the computer algebra system Gap [37].

Finally, we prove Theorem B in §6, assuming all the results in the previously mentioned articles.

We give below, in separate sections I–V, condensates of the contents of these articles following their numbering, with relevant definitions and sketches of proofs, and conclude this article with, in §6, the skeleton of the proof of Theorem B and, in §7, a series of examples seen from the topological, group-theoretical and algebraic perspectives.

0.3. Pilgrim's decomposition

Pilgrim develops in [28] a decomposition theory for branched coverings. In particular, he constructs a *canonical obstruction*, which is a multicurve Γ_f associated to a branched self-covering $f\colon (S^2, A) \circlearrowleft$ that is not doubly covered by a torus endomorphism and which has the property that f is combinatorially equivalent to a rational map if and only if $\Gamma_f = \varnothing$.

Let us review the construction, omitting on purpose the case of maps doubly covered by torus endomorphisms. The *Teichmüller space* $\mathscr{T}_A$ of (S^2, A) is the space of complex structures on the marked sphere (S^2, A), or equivalently Riemannian metrics on $S^2 \backslash A$ of curvature -1. Thurston associates with $f\colon (S^2, A) \circlearrowleft$ a self-map $\sigma_f\colon \mathscr{T}_A \circlearrowleft$ defined by pulling back complex structures through f. He shows (see [17]) that σ_f is weakly contracting for the Teichmüller metric on $\mathscr{T}_A$, so that (starting from an arbitrary point $\tau \in \mathscr{T}_A$) either $\sigma_f^n(\tau)$ converges to a fixed point (which is then a complex structure preserved by f, so f is combinatorially equivalent to a rational map) or degenerates to the boundary of $\mathscr{T}_A$, in which case some curves on $S^2\backslash A$ become very short in the hyperbolic metric defined by $\sigma_f^n(\tau)$. The canonical obstruction Γ_f is simply defined as the collection of simple closed curves on $S^2\backslash A$ whose length goes to 0 as $n \to \infty$.

Selinger gives in [32, Theorem 5.6] a topological characterization of the canonical obstruction. As a consequence, we deduce that Pilgrim's canonical obstruction is the union of the *Levy obstruction* (the multicurve along which S^2 is pinched to produce the Levy decomposition) and the *rational obstruction* (the multicurve along which the expanding maps of the Levy decomposition should be further pinched to give rational maps). Selinger and Yampolsky show in [33, Main Theorem I] that the canonical obstruction is computable.

0.4. Remarks

The main objects of study are *bisets* (called *combinatorial bimodules* in [26]), which we generalize by allowing different groups H, G to to act on the left and right respectively. Bisets may be thought of as generalizations of group homomorphisms, up to pre- and post-composition by inner automorphisms. Indeed, if $\phi\colon H \to G$ is a group homomorphism, written $h \mapsto h^\phi$, one associates with it the H-G-set B_ϕ, which, qua right G-set, is plainly G; the left H-action is by

$$h \cdot b = h^\phi b\,.$$

Conversely, if B is a transitive H-G biset (a general biset splits as a disjoint union of its transitive components), then there is a group K and there are homomorphisms $\phi\colon K \to G$ and $\psi\colon K \to H$ such that $B \cong B_\psi^\vee \otimes B_\phi$, where $B_\psi^\vee$ is the contragredient of B_ψ, see §I, so bisets may also be thought as correspondences of groups. In fact, to every topological correspondence $Y \leftarrow Z \to X$ there is a naturally associated $\pi_1(Y)$-$\pi_1(X)$-biset, independent of basepoints up to isomorphism.

The main decision problems we study are, in the context of bisets, the *conjugacy problem* that can be asked in any category with multiplication (given B, C, does there exist X with $XB = CX$?), *witnessed conjugacy problem* (given B, C, find an X with $XB = CX$ or prove that there are none), and the *centralizer problem* (given B, describe the set of X with $BX = XB$).

We proceed from the best-behaved bisets (that of rational Thurston maps) to the general case by following diverse reductions, in particular introducing *portraits of bisets* to add and erase marked points, and *trees of bisets* to glue and cut along multicurves.

In fact, by a general trick, only the centralizer problem needs to be considered: assuming that the centralizer problem is solvable and given B, C, compute the centralizer of $B \sqcup C$, and check whether it contains an element that switches B and C; if so, its restriction to C is a witness for conjugacy of B and C. This is easy to check e.g. if centralizers are finitely generated subgroups of well-understood groups. We show, however, that centralizers are only computable in the weaker sense of being expressible as kernels of maps from well-understood groups to Abelian groups. In particular, they can be infinitely generated, see Example 7.9. For extra clarity, we treat all three decision problems in parallel.

We restrict ourselves to studying actions of the pure mapping class group: punctures on our marked spheres may be permuted by Thurston maps, but are fixed by the mapping classes. In this manner, the *portrait* of Thurston maps, namely the dynamics on their marked points, is preserved by pre- and post-composition by mapping classes. One could extend the statements and algorithms to non-pure mapping class groups, at the cost of introducing finite groups in a few places. The action of non-pure mapping classes would better capture the notions of conjugation, centralizer and combinatorial equivalence of maps. In particular, the action of non-pure (i.e. fractional) Dehn twists could lead to valuable systematic constructions of Thurston maps, in the spirit of near-Euclidean Thurston maps [13].

0.5. Acknowledgments

We are grateful to Kevin Pilgrim and Thomas Schick for enlightening discussions.

0.6. Notations

Here are some notations that shall be used throughout the texts:

- The symmetric group on a set S is written $S\wr$
- Concatenation of paths is written $\gamma\#\delta$ for "first γ, then δ"; inverses of paths are written γ^{-1}

- The identity map is written $\mathbb{1}$. Composition of maps, permutations etc. is in the algebraic, left-to-right order, unless explicitly written as $(f \circ g)(x) = f(g(x))$. The restriction of a map $f\colon A \to B$ to a subset $C \subseteq A$ is written $f{\restriction}_C$. Self-maps are written $f\colon X \circlearrowleft$ in preference to $f\colon X \to X$
- We write $\approx$ for isotopy of paths, maps etc, $\sim$ for conjugacy or combinatorial equivalence, and $\cong$ for isomorphism of algebraic objects
- We try to use similar fonts for similar objects: script $\mathscr{C}$ for multicurves, $\mathfrak{X}$ for graphs of groups, G_z for its vertex and edge groups (even if the graph of groups is not called G), $\mathfrak{B}$ for graph of bisets, B_z for its vertex and edge bisets, usually Greek letters for functions.

We also establish a kind of "dictionary" between topological and algebraic notions:

Topology	Algebra
Continuous map	Right-principal biset $B(f)$
Covering map	Left-free right-principal biset
Composition of maps $g \circ f$	Tensor product of bisets $B(f) \otimes B(g)$
Topological correspondence (f, i)	Biset $B(f, i) = B(i)^\vee \otimes B(f)$
Covering pair (f is a covering)	Left-free biset
Decomposition of correspondence	Graph of bisets
Punctured sphere (S^2, A)	Sphere group $G = \langle \gamma_1, \dots, \gamma_n \mid \gamma_1 \cdots \gamma_n \rangle$
Puncture $a_i \in A$	Peripheral conjugacy class γ_i^G in G
Mapping class group $\mathbf{Mod}(S^2, A)$	Outer automorphism group $\mathbf{Mod}(G)$
Multicurve $\mathscr{C}$	Family of essential conjugacy classes in G
Decomposition of S^2 along $\mathscr{C}$	Decomposition of G as sphere tree of groups $\mathfrak{X}$
$\mathscr{C} \sqcup A$	Distinguished conjugacy classes X of $\mathfrak{X}$
Homeomorphism $(S^2, B, \mathscr{D}) \to (S^2, A, \mathscr{C})$	Conjugator ${}_{\mathfrak{Y}}\mathfrak{I}_{\mathfrak{X}}$ between trees of groups
Group of Dehn twists along $\mathscr{C}$	$\mathbb{Z}^{\mathscr{C}}$
Stabilizer $\mathbf{Mod}(S^2, A, \mathscr{C})$ of $\mathscr{C}$	Stabilizer $\mathbf{Mod}(\mathfrak{X})$ of $\mathfrak{X}$
Branched covering $f\colon (S^2, B) \to (S^2, A)$	Sphere H-G-biset $B(f)$
Isotopy rel A	Isomorphism of sphere bisets
Mapping class biset $M(f)$	Mapping class biset $M(B)$
Thurston map $f\colon (S^2, A) \circlearrowleft$	Sphere G-G-biset $B(f)$
Expanding map f	Contracting biset $B(f)$
Restriction $f\colon A \circlearrowleft$	Portrait $B_*\colon A \circlearrowleft$
Combinatorial equivalence	Conjugacy of sphere bisets
Centralizer $Z(f)$	Centralizer $Z(B)$
Extra marked points of expanding map f	Portrait of bisets for $B(f)$
Torus endomorphism $\mathbb{R}^2/\mathbb{Z}^2 \circlearrowleft$	Biset ${}_{\mathbb{Z}^2}B_{\mathbb{Z}^2}$ of a linear map
Map doubly covered by $\mathbb{R}^2/\mathbb{Z}^2 \circlearrowleft$	Crossed product ${}_{\mathbb{Z}^2}B_{\mathbb{Z}^2} \rtimes \{\pm 1\}$
Decomposition of f along multicurve	Decomposition of $B(f)$ as tree of bisets $\mathfrak{B}$
Sub-mapping class biset $M(f, B, A, \mathscr{D}, \mathscr{C})$	$M(\mathfrak{B})$
Renormalization of f w.r.t. $\mathscr{C}$	Return bisets of $\mathfrak{B}$

I. Bisets and van Kampen's theorem [3]

Let $f\colon Y \to X$ be a continuous map between topological spaces. Fix basepoints $\dagger \in Y$ and $* \in X$, and consider the fundamental groups $H = \pi_1(Y,\dagger)$ and $G = \pi_1(X,*)$. Then the map f may be encoded into an H-G-*biset*, namely a set $B(f)$ with commuting left H-action and right G-action. As a set, $B(f)$ is the set of homotopy classes of paths, in X, from $f(\dagger)$ to $*$. The actions of H and G are respectively given by pre-catenation of the f-image and by post-catenation. To recall the acting groups, we sometimes write ${}_H B(f)_G$. Bisets can be multiplied; the *product* $B \otimes_G C$ of an H-G-biset B with a G-F-biset C is the H-F-biset $(B \times C)/\{(bg,c) = (b,gc)\}$. The *contragredient* of the H-G-biset B is the G-H-biset $B^\vee$, which is B as a set with actions $g \cdot (b^\vee) \cdot h = (h^{-1}bg^{-1})^\vee$.

An *intertwiner* from an H-G-biset B to an H'-G'-biset B' is a map $\beta\colon B \to B'$ and a pair of homomorphisms $\gamma\colon H \to H'$ and $\alpha\colon G \to G'$ with

$$\beta(hbg) = \gamma(h)\beta(b)\alpha(g) \text{ for all } h \in H, b \in B, g \in G\,.$$

If $H = G$, $H' = G'$ and $\gamma = \alpha$, then the intertwiner is a *semiconjugacy.* If $H = H'$ and $G = G'$ and $\gamma = \mathbb{1}$ and $\alpha = \mathbb{1}$, then the intertwiner is a *morphism. Congruences*, *conjugacies*, and *isomorphisms* are invertible intertwiners, semiconjugacies, and morphisms respectively.

$$\begin{array}{ccccc}
\begin{array}{c}\text{semiconjugacies}\\ (\alpha,\beta,\alpha)\\ G = H, G' = H'\end{array} & \subset & \begin{array}{c}\text{intertwiners}\\ (\gamma,\beta,\alpha)\end{array} & \supset & \begin{array}{c}\text{morphisms}\\ (\mathbb{1},\beta,\mathbb{1})\\ G = G', H = H'\end{array}\\
\cup & & \cup & & \cup\\
\text{conjugacies} & \subset & \begin{array}{c}\text{congruences}\\ \exists(\gamma^{-1},\beta^{-1},\alpha^{-1})\end{array} & \supset & \text{isomorphisms}
\end{array}$$

Note that if $G = H$ then every morphism is also a semiconjugacy.

Let X, Y be path connected topological spaces. Bisets are well adapted to encode more general objects than continuous maps $Y \to X$, namely *topological correspondences*. These are triples (Z, f, i) consisting of a topological space Z and continuous maps $f\colon Z \to X$ and $i\colon Z \to Y$, and are simply written $Y \leftarrow Z \to X$. If Z is path connected, then the biset of the correspondence is $B(f,i) = B(i)^\vee \otimes B(f)$; in general, it is the disjoint union of the bisets on all path connected components of Z.

I.1. Coverings and left-free bisets

The biset ${}_H B(f)_G$ of a continuous map $f\colon Y \to X$ is, by construction, isomorphic to G_G qua right G-set. If furthermore f is a covering, say of degree d, then ${}_HB(f)$ is free of degree d qua left H-set. In particular, if in the topological correspondence $Y \leftarrow Z \rightarrow X$ the map f is a degree-d covering, then $B(f,i)$ is left-free of degree d. The correspondence is called a *covering pair.*

Choose then a subset $S \subseteq B(f,i)$ of cardinality d that intersects once every left H-orbit; such a subset is called a *basis.* Using the isomorphism ${}_HB(f,i) = H \times S$, we may write the right G-action on $B(f,i)$ in the form

$$s \cdot g = h \cdot s' \text{ for some } h \in H,\ s' \in S\,.$$

This is a map $S \times G \to H \times S$, which (writing $S\wr$ for the permutation group on S) yields a group homomorphism

$$\psi \colon G \mapsto H^S \rtimes S\wr$$

called the *wreath recursion* of $B(f,i)$. Writing $S = \{\ell_1, \dots, \ell_d\}$, we may write ψ as

$$g \mapsto \langle\!\langle h_1, \dots, h_d \rangle\!\rangle \pi$$

for elements $h_1, \dots, h_d$ and a permutation π. It is sufficient to specify ψ on generators of g, and these data are called a *presentation* of the biset.

We shall see in §7 many examples of biset presentations. We stress here that they are eminently computable, and in particular with pencil and paper. Here is the concrete recipe, for a covering correspondence $Y \leftarrow Z \rightarrow X$. Fix basepoints $* \in X$ and $\dagger \in Y$, and write $G = \pi_1(X,*)$ and $H = \pi_1(Y,\dagger)$. Choose for each $z \in f^{-1}(*)$ a path ℓ_z in Y from $i(z)$ to $\dagger$, and set $S = \{\ell_z \mid z \in f^{-1}(*)\}$. For every (generator) $g \in G$, represented as a curve $g\colon [0,1] \to X$, and for every $\ell_z \in S$, there exists a unique lift $\tilde{g}_z \colon [0,1] \to Z$ of g that starts at z. Let z' be the endpoint of $\tilde{g}_z$. Then the concatenation $h_z := \ell_z^{-1} \# (i \circ \tilde{g}_z) \# \ell_{z'}$ is a loop at $\dagger$, and its class in H is independent of the choice of representative for g. The wreath recursion of $B(f,i)$ is the map $g \mapsto \langle\!\langle h_{z_1}, \dots, h_{z_d} \rangle\!\rangle \pi$ with $\pi \in S\wr$ the permutation $z \mapsto z'$.

I.2. Graphs of bisets

The van Kampen theorem expresses the fundamental group of a topological space in terms of the fundamental groups of subspaces. A convenient algebraic object that captures the data is a graph of groups — a graph decorated with groups such that each edge group has two morphisms into the neighboring vertex groups. It is therefore convenient [34] to double each edge — to replace it with a pair of directed edges of the opposite orientation.

We view *graphs* as sets $\mathfrak{X}$ endowed with two maps $x \mapsto x^-$ and $x \mapsto \overline{x}$, with axioms $\overline{\overline{x}} = x$ and $(x^-)^- = x^-$ and $x^- = x \Leftrightarrow \overline{x} = x$. The vertex set

$V(\mathfrak{X})$ is then $\{x \mid x^- = x\}$, and the edge set $E(\mathfrak{X})$ is $\mathfrak{X}\backslash V(\mathfrak{X})$. The object $\overline{x}$ is called the *reverse* of x. Setting $x^+ := (\overline{x})^-$, the vertices x^- and x^+ are respectively the *origin* and *terminus* of x. A *graph morphism* is a map $h\colon \mathfrak{X} \to \mathfrak{X}'$ such that $h(x^-) = h(x)^-$ and $h(\overline{x}) = \overline{x}$ for all $x \in \mathfrak{X}$. Note that the image of a vertex is a vertex while the image of an edge is either an edge or a vertex.

Recall (e.g. from [34, §4]) that a graph of groups is a graph $\mathfrak{X}$ with a group G_x associated with each $x \in \mathfrak{X}$, and for each edge $x \in \mathfrak{X}$ homomorphisms $G_x \to G_{x^-}$ and $G_x \to G_{\overline{x}}$, written respectively $g \mapsto g^-$ and $g \mapsto \overline{g}$. Its *fundamental group* $\pi_1(\mathfrak{X}, v)$ is the set of group-decorated loops $g_0 e_1 g_1 \cdots e_n g_n$ with $e_1 \cdots e_n$ a loop at $v \in V(\mathfrak{X})$ in $\mathfrak{X}$ and $g_i \in G_{e_i^+}$ for all i, up to the relations $eg^+ = g^- e$ for all edges $e \in E$ and all $g \in G_e$. More generally, given $v, w \in \mathfrak{X}$ we define $\pi_1(\mathfrak{X}, v, w)$ as the set of group-decorated paths $g_0 e_1 g_1 \cdots e_n g_n$ with $e_1 \cdots e_n$ a path from v to w and $g_i \in G_{e_i^+}$ for all i, up to the relations $eg^+ = g^- e$ as above. The set $\pi_1(\mathfrak{X}, v, w)$ is naturally a $\pi_1(\mathfrak{X}, v)$-$\pi_1(\mathfrak{X}, w)$ biset. Given $p \in \pi_1(\mathfrak{X}, u, v)$ and $q \in \pi_1(\mathfrak{X}, v, w)$ their product pq is in $\pi_1(\mathfrak{X}, u, w)$.

Definition I.1 (Graph of bisets). — *Let $\mathfrak{X}, \mathfrak{Y}$ be two graphs of groups. A* graph of bisets ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ *between them is the following data:*

- *a graph $\mathfrak{B}$;*
- *graph morphisms $\lambda\colon \mathfrak{B} \to \mathfrak{Y}$ and $\rho\colon \mathfrak{B} \to \mathfrak{X}$;*
- *for every $z \in \mathfrak{B}$, a $G_{\lambda(z)}$-$G_{\rho(z)}$-biset B_z, an intertwiner $()^-\colon B_z \to B_{z^-}$ with respect to the homomorphisms $G_{\lambda(z)} \to G_{\lambda(z)^-}$ and $G_{\rho(z)} \to G_{\rho(z)^-}$, and an intertwiner $\overline{()}\colon B_z \to B_{\overline{z}}$ with respect to the homomorphisms $G_{\lambda(z)} \to G_{\overline{\lambda(z)}}$ and $G_{\rho(z)} \to G_{\overline{\rho(z)}}$. These intertwiners satisfy natural axioms: the composition $B_z \to B_{\overline{z}} \to B_{\overline{\overline{z}}} = B_z$ is the identity for every $z \in \mathfrak{X}$, and if $z \in V(\mathfrak{X})$, then the homomorphisms $B_z \to B_{z^-}$ and $B_z \to B_{\overline{z}}$ are the identity. For $b \in B_z$ we write $b^+ = \overline{b}^-$.*

*We call $\mathfrak{B}$ a $\mathfrak{Y}$-$\mathfrak{X}$-*biset.

Definition I.2 (Fundamental biset of graph of bisets). — *Let $\mathfrak{B}$ be a $\mathfrak{Y}$-$\mathfrak{X}$-biset; choose $* \in V(\mathfrak{X})$ and $\dagger \in V(\mathfrak{Y})$. Write $G = \pi_1(\mathfrak{X}, *)$ and $H = \pi_1(\mathfrak{Y}, \dagger)$. The* fundamental biset *of $\mathfrak{B}$ is an H-G-biset $B = \pi_1(\mathfrak{B}, \dagger, *)$, constructed as follows.*

$$B = \frac{\bigsqcup_{z \in V(\mathfrak{B})} \pi_1(\mathfrak{Y}, \dagger, \lambda(z)) \otimes_{G_{\lambda(z)}} B_z \otimes_{G_{\rho(z)}} \pi_1(\mathfrak{X}, \rho(z), *)}{\left\{ qb^- p = q\lambda(z) b^+ \overline{\rho(z)} p \quad \forall \begin{array}{c} q \in \pi_1(\mathfrak{Y}, \dagger, \lambda(z)^-), b \in B_z, \\ p \in \pi_1(\mathfrak{X}, \rho(z)^-, *), z \in E(\mathfrak{B}) \end{array} \right\}}. \tag{I.1}$$

In other words, elements of B are sequences $h_0 y_1 h_1 \cdots y_n\, b\, x_1 \cdots g_{m-1} x_n g_n$ subject to the equivalence relations used previously to define $\pi_1(\mathfrak{X})$, as well as $y_n h b^- g x_1 \leftrightarrow y_n h \lambda(z) b^+ \overline{\rho(z)} g x_1$ for all $z \in \mathfrak{B}$, $b \in B_z$, $h \in G_{\lambda(z)^-}$, $g \in G_{\rho(z)^-}$.

Up to congruence, the fundamental biset is independent on the choice of basepoints: $\pi_1(\mathfrak{B}, \dagger, *) = \pi_1(\mathfrak{Y}, \dagger, \dagger') \otimes \pi_1(\mathfrak{B}, \dagger', *') \otimes \pi_1(\mathfrak{X}, *', *)$.

The definition is a bit unwieldy, but it has a simpler version in case the graph of bisets is *left-fibrant*, see [3, Definition 3.16]. Such graphs of bisets arise from correspondences where one of the maps is a fibration (such as a covering). A left-fibrant graph of bisets possesses a lifting property: any $pbq \in \pi_1(\mathfrak{B}, \dagger, *)$ can be rewritten in an essentially unique way as $pq'b''$ for some b' in a vertex biset. Thus (I.1) takes the form [3, (9)]

$$\pi_1(\mathfrak{B}, \dagger, *) = \bigsqcup_{z \in \rho^{-1}(*)} \pi_1(\mathfrak{Y}, \dagger, \lambda(z)) \otimes_{G_{\lambda(z)}} B_z\,, \tag{I.2}$$

with right action given by lifting of paths in $\pi_1(\mathfrak{X}, *)$.

A biset ${}_H B_G$ is *biprincipal* if both actions are free and transitive. A graph of bisets ${}_{\mathfrak{Y}}\mathfrak{I}_{\mathfrak{X}}$ is *biprincipal* if

(1) $\lambda\colon \mathfrak{I} \to \mathfrak{Y}$ and $\rho\colon \mathfrak{I} \to \mathfrak{X}$ are graph isomorphisms; and
(2) B_z are biprincipal for all objects $z \in \mathfrak{I}$.

We use this notion to define congruence and conjugacy of graphs of bisets:

Definition I.3. — *Two graphs of groups $\mathfrak{Y}$, $\mathfrak{X}$ are called* congruent *if there is a biprincipal graph of bisets ${}_{\mathfrak{Y}}\mathfrak{I}_{\mathfrak{X}}$.*

Isomorphism of graphs of bisets is meant in the strongest possible sense: isomorphism of the underlying graphs, and isomorphisms of the respective bisets. There is a general notion of tensor product of graphs of bisets, which in the cases below simply amounts to tensoring the vertex and edge bisets together.

Two graphs of bisets ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ and ${}_{\mathfrak{Y}'}\mathfrak{C}_{\mathfrak{X}'}$ are congruent *if there are biprincipal graph of bisets ${}_{\mathfrak{Y}}\mathfrak{I}_{\mathfrak{Y}'}$ and ${}_{\mathfrak{X}}\mathfrak{L}_{\mathfrak{X}'}$ such that ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}} \otimes \mathfrak{L}$ and $\mathfrak{I} \otimes {}_{\mathfrak{Y}'}\mathfrak{C}_{\mathfrak{X}'}$ are isomorphic.*

Two graphs of bisets ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ and ${}_{\mathfrak{Y}}\mathfrak{C}_{\mathfrak{Y}}$ are conjugate *if there is a biprincipal graph of bisets ${}_{\mathfrak{X}}\mathfrak{I}_{\mathfrak{Y}}$ such that $\mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{I}$ and $\mathfrak{I} \otimes_{\mathfrak{Y}} \mathfrak{C}$ are isomorphic.*

I.3. Graphs of bisets from 1-dimensional covers

Definition I.4 (Finite 1-dimensional covers). — *Consider a path connected space X, covered by a finite collection of path connected (not necessarily open) subspaces $(X_v)_{v\in V}$. It is a* finite 1-dimensional cover *of X if*

- *for every $u,v \in V$ and for every path connected component X' of $X_u \cap X_v$ there are an open neighbourhood $\widetilde{X}' \supset X'$ and an $X_w \subset X'$ such that $X_w \hookrightarrow \widetilde{X}'$ is a homotopy equivalence;*
- *if $X_u \subseteq X_v \subseteq X_w$ then $u = v$ or $v = w$.*

We order V by writing $u < v$ if $X_u \subsetneqq X_v$.

Definition I.5 (Graphs of groups from covers). — *Consider a path connected space X with a 1-dimensional cover $(X_v)_{v\in V}$. It has an associated graph of groups $\mathfrak{X}$, defined as follows. The vertex set of $\mathfrak{X}$ is V. For every pair $u < v$ there are edges e and $\overline{e}$ connecting $u = e^- = \overline{e}^+$ and $v = e^+ = \overline{e}^-$, and we let E be the set of these edges. Set $\mathfrak{X} = V \sqcup E$.*

Choose basepoints $*_v \in X_v$ for all $v \in V$. Choose for each edge e a path ℓ_e from $*_{e^-}$ to $*_{e^+}$ such that $\ell_{\overline{e}} = \ell_e^{-1}$. Set $G_v := \pi_1(X_v, *_v)$ for every $v \in V$. For every edge e with $e^- < e^+$ set $G_e := G_{e^-}$; define $G_e \to G_{e^-} := \mathbb{1}$ and $G_e \to G_{e^+}$ by $\gamma \mapsto \ell_e^{-1}\#\gamma\#\ell_e$. For every edge e with $e^- > e^+$ define $G_e := G_{\overline{e}}$ and define morphisms $G_e \to G_{e^-}$ and $G_e \to G_{e^+}$ as $G_{\overline{e}} \to G_{\overline{e}^+}$ and $G_{\overline{e}} \to G_{\overline{e}^-}$ respectively.

Consider now a correspondence (Z,f,i), with $f\colon Z \to X$ and $i\colon Z \to Y$, between path connected spaces X and Y. Suppose that (U_α), (V_β), and (W_γ) are finite 1-dimensional covers of X, Y, and Z respectively, *compatible with* f and i: for every γ there are $\lambda(\gamma)$ and $\rho(\gamma)$ such that $f(W_\gamma) \subset U_{\rho(\gamma)}$ and $i(W_\gamma) \subset V_{\lambda(\gamma)}$. Then the graph of bisets ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ of (f,i) with respect to the above data is as follows:

- the graphs of groups $\mathfrak{X}$ and $\mathfrak{Y}$ are constructed as in Definition I.5 using the covers (U_α) and (V_β) of X,Y respectively. Choices of paths ℓ_e, m_e were made for edges e in $\mathfrak{X}$, $\mathfrak{Y}$ respectively;
- the underlying graph of $\mathfrak{B}$ is similarly constructed using the cover (W_γ) of Z. For every vertex $z \in \mathfrak{B}$ the biset ${}_{G_{\lambda(z)}}(B_z)_{G_{\rho(z)}}$ is $B(f\restriction_{W_z}, i\restriction_{W_z})$;
- for every edge $e \in \mathfrak{B}$ representing the embedding $W_{z'} \subsetneqq W_z$ the biset B_e is $B_{z'}$, and if e is oriented so that $e^- = z'$ then the intertwiners $()^\pm$ are the maps $()^- = \mathbb{1}\colon B_e \to B_{z'}$ and $()^+\colon B_e \to B_z$

given by $(\gamma^{-1},\delta)\mapsto(m_{\lambda(e)}^{-1}\#\gamma^{-1},\delta\#\ell_{\rho(e)})$ in the description of B_e as $B(i\downharpoonright_{W_{e^-}})^\vee\otimes B(f\downharpoonright_{W_{e^-}})$.

The graphs of groups $\mathfrak{X},\mathfrak{Y}$ and the graph of bisets $\mathfrak{B}$ are independent of the choices of basepoints and connecting paths ℓ_e,m_e, up to congruence.

Theorem I.6 (Van Kampen's theorem for correspondences). — *Let (f,i) be a topological correspondence from a path connected space Y to a path connected space X, and let ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ be the graph of bisets subject to compatible finite 1-dimensional covers of spaces in question.*

Then for every $v\in\mathfrak{Y}$ and $u\in\mathfrak{X}$ we have an isomorphism

$$B(f,i,\dagger_v,*_u)\cong\pi_1(\mathfrak{B},v,u),$$

where $\dagger_v$ and $*_u$ are basepoints.

If in Theorem I.6 the map $f\colon Z\to Y$ is a covering and all restrictions $f\colon W_\gamma\to U_{\rho(\gamma)}$ are covering maps, then $\mathfrak{B}$ is left-fibrant. In this case its fundamental biset is computed by (I.2).

I.4. Hubbard trees

Consider a polynomial $p(z)\in\mathbb{C}[z]$. We will see how to construct a graph of bisets out of p's "Hubbard tree", see Figure I.1. We first recall some basic definitions and properties; see [15, 16] for details.

The *post-critical set* $P(p)$ is the forward orbit of p's critical values:

$$P(p):=\{p^n(z)\mid p'(z)=0,\ n\geqslant 1\}.$$

The polynomial p is *post-critically finite* if $P(p)$ is finite. The *Julia set* $J(p)$ of p is the boundary of the filled-in Julia set $K(p)$, and the *Fatou set* is its complement:

$$K(p):=\{z\in\mathbb{C}\mid\{p^n(z)\mid n\in\mathbb{N}\}\text{ is bounded}\},\quad J_c=\partial K_c,\quad F(p)=\mathbb{C}\backslash J(p).$$

The *Hubbard tree* of p is the smallest tree in $K(p)$ that contains $P(p)$ and all of p's critical points; it intersects $F(p)$ along radial arcs. It is a simplicial graph, with some distinguished vertices corresponding to $P(p)$. All its vertices have an *order* in $\mathbb{N}\cup\{\infty\}$: by definition, p behaves locally as $z\mapsto z^{\deg_z(p)}$ at a point $z\in\mathbb{C}$, and

$$\operatorname{ord}(v)=\mathrm{l.c.m.}\{\deg_z(p^n)\mid n\geqslant 0,z\in p^{-n}(v)\}.$$

Thus in particular $\operatorname{ord}(v)=\infty$ if v is critical and periodic.

This order function defines an *orbispace* structure on $\mathbb{C}$, see II.5: a topological space with the extra data of a non-trivial group G_v attached at a

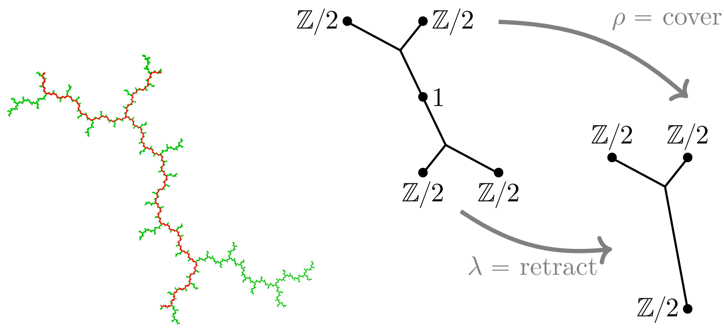


Figure I.1. The Julia set of $p(z) = z^2+i$, its Hubbard tree (in red), and its associated graph of bisets above graph of groups. The vertex groups and bisets are indicated on the picture, all edge bisets and groups are trivial, and the embeddings of edge bisets into vertex bisets are irrelevant.

discrete set of points v, in canonical neighbourhoods of which the fundamental group is isomorphic to G_v. In our situation, the group attached to $v \in P(p)$ is cyclic of order $\operatorname{ord}(v)$.

For each $z \in P(p)$, let γ_z denote a small loop around z, and identify γ_z with a representative of a conjugacy class in $\pi_1(\mathbb{C}\backslash P(p), *)$, see §II.1. It follows that the fundamental group of the orbispace defined by ord is given as follows:

$$G_p = \pi_1(\mathbb{C}\backslash P(p), *)/\langle \gamma_z^{\operatorname{ord}(z)} : z \in P(p)\rangle .$$

The biset $B(p)$ of p is the biset of the orbispace-correspondence

$$\big(p\colon (\mathbb{C}, p^{-1}(P(p))) \to (\mathbb{C}, P(p)), (\mathbb{C}, p^{-1}(P(p))) \hookrightarrow (\mathbb{C}, P(p))\big) .$$

Since p is an orbispace-covering, the biset $B(p)$ is left-free, see §I.1.

Out of the Hubbard tree T of p, we may construct a graph of groups $\mathfrak{X}$. Qua graph, $\mathfrak{X}$ is T. A cyclic group of order $\operatorname{ord}(v)$ is attached to every vertex $v \in T$, and the edges of $\mathfrak{X}$ all carry trivial groups.

We may also construct a graph of bisets ${}_{\mathfrak{X}}\mathfrak{T}_{\mathfrak{X}}$ as follows, using the Hubbard tree T. The underlying graph of $\mathfrak{T}$ is $p^{-1}(T)$ and $\rho\colon \mathfrak{T} \to \mathfrak{X}$ is given by the covering map $p\colon p^{-1}(T) \to T$. The map $\lambda\colon \mathfrak{T} \to \mathfrak{X}$ is the canonical retraction of $p^{-1}(T)$ to its subtree T. There is a degree-$\deg_z(p)$ cyclic biset attached to each vertex $z \in \mathfrak{T}$, and trivial bisets attached to edges of $\mathfrak{T}$. The biset inclusions are determined by an additional piece of information: *angles* between incoming edges at vertices of T. We shall give in Algorithm V.3 a procedure that constructs the graph of cyclic bisets directly out of the combinatorial data of T, see also [30].

Proposition I.7. — *Let p be a complex polynomial. Then the groups G_p and $\pi_1(\mathfrak{X},*)$ are isomorphic, and the bisets $B(p)$ and $\pi_1(\mathfrak{T})$ are conjugate via the group isomorphism between G_p and $\pi_1(\mathfrak{X},*)$.*

II. Spheres and their decompositions [4]

We specialize the maps we consider to branched coverings $(S^2,B) \to (S^2,A)$ between spheres with finitely many marked points. Decompositions of S^2 are given by *multicurves*, namely collections of disjoint simple closed curves on S^2. Once these small curves are pinched to points, one obtains a new collection of topological spheres attached at these points.

The main results of this part are decidability statements: conjugacy and isotopy questions for sphere maps can be translated to group theory. In the presence of a multicurve, these questions can be reduced to simpler questions on the restrictions of the maps to the topological spheres in the complement of the multicurve.

II.1. Sphere groups and maps

Spheres with marked points are described, within group theory, by their fundamental group. Let (S^2,A) be a topological sphere, marked by a finite subset $A \subset S^2$ with $\#A \geqslant 2$. Choose a basepoint $* \in S^2\backslash A$. Then the fundamental group of $S^2\backslash A$ may be computed as follows. Order $A = \{a_1,\dots,a_n\}$. Choose for each $a_i \in A$ a path $\overline{\gamma}_i$ from $*$ to a_i, such that $\overline{\gamma}_i$ and $\overline{\gamma}_j$ intersect only at $*$ for $i \neq j$, and such that the $\overline{\gamma}_i$ are cyclically ordered as $\overline{\gamma}_1,\dots,\overline{\gamma}_n$ counterclockwise around $*$. Let γ_i be the loop at $*$ that travels on the right of $\overline{\gamma}_i$, circles once counterclockwise around a_i, and returns to $*$ on the right of $\overline{\gamma}_i^{-1}$. We then have

$$G = \pi_1(S^2\backslash A,*) = \langle \gamma_1,\dots,\gamma_n \mid \gamma_1\cdots\gamma_n\rangle. \tag{II.1}$$

A cycle (loop without basepoint) in $S^2\backslash A$ is represented by a conjugacy class in G. The group G comes with extra data: the collection $\{\gamma_1^G,\dots,\gamma_n^G\}$ of conjugacy classes, called *peripheral conjugacy classes*, defined by the property that γ_i^G represents can be homotoped to a loop circling a_i once counterclockwise.

A *sphere map* $f\colon (S^2,B) \to (S^2,A)$ is a branched covering $f\colon S^2 \to S^2$ such that $f(B \cup \{\text{critical points}\}) \subseteq A$. The *biset* $B(f)$ is the biset $B(f,i)$ of the correspondence

$$(f\colon S^2\backslash f^{-1}(A) \to S^2\backslash A, i\colon S^2\backslash f^{-1}(A) \hookrightarrow S^2\backslash B).$$

Fixing basepoints $\dagger \in S^2\backslash B$ and $* \in S^2\backslash A$ for the fundamental groups $H := \pi_1(S^2\backslash B, \dagger)$ and $G := \pi_1(S^2\backslash A, *)$, the H-G-biset $B(f)$ may be concretely seen as

$$B(f) = \{\gamma\colon [0,1] \to S^2\backslash B \mid \gamma(0) = \dagger, f(\gamma(1)) = *\}\,.$$

The left- and right-actions are respectively by pre-catenation and post-catenation of the appropriate f-lift.

Two sphere maps $f_0, f_1\colon (S^2, B) \to (S^2, A)$ are *isotopic*, written $f_0 \approx f_1$, if there exists a path $(f_t\colon (S^2, B) \to (S^2, A))_{t\in[0,1]}$ of sphere maps connecting f_0 to f_1. Clearly, all $B(f_t)$ are isomorphic.

A *Thurston map* is a self-sphere map $f\colon (S^2, A) \circlearrowleft$. In that dynamical setting, we naturally assume that the basepoints $*$ and $\dagger$ coincide. Two Thurston maps $f\colon (S^2, A) \circlearrowleft$ and $g\colon (S^2, B) \circlearrowleft$ are *combinatorially equivalent* if there exists a homeomorphism $\phi\colon (S^2, A) \to (S^2, B)$ with $\phi \circ f \approx g \circ \phi$.

We consider algebraic counterparts to these notions. In [22], Hurwitz describes an elegant classification of degree-d branched coverings $S^2 \circlearrowleft$ with critical values contained in $\{a_1, \dots, a_n\}$ in terms of *admissible* n-tuples of permutations $(\sigma_i \in d\wr)_{i=1,\dots,n}$. A n-tuple is *admissible* if $\sigma_1 \cdots \sigma_n = 1$ and $\langle \sigma_1, \dots, \sigma_n\rangle$ is a transitive subgroup of $d\wr$ and the cycle lengths of the σ_i satisfy the condition

$$\sum_{i=1}^{n} \sum_{\substack{c \text{ cycle} \\ \text{of } \sigma_i}} \big(\text{length}(c) - 1\big) = 2d - 2\,. \tag{II.2}$$

Definition II.1 (Sphere groups). — *A* sphere group *is a tuple* $(G, \Gamma_1, \dots, \Gamma_n)$ *consisting of a group and n conjugacy classes Γ_i in G, such that G admits a presentation as in* (II.1) *for some choice of $\gamma_i \in \Gamma_i$. The Γ_i are called* peripheral conjugacy classes.

If (S^2, A) is a marked sphere, we note that $\pi_1(S^2\backslash A, *)$ is a sphere group for each $* \in S^2\backslash A$. For every $d \in \mathbb{N}$ we denote by Γ_i^d the subset $\{g^d \mid g \in \Gamma_i\}$.

Let ${}_H B_G$ be a left-free biset of finite degree, and choose a basis S of B, namely a set of representatives for the left action. Consider $g \in G$. Then $S \cong \{\cdot\} \otimes_H B$ decomposes into orbits $S_1 \sqcup \cdots \sqcup S_\ell$ under the action of g, of respective cardinalities $d_1, \dots, d_\ell$; and for all $i = 1, \dots, \ell$, choosing $s_i \in S_i$ there are elements $h_i \in H$ with $h_i s_i = s_i g^{d_i}$. The multiset $\{(d_i, h_i^H) \mid i = 1, \dots, \ell\}$ consisting of degrees and conjugacy classes in H is independent of the choice of S, and depends only on the conjugacy class of g; it is called the *lift* of g^G.

Definition II.2 (Sphere bisets). — *Let $(G, \{\Gamma_i\})$ and $(H, \{\Delta_j\})$ be sphere groups. A* sphere biset *is an H-G-biset B such that the following hold:*

(1) *B is left-free and right-transitive;*
(2) *the permutations of $\{\cdot\} \otimes_H B$ induced by the right action of representatives of $\Gamma_1, \dots, \Gamma_n$ form an admissible tuple as in* (II.2)*;*
(3) *the multiset of all lifts of $\Gamma_1, \dots, \Gamma_n$ contains exactly once every Δ_j, the other conjugacy classes being all trivial.*

By the last condition, to every peripheral conjugacy class Δ_j in H is associated a well-defined degree $\deg_{\Delta_j}(B) \in \mathbb{N}$ *and conjugacy class $\Gamma_i =: B_*(\Delta_j)$, such that $(\deg_{\Delta_j}(B), \Delta_j)$ belongs to the lift of Γ_i. We define in this manner a map B_* from the peripheral conjugacy classes in H to those of G, called the* portrait *of B.*

In case the peripheral classes of G, H are indexed as $(\Gamma_a)_{a\in A}$ and $(\Delta_c)_{c\in C}$ respectively, we write $B_*(c) = a$ rather than $B_*(\Delta_c) = \Gamma_a$, defining in this manner a map $B_* \colon C \to A$.

If $G = \pi_1(S^2\backslash A)$ and $H = \pi_1(S^2\backslash B)$ and $f\colon (S^2, B) \to (S^2, A)$ is a sphere map, then $B(f)$ is a sphere H-G-biset. The following result extends the Dehn-Nielsen-Baer Theorem II.12 to non-invertible maps; its first part (in the dynamical setting $A = B$) is due to Kameyama [24]. Recall that an isomorphism between sphere bisets is required to preserve the peripheral conjugacy classes.

Theorem II.3. — *The isomorphism class of $B(f)$ depends only on the isotopy class of f, and conversely every sphere H-G-biset is of the form $B(f)$ for a sphere map $f\colon (S^2, B) \to (S^2, A)$.*

In summary, there is a bijective correspondence between isotopy classes of sphere maps and isomorphism classes of sphere bisets.

II.2. Multicurves

A *multicurve* on (S^2, A) is a disjoint collection $\mathscr{C}$ of non-trivial, non-peripheral simple closed curves on $S^2\backslash A$. Algebraically, each curve in $\mathscr{C}$ is expressed as a conjugacy class in $\pi_1(S^2\backslash A)$; and one may choose in each $\Gamma \in \mathscr{C}$ a representative $c_\Gamma \in \Gamma$ such that $\pi_1(S^2\backslash A)$ decomposes as a graph of groups, with one vertex per connected component S of $S^2\backslash\mathscr{C}$, with group $\pi_1(S)$, and one edge per curve $\Gamma \in \mathscr{C}$, with group $\langle c_\Gamma\rangle$. The underlying graph of the graph of groups is a tree. Following Definition I.5, we consider the barycentric subdivision $\mathfrak{X}$ of this graph of groups, with one vertex per curve in $\mathscr{C}$ and one per connected component of $S^2\backslash\mathscr{C}$.

Let $f\colon (S^2,A)\circlearrowleft$ be a Thurston map. A multicurve $\mathscr{C}$ is *f-invariant*[1] if every component of $f^{-1}(\mathscr{C})$ is either trivial, peripheral, or homotopic to a curve in $\mathscr{C}$, and every curve in $\mathscr{C}$ appears in this manner; i.e. $f^{-1}(\mathscr{C})=\mathscr{C}$ up to isotopy.

As soon as $f^{-1}(\mathscr{C})\subseteq\mathscr{C}$ up to isotopy, one may construct the *transition matrix* of f with respect to $\mathscr{C}$, also called its *Thurston matrix*. It is the endomorphism T_f of $\mathbb{Q}\mathscr{C}$ defined by

$$T_f(\gamma)=\sum_{\substack{\delta\in f^{-1}(\gamma)\\ \delta\approx\varepsilon\in\mathscr{C}}}\frac{1}{\deg(f\downharpoonleft_\delta\colon\delta\to\gamma)}\varepsilon. \tag{II.3}$$

Here by deg one means the usual positive degree of f; i.e. the degree of $z^d\colon\{|z|=1\}\circlearrowleft$ is $|d|$. A multicurve $\mathscr{C}$ is called an *annular obstruction* if the spectral radius of its Thurston matrix is $\geqslant 1$; see Theorem V.7.

Let $f\colon (S^2,A)\circlearrowleft$ be a Thurston map and let $\mathscr{C}$ be an f-invariant multicurve. The van Kampen theorem lets us decompose $B(f)$ as a sphere tree of bisets. Denote by $S'_1,\dots,S'_n$ the connected components of $S^2\backslash\mathscr{C}$, set $S_j:=\overline{S'_j}\backslash A$ and call S_j a *small sphere*. View each S_j as a punctured sphere, so $\pi_1(S_j)$ is a sphere group. Observe that $\{S_j\}\sqcup\mathscr{C}$ is a finite 1-dimensional cover of $S^2\backslash A$ and denote by $\mathfrak{X}$ the associated *sphere tree of groups*, see Definition I.5, with the sphere structure given by the set of peripheral conjugacy classes in every $\pi_1(S_j)$. By construction, each vertex of $\mathfrak{X}$ represents either a sphere S_j or a curve in $\mathscr{C}$. The former is called a *sphere vertex* and the latter is called a *curve vertex*.

Let $T'_1,\dots,T'_m$ be the connected components of $S^2\backslash f^{-1}(\mathscr{C})$ and set $T_j:=\overline{T'_j}\backslash f^{-1}(A)$. Then $\{T_j\}\sqcup f^{-1}(\mathscr{C})$ is a 1-dimensional cover of $S^2\backslash f^{-1}(A)$. Using an isotopy rel A modify the inclusion $S^2\backslash f^{-1}(A)\hookrightarrow S^2\backslash A$ so that the new map $i\colon S^2\backslash f^{-1}(A)\to S^2\backslash A$ squeezes all annuli between the essential curves in $f^{-1}(\mathscr{C})$ that are isotopic rel A and maps them to the corresponding curve in $\mathscr{C}$. If $i(T_j)\subset\gamma\in\mathscr{C}$, then define $\lambda(T_j):=\gamma$; otherwise there is a unique S_k such that $\lambda(T_j)\subset S_k$, and define $\lambda(T_j):=S_k$. The map λ is defined similarly for curves in $f^{-1}(\mathscr{C})$. Since $f\colon S^2\backslash f^{-1}(A)\to S^2\backslash A$ is a covering, there is a unique $\rho\colon\{T_j\}\sqcup f^{-1}(\mathscr{C})\to\{S_k\}\sqcup\mathscr{C}$ such that $f\colon T_j\to\rho(T_j)$ and $f\colon\gamma\to\rho(\gamma)$ are coverings. In this way we obtain a covering correspondence $f,i\colon S^2\backslash f^{-1}(A)\rightrightarrows S^2\backslash A$ compatible with the 1-dimensional covers. The *sphere tree of bisets* ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ is the associated graph of bisets. A *conjugacy* of a sphere tree of bisets ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ is required to respect the sphere structure; namely the $\mathfrak{I}$ in Definition I.3 is a sphere tree of bisets. As with sphere trees of groups, vertices of $\mathfrak{B}$ representing spheres T_j are called *sphere vertices* and vertices representing curves in $f^{-1}(\mathscr{C})$ are *curve vertices*.

[1] It is sometimes called "completely invariant".

We prove that the decomposition of $B(f)$ as a sphere tree of bisets is computable:

Algorithm II.4. — Given $f\colon (S^2, A) \circlearrowleft$ *a Thurston map by its sphere biset, and given $\mathscr{C}$ an f-invariant multicurve as a collection of conjugacy classes,*
Compute *the decomposition of $B(f)$ as a sphere tree of bisets.*

We return to small spheres $T_i \subset S^2 \setminus A$ defining sphere vertices of $\mathfrak{B}$. Each $\overline{T}_i$ can be homotoped rel A either to a point (i.e. the map $\overline{T}_i \hookrightarrow S^2$ is homotopic rel A to a constant map), or to a curve in $\mathscr{C}$, or to a component S_j (after filling in trivial rel A discs), and is respectively called *trivial*, *annular* or *essential*. Every S_j contains up to homotopy a single essential T_i, and T_i covers via f a single piece S_k, so that we have a map $S_j \to S_k$ induced by f, well-defined up to isotopy. We also write $k = f(j)$ so that $f\colon \{1, \dots, n\} \circlearrowleft$ describes also how the components of $S^2 \setminus \mathscr{C}$ are mapped by f. We define finally the set of *return maps*

$$R(f, \mathscr{C}) := \{f^e\colon S_j \circlearrowleft \mid f^e(j) = j \text{ and } f^{e'}(j) \neq j \text{ for all } e' < e\}.$$

All these notions have algebraic counterparts: consider a sphere biset B. A multicurve $\mathscr{C}$ is B-*invariant* if, for every conjugacy class $\Gamma \in \mathscr{C}$ and every $b \in B$ there exist $d \in \mathbb{N}$ and $\Delta \in \mathscr{C} \cup \{\Gamma_i\} \cup \{1\}$ such that $\Delta^{\pm} b \subseteq b\Gamma^{\pm d}$, and if all curves in $\mathscr{C}$ occur as such a Δ. Note the similarity to Definition II.2. Consider the sphere tree of bisets decomposition ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ of B along $\mathscr{C}$, and denote again by $B_*\colon \{1, \dots, n\} \circlearrowleft$ the dynamics on the essential sphere vertices of $\mathfrak{B}$. A sphere vertex biset B_z of $\mathfrak{B}$ is called *trivial* or *annular* if it is of the form $G_{\lambda(z)} \otimes_P B'$ for a P-$G_{\rho(z)}$-set B' and a subgroup $P \leqslant G_{\lambda(z)}$ generated by a representative of a peripheral or trivial conjugacy class, respectively a class in $\mathscr{C}$; and is called *essential* otherwise. Let us denote by $B_1, \dots, B_n$ the bisets associated with essential vertices in $\mathfrak{B}$, and let

$$\begin{aligned} R(\mathfrak{B}) &= R(B, \mathscr{C}) \\ &:= \{B_j \otimes B_{B_*(j)} \otimes \cdots \otimes B_{B_*^{e-1}(j)} \circlearrowleft \mid B_*^e(j) = j \text{ with } e \text{ minimal}\} \end{aligned}$$

denote the return bisets of B, namely the bisets obtained by following a cycle in the sphere tree of bisets into which B decomposes.

We generalize a result by Kameyama [24] to marked spheres with multicurves:

Theorem II.5. — *Let $f\colon (S^2, A, \mathscr{C}) \circlearrowleft$ and $g\colon (S^2, B, \mathscr{D}) \circlearrowleft$ be two maps with respective invariant multicurves $\mathscr{C}$ and $\mathscr{D}$. Then f and g are combinatorially equivalent along an isotopy carrying $\mathscr{C}$ to $\mathscr{D}$ if and only if the sphere tree of bisets decompositions of $B(f)$ and $B(g)$ are conjugate (by a sphere tree of bisets, see Definition I.3).*

If $A = B$ and $\mathscr{C} = \mathscr{D}$, then f and g are isotopic rel $A \cup \mathscr{C}$ if and only if their sphere trees of bisets decompositions are isomorphic.

II.3. Mapping class bisets

Let (S^2, A) be a marked sphere, and denote by $\mathbf{Mod}(S^2, A)$ the pure mapping class group of (S^2, A), namely the set of isotopy classes of homeomorphisms of S^2 fixing A. For $\mathscr{C}$ a multicurve on (S^2, A), denote by $\mathbf{Mod}(S^2, A, \mathscr{C})$ the subgroup of $\mathbf{Mod}(S^2, A)$ fixing each curve (together with its orientation) of $\mathscr{C}$ up to isotopy. Similarly, if G is a sphere group $\pi_1(S^2 \backslash A)$ then we write $\mathbf{Mod}(G)$ for $\mathbf{Mod}(S^2, A)$; by the Dehn–Nielsen–Baer theorem (see [18, Theorem 8.8] and §II.5), the group $\mathbf{Mod}(G)$ is actually a group of outer automorphisms of G. If $\mathfrak{X}$ is a sphere tree of groups decomposition of G along a multicurve $\mathscr{C}$, then we write $\mathbf{Mod}(\mathfrak{X})$ for $\mathbf{Mod}(S^2, A, \mathscr{C})$; it is a group of self-conjugators of $\mathfrak{X}$ in the sense of Definition I.3.

Let $f\colon (S^2, B) \to (S^2, A)$ be a sphere map, and let $\mathscr{D}, \mathscr{C}$ be multicurves on $S^2 \backslash B, S^2 \backslash A$ respectively with $\mathscr{D} \subseteq f^{-1}(\mathscr{C})$.

Definition II.6 (Mapping class bisets). — *The $\mathbf{Mod}(S^2, B)$-$\mathbf{Mod}(S^2, A)$-biset $M(f, B, A)$ is defined as*

$$M(f, B, A) = \{m' f m'' \mid m' \in \mathbf{Mod}(S^2, B), m'' \in \mathbf{Mod}(S^2, A)\} / \approx .$$

It admits as a subbiset the $\mathbf{Mod}(S^2, B, \mathscr{D})$-$\mathbf{Mod}(S^2, A, \mathscr{C})$-biset

$$M(f, B, A, \mathscr{D}, \mathscr{C}) = \{m' f m'' \mid m' \in \mathbf{Mod}(S^2, B, \mathscr{D}), m'' \in \mathbf{Mod}(S^2, A, \mathscr{C})\} / \approx .$$

The left- and right-actions are given by $m' f m'' = m'' \circ f \circ m'$, in keeping with using the algebraic order of operations in bisets.

By Theorem II.5, the biset $M(f, B, A)$ is also the set of isomorphism classes of bisets of the form $B(m') \otimes B(f) \otimes B(m'')$ with $m' \in \mathbf{Mod}(S^2, B)$, $m'' \in \mathbf{Mod}(S^2, A)$, and similarly for $M(f, B, A, \mathscr{D}, \mathscr{C})$. Therefore, for a sphere biset ${}_H B_G$ we introduce the notation

$$M(B) = \{B_\psi \otimes B \otimes B_\phi \mid \psi \in \mathbf{Mod}(H), \phi \in \mathbf{Mod}(G)\} / \cong , \tag{II.4}$$

and for a sphere tree of bisets ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ we define

$$M(\mathfrak{B}) = \{\mathfrak{N}' \otimes \mathfrak{B} \otimes \mathfrak{N}'' \mid \mathfrak{N}' \in \mathbf{Mod}(\mathfrak{Y}), \mathfrak{N}'' \in \mathbf{Mod}(\mathfrak{X})\} / \cong . \tag{II.5}$$

We prove that $M(f, B, A)$ and $M(f, B, A, \mathscr{D}, \mathscr{C})$ are left-free of finite degree. The groups $\mathbf{Mod}(S^2, A)$ and $\mathbf{Mod}(S^2, A, \mathscr{C})$ are computable: by the Dehn–Nielsen–Baer theorem (see [18, Theorem 8.8] and §II.5), the group

$\mathbf{Mod}(S^2, A)$ is the group of outer automorphisms of $\pi_1(S^2\backslash A, *)$ that preserve peripheral conjugacy classes, and $\mathbf{Mod}(S^2, A, \mathscr{C})$ is the subgroup that also preserves the classes in $\mathscr{C}$.

Algorithm II.7. — Given *f a sphere map $(S^2, B, \mathscr{D}) \to (S^2, A, \mathscr{C})$*, Compute *the biset $M(f, B, A, \mathscr{D}, \mathscr{C})$.*

The mapping class group $\mathbf{Mod}(S^2, A, \mathscr{C})$ naturally fits into a split exact sequence

$$1 \longrightarrow \mathbf{eMod}(S^2, A, \mathscr{C}) \xrightarrow{\iota} \mathbf{Mod}(S^2, A, \mathscr{C}) \xrightarrow{\pi} \mathbf{vMod}(S^2, A, \mathscr{C}) \longrightarrow 1 \tag{II.6}$$

whose kernel $\mathbf{eMod}(S^2, A, \mathscr{C})$ is generated by Dehn twists about curves in $\mathscr{C}$, and thus is isomorphic to $\mathbb{Z}^{\#\mathscr{C}}$, and whose quotient is isomorphic to the direct product of the mapping class groups of the path connected components of $S^2\backslash(\mathscr{C} \sqcup A)$, where all removed discs are shrunk to punctures.

The structure of $M(f, B, A, \mathscr{D}, \mathscr{C})$ qua $\mathbf{eMod}(S^2, B, \mathscr{D})$-$\mathbf{eMod}(S^2, A, \mathscr{C})$-biset is described by the Thurston matrix T_f of f, see (II.3), in the sense that if $fm = m'f$ with $m \in \mathbf{eMod}(S^2, A, \mathscr{C})$, then $m' \in \mathbf{eMod}(S^2, B, \mathscr{D})$ is the image of m under the Thurston matrix.

We arrive finally at the main results of this part: decision problems for mapping class bisets. Consider a finitely generated group P, and a P-P-biset B that is left-free of finite degree. We study the following decision problems:

The conjugacy problem. — Given $b, c \in B$, are they conjugate? If so, give a witness $g \in P$ with $bg = gc$.

The centralizer problem. — Given $b \in B$, compute its centralizer $Z(b) := \{g \in P \mid gb = bg\}$.

By a *computable group* we mean a finitely generated group with solvable word problem. A subgroup H of a computable group P is *computable* if H is finitely generated and has solvable membership problem (i.e. there is an algorithm that decides, given $g \in G$, whether $g \in H$). We say that a subgroup $L \leqslant P$ is *sub-computable* if there is a computable subgroup $H \leqslant P$ and a computable homomorphism $H \to A$ to an Abelian group such that $L = \ker(H \to A)$. (It follows from the definition that L also has solvable membership problem because it is decidable if $h \in H$ is in $\ker(H \to A)$.) We say that a centralizer problem is *solvable*, respectively *sub-solvable*, if there is an algorithm calculating the centralizer group as a computable, respectively sub-computable, subgroup. When elements of a left-free H-G-biset $B \cong H \times S$ are supplied to an algorithm, they are given in the form hs with $h \in H$ and $s \in S$; a *computable biset* is one such that the groups G, H and the map $S \times G \to H \times S$ are computable.

Let $f\colon (S^2, A, \mathscr{C}) \circlearrowleft$ be a Thurston map, and set for brevity $M := M(f, A, A, \mathscr{C}, \mathscr{C})$. From the exact sequence (II.6) we derive an "exact sequence" of bisets

$$ {}_{\mathbf{eMod}(S^2,A,\mathscr{C})}M_{\mathbf{eMod}(S^2,A,\mathscr{C})} \hookrightarrow M \twoheadrightarrow {}_{\mathbf{eMod}(S^2,A,\mathscr{C})}\backslash M/_{\mathbf{eMod}(S^2,A,\mathscr{C})} \tag{II.7} $$

in which the first term is M with restricted left and right actions, and the third term is the quotient of M by the left and right actions of $\mathbf{eMod}(S^2, A, \mathscr{C})$. There is a finite-to-one map from this third term to the product of mapping class bisets of spheres in $(S^2, A)\backslash\mathscr{C}$. We prove a general result about conjugacy and centralizer problems in extensions, which gives the

Theorem II.8. — *Let M be a mapping class biset as in* (II.7)*. There is an algorithm that computes the following. It receives as input two elements $b, c \in M$, a conjugator $g \in \mathbf{vMod}(S^2, A, \mathscr{C})$ with $\bar{b}g = g\bar{c}$ in ${}_{\mathbf{eMod}(S^2,A,\mathscr{C})}\backslash M/_{\mathbf{eMod}(S^2,A,\mathscr{C})}$, and the centralizer $Z(\bar{b}) \leqslant \mathbf{vMod}(S^2, A, \mathscr{C})$ as a computable group. It computes whether $b \sim c$ in M, if so finds a conjugator, and produces $Z(b)$ as a sub-computable group. If furthermore $Z(\bar{b})$ is finite, then $Z(b)$ is computable.*

It is therefore sufficient, to solve conjugacy problems in M, to solve them for return bisets in $R(B, \mathscr{C})$. We note that the centralizer problem in $\mathbf{Mod}(S^2, A)$ is solvable while the centralizer problem in M is only sub-solvable, see the example in §7.9.

II.4. Distinguished conjugacy classes

Let (S^2, A) be a marked sphere, and let $\mathscr{C}$ be a multicurve. It is often convenient to treat similarly the conjugacy classes describing elements of A and of $\mathscr{C}$. Consider a sphere tree of groups $\mathfrak{X}$. The set of *distinguished conjugacy classes* X of $\mathfrak{X}$ is the set of all peripheral conjugacy classes of all sphere vertex groups with two conjugacy classes identified if they are related by an edge. Equivalently, if $\mathfrak{X}$ is the sphere tree of groups decomposition of $(S^2, A, \mathscr{C})$, then X is in bijection with $A \cup \mathscr{C}$. Note that the set of geometric edges of $\mathfrak{X}$ is naturally a subset of the distinguished conjugacy classes.

The following algorithm determines when a bijection between (possibly peripheral) multicurves is induced by a homeomorphism between the underlying spheres:

Algorithm II.9. — Given $\mathfrak{X}$ *and* $\mathfrak{Y}$ *two sphere trees of groups with distinguished conjugacy classes X and Y, and given a bijection $h\colon X \to Y$,* Decide *whether $h\colon X \to Y$ promotes to a conjugator $\mathfrak{I}$ from $\mathfrak{X}$ to $\mathfrak{Y}$, and if so* construct $\mathfrak{I}$ as follows:

(1) Check whether h restricts to an isomorphism between the geometric edge sets of $\mathfrak{X}$ and $\mathfrak{Y}$. If not return `fail`.
(2) Check whether the isomorphism between the edge sets promotes into a graph-isomorphism $h\colon \mathfrak{X} \to \mathfrak{Y}$. If not, return `fail`.
(3) For a sphere vertex $v \in \mathfrak{X}$ let $\Gamma_v \subset X$ be the set of peripheral conjugacy classes of G_v. Check whether $h(\Gamma_v) = \Gamma_{h(v)}$ for all vertices $v \in \mathfrak{X}$. If not, return `fail`.
(4) For every sphere vertex $v \in \mathfrak{X}$ choose an isomorphism $\phi(v)\colon G_v \to G_{h(v)}$ compatible with $h\colon \Gamma_v \to \Gamma_{h(v)}$. For every edge $e \in \mathfrak{Y}$ choose an isomorphism $\phi(e)\colon G_e \to G_{h(e)}$.
(5) Set $\mathfrak{I} := \mathfrak{X}$, $\lambda := \mathbb{1}$, $\rho := h$ and $B_z := G_{h(z)}$ for all $z \in \mathfrak{I}$; the left action of G_z on B_z is via $\phi(z)$, the right action is natural, and the inclusion of B_e into B_{e^-} is via $1 \mapsto g$, for any $g \in G_{h(z)}$ with $()^g \circ \phi(e) = \phi(e^-)$, if we identify G_e with a subgroup of G_{e^-}.
(6) Return $\mathfrak{I}$.

Let ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ and ${}_{\mathfrak{Y}}\mathfrak{C}_{\mathfrak{Y}}$ be two sphere trees of bisets. If $\mathfrak{I}$ is a conjugator between $\mathfrak{X}$ and $\mathfrak{Y}$, then

$$(\mathfrak{C})^{\mathfrak{I}} := \mathfrak{I} \otimes \mathfrak{C} \otimes \mathfrak{I}^\vee$$

is an $\mathfrak{X}$-tree of bisets. Recall from (II.5) the notations $\mathbf{Mod}(\mathfrak{X})$ and $M(\mathfrak{B})$. The following two algorithms determine, given two trees of bisets that stabilize the same multicurve, whether they are twists of one another by mapping classes respecting the multicurve.

The first algorithm expresses, if possible, a sphere tree of bisets as a left multiple of another one. It relies on the following observation. Suppose that we want to construct a biprincipal sphere tree of bisets $\mathfrak{T}$ over a sphere tree of groups $\mathfrak{X}$, and that its vertex and edge bisets are already given, so that only the intertwiners $T_e \to T_{e^-}$ need be specified at edges of $\mathfrak{T}$. Consider an edge pair $\{e, \overline{e}\}$. The bisets T_e and $T_{e^\pm}$ may be identified with the groups G_e and $G_{e^\pm}$ respectively; then the intertwiners $T_e \to T_{e^\pm}$ are defined by $1 \mapsto g_\pm$ for some $g_\pm \in G_{e^\pm}$ which commutes with the image of G_e. Since the $G_{e^\pm}$ are free while G_e is Abelian, the element $g_\pm$ may be chosen arbitrarily in $(G_e)^\pm$. All resulting choices of maps $T_e \to T_{e^\pm}$ are called *legal intertwiners*. In fact, writing $g_\pm = (h_\pm)^\pm$ for some $h_\pm \in G_e$, the isomorphism class of $\mathfrak{T}$ depends only on $h_+(h_-)^{-1}$. See Example 7.2 for the interpretation of these intertwiners as "Dehn twists".

Algorithm II.10. — Given ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ *and* ${}_{\mathfrak{X}}\mathfrak{C}_{\mathfrak{X}}$ *two sphere $\mathfrak{X}$-trees of bisets,* Decide *whether there is an $\mathfrak{M} \in \mathbf{Mod}(\mathfrak{X})$ such that $\mathfrak{C} \cong \mathfrak{M} \otimes \mathfrak{B}$, and if so* construct $\mathfrak{M}$ *and the isomorphism* as follows:

(1) Try to construct an isomorphism of trees $h\colon \mathfrak{B} \to \mathfrak{C}$ mapping essential vertices into essential vertices such that $\lambda_{\mathfrak{B}}(z) = \lambda_{\mathfrak{C}} \circ h(z)$ and

$\rho_{\mathfrak{B}}(z) = \rho_{\mathfrak{C}} \circ h(z)$ for every $z \in \mathfrak{B}$. If h does not exist, then return `fail`. Otherwise h is unique.

(2) Choose an essential sphere vertex $v \in \mathfrak{B}$. Try to find $M_{\lambda(v)} \in \mathbf{Mod}(G_{\lambda(v)})$ such that $M_{\lambda(v)} \otimes B_v \cong C_{h(v)}$. If such $M_{\lambda(v)}$ do not exist, return `fail`. Otherwise set $S := \{v\}$ and run Steps 3–6 over all pairs $\{e, \overline{e}\} \notin S$ but with $e^- \in S$.

(3) If $\lambda(e) \notin \lambda(S)$, then do the following. (Note that in this case $\lambda(e)$ is an edge in $\mathfrak{X}$.) Add e and $\overline{e}$ to S, let $M_{\lambda(e)}$ be a principal $\mathbb{Z}$-biset, choose any legal intertwiner $()^-\colon M_{\lambda(e)} \to M_{\lambda(e)^-}$, and define $M_{\overline{e}}$ similarly.

(4) Try to find an isomorphism between $M_{\lambda(e)} \otimes B_e$ and $C_{h(e)}$ compatible with the intertwiner maps. If it does not exist, return `fail`.

(5) If e^+ is not an essential sphere vertex, then do the following. If $\lambda(e^+) \notin S$, then (in this case $\lambda(e^+)$ is a curve vertex) choose a biprincipal $\mathbb{Z}$-biset $M_{\lambda(e^+)}$, choose a legal intertwiner from $M_{\lambda(e)}$ to $M_{\lambda(e^+)}$, and add e^+ to S. Try to find an isomorphism between $M_{\lambda(e^+)} \otimes B_{e^+}$ and $C_{h(e^+)}$ that is compatible with the isomorphism between $M_{\lambda(e)} \otimes B_e$ and $C_{h(e)}$ via the intertwiner maps. If it does not exist, return `fail`.

(6) If e^+ is an essential sphere vertex, then do the following. Try to find $M_{\lambda(z)} \in \mathbf{Mod}(G_{\lambda(z)})$ such that $M_{\lambda(z)} \otimes B_z \cong C_{h(z)}$. If such $M_{\lambda(z)}$ do not exist, return `fail`. Try to find a legal intertwiner $()^-\colon M_{\lambda(e)} \to M_{\lambda(e^+)}$ such that the isomorphism between $M_{\lambda(e)} \otimes B_e$ and $C_{h(e)}$ is compatible with the isomorphism between $M_{\lambda(e)} \otimes B_e$ and $C_{h(e)}$ via the intertwiner maps. If no such intertwiner exists, return `fail`. Add e^+ to S.

(7) Return the principal sphere tree of bisets $\mathfrak{M}$ and the isomorphism between $\mathfrak{C}$ and $\mathfrak{M} \otimes \mathfrak{B}$ constructed via h.

Algorithm II.11. — Given ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ *and* ${}_{\mathfrak{X}}\mathfrak{C}_{\mathfrak{X}}$ *two sphere* $\mathfrak{X}$*-trees of bisets,* Decide *whether* $\mathfrak{C} \in M(\mathfrak{B})$*, and if so* construct $\mathfrak{M}, \mathfrak{N} \in \mathbf{Mod}(\mathfrak{X})$ *such that* $\mathfrak{C} \cong \mathfrak{M} \otimes \mathfrak{B} \otimes \mathfrak{N}$ as follows:

(1) Follow Algorithm II.7 to compute a basis of the mapping class biset $M(\mathfrak{B})$.

(2) For each $\mathfrak{N}$ in the basis, do the following. Run Algorithm II.10 on $\mathfrak{C}$ and $\mathfrak{B} \otimes \mathfrak{N}$. If there exists $\mathfrak{M} \in \mathbf{Mod}(\mathfrak{X})$ with $\mathfrak{C} \cong \mathfrak{M} \otimes (\mathfrak{B} \otimes \mathfrak{N})$, return $(\mathfrak{M}, \mathfrak{N})$.

(3) Return `fail`.

II.5. Orbispheres

In fact, a slightly more general situation than marked spheres, that of *orbispheres*, can be considered at almost no cost. Let there also be given a function $\mathrm{ord}\colon A \to \{2,3,\dots,\infty\}$, assigning a positive or infinite order to each marked point. This describes an *orbispace* structure: if $\mathrm{ord}(a)=\infty$, the point $a \in A$ is punctured, while if $\mathrm{ord}(a)=n$ then the space has a cone-type singularity of angle $2\pi/n$ at a. It is convenient to extend ord to S^2 so that $\mathrm{ord}(p)=1 \Leftrightarrow p \notin A$. We call (S^2, A, ord) an *orbisphere*, and write its fundamental group, called an *orbisphere group*, as

$$G = \pi_1(S^2, A, \mathrm{ord}, *) = \langle \gamma_1,\dots,\gamma_n \mid \gamma_1^{\mathrm{ord}(a_1)},\dots,\gamma_n^{\mathrm{ord}(a_n)}, \gamma_1\cdots\gamma_n\rangle. \quad \text{(II.8)}$$

A *map* $f\colon (S^2, B, \mathrm{ord}_B) \to (S^2, A, \mathrm{ord}_A)$ between orbispheres is a branched covering between the underlying spheres, with $f(B) \cup \{\text{critical values}(f)\} \subseteq A$. It is locally modeled at $p \in S^2$ by $z \mapsto z^{\deg_p(f)}$, in charts respectively centered at p and $f(p)$, and the orbispace structures satisfy $\mathrm{ord}_B(p)\deg_p(f) \mid \mathrm{ord}_A(f(p))$ for all $p \in S^2$.

Sphere groups and maps are subsumed in these definitions, by setting $\mathrm{ord}(a)=\infty$ for all $a \in A$. On the other hand, let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map. There is then a *minimal* orbisphere structure $(S^2, P_f, \mathrm{ord}_f)$, with $P_f \subseteq A$ the post-critical set of f, given by

$$\mathrm{ord}_f(p) = \mathrm{l.c.m.}\{\deg_q(f^n) \mid n \geqslant 0, q \in f^{-n}(p)\}.$$

These notions have algebraic counterparts. Let G be an orbisphere group with peripheral conjugacy classes $\{\Gamma_a\}_{a\in A}$. For $a \in A$ set

$$\mathrm{ord}_G(a) = \min\{d \in \mathbb{N} \mid \Gamma_a^d = \{1\}\}.$$

Orbisphere bisets are defined exactly as in Definition II.2. Let ${}_GB_G$ be an orbisphere biset. It has a *portrait* $B_*\colon A \circlearrowleft$ induced by the map on peripheral conjugacy classes, and a *local degree* $\deg_a(B)$. There is a *minimal orbisphere quotient* of G associated with B, which is the quotient $\overline{G}$ of G by the additional relations $\Gamma_a^{\mathrm{ord}_B(a)}=1$, for

$$\mathrm{ord}_B(a) = \mathrm{l.c.m.}\{d \mid n \geqslant 0 \text{ and } \Gamma_a \text{ has a lift of degree } d \text{ under } B^{\otimes n}\}, \quad \text{(II.9)}$$

and clearly $\mathrm{ord}_B(a) \mid \mathrm{ord}_G(a)$. We call the quotient biset $\overline{G} \otimes_G B \otimes_G \overline{G}$ the *minimal orbisphere biset* of ${}_GB_G$.

The mapping class group $\mathbf{Mod}(G)$ of an orbisphere group is naturally defined as the group of outer automorphisms of G that preserve its peripheral conjugacy classes classwise. It turns out that $\mathbf{Mod}(G)$ is just a usual mapping class group. More precisely, consider a marked sphere (S^2, A) with

$\widetilde{G} = \pi_1(S^2, A, *)$ and an orbisphere structure (S^2, A, ord) with non-positive Euler characteristic and with corresponding orbisphere group G. There is a natural map $\widetilde{G} \to G$ mapping each generator $\gamma_i \in \widetilde{G}$ to $\gamma_i \in G$.

Theorem II.12 (Dehn–Nielsen–Baer–Zieschang–Vogt–Coldewey [40, Theorem 5.8.3]). — *If G has at least one peripheral class of order $\geqslant 3$, then the natural map $\widetilde{G} \to G$ induces an isomorphism $\mathbf{Mod}(\widetilde{G}) \to \mathbf{Mod}(G)$.* □

There is a small difficulty if all peripheral classes in G have order 2, because then the orientation of a mapping class is difficult to read in $\mathbf{Mod}(G)$. In that case, we replace $\mathbf{Mod}(G)$ by its orientation-preserving index-2 subgroup. In the $(2,2,2,2)$-case, every mapping class is of the form $M^{(0,0)}$, see (III.5), with $\det(M) = 1$ and M fixing peripheral conjugacy classes. (See [4, §7] for details.)

An equivalent and useful formulation of orbispheres is via *planar discontinuous group actions*. To every orbisphere (S^2, A, ord) there corresponds a universal cover which, if $\sum_{a\in A}(1 - \mathrm{ord}(a)^{-1}) \geqslant 2$, is a topological plane Π, and an action of $G = \pi_1(S^2, A, \mathrm{ord}, *)$ on Π by homeomorphisms such that $(S^2, A, \mathrm{ord}) = \Pi/G$; for every $p \in \Pi$ that projects to a marked point $a_i \in A$, its stabilizer G_p is a cyclic group of order $\mathrm{ord}(a_i)$, conjugate to $\langle\gamma_i\rangle$. Maps between orbispace can be lifted to equivariant maps between their universal covers.

Theorem II.13 (Baer, [40, Theorem 5.14.1]). — *An orientation preserving homeomorphism of a plane commuting with planar discontinuous group is isotopic to the identity relative to the group action.* □

We consider now in more detail the operation on sphere groups and sphere bisets consisting of changing the orbispace structure while retaining the marked points. The less innocuous operation of erasing punctures (i.e. setting their order to 1) will be treated in §III.

Let $\widetilde{G}$, G be two orbisphere groups with respective peripheral conjugacy classes $(\widetilde{\Gamma}_a)_{a\in A}$ and $(\Gamma_a)_{a\in A}$. An *inessential forgetful morphism* $\widetilde{G} \to G$ is a homomorphism $\iota\colon \widetilde{G} \to G$ such that for every $a \in A$ we have $\iota(\widetilde{\Gamma}_a) = \Gamma_a$. Therefore the order of a in G divides the order of a in $\widetilde{G}$ for all $a \in A$.

Let G be an orbisphere group and let B be a G-G-biset. We naturally get a right action of G on

$$T(B) := \bigsqcup_{n\geqslant 0} \{\cdot\} \otimes_G B^{\otimes n}.$$

If B is left-free of degree d then $T(B)$ naturally has the structure of a d-regular rooted tree: if S is a basis of B, then $T(B)$ is in bijection with the set of words S^*, which forms a $\#S$-regular tree if one puts an edge between $s_1 \dots s_n$ and $s_1 \dots s_{n+1}$ for all $s_i \in S$. The action of G on $T(B)$ need not be free; following [26, 5.1.1] we denote by $\mathrm{IMG}_B(G)$ the quotient of G by the kernel of this action.

Lemma II.14. — *Let ${}_{\widetilde G}\widetilde B_{\widetilde G}$ be an orbisphere biset and let $\iota\colon \widetilde G \to G$ be an inessential forgetful morphism between orbisphere groups. Then*

$$B := G \otimes_{\widetilde G} \widetilde B_{\widetilde G} \otimes G \tag{II.10}$$

is an orbisphere G-G-biset if and only if the kernel of $\iota\colon \widetilde G \to G$ is contained in the kernel of $\widetilde G \to \mathrm{IMG}_{\widetilde B}(\widetilde G)$ if and only if the kernel of $\iota\colon \widetilde G \to G$ is contained in the kernel of $\widetilde G \to \overline G$, where $\overline G$ is the minimal orbisphere quotient *of G associated with B.*

For an orbisphere biset ${}_G B_G$ its mapping class biset $M(B)$ is defined in the same way as in the sphere case (II.4); namely $M(B)$ is the set of isomorphism classes of twists $B_\psi \otimes B \otimes B_\phi$ under all $\psi \in \mathbf{Mod}(H), \phi \in \mathbf{Mod}(G)$.

Theorem II.15. — *Suppose that $\widetilde G \to G$ is an inessential forgetful morphism such that $B := G \otimes_{\widetilde G} \widetilde B_{\widetilde G} \otimes G$ as in* (II.10) *is an orbisphere biset. Then the natural map $\widetilde b \mapsto 1 \otimes \widetilde b \otimes 1$ induces an isomorphism between the mapping class bisets $M(\widetilde B)$ and $M(B)$.*

III. Forgetful morphisms and geometric maps [5]

We consider in this part the operation of erasing punctures from a sphere. Consider $f\colon (S^2, A \sqcup E, \widetilde{\mathrm{ord}}) \circlearrowleft$, and assume that f induces a branched covering $f\colon (S^2, A, \mathrm{ord}) \circlearrowleft$ such that $\mathrm{ord}\colon A \to \{2,3,\dots,\infty\}$ satisfies $\mathrm{ord}(a) \mid \widetilde{\mathrm{ord}}(a)$ for all $a \in A$. There is a natural *forgetful epimorphism*

$$\widetilde G := \pi_1(S^2, A \sqcup E, \widetilde{\mathrm{ord}}, *) \to \pi_1(S^2, A, \mathrm{ord}, *) =: G\,, \tag{III.1}$$

as well as a natural *forgetful biset epimorphism*

$${}_{\widetilde G}\widetilde B_{\widetilde G} := B(f\colon (S^2, A \sqcup E, \widetilde{\mathrm{ord}}) \circlearrowleft) \to B(f\colon (S^2, A, \mathrm{ord}) \circlearrowleft) =: {}_G B_G \tag{III.2}$$

given by

$$\begin{cases} {}_{\widetilde G}\widetilde B_{\widetilde G} & \to G \otimes_G \widetilde B \otimes_G G = {}_G B_G\,, \\ \quad b & \mapsto 1 \otimes b \otimes 1\,. \end{cases} \tag{III.3}$$

We say that (III.1) and (III.2) are *essential* if $A \subsetneq \widetilde A$; otherwise (III.1) and (III.2) are *inessential.* A forgetful morphism (III.2) is *maximal* if $(S^2, A, \mathrm{ord}) = (S^2, P_f, \mathrm{ord}_f)$.

We show that the geometric (see the next §) biset ${}_{\widetilde{G}}\widetilde{B}_{\widetilde{G}}$ of degree > 1 can be described, and recovered, in terms of ${}_GB_G$ and extra data which we call a *portrait of bisets*. This allows us, in particular, to understand algorithmically maps doubly covered by torus endomorphisms. In that case we note that $(S^2, A, \mathrm{ord}) := (S^2, P_f, \mathrm{ord}_f)$ is a $(2,2,2,2)$-orbisphere and we show that ${}_GB_G$ is a crossed product of an Abelian biset with an order-2 group, see §III.3.

III.1. Geometric maps

Let us specify, by geometric conditions, a class of maps that will be central to our study; see Figure 0.1. Ultimately, we will show that it is equivalent to a topological condition, being "Levy-free", see Definition IV.3.

Definition III.1. — *A homeomorphism $f\colon (S^2, A) \circlearrowleft$ is* geometric *if f is either*

{FO} *of finite order: there is an $n > 0$ such that $f^n = \mathbb{1}$; or*
{PA} *pseudo-Anosov* [39]*: there are two transverse measured foliations preserved by f such that one foliation is expanded by f while another is contracted.*

Consider now a non-invertible map $f\colon (S^2, A) \circlearrowleft$. Let $A^\infty \subseteq A$ denote the forward orbit of the periodic critical points of f. The map f is Böttcher expanding *if there exists a metric on $S^2 \setminus A^\infty$ that is expanded by f, and such that f is locally conjugate to $z \mapsto z^{\deg_a(f)}$ at all $a \in A^\infty$. The map f is* geometric *if f is either*

{Exp} *Böttcher expanding; or*
{GTor/2} *a quotient of a torus endomorphism $Mz + v\colon \mathbb{R}^2 \circlearrowleft$ by the involution $z \mapsto -z$, such that the eigenvalues of M are different from ± 1.*

The two cases are not mutually exclusive. A map $f \in$ {GTor/2} is expanding if and only if the absolute values of the eigenvalues of M are greater than 1.

A distinguished property of a geometric map is *rigidity*: two geometric maps are combinatorially equivalent if and only if they are topologically conjugate.

An orbisphere biset ${}_GB_G$ is *geometric* if it is the biset of a geometric map, and {GTor/2} and {Exp} bisets are defined similarly. By rigidity there is a map $f_B\colon (S^2, A, \mathrm{ord}) \circlearrowleft$, unique up to conjugacy, such that the biset of f_B is B.

If ${}_G B_G$ is geometric and ${}_{\widetilde{G}}\widetilde{B}_{\widetilde{G}} \to {}_G B_G$ is a forgetful morphism as in (III.2), then elements of E (which *a priori* are defined up to homotopy) can be interpreted dynamically as extra marked points on $S^2\backslash A$. This is an instance of *homotopy shadowing*, see [23].

More precisely, suppose that ${}_{\widetilde{G}}\widetilde{B}_{\widetilde{G}} \to {}_G B_G$ is a forgetful morphism as in (III.2); suppose that ${}_G B_G$ is the biset of a geometric map $f_B : (S^2, A, \mathrm{ord}) \circlearrowleft$; and suppose that ${}_{\widetilde{G}}\widetilde{B}_{\widetilde{G}}$ is a Levy-free biset, see Definition IV.3. Then a finite set $E \subset S^2\backslash A$ with $f_B(E) \subset A \sqcup E$ can be added to A in such a way that ${}_{\widetilde{G}}\widetilde{B}_{\widetilde{G}}$ is conjugate to the biset of $f_B\colon (S^2, A \sqcup E, \widetilde{\mathrm{ord}}) \circlearrowleft$. Moreover, if G is not a cyclic group, then the set E is unique.

III.2. Portraits of groups and bisets

Let ${}_G B_G$ be an orbisphere biset with peripheral conjugacy classes $(\Gamma_a)_{a\in A}$ and portrait $B_*\colon A \circlearrowleft$. Suppose that E is a finite set and suppose that $B_*\colon A \sqcup E \circlearrowleft$ is an extension of $B_*\colon A \circlearrowleft$.

Definition III.2 (Portraits of groups and bisets). — *Let G be an orbisphere group with peripheral conjugacy classes $(\Gamma_a)_{a\in A}$ and let E be a finite set. A* portrait of groups *$(G_a)_{a\in A\sqcup E}$ in G is a collection of cyclic subgroups $G_a \leqslant G$ such that*

$$G_a = \begin{cases} \langle g \rangle & \text{for some } g \in \Gamma_a \text{ if } a \in A,\\ \langle 1 \rangle & \text{if } a \in E.\end{cases}$$

If $E = \varnothing$, then $(G_a)_{a\in A}$ is a minimal *portrait of groups.*

Let ${}_G B_G$ be an orbisphere biset and let $B_*\colon A \sqcup E \circlearrowleft$ be an extension of $B_*\colon A \circlearrowleft$. A portrait of bisets *in ${}_G B_G$* parameterized *by $B_*\colon A \sqcup E \circlearrowleft$ is a collection $(G_a, B_a)_{a\in A\sqcup E}$ such that*

(1) *$(G_a)_{a\in A\sqcup E}$ is a portrait of groups in G; and*
(2) *B_a is a transitive G_a-$G_{B_*(a)}$-subbiset of ${}_G B_G$ such that if $B_*(a) = B_*(c)$ and $G \otimes_{G_a} B_a = G \otimes_{G_c} B_c$ as subsets of B, then $a = c$.*

If $E = \varnothing$, then $(G_a, B_a)_{a\in A\sqcup E}$ is a minimal *portrait of bisets.*

Two portraits of bisets $(G_a, B_a)_{a\in A\sqcup E}$ and $(G'_a, B'_a)_{a\in A\sqcup E}$ parameterized by $B_*\colon A \sqcup E \circlearrowleft$ are conjugate *if there exist $(\ell_a)_{a\in A\sqcup E}$ such that*

$$G_a = \ell_a^{-1} G'_a \ell_a \text{ and } B_a = \ell_a^{-1} B'_a \ell_{B_*(a)}. \tag{III.4}$$

Every biset admits a minimal portrait, unique up to conjugacy.

Let us give a geometric interpretation of portraits of bisets. Consider a branched covering $f\colon (S^2, A) \circlearrowleft$. For every $a \in A$ choose a small neighbourhood D_a of it; up to isotopy we may assume that $f\colon D_a\backslash\{a\} \to D_{f(a)}\backslash\{f(a)\}$ is a covering. Making appropriate choices we may embed $\pi_1(D_a\backslash\{a\})$ and $B(f\colon D_a\backslash\{a\} \to D_{f(a)}\backslash\{f(a)\})$ into $\pi_1(S^2, A)$ and $B(f)$ respectively; calling the images of these embeddings G_a and B_a we get a minimal portrait of bisets $(G_a, B_a)_{a\in A}$ in $B(f)$.

Recall from §I that a self-conjugacy of a G-G-biset B is a pair of maps $(\phi\colon G \circlearrowleft, \beta\colon B \circlearrowleft)$ with $\beta(hbg) = \phi(h)\beta(b)\phi(g)$; and an automorphism of B is a self-conjugacy with $\phi = \mathbb{1}$. We show that, for sphere bisets, every self-conjugacy (ϕ, β) is determined by its map ϕ, or equivalently that B admits no non-trivial automorphism:

Theorem III.3 (No automorphisms). — *Suppose ${}_G B_G$ is a non-cyclic orbisphere biset. Then the automorphism group of ${}_G B_G$ is trivial.*

Definition III.4. — *Let $\iota\colon {}_{\widetilde G}\widetilde B_{\widetilde G} \to {}_G B_G$ be a forgetful morphism of orbisphere bisets as in* (III.2). *Let $(\widetilde G_a, \widetilde B_a)_{a\in A\sqcup E}$ be a minimal portrait of bisets in $\widetilde B$. Then*

$$(G_a, B_a)_{a\in A\sqcup E} := (\iota(\widetilde G_a), \iota(\widetilde B_a))_{a\in A\sqcup E}$$

is the induced portrait of bisets*. It is parameterized by $B_* := \widetilde B_*\colon A \sqcup E \circlearrowleft$.*

Consider a forgetful morphism $\widetilde G \to G$ as in (III.1). For every $e \in E$ set

$$\widetilde G_e := \pi_1(S^2, A \sqcup \{e\}, \widetilde{\operatorname{ord}}\,|_{A\sqcup\{e\}}, *)$$

so that the forgetful morphism $\widetilde G \to G$ factors as $\widetilde G \to \widetilde G_e \to G$. We say that a biset ${}_{\widetilde G}K_{\widetilde G} \in \mathbf{Mod}(\widetilde G)$ is *knitting* if for every $e \in E$ we have

$$\widetilde G_e \otimes_{\widetilde G} K \otimes_{\widetilde G} \widetilde G_e \cong {}_{\widetilde G_e}(\widetilde G_e)_{\widetilde G_e}.$$

If $m \in \mathbf{Mod}(S^2, A \sqcup E)$, then the biset of m is knitting if and only if m is trivial in $\mathbf{Mod}(S^2, A \sqcup \{e\})$ for all $e \in E$.

Theorem III.5. — *Let $\widetilde G \to G$ be a forgetful morphism as in* (III.1). *Suppose that ${}_G B_G$ is an orbisphere biset and that $(G_a, B_a)_{a\in A\sqcup E}$ is a portrait of bisets parameterized by $B_*\colon A \sqcup E \circlearrowleft$.*

Then there is an orbisphere biset ${}_{\widetilde G}\widetilde B_{\widetilde G}$ such that $(G_a, B_a)_{a\in A\sqcup E}$ is induced by the forgetful morphism ${}_{\widetilde G}\widetilde B_{\widetilde G} \to {}_G B_G$ defined by (III.3)*, and if ${}_{\widetilde G}\widetilde B'_{\widetilde G}$ is another such biset then there is a knitting biset ${}_{\widetilde G}K_{\widetilde G}$ with*

$$\widetilde B' \cong K \otimes_{\widetilde G} \widetilde B.$$

Conjugacy and centralizers of bisets of the form $\widetilde{B}$ may be studied as follows:

Theorem III.6. — *Let $\widetilde{G} \to G$ be a forgetful morphism of groups as in* (III.1) *and let*

$${}_{\widetilde{G}}\widetilde{B}_{\widetilde{G}} \to {}_G B_G \quad \text{and} \quad {}_{\widetilde{G}}\widetilde{C}_{\widetilde{G}} \to {}_G C_G$$

be two forgetful biset morphisms as in (III.2)*. Suppose furthermore that $\widetilde{B}$ is geometric of degree > 1. Denote by $(G_a, B_a)_{a \in A \sqcup E}$ and $(G_a, C_a)_{a \in A \sqcup E}$ the portraits of bisets induced by $\widetilde{B}$ and $\widetilde{C}$ in B and C respectively.*

Then $\widetilde{B}, \widetilde{C}$ are conjugate by $\mathbf{Mod}(\widetilde{G})$ if and only if there exists $\phi \in \mathbf{Mod}(G)$ such that $B^\phi \cong C$ and the portraits $(G_a^\phi, B_a^\phi)_{a \in A \sqcup E}$ and $(G_a, C_a)_{a \in A \sqcup E}$ are conjugate qua portraits in C.

Furthermore, the centralizer $Z(\widetilde{B})$ of $\widetilde{B}$ is isomorphic, via the forgetful map $\mathbf{Mod}(\widetilde{G}) \to \mathbf{Mod}(G)$, to

$$\left\{\phi \in Z(B) \,\middle|\, (G_a^\phi, B_a^\phi)_{a \in A \sqcup E} \sim (G_a, B_a)_{a \in A \sqcup E}\right\}$$

and is a finite-index subgroup of $Z(B)$.

It follows that the conjugacy and centralizer problems for $\widetilde{B}$ are decidable as soon as they are decidable for $(B, \text{portrait})$.

Proposition III.7 (Contracting case). — *Suppose that B is an orbisphere contracting G-biset, see Definition IV.5. Then for every $B_* \colon A \sqcup E \circlearrowleft$ the number of conjugacy classes of portraits of bisets parameterized by B_* is finite.*

Moreover, there is an algorithm that, given $B_* \colon A \sqcup E \circlearrowleft$, decides whether two portraits of bisets parameterized by B_* are conjugate.

The algorithm of Proposition III.7 reduces the conjugacy problem for portraits to conjugacy problems of elements in $B^{\otimes n}$. Here is a simple example illustrating this main step in the algorithm:

Example III.8. — Suppose $E = \{e\}$ with $B_*(e) = e$, and let $(G_a, B_a)_{a \in A \sqcup E}$ and $(G_a, C_a)_{a \in A \sqcup E}$ be two portraits of bisets. Then $G_e = 1$ and $B_e = \{b\}$ and $C_e = \{c\}$.

The portraits $(G_a, B_a)_{a \in A \sqcup E}$ and $(G_a, C_a)_{a \in A \sqcup E}$ are conjugate if and only if there exists $\ell \in G$ such that $\ell b \ell^{-1} = c$.

This is a conjugacy problem in the biset B which can effectively be solved using contraction in B, since the length of ℓ can be bounded in terms of the lengths of b and c.

Let B be the biset of a map f. In case $E = \{e_1, \dots, e_n\}$ consists of a single cycle for B_*, all the bisets B_e are singletons and the portrait of bisets contains precisely the same information as a "homotopy pseudo orbit", namely a sequence of points $z_1, \dots, z_n$ with homotopy classes of paths γ_n from z_n to an f-preimage of z_{n+1}, indices read modulo n; see [23]. In case f is expanding, these authors prove that $(z_1, \dots, z_n)$ is homotopic to a unique period-n cycle.

Algorithm III.9. — Given *bisets* $\widetilde{B}$, $\widetilde{C}$ *together with the forgetful morphisms onto* B *and* C *as in Theorem III.6 such that, in addition,* B *is contracting; and* Given $\phi \in \mathbf{Mod}(G)$ *such that* $B^\phi \cong C$ *and a finite generating set of* $Z(B)$,
Decide *whether* $\widetilde{B}$ *and* $\widetilde{C}$ *are conjugate, and* compute $Z(\widetilde{B})$.

III.3. Bisets of minimal $(2,2,2,2)$-maps

A $(2,2,2,2)$-orbisphere is (S^2, A, ord) with $\#A = 4$ and $\mathrm{ord}(a) = 2$ for all $a \in A$. Let us denote by $\mathbb{Z}/2$ the group of integers modulo 2.

Lemma III.10. — *If* (S^2, A, ord) *is a* $(2,2,2,2)$*-orbisphere, then* $\pi_1(S^2, A, \mathrm{ord}) \cong \mathbb{Z}^2 \rtimes \{\pm 1\}$. *There are exactly four order-2 conjugacy classes in* $\pi_1(S^2, A, \mathrm{ord}) \cong \mathbb{Z}^2 \rtimes \{\pm 1\}$*; these classes are identified with* A *and are of the form*

$$(n, 1)^{\mathbb{Z}^2 \rtimes \{\pm 1\}} = \{(n + 2m, 1) \mid m \in \mathbb{Z}^2\} \text{ for all } n \in \{0,1\}^2.$$

Let us fix a $(2,2,2,2)$-orbisphere (S^2, A, ord) and let us set $G := \pi_1(S^2, A, \mathrm{ord})$. Thanks to Lemma III.10 we identify A with the set of all order-2 conjugacy classes of G. By Euler characteristic, every branched covering $f\colon (S^2, A, \mathrm{ord}) \circlearrowleft$ is a self-covering. Therefore, the biset of f is right principal.

We denote by $\mathbf{Mat}_2^+(\mathbb{Z})$ the set of 2×2 integer matrices M with $\det(M) > 0$. For a matrix $M \in \mathbf{Mat}_2^+(\mathbb{Z})$ and a vector $v \in \mathbb{Z}^2$ there is an injective endomorphism $M^v\colon \mathbb{Z}^2 \rtimes \{\pm 1\} \circlearrowleft$ given by the following "crossed product" structure:

$$M^v(n, 0) = (Mn, 0) \text{ and } M^v(n, 1) = (Mn + v, 1)\,. \tag{III.5}$$

Furthermore, M^v induces a map $(M^v)_*\colon A \circlearrowleft$ on conjugacy classes. We write $B_{M^v} = B_M \rtimes \{\pm 1\}$ the crossed product decomposition of the biset of M_v.

Proposition III.11. — *Every* $(2,2,2,2)$*-orbisphere biset* B *is of the form* $B = B_{M^v}$ *for some* M^v *as in* (III.5)*, and* M^v *is computable from the biset* B*.*

Conversely, B_{M^v} *is an orbisphere biset for every* M^v *as in* (III.5)*. Two bisets* B_{M^v} *and* B_{N^w} *are isomorphic if and only if* $M = \pm N$ *and* $(M^v)_* = (N^w)_*$ *as maps on* A*.*

The biset B_{M^v} *is geometric if and only if both eigenvalues of* M *are different from* ± 1*. If* B_{M^v} *is geometric, then for every* $(B_{M^v})_* \colon A \sqcup E \circlearrowleft$ *the number of conjugacy classes of portraits of bisets in* B_{M^v} *parameterized by* $(B_{M^v})_*$ *is finite, and it is algorithmically decidable whether for a given* $B_* \colon A \sqcup E \circlearrowleft$ *two portraits of bisets within* B_{M^v} *are conjugate.*

We need the following fact.

Theorem III.12 (Corollary of [19]). — *There is an algorithm deciding whether two* $M, N \in \mathbf{Mat}_2^+(\mathbb{Z})$ *are conjugate by an element* $X \in \mathbf{SL}_2(\mathbb{Z})$*, and produces such an* X *if it exists.*

There is an algorithm computing, as a finitely generated subgroup of $\mathbf{SL}_2(\mathbb{Z})$*, the centralizer of* $M \in \mathbf{Mat}_2^+(\mathbb{Z})$*.*

Algorithm III.13. — Given ${}_{\widetilde G}\widetilde B_{\widetilde G}$, ${}_{\widetilde G}\widetilde C_{\widetilde C}$ *two* {GTor/2} *bisets*
Compute *the centralizer* $Z(\widetilde B)$*, and* decide *whether* $\widetilde B$ *and* $\widetilde C$ *are conjugate by an element of* $\mathbf{Mod}(\widetilde G)$*, and if so* construct *a conjugator* as follows:

1. (1) If $\widetilde B_* \neq \widetilde C_*$ as maps on peripheral conjugacy classes, then return `fail`.
2. (2) Let ${}_{\widetilde G}\widetilde B_{\widetilde G} \to {}_G B_G$ and ${}_{\widetilde G}\widetilde C_{\widetilde G} \to {}_{G'} C_{G'}$ be two maximal forgetful morphisms. If $G \neq G'$, then return `fail`. Otherwise by Lemma III.10 identify $G = G' \cong \mathbb{Z}^2 \rtimes \{\pm 1\}$ and by Proposition III.11 present B and C as B_{M^v} and B_{N^w} respectively.
3. (3) Using Theorem III.12 check whether M and N are conjugate. If not, return `fail`; otherwise find a conjugator X and compute the centralizer subgroup K of M.
4. (4) Check whether there is a $Y \in K$ such that $(YX)^0$ is a conjugator between M^v and N^w. If there is none, return `fail`; otherwise set $X := YX$ and replace K by $\{Y \in K \mid Y^0 \text{ centralizes } M^v\}$, a subgroup of finite index in K.
5. (5) Let $(G_a, B_a)_{a \in A \sqcup E}$ and $(G_a, C_a)_{a \in A \sqcup E}$ be induced by $\widetilde B$ and $\widetilde C$ portrait of bisets in $B \cong B_{M^v}$ and in $C \cong B_{N^w}$. Using Proposition III.11 check whether the is an $Y \in K$ such that
$$\left(G_a^{(YX)^0}, B_a^{(YX)^0}\right)_{a \in A \sqcup E} \sim (G_a, C_a)_{a \in A \sqcup E}\,.$$

If not, return `fail`. Otherwise use Theorem III.6 to promote $(YX)^0$ into a conjugacy between $\widetilde{B}$ and $\widetilde{C}$.

(6) The centralizer of $\widetilde{B}$ is computed using Theorem III.6.

Corollary III.14. — *There is an algorithm that, given two* {GTor/2} *bisets* ${}_GB_G$ *and* ${}_HC_H$*, decides whether* B *and* C *are conjugate, and computes the centralizer of* B*.*

IV. Expanding maps and the Levy decomposition [6]

Consider a Thurston map $f\colon (S^2, A) \circlearrowleft$. We give a criterion for the existence of a Riemannian metric on (S^2, A) such that f is isotopic to an expanding map. This criterion is in terms of multicurves on $S^2\backslash A$. We then give an application to the study of *matings* of polynomials.

IV.1. Levy, anti-Levy, Cantor, and anti-Cantor multicurves

Recall that a multicurve $\mathscr{C}$ is invariant if $f^{-1}(\mathscr{C}) = \mathscr{C}$, up to isotopy and removing peripheral and trivial components. If C is a multicurve and $C \subseteq f^{-1}(C)$, then there is a unique invariant multicurve $\mathscr{C}$ *generated* by C, namely $\mathscr{C} = \bigcup_{n\geqslant 0} f^{-n}(C)$.

Consider the following graph called the *curve graph* of (S^2, A). Its vertex set is the set of isotopy classes of essential curves on $S^2\backslash A$. For every simple closed curve γ and for every component δ of $f^{-1}(\gamma)$ there is an edge from γ to δ labeled $\deg(f\restriction_\delta)$.

Let $\mathscr{C}$ be an invariant multicurve, and consider the directed subgraph of the curve graph that it spans. A *strongly connected component* is a maximal subgraph spanned by a subset $C \subseteq \mathscr{C}$ such that, for every $\gamma, \delta \in C$, there exists a non-trivial path from γ to δ in C. A strongly connected component C is *primitive* if all its incoming edges come from C itself. We call C a *bicycle* if for every $\gamma, \delta \in C$ there exists $n \in \mathbb{N}$ such that at least two paths of length n join γ to δ in C, and a *unicycle* otherwise; see Figure IV.1 for an illustration. Clearly, every invariant multicurve is generated by its primitive strongly connected components.

We remark that bicycles contain at least two cycles, so that the number of paths of length n grows exponentially in n. On the other hand, every unicycle is an actual *periodic cycle*: it can be written as $C = (\gamma_0, \gamma_1, \dots, \gamma_n = \gamma_0)$ in such a manner that γ_{i+1} has an f-preimage γ_i' isotopic to γ_i. If furthermore

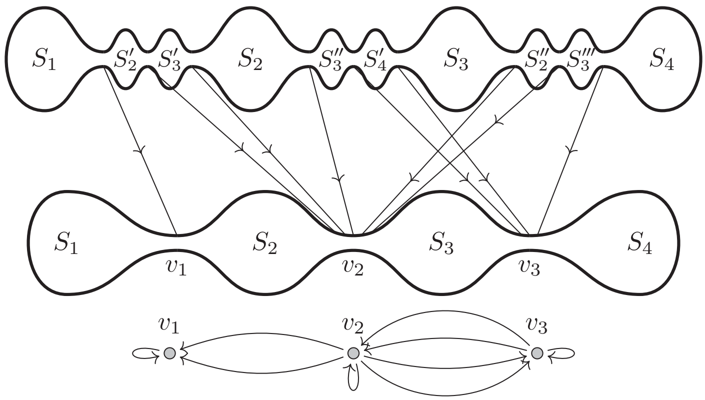
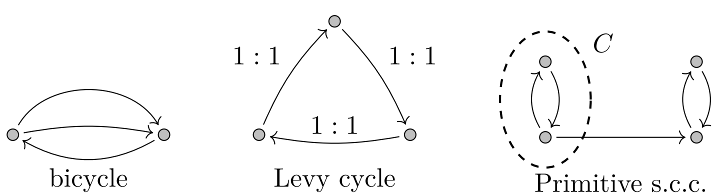


Figure IV.1. A bicycle $\{v_2, v_3\}$ generates a Cantor multicurve $\{v_1, v_2, v_3\}$. The action of the map f is indicated on the preimages of $\{v_1, v_2, v_3\}$. If annuli are mapped by degree 1, then it is also a Levy cycle. The graph below is the corresponding portion of the curve graph.

the γ'_i may be chosen so that f maps each γ'_i to γ_{i+1} by degree 1, then C is called a *Levy cycle*.



Definition IV.1 (Types of invariant multicurves). — *Let $\mathscr{C}$ be an invariant multicurve. Then $\mathscr{C}$ is*

Cantor *if $\mathscr{C}$ is generated by its bicycles;*
anti-Cantor *if $\mathscr{C}$ does not contain any bicycle;*
Levy *if $\mathscr{C}$ is generated by its Levy cycles;*
anti-Levy *if $\mathscr{C}$ does not contain any Levy cycle.*

Note that $\mathscr{C}$ being Cantor / anti-Cantor depends on the mappings of curves of $\mathscr{C}$ to themselves, and being Levy / anti-Levy depends on the degrees under which the curves map.

Suppose $f\colon (S^2,A) \circlearrowleft$ is a Thurston map with an invariant multicurve $\mathscr{C}$. Recall from §II.2 that by $R(f,\mathscr{C})$ we denote the return maps induced by f on $S^2\backslash\mathscr{C}$.

Proposition IV.2. — *Suppose $f\colon (S^2,A) \circlearrowleft$ is a Thurston map with an invariant multicurve $\mathscr{C}$. Then:*

(1) *there is a unique maximal invariant Cantor sub-multicurve $\mathscr{C}_{Cantor} \subseteq \mathscr{C}$ such that the multicurves induced by $\mathscr{C}$ on pieces in $R(f,\mathscr{C}_{Cantor})$ are anti-Cantor invariant multicurves;*
(2) *there is a unique maximal invariant Levy sub-multicurve $\mathscr{C}_{Levy} \subseteq \mathscr{C}$ such that multicurves induced by $\mathscr{C}$ on pieces in $R(f,\mathscr{C}_{Levy})$ are anti-Levy invariant multicurves.*

Definition IV.3 (Levy-free). — *Let $f\colon (S^2,A) \circlearrowleft$ be a Thurston map. It is* Levy-free *if* $\deg(f) > 1$ *and f does not admit a Levy cycle.*

We say that an invariant Levy multicurve $\mathscr{C}$ is *complete* if every piece in $R(f,\mathscr{C})$ is either Levy-free or has degree 1. We show that for a Thurston map $f\colon (S^2,A) \circlearrowleft$ there is a unique minimal invariant complete Levy multicurve, which we denote by $\mathscr{C}_{f,\mathrm{Levy}}$ and call *the canonical Levy obstruction.* Any other invariant complete Levy multicurve $\mathscr{C}$ contains $\mathscr{C}_{f,\mathrm{Levy}}$ as a sub-multicurve. For B a biset, we define in a similar manner $\mathscr{C}_{B,\mathrm{Levy}}$.

Definition IV.4 (Levy decomposition). — *The Levy decomposition of $f\colon (S^2,A) \circlearrowleft$ is the decomposition along the canonical Levy obstruction $\mathscr{C}_{f,Levy}$.*

The Levy decomposition of a biset B is the sphere tree of bisets decomposition of B along $\mathscr{C}_{B,Levy}$.

IV.2. Expanding maps

A *length metric with singularities* on S^2 is a length orbifold metric that is allowed to have finitely many points, called *singularities*, at infinite distance such that points topologically close to singularities are far away from usual points. A basic example is the hyperbolic metric on the Riemann sphere with finitely many removed points. We will also refer to singularities as *points at infinity.*

A Thurston map $f\colon (S^2,A) \circlearrowleft$ is *metrically expanding* if there is a length metric μ, with singularities, on S^2 such that f is expanding with respect to μ and all points sufficiently close topologically to infinity escape to infinity. A combinatorial equivalence class of Thurston maps is called *expanding* if it contains an expanding map.

It follows from the definition that the set of singularities of μ is forward invariant, and a periodic point is singular if and only if it is topologically attracting.

We say that an expanding map is *Böttcher* if the first return map near every critical periodic point is conjugate to $z \to z^d$, where $d > 1$ is the degree of the first return map. Two Böttcher expanding maps are conjugate if and only if they are combinatorially equivalent. (This is an application of the "pullback argument": if we have $f \circ \phi = g\colon (S^2, A) \circlearrowleft$ with ϕ isotopic rel A to the identity, then ϕ can be normalized to be the identity near every periodic critical cycle; by expansion the lifts ϕ_n of ϕ through f^n converge exponentially fast to the identity, so their product $\psi := \cdots \circ \phi_1 \circ \phi_0$ conjugates f into g.)

Definition IV.5 ([26, Definition 2.11.8]). — *Let B be a G-G-biset. It is called* contracting *if for every finite $S \subseteq B$ there exists a finite subset $N \subseteq G$ with the following property: for every $g \in G$ and every $n \gg 0$ we have $\{h \in G \mid hS^n \cap S^n g \neq \varnothing\} \subseteq N$.*

For a finitely generated group G, the biset B is contracting if there is a proper metric $|\cdot|$ on G and constants $\lambda < 1, C$ such that $|h| \leqslant \lambda|g| + C$ whenever $hS \cap Sg \neq \varnothing$.

If B is left-free and we chose a basis $S \subseteq B$ for the left action, we obtain a wreath recursion $\psi\colon G \to G^S \rtimes S\wr$, see §I.1; and B is contracting if for every $g \in G$ one obtains only elements of N as coördinates when one iterates ψ long enough.

The set N in Definition IV.5 is not unique; but for every $S \subseteq B$ there exists a minimal such N, written $N(S)$ and called the *nucleus* of (B, S). It gives rise to a labeled graph, called the *nucleus machine* of (B, S): its vertex set is $N(S)$, and there is an edge from g to h with *input* and *output* labels $s \in S$ and $t \in S$ respectively whenever $sg = ht$ holds in B.

We slightly modify the definition of "contracting" for sphere bisets, because of the orbisphere structures. Let ${}_G B_G$ be a sphere biset with $G = \pi_1(S^2, A)$. Recall from (II.9) that there is a minimal orbisphere structure ord_B given by B. We call an orbisphere structure $\mathrm{ord}\colon A \to \{2, 3, \ldots, \infty\}$ *bounded* if $\mathrm{ord}(a) = \infty \Leftrightarrow \mathrm{ord}_B(a) = \infty$ and $\mathrm{ord}(a)\deg_a(B) \mid \mathrm{ord}(B_*(a))$ for all $a \in A$. Let $\overline{G}$ denote the quotient orbisphere group $G/\langle \gamma_a^{\mathrm{ord}(a)} : a \in A\rangle$. Then we call B an *orbisphere contracting* biset if $\overline{G} \otimes_G B \otimes_G \overline{G}$ is contracting for some bounded orbisphere structure on (S^2, A).

The main result of this part is the following criterion:

Theorem C. — *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map, not doubly covered by a torus endomorphism. The following are equivalent:*

(1) *f is isotopic to a Böttcher expanding map;*
(2) *f is Levy-free;*
(3) *$B(f)$ is an orbisphere contracting biset.*

Furthermore, if these properties hold, the metric μ on S^2 that is expanded by f may be assumed to be Riemannian of pinched negative curvature.

Haïssinsky and Pilgrim ask in [20] whether every everywhere-expanding map is isotopic to a smooth map. By Theorem C, a combinatorial equivalence class contains a Böttcher smooth expanding map if and only if it is Levy free.

It was already proven in [33] that every Levy-free map that is doubly covered by a torus endomorphism is in $\{\mathrm{GTor}/2\}$. Combining this with the results [27, 39] on geometrization of surface self-homeomorphisms we obtain the

Corollary IV.6. — *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map. Then every map in $R(f, \mathscr{C}_{Levy})$ has a geometric structure: it is either expanding (i.e. in $\{\mathrm{Exp}\}$), of degree 1, or a non-expanding irrational map doubly covered by a torus endomorphism (i.e. in $\{\mathrm{GTor}/2\}\backslash\{\mathrm{Exp}\}$).*

Furthermore, the property of being geometric is algorithmically recognizable: bisets double covered by a torus endomorphism are of a very particular nature (III.5); contracting bisets are recognized by their nucleus, and non-contracting bisets are recognized by their Levy obstruction.

Algorithm IV.7. — Given *a Thurston map $f\colon (S^2, A) \circlearrowleft$ by its biset,* Compute *the Levy decomposition of f* as follows:

(1) For an enumeration of all multicurves $\mathscr{C}$ on (S^2, A), that never reaches a multicurve before reaching its proper submulticurves, do the following steps:
(2) If the multicurve $\mathscr{C}$ is not invariant, or is not Levy, continue in (1) with the next multicurve.
(3) Compute the decomposition of f using Algorithm II.4.
(4) If not all pieces are expanding or $\{\mathrm{GTor}/2\}$ or degree-1 maps, continue in (1) with the next multicurve;
(5) Return $\mathscr{C}$.

IV.3. Matings of higher degree polynomials

We turn to an application to matings of polynomials. Let $p, q\colon \mathbb{C} \circlearrowleft$ be two monic polynomials of same degree $d \geqslant 2$. Identify $\mathbb{C}$ with the disk $\mathbb{D}$

by the map $\nu_+ : z \mapsto z/\sqrt{1+|z|^2}$ and with its complement $\widehat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ by the map $\nu_-(z) = 1/\nu_+(z)$. Consider the continuous map

$$p \uplus q \colon \begin{cases} \widehat{\mathbb{C}} & \to \widehat{\mathbb{C}}, \\ z \text{ with } |z| < 1 & \mapsto \nu_+(p(\nu_+^{-1}(z))), \\ z \text{ with } |z| = 1 & \mapsto z^d, \\ z \text{ with } |z| > 1 & \mapsto \nu_-(q(\nu_-^{-1}(z))). \end{cases}$$

It is called the *formal mating* of p and q, and is a degree-d branched covering of S^2.

Recall also that there are *external rays* associated to the polynomials p, q. First, the *filled-in Julia set* K_p of p is

$$K_p = \{z \in \mathbb{C} \mid f^n(z) \not\to \infty \text{ as } n \to \infty\}.$$

Assume that K_p is connected. There exists then a unique holomorphic isomorphism $\phi_p \colon \widehat{\mathbb{C}}\backslash K_p \to \widehat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ satisfying $\phi_p(p(z)) = \phi_p(z)^d$ and $\phi_p(\infty) = \infty$ and $\phi_p'(\infty) = 1$. It is called a *Böttcher coördinate*, and conjugates p to z^d in a neighbourhood of ∞. For $\theta \in \mathbb{R}/\mathbb{Z}$, the associated *external ray* is

$$R_p(\theta) = \{\phi_p^{-1}(re^{2i\pi\theta}) \mid r \geqslant 1\}.$$

Let $X_{p,q}$ be the quotient of $\widehat{\mathbb{C}}$ in which each $\overline{\nu_+(R_p(\theta))}$ has been identified to one point for each $\theta \in \mathbb{R}/\mathbb{Z}$, and similarly each $\overline{\nu_-(R_q(\theta))}$ has been identified to one point. Note that $X_{p,q}$ is a quotient of $K_p \sqcup K_q$, and need not be a Hausdorff space. A classical criterion (due to Moore) determines when $X_{p,q}$ is homeomorphic to S^2. If this occurs, p and q are said to be *topologically mateable*, and the map induced by $p \uplus q$ on $X_{p,q}$ is called the *topological mating* of p and q and denoted $p \amalg q \colon X_{p,q} \circlearrowleft$.

Furthermore, if there exists a homeomorphism $\phi \colon X_{p,q} \to \widehat{\mathbb{C}}$ that is conformal on $\nu_+(K_p^o) \cup \nu_-(K_q^o)$ and such that $f := \phi \circ (p \amalg q) \circ \phi^{-1}$ is a rational map, then p, q are said to be *geometrically mateable*, and any such f is called a *geometric mating* of p and q.

Mary Rees and Tan Lei [36] proved that two post-critically finite quadratic polynomials p, q are geometrically mateable if and only if p and q do not belong to conjugate limbs of the Mandelbrot set; see [12, Theorem 2.1]. To be more precise, let us assume that p and q are hyperbolic. (The subhyperbolic case requires a slight clarification because points in the post-critical set may get glued during the geometric mating.) Then the following are equivalent for $d = 2$:

(1) $p \uplus q$ is isotopic to a rational map $p \mathbin{\square} q$;
(2) $p \amalg q$ is a sphere map (necessarily conjugate to $p \mathbin{\square} q$);
(3) p, q are not in conjugate primary limbs of the Mandelbrot set.

This theorem relies on the fact that any annular obstruction in degree 2 is Levy; indeed in degree $\geqslant 3$ there are topological matings that are not conjugate to rational maps, see [35] and §7.8. Since the obstruction to be an expanding map is also Levy, we can generalize the degree 2 criterion in the class of expanding maps as follows.

Definition IV.8. — *Let p, q be two hyperbolic post-critically finite polynomials of same degree d. We say that p, q have a* pinching cycle of periodic angles *if there are angles $\phi_0, \phi_1, \dots, \phi_{2n-1} \in \mathbb{R}/\mathbb{Z}$, indices treated modulo $2n$, such that for all i,*

- *the angle ϕ_i is periodic under multiplication by d;*
- *the rays $R_p(\phi_{2i})$ and $R_p(\phi_{2i+1})$ land together;*
- *the rays $R_q(-\phi_{2i-1})$ and $R_q(-\phi_{2i})$ land together.*

In degree 2, the pair p, q has a pinching cycle of periodic angles if and only if p and q are in the conjugate primary limbs of the Mandelbrot set. In degree $\geqslant 3$, there is still no well-accepted notion of limbs, and they should be rather defined as sets of parameters in which certain periodic rays land together.

If p, q admit a pinching cycle, then a fibre of the map $\widehat{\mathbb{C}} \to X_{p,q}$ is a cycle, so $X_{p,q}$ cannot be homeomorphic to S^2. We obtain the following criterion:

Theorem IV.9. — *Let p, q be two monic hyperbolic post-critically finite polynomials. Then the following are equivalent:*

1. *$p \uplus q$ is isotopic to an expanding map $p \mathbin{\square} q$;*
2. *$p \amalg q$ is a sphere map (necessarily conjugate to $p \mathbin{\square} q$);*
3. *p, q do not have a pinching cycle of periodic angles.*

Furthermore, pinching cycles of periodic angles are effectively enumerable using the nuclei of the bisets $B(p)$ and $B(q)$.

V. Symbolic and floating-point algorithms [7]

In this part we describe in more details the computational techniques available to manipulate Thurston maps. In particular, we give a range of symbolic algorithms, converters from one formalism into another, and describe hybrid algorithms that bridge between the group theory side and the complex analytic sides of Thurston maps. Many of these algorithms are already implemented in a GAP package Img available from the GAP website.

The main objects that the algorithms manipulate are bisets. They may be constructed using classical data such as external angles, floating-point approximations of maps, Hubbard trees, subdivision rules, etc. Many operations such as matings, tunings and composition with twists may be naturally implemented on them.

On the one hand, bisets are often constructed by choosing a basis and expressing its wreath recursion in that basis. Bisets, however, are fundamentally basis-free objects, and isomorphism, congruence etc. of bisets is discovered by finding bases in which their recursions coincide. Classical notions such as supporting rays, Hubbard trees etc. are viewed as invariants of the biset. Twisting of bisets, or more generally compositions with homeomorphisms, are implemented by changing the bases.

V.1. Converters

In order to compute with Thurston maps, it is important to represent them, and the objects they are related to, in an efficient manner.

Consider a Thurston map $f\colon (S^2, A) \circlearrowleft$. In case f is a topological polynomial, namely there exists a point $\infty \in A$ with $f^{-1}(\infty) = \{\infty\}$, then, on the one hand, the biset of f can be expressed in an adapted basis, in which its structure is essentially equivalent to a description by external angles. In case $\#A = 4$ and all critical points of f are simple, it is possible to express f as a "near-Euclidean map" (NET), see [13]. We consider here the general case.

Sphere groups themselves are represented by their number of generators and cyclic ordering of peripheral generators, as in (II.1). For each sphere group, an isomorphism with an underlying free group is chosen so as to allow fast comparison between elements. Elements in sphere groups are represented as words in the peripheral generators and their inverses.

Mapping class groups are described as outer automorphisms of sphere groups. They are represented by keeping track of the images of generators. This list of images is reduced to a lexicographically minimal one by conjugating it diagonally by the sphere group. Stallings' graphs are used to manipulate them, and in particular compute their inverse.

All our bisets are left-free, so are represented by their associated wreath recursions: if the right-acting sphere group has n generators and the biset has degree d, the bisets are represented as a list of n permutations in $d\wr$ and an $n \times d$ table of group elements.

ALGORITHM V.1. — GIVEN *a degree d and a list of tuples of rational numbers representing supporting rays à la Poirier* [29],
COMPUTE *the biset of the polynomial that it describes.*

ALGORITHM V.2. — GIVEN *a sphere biset B,*
COMPUTE *the supporting rays à la Poirier* [29]*, if B is combinatorially equivalent to a polynomial, and* `fail` *else.*

ALGORITHM V.3. — GIVEN *an angled Hubbard tree endowed with a self-map,*
COMPUTE *the biset of the polynomial that it describes.*

Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map. It may be conveniently encoded combinatorially as follows:

(1) A triangulation $\mathcal{T}_0$ of S^2 with vertex set A is chosen;
(2) The lift $\mathcal{T}_1$ of $\mathcal{T}_0$ under f is computed; it is a triangulation of S^2 with vertex set $f^{-1}(A)$, and f is expressed as a simplicial map $\mathcal{T}_1 \to \mathcal{T}_0$;
(3) A refinement $\mathcal{T}_1'$ of $\mathcal{T}_0$, also with vertex set $f^{-1}(A)$, is chosen, by subdividing in $\mathcal{T}_0$ all triangles containing elements of $f^{-1}(A) \backslash A$;
(4) The relation between $\mathcal{T}_1$ and $\mathcal{T}_1'$ is encoded by computing for each edge of $\mathcal{T}_1$ the sequence of edges of $\mathcal{T}_1'$ that it crosses.

The triangulation $\mathcal{T}_0$ expresses in a combinatorial manner the fundamental group of $S^2 \backslash A$: there is a retraction from $S^2 \backslash A$ to the dual graph $\mathcal{T}_0^{\perp}$ of $\mathcal{T}_0$, so $\pi_1(S^2 \backslash A) \cong \pi_1(\mathcal{T}_0^{\perp})$. In (4) we give a homotopy equivalence between $\mathcal{T}_1^{\perp}$ and $\mathcal{T}_1'^{\perp}$, called a *retriangulation*.

Conversely, a Thurston map is specified uniquely by the above data: a triangulation $\mathcal{T}_0$, a simplicial map $f\colon \mathcal{T}_1 \to \mathcal{T}_0$, a refinement $\mathcal{T}_1' \supseteq \mathcal{T}_0$, and a retriangulation of $\mathcal{T}_1'$ into $\mathcal{T}_1$ as specified in (4).

De facto, we are viewing f as a covering pair $f, i\colon (S^2, f^{-1}(A)) \rightrightarrows (S^2, A)$; we consider a triangulation $\mathcal{T}_0$ of the range, compute its preimages $\mathcal{T}_1$ and $\mathcal{T}_1'$ under f and i respectively, and encode combinatorially the relation between these triangulations at the source.

In fact, in some situations we already start with a combinatorial description of a map, rather than an actual Thurston map on a topological sphere. Most classical combinatorial descriptions are very close to the one given above. We argue in this text that a description in terms of bisets is algorithmically the most useful; it can be easily obtained from the combinatorial description sketched above:

ALGORITHM V.4. — GIVEN *triangulations $\mathcal{T}_1, \mathcal{T}_1', \mathcal{T}_0$ of S^2 such that $\mathcal{T}_1'$ is a refinement of $\mathcal{T}_0$, and given a simplicial map $f\colon \mathcal{T}_1 \to \mathcal{T}_0$ as well as a*

retriangulation $\mathcal{T}_1 \leftrightarrow \mathcal{T}_1'$,
COMPUTE *the biset of* f.

V.2. Floating-point algorithms

When we write that something is computable over $\mathbb{C}$, we mean, in the following strong sense, that all elements of $\mathbb{C}$ that we refer to are algebraic numbers ξ, that a minimal polynomial of ξ over $\mathbb{Z}$ is computable, and that a rectangle with rational corners is computable, that separates ξ from its Galois conjugates.

On the other hand, a *floating-point approximation* of $z \in \mathbb{C}$ is an algorithm producing an infinite data stream $(x_n + y_n i, \epsilon_n)$, with $x_n, y_n, \epsilon_n \in \mathbb{Q}$ and $|x_n + y_n i - z| < \epsilon_n$ and $\epsilon_n \to 0$.

Every computable number over $\mathbb{C}$ admits a floating-point approximation. Conversely, given a floating-point approximation $(x_n + y_n i, \epsilon_n)$ and the knowledge that its limit ξ is algebraic with given degree and bound on the coefficients of its minimal polynomial, the number ξ is computable over $\mathbb{C}$. This only uses standard algorithms for root finding, root isolation and lattice reduction (LLL).

A floating-point approximation of a rational map $f\colon (S^2, A) \circlearrowleft$ is an algorithm producing an infinite data stream (f_n, A_n, ϵ_n) with f_n a rational map with rational coefficients, A_n a set in fixed bijection $a_n \leftrightarrow a$ with A, and $\epsilon_n \to 0$ rational numbers, such that the coefficients of f_n are ϵ_n-close to those of f, and the spherical distances between pairs a, a_n and between pairs $f(x), f_n(x)$ are less than ϵ_n for all $a \in A$ and all $x \in \widehat{\mathbb{C}}$.

Consider a portrait $(f\colon A \cup C \to A, \deg\colon A \cup C \to \mathbb{N})$ of rational maps with hyperbolic orbifold; C is the set of critical points, and A is a forward-invariant set. The set of rational Thurston maps agreeing with f on $A \cup C$ and having required degree at $A \cup C$ is finite, and explicitly describable by a set of equations with integer coefficients — for example, with variables the coefficients of rational map and the position of $A \cup C$ on $\widehat{\mathbb{C}}$. It follows that, for every rational Thurston map with hyperbolic orbifold, its coefficients are computable over $\mathbb{C}$ in the sense above.

ALGORITHM V.5. — GIVEN *a floating-point approximation of a rational Thurston map* f*, as well as its portrait,* COMPUTE *the biset* $B(f)$ AS FOLLOWS:

(1) Find the post-critical set of f on a sufficiently close approximation of f, using the given portrait.

(2) Compute the Delaunay triangulation on the post-critical set.
(3) Add sufficiently many points to the triangulation, keeping the Delaunay condition, so that f-lifts of triangles are small enough; e.g. don't touch more than one post-critical point.
(4) Compute the f-preimage of this triangulation.
(5) Apply Algorithm V.4.

Algorithm V.6 ([2]). — Given *an admissible n-tuple of permutations $\pi_1, \dots, \pi_n \in d\wr$ and an n-tuple of distinct points $z_1, \dots, z_n \in \widehat{\mathbb{C}}$,*
Compute *a floating-point approximation of a rational map whose critical values z_i have monodromy π_i.*

We recall Thurston's fundamental "annular obstruction" theorem:

Theorem V.7 (Thurston [17]). — *Let $f\colon (S^2, P_f) \circlearrowleft$ be a Thurston map with hyperbolic orbifold. Then f is combinatorially equivalent to a rational map if and only if f admits no annular obstruction, namely no invariant multicurve whose Thurston matrix has spectral radius $\geqslant 1$, see* (II.3). *Furthermore, in that case the rational map is unique up to conjugation by Möbius transformations.*

Given a sphere biset ${}_G B_G$ with hyperbolic orbifold, either an annular obstruction for B or a rational map f with algebraic coefficients and $B(f) \sim B$ may be computed: as we noted above, there are finitely many rational maps $f_1, \dots, f_N$ with given degree and portrait, and they can be computed, e.g. by solving algebraic equations over $\mathbb{Z}$. Their bisets may be computed using Algorithm V.5. We may then in parallel search through the countably many multicurves in G, seeking an annular obstruction for B, and the countably many bijections between B and $B(f_i)$ for all $i \in \{1, \dots, N\}$, seeking a conjugacy. By Theorem V.7, one of these searches will eventually succeed.

If one knows beforehand that at least one critical point of f is periodic, and one is only interested in knowing whether f admits an annular obstruction, then a more straightforward algorithm is available: one may run in parallel a search for the annular obstruction and for a realization of f as a self-map on an "elastic graph" with appropriate looseness; such a self-map is a certificate for f being isotopic to a rational map, by a recent result of Dylan Thurston [38].

The degree of the equations over $\mathbb{Z}$ to consider is so high that this approach fails except in the most trivial cases. The following algorithm does work in practice:

ALGORITHM V.8. — GIVEN *a sphere biset* ${}_GB_G$ *with hyperbolic orbifold,* COMPUTE *either a rational map* $f \in \mathbb{C}(z)$*, with computable algebraic coefficients, such that* $B(f) \sim B$*, or a* B*-invariant annular obstruction* AS FOLLOWS:

(1) We consider *Teichmüller space modeled on* G as the space of triangulations of $\widehat{\mathbb{C}}$ with a distinguished n-tuple of vertices, as well as markings of its dual edges by elements of G. Two marked triangulations represent the same point in Teichmüller space if their distinguished vertices are images of each other under a Möbius transformation, and after their distinguished vertices are identified they admit a common refinement, up to isotopy, compatible with the group markings.
(2) Start by an arbitrary point S_0 in Teichmüller space. We iterate the following steps, starting with $i = 0$ and increasing it each time by 1.
(3) Apply Algorithm V.6 to the permutations in B and the marked points in S_i, find a rational map f_i.
(4) Compute the biset $B(f_i)$ by Algorithm V.5. Its right-acting group is G and its left-acting group is H_i.
(5) Match the bisets B and $B(f_i)$ by finding a group homomorphism $\phi_i \colon H_i \to G$ such that $B^\vee_{\phi_i} \otimes B(f_i) \cong B$. Note that ϕ_i is unique up to conjugation by inner automorphisms. The group H_i marks a triangulation of $\widehat{\mathbb{C}}$ with distinguished vertices $\widetilde{V}_i$ in bijection with peripheral classes in H_i. Those peripheral classes in H_i that map under ϕ_i to non-trivial peripheral classes in G determine a subset $V_i \subseteq \widetilde{V}_i$, and ϕ_i may be applied to the markings of the triangulation to produce a new triangulation, marked by G and with distinguished vertices V_i. This defines a new element S_{i+1} in Teichmüller space.
(6) Normalize the marking of S_{i+1}, viewed as $\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$, so that the barycentre of its marked points is $(0, 0, 0) \in \mathbb{R}^3$, the first of them (in some predetermined order) is at $(0, 0, 1)$, and the second of them lies in $\{0\} \times \mathbb{R}_+ \times \mathbb{R}$.
(7) By Thurston's Theorem V.7, either the maps f_i converge, and give approximations of the desired f, or points in S_i cluster.
To detect this, compute all cross-ratios of points in S_i. Partition the points in S_i by clustering together points with degenerating cross-ratios. Produce conjugacy classes in G by following labels on dual edges in the triangulation around clusters. Complete these conjugacy classes by adding their B_*-preimages till the collection becomes either B-invariant or non-(non-crossing). If it became invariant, compute its transition matrix to check whether it is an annular obstruction. In that case, return the annular obstruction.
(8) Simultaneously, seek algebraic numbers of low degree, small height (= maximal absolute value of coefficients over $\mathbb{Z}$), and close to the

estimated position of the set S_i (now normalizing it in $\widehat{\mathbb{C}}$ by putting three of its points at $0, 1, \infty$ respectively), and obtain in this manner a probable position $\tilde{S}$ of S_i consisting of algebraic numbers. Find the corresponding rational map $\tilde{f}$ with algebraic coefficients. Run Steps (4)–(5) with $\tilde{f}, \tilde{S}$ *en lieu* of f_i, S_i, checking at the same time that the permutations of $B(\tilde{f})$ match those of B; this produces a lift S_{i+1} of $\tilde{S}$. If S_{i+1}, suitably normalized as above, coincides with $\tilde{S}$, then return $\tilde{f}$, while if not increase i and return to step (3).

We note that the resort to algebraic numbers in Step (8) of Algorithm V.8 is only necessary to prove its validity; in practice it may be replaced by careful interval arithmetic; details are postponed to [7]. If we are interested in good floating-point approximations of the position of $\tilde{S}$, then we may iterate Steps (4)–(5), without changing the combinatorics of the marked spheres and only improving the position of the marked set S_i.

Corollary V.9. — *There is an algorithm that, given two sphere bisets ${}_GB_G$ and ${}_HC_H$ of rational non-*{GTor/2} *maps, decides whether B and C are conjugate, and if so produces a conjugator.*

The centralizers of ${}_GB_G$ and ${}_HC_H$ are trivial.

Proof. — The rational maps may be constructed using Algorithm V.8, and their coefficients compared once three points in each post-critical set are fixed. ☐

V.3. The canonical decomposition of Levy free maps

Recall the canonical decomposition of a Thurston map, introduced in §0.3. We show that Levy-free maps have quite restricted canonical decompositions:

Lemma V.10. — *Let f be a Levy free map. Either small maps in the canonical decomposition of f are rational or f is a* {GTor/2} *map with trivial canonical obstruction.*

As a consequence, the canonical obstruction is the union of the *canonical Levy obstruction* (the minimal multicurve whose return maps are Levy-free) and the *rational obstruction*: for an expanding map not doubly covered by a torus endomorphism, the minimal multicurve whose return maps are rational (for {GTor/2} maps, that multicurve is characterized in [32]).

Algorithm V.11. — Given *a Levy-free non-*{GTor/2} *sphere biset ${}_GB_G$,*
Compute *the canonical obstruction of B* as follows:

(1) *Enumerate in increasing order all multicurves $\mathscr{C}$ on the sphere marked by G;*
(2) *If $\mathscr{C}$ is not fully B-invariant, discard it;*
(3) *Run Algorithm V.8 on the small maps $R(B,\mathscr{C})$. If all small maps are rational, then return $\mathscr{C}$, otherwise discard it.*

Selinger and Yampolsky already showed in [33] that the canonical obstruction of a Thurston map is computable.

Corollary V.12. — *There is an algorithm that, given a sphere biset ${}_GB_G$ of type $\{\mathrm{Exp}\}\backslash\{\mathrm{GTor}/2\}$, computes the canonical decomposition ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$. All bisets in $R(\mathfrak{B})$ are the bisets of rational maps.*

Algorithm V.13. — Given *two contracting sphere bisets ${}_GB_G$ and ${}_HC_H$ that are not $\{\mathrm{GTor}/2\}$,*
Decide *whether B and C are conjugate, and* compute *the centralizer $Z(B)$* as follows:

(1) If there are more peripheral conjugacy classes in B, respectively C, than in their post-critical set, then these conjugacy classes can be expressed into a portrait of bisets, as in §III.2. Conjugacy of portraits of bisets is decidable, since B and C are contracting, see Proposition III.7. We therefore assume that B, C are marked by their post-critical conjugacy classes.
(2) Using Corollary V.12, compute the canonical decompositions ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ and ${}_{\mathfrak{Y}}\mathfrak{C}_{\mathfrak{Y}}$ of B and C respectively.
(3) Let X and Y be the set of distinguished conjugacy classes of $\mathfrak{X}$ and $\mathfrak{Y}$ respectively, see §II.4. Enumerate all possible bijections $h\colon X \to Y$.
(4) For every $h\colon X \to Y$ try to do the following steps. Return `fail` if there is no success.
(5) Using Algorithm II.9 try to promote $h\colon X \to Y$ into a biprincipal sphere $\mathfrak{X}$-$\mathfrak{Y}$-tree of bisets $\mathfrak{I}$. Discard h if there is no promotion.
(6) Check by Algorithm II.11 whether $\mathfrak{C}^{\mathfrak{I}} \in M(\mathfrak{B})$; if not, discard h.
(7) For every $\mathfrak{I}$ check, using Corollary V.9, whether bisets in $R(\mathfrak{B})$ and $R(\mathfrak{C}^{\mathfrak{I}})$ are conjugate; if not discard h.
(8) For every $\mathfrak{I}$, using Theorem II.8 check, whether the conjugacies between $R(\mathfrak{B})$ and $R(\mathfrak{C}^{\mathfrak{I}})$ promote into a conjugacy between $\mathfrak{B}$ and $\mathfrak{C}$.
(9) Using Theorem II.8 compute the centralizer $Z(B)$ of B, which is computable because by Corollary V.9 all bisets in $R(\mathfrak{B})$ have trivial centralizers.

Corollary V.14. — *Let B, C be contracting sphere bisets that are not $\{\mathrm{GTor}/2\}$. Then it is decidable whether B, C are conjugate, and the centralizer of B is computable.*

6. Decidability of combinatorial equivalence

In this brief section we put together results from §§ I–V to prove Theorem B: "it is decidable whether two Thurston maps are combinatorially equivalent".

Theorem 6.1. — *Let B, C be two sphere bisets. Assume that B and C are either degree-1 or Levy-free.*

Then it is decidable whether B and C are conjugate, and the centralizer of B is computable as a finitely generated subgroup of a product of pure mapping class groups.

Proof. — It is decidable whether B and C have the same degree, and have isomorphic acting groups; if not, they are not conjugate.

If B, C have degree 1, then they may be written as B_ϕ, B_ψ respectively; then $B \sim C$ if and only if ϕ and ψ are conjugate in a pure mapping class group; this is decidable by [21]. The centralizer of a mapping class is also computable; for example, train tracks [10] can be used to compute the centralizer of a pseudo-Anosov mapping class.

If B, C have degree > 1 and their orbisphere quotient groups are both $(2, 2, 2, 2)$-orbisphere groups, then their conjugacy and centralizer problems are solvable by Corollary III.14.

In the remaining case, B, C are contracting by Theorem C, so their conjugacy and centralizer problems are solvable by Corollary V.14. □

We note that the condition that the bisets be Levy-free is essential: in general, the centralizer need not be finitely generated, see Example 7.9.

We are ready to prove Theorem B. Its proof consists of the following

Algorithm 6.2. — Given *two sphere bisets ${}_GB_G$ and ${}_HC_H$,*
Decide *whether B and C are conjugate* as follows:

(1) Using Algorithm IV.7, compute the Levy decompositions ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ and ${}_{\mathfrak{Y}}\mathfrak{C}_{\mathfrak{Y}}$ of B and C respectively.
(2) Let X and Y be the set of distinguished conjugacy classes of $\mathfrak{X}$ and $\mathfrak{Y}$ respectively, see §II.4. Enumerate all possible bijections $h\colon X \to Y$.
(3) For every $h\colon X \to Y$ try to do the following steps. Return `fail` if there is no success.
(4) Using Algorithm II.9, try to promote $h\colon X \to Y$ into a biprincipal sphere $\mathfrak{X}$-$\mathfrak{Y}$-tree of bisets $\mathfrak{I}$. Discard h if there is no promotion.
(5) Using Algorithm II.11, check whether $\mathfrak{C}^{\mathfrak{I}} \in M(\mathfrak{B})$; if not, discard $\mathfrak{I}$.

(6) For every $\mathfrak{I}$ check, using Theorem 6.1, whether bisets in $R(\mathfrak{B})$ and $R(\mathfrak{C}^{\mathfrak{I}})$ are conjugate; if not discard $\mathfrak{I}$.
(7) Using Theorem 6.1 compute the centralizer of bisets in $R(\mathfrak{B})$.
(8) For every $\mathfrak{I}$ check, using Theorem II.8, whether the conjugacies between $R(\mathfrak{B})$ and $R(\mathfrak{C}^{\mathfrak{I}})$ promote into a conjugacy between $\mathfrak{B}$ and $\mathfrak{C}$.
(9) Using Theorem II.8, compute the centralizer $Z(B)$ of B.

7. Algebraic realizations

The previous sections explain how a Thurston map can canonically, and computably, be decomposed into pseudo-Anosov and finite-order homeomorphisms, rational maps of degree $\geqslant 2$ and maps doubly covered by torus endomorphisms.

Finite-order homeomorphisms are isotopic to Möbius transformations, which are rational maps. If $f\colon (S^2, A) \circlearrowleft$ has only rational maps as pieces, then each of its small spheres may be given a complex structure so as to make all pieces rational. The structure encoding f is then a *map of noded spheres.*

A *complex stable curve* is an algebraic variety with the topology of a noded sphere. It may be given as $X = (\widehat{\mathbb{C}}_1, P_1) \sqcup \cdots \sqcup (\widehat{\mathbb{C}}_n, P_n)/{\sim}$ with $\#P_i \geqslant 3$ for all i and an equivalence relation $\sim$ on $P_1 \sqcup P_2 \sqcup \cdots \sqcup P_k$ with classes of size $\leqslant 2$, such that the space obtained by replacing each $\widehat{\mathbb{C}} \cong S^2$ by a closed ball is contractible. It is thus a collection of $\widehat{\mathbb{C}} = \mathbb{P}^1(\mathbb{C})$'s glued to each other at single points, in a tree-like manner. The points in non-trivial $\sim$-classes are called *nodes*. Listing the non-trivial equivalence classes as $p_k \sim q_k$ for $i = 1, \dots, \ell$, with $p_k \in P_{i(k)}$ and $q_k \in P_{j(k)}$, one may describe X as an algebraic singular curve

$$\mathbb{X} = \{(x_1, \dots, x_n) \in \widehat{\mathbb{C}}^n \mid (x_{i(k)} - p_k)(x_{j(k)} - q_k) = 0 \text{ for } k = 1, \dots, \ell\}.$$

If $\mathscr{C}$ is a multicurve on a topological sphere (S^2, A), then shrinking all components of $\mathscr{C}$ to points gives a noded topological sphere. Conversely, given a complex stable curve, each node may be "opened", namely replaced by a small cylinder, so as to give a topological sphere and a multicurve.

Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map, and let $\mathscr{C}$ be an f-invariant multicurve. Let X be the corresponding noded topological sphere, and let Y be the noded topological sphere corresponding to $(S^2, f^{-1}(A), f^{-1}(\mathscr{C}))$. Then f induces a branched covering still written $f\colon Y \to X$, while the inclusion $f^{-1}(A) \supseteq A$ induces a continuous map $i\colon Y \to X$. We thus have a topological self-correspondence $f, i\colon Y \rightrightarrows X$. Topologically, i is a *blow-down*: it shrinks some spheres to points, and erases some marked points.

Note that the correspondence $f, i\colon Y \rightrightarrows X$ does not quite determine $f\colon (S^2, A) \circlearrowleft$: once nodes of X are opened to a multicurve $\mathscr{C}$ on (S^2, A), the map f is defined on (S^2, A) only up to an integer number of full twists along $\mathscr{C}$.

If X, Y may be given structures of complex stable curves $\mathbb{X}, \mathbb{Y}$ so that f, i become rational maps $\mathbb{Y} \to \mathbb{X}$, we say f is *realized* by $f, i\colon \mathbb{Y} \to \mathbb{X}$.

A periodic cycle of curves is called an *annular obstruction* if the spectral radius of its Thurston matrix (II.3) is $\geqslant 1$; and a Thurston map is called *obstructed* if it admits an annular obstruction. We show that many examples of Thurston maps, even if they are obstructed and therefore not combinatorially equivalent to a rational map (see Theorem V.7), may be described algebraically, using the following result:

Theorem D. — *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map. Then f may be realized on a complex stable curve by pinching along a multicurve generated by annular obstructions if and only if all pieces of f's canonical decomposition are rational, non-irrational* {GTor/2} *maps and homeomorphisms not containing pseudo-Anosov maps.*

Furthermore, it is computable whether all pieces of f are this form, and in that case what the correspondence of complex stable curves is.

In case all return maps are rational with hyperbolic orbifold, it follows from [31, Theorem 10.4] that such a realization exists, given by a fixed point in augmented Teichmüller space.

Proof of Theorem D. — If there are irrational maps doubly covered by torus endomorphisms, or homeomorphisms containing pseudo-Anosov maps, in the canonical decomposition, then these maps appear in every decomposition along an annular obstruction. They preserve transverse measured foliations with different stretch factors; this prevents them from being rational. Furthermore, since the eigenvalues are irrational, these pieces cannot be further decomposed.

On the other hand, given a decomposition in which the return maps are rational, non-irrational {GTor/2} maps and homeomorphisms not containing pseudo-Anosov maps, it is straightforward to assemble them together in a complex stable curve. Maps doubly covered by torus endomorphisms with integer eigenvalues decompose into Chebyshev polynomials, see §7.5. Homeomorphisms with no pseudo-Anosov piece decompose along an annular obstruction into finite-order homeomorphisms, which are isotopic to Möbius transformations. □

We conclude this section, and this article, with some fundamental examples illustrating decompositions, and of possible realizations on complex stable curves:

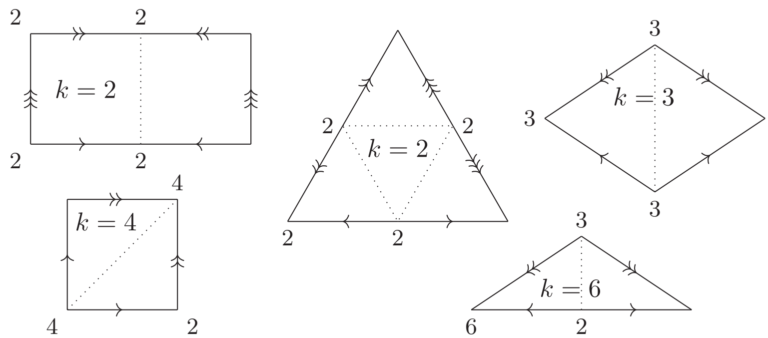
- **§7.3:** various maps obtained from composing the polynomial $z^2 + i$ with Dehn twists;
- **§7.4:** the mating of $z^2 - 1$ with itself: it is the simplest example of an obstructed map;
- **§7.6:** Douady–Hubbard's mating of the quadratic lamination 5/12 with itself. This map has intersecting annular obstructions, and we show which maps are obtained by cutting along the different multicurves;
- **§7.5:** maps doubly covered by diagonalizable torus endomorphisms;
- **§7.7:** A degree-5 rational map considered by Pilgrim in [28, §1.3.4]. It is obtained by blowing up an arc on torus endomorphism. (We solve the hitherto-open problem of determining whether this map is obstructed; it is so);
- **§7.8:** a degree-3 mating studied by Tan Lei and Shishikura [35], which is obstructed but does not admit Levy cycles. (We describe the mating both as a biset and as an algebraic self-map of a doubly-noded sphere, and answer [14, Question 2] by Chéritat);
- **§7.9:** a degree-4 Thurston map whose centralizer is infinitely generated.

7.1. $(2, 3, 6)$-maps

We give a brief description of rational maps multiply covered by a torus endomorphism in terms of their orbispace. Orbispaces of Euler characteristic 0 have marked points of respective orders $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 6)$. They may be treated uniformly as follows: let ζ be a kth root of unity, for $k \in \{2, 3, 4, 6\}$. If $k \geqslant 3$, let $\Lambda := \mathbb{Z}[\zeta]$ be the lattice spanned by ζ in $\mathbb{C}$; while if $k = 2$ choose at will $\Lambda = \mathbb{Z}[i]$ or $\Lambda = \mathbb{Z}[(-1 + \sqrt{-3})/2]$. Set then

$$G := \Lambda \rtimes \langle\zeta\rangle.$$

The orbispace is $\mathbb{C}/G$, with marked points of orders $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$ for $k = 2, 3, 4, 6$ respectively. Graphically,



Endomorphisms of the orbisphere $\mathbb{C}/G$ are quotients $\overline{f}$ of maps $f\colon \mathbb{C} \circlearrowleft$ with $f \circ G \subseteq G \circ f$. It immediately follows that f is isotopic to an affine map, say $f(z) = az + b$ with $a, b \in \mathbb{C}$. Then $f \circ (z \mapsto z + 1) = (z \mapsto z + a) \circ f$, so $a \in \Lambda$; and $f \circ (z \mapsto \zeta z) = (z \mapsto \zeta z + (1 - \zeta)b) \circ f$, so $b \in (1 - \zeta)^{-1}\Lambda$. Since $z \mapsto z + 1$ and $z \mapsto \zeta z$ generate G, these conditions are also sufficient. Therefore, every rational map on $\mathbb{C}/G$ is determined by parameters $a \in \Lambda$ and $b \in \frac{1}{1-\zeta}\Lambda/\Lambda$.

The biset of $\overline{f}$ is then easy to compute: since f is invertible, there exists, for all $g \in G$, a unique element, written $g^f \in G$, such that $gf = fg^f$. We obtain:

Proposition 7.1. — *The biset $B(\overline{f})$ of $f(z) = az + b$ is G as a right G-set, with left action given by $g \cdot b = g^f b$.*

Note also $g^f \in \Lambda$ for all $g \in \Lambda$, so $\Lambda \subseteq B(\overline{f})$ is a Λ-subbiset. Its structure depends only on the linear part a of f. We may then write $B(\overline{f})$ as a crossed product biset $\Lambda \rtimes \langle\zeta\rangle$ of this Λ-subbiset with the group $\langle\zeta\rangle$ acting by automorphisms on Λ. This extends the discussion in §III.3, and in particular (III.5).

7.2. A Dehn twist

Before studying the twisted cousins of the polynomial $z^2 + i$ in §7.3, we consider the simple example of a Dehn twist on a sphere with four punctures. Consider a basepoint $*$, and arrange the punctures in counterclockwise order around $*$. Let a, b, c, d denote the "lollipop" generators about the punctures, giving rise to the sphere group

$$G = \langle a, b, c, d \mid dcba \rangle .$$

Set $r = cb$, and let T denote the Dehn twist about the curve r^G. Note that T fixes $*$. The action of T on G is given by

$$a \mapsto a\,, \quad b \mapsto b^r\,, \quad c \mapsto c^r\,, \quad d \mapsto d\,,$$

so the biset of T has in basis $\{*\}$ the wreath recursion

$$a = \langle\!\langle a \rangle\!\rangle, \quad b = \langle\!\langle b^r \rangle\!\rangle, \quad c = \langle\!\langle c^r \rangle\!\rangle, \quad d = \langle\!\langle d \rangle\!\rangle.$$

We have $G = G_1 *_{\mathbb{Z}} G_2$, with $G_1 = \langle a, r, d \mid dra \rangle$ and $G_2 = \langle b, c, r^{-1} \mid bcr^{-1} \rangle$, glued along $\langle r \rangle \cong \mathbb{Z}$. The tree of groups decomposition of G therefore consists of a single segment. The decomposition $\mathfrak{T}$ of $B(T)$ as a tree of bisets is also a single segment, with $\rho = \lambda = \mathbb{1}$ and vertex and edge bisets B_1, B_2, E_1 above $G_1, G_2, \langle r \rangle$ respectively:

$$\begin{tikzcd}
B_1 \arrow[rr, no head, "E_1"'] \arrow[d, "\rho"] \arrow[d, squiggly, bend right, "\lambda"'] & & B_2 \arrow[d, "\rho"] \arrow[d, squiggly, bend right, "\lambda"'] \\
G_1 \arrow[rr, no head, "\langle r \rangle"] & & G_2 .
\end{tikzcd}$$

These bisets are obtained as subbisets of $B(T)$ by restricting the wreath recursion of $B(T)$ to the subgroups G_1, G_2 and $\langle r \rangle$ respectively. If we use for them the same basis $\{*\}$, they are given as follows:

- the G_1-G_1-biset B_1 is the biset of the identity
$$a = \langle\!\langle a \rangle\!\rangle, \quad r = \langle\!\langle r \rangle\!\rangle, \quad d = \langle\!\langle d \rangle\!\rangle,$$
because the basepoint $*$ belongs to the sphere marked by a, d, r;
- the G_2-G_2-biset B_2 has wreath recursion
$$b = \langle\!\langle b^r \rangle\!\rangle, \quad c = \langle\!\langle c^r \rangle\!\rangle, \quad r = \langle\!\langle r \rangle\!\rangle;$$
- the biset E_1 is the identity $\mathbb{Z}$-$\mathbb{Z}$-biset. It embeds naturally in its respective source and target vertex bisets under $g \cdot * \mapsto g \cdot *$ for all $g \in \langle r \rangle$.

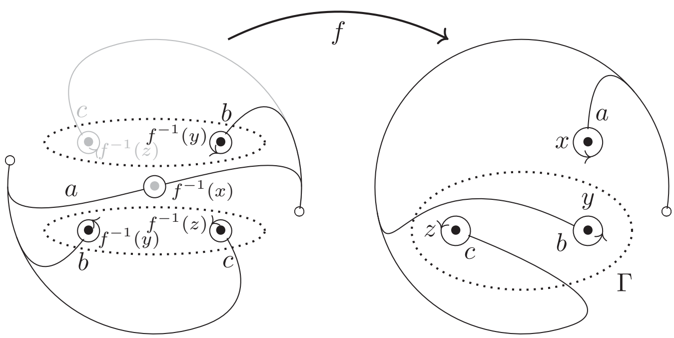
Note that, if one changes B_2's basis to $\{*' := r \cdot *\}$, one obtains for B_2 the wreath recursion of the identity map; but then the embeddings of the edge $\mathbb{Z}$-$\mathbb{Z}$-biset E_1 in B_1 and B_2 change: the basis element $*$ of E_1 maps to the basis element $*$ of B_1, but to $r^{-1} \times$ the basis element $*'$ of B_2.

Note also that the biset of T is biprincipal and that $\mathfrak{T}$ is a biprincipal tree of bisets. The biset of T^n is $B(T)^{\otimes n}$, and its decomposition $\mathfrak{T}^{\otimes n}$ also consists of a single edge, with identity bisets at vertices and the edge, and embeddings $b \mapsto b$ and $b \mapsto r^{-n} \cdot b$ of the edge biset into the vertex bisets B_1 and B_2 respectively.

This example illustrates the importance of the edge biset embeddings. For all $\mathfrak{T}^{\otimes n}$ the trees of groups are the same, the trees of bisets have the same underlying tree and the same vertex and edge bisets; the algebraic realizations are the same (a noded sphere with identity self-map), but the embeddings of the edge biset in the vertex bisets depend on the twist parameter n.

7.3. All the twisted cousins of $z^2 + i$

The polynomial z^2+i has the following critical graph: $0 \Rightarrow i \to i-1 \leftrightarrow -i$. There exist maps with the same post-critical graph as z^2+i that are obstructed, and one such map, f may be constructed as follows [9, §6.1]:



The post-critical set of f is $P_f := \{\infty, x, y, z\}$ with post-critical graph $\cdot \Rightarrow x \to y \leftrightarrow z$. Let Γ denote the simple closed curve Γ encircling y and z.

We first describe the biset $B(f)$. For this, we choose a basepoint close to ∞ on the positive real axis (indicated by a white dot on the figure above), and we consider the simple ("lollipop") loops a, b, c, d around x, y, z, ∞ respectively. This gives the sphere group (see §II.1)

$$G = \langle a, b, c, d \mid dcba \rangle .$$

The wreath recursion (see §I.1) of $B(f)$ may be computed as follows. By our choice of basepoint $*$, one preimage $*_1$ is close to $+\infty$ and the other one $*_2$ is close to $-\infty$. We choose as basis of $B(f)$ the set $Q = \{\ell_1, \ell_2\}$ with ℓ_1 a very short path from $*_1$ to $*$ and ℓ_2 a half-turn in the upper half-plane from $*_2$ to $*$. Then, tracing f-lifts of the lollipop generators, we obtain a presentation of $B(f)$ as

$$a = \langle\!\langle a^{-1}, a \rangle\!\rangle(1,2), \quad b = \langle\!\langle a, c \rangle\!\rangle, \quad c = \langle\!\langle 1, cbc^{-1} \rangle\!\rangle, \quad d = \langle\!\langle d, 1 \rangle\!\rangle(1,2).$$

One checks easily that $r = cb$, representing the conjugacy class Γ, is an annular obstruction, and furthermore is a Levy cycle: indeed $r = \langle\!\langle a, r\rangle\!\rangle$, so its transition matrix is (1).

Let T denote the Dehn twist about Γ as in 7.2. We also consider all the maps $f_n = T^n \circ f$; they are also obstructed, and are all combinatorially inequivalent (see [28, Theorem 8.2] or [9, Proposition 6.10]). The action of T on G is given by

$$a \mapsto a\,, \quad b \mapsto b^r\,, \quad c \mapsto c^r\,, \quad d \mapsto d$$

so the biset of f_n has wreath recursion

$$a = \langle\!\langle a^{-1}, a\rangle\!\rangle(1,2)\,, \quad b = \langle\!\langle a, c^{r^n}\rangle\!\rangle\,, \quad c = \langle\!\langle 1, b^{r^{n-1}}\rangle\!\rangle\,, \quad d = \langle\!\langle d, 1\rangle\!\rangle(1,2)\,. \tag{7.1}$$

Another construction of f_n may be given as follows. Start with the biset $B(z^2+i)$; it can be computed by drawing paths in $\mathbb{C}\backslash\{i, i-1, -i\}$ and lifting them by $\sqrt{z-i}$, but can also be obtained by Algorithm V.1 starting from the external angle $1/6$ of the map z^2+i. The wreath recursion of $B(z^2+i)$ is

$$a = \langle\!\langle dc, ba\rangle\!\rangle(1,2)\,, \quad b = \langle\!\langle a, c\rangle\!\rangle\,, \quad c = \langle\!\langle b, 1\rangle\!\rangle\,, \quad d = \langle\!\langle d, 1\rangle\!\rangle(1,2)\,.$$

Consider next the Dehn twist U about the simple closed curve encircling i and $i-1$; its action on G is

$$a \mapsto a^{ba}\,, \quad b \mapsto b^a\,, \quad c \mapsto c\,, \quad d \mapsto d\,.$$

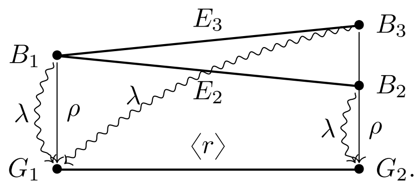
Then f_1 is combinatorially equivalent (see §II.1) to $(z^2+i)\circ U^{-1}$, so $B(f_1) \cong B(U)^\vee \otimes B(z^2+i)$, and indeed the wreath recursion of $B(U)^\vee \otimes B(z^2+i)$ is

$$a = \langle\!\langle cb, ad\rangle\!\rangle(1,2)\,, \quad b = \langle\!\langle a^d, c^b\rangle\!\rangle\,, \quad c = \langle\!\langle 1, b\rangle\!\rangle\,, \quad d = \langle\!\langle 1, d\rangle\!\rangle(1,2)\,,$$

which in basis $\{\ell_2, d^{-1}\ell_1\}$ coincides with (7.1) for $n = 1$.

We have $G = G_1 *_{\mathbb{Z}} G_2$, with $G_1 = \langle a, r, d \mid dra\rangle$ and $G_2 = \langle b, c, r^{-1} \mid bcr^{-1}\rangle$, glued along $\langle r\rangle \cong \mathbb{Z}$. The tree of groups therefore consists of a single segment.

Since no small sphere in $(S^2, f_n^{-1}(P_f), f_n^{-1}(\Gamma))$ maps to an annulus, we do not need to subdivide the tree of groups barycentrically. Thus the decomposition $\mathfrak{B}(f_n)$ of $B(f_n)$ as a tree of bisets has three vertex bisets, corresponding to the three components B_1, B_2, B_3 of $S^2\backslash f^{-1}(\Gamma)$. They are arranged as follows:



(7.2)

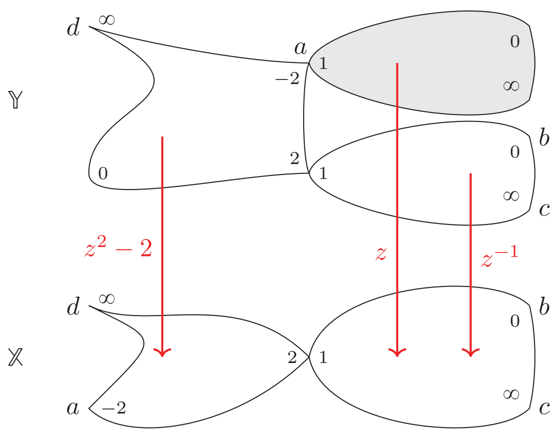
By convention, the covering map ρ is given by vertical projection and drawn in plain lines, while the map λ is drawn in squiggly lines; it sends B_1, E_3 and B_3 to G_1 while sending E_2 to $\langle r\rangle$ and B_2 to G_2.

The vertex bisets are obtained as subbisets of $B(f_n)$ by restricting the wreath recursion of $B(f_n)$ to the subgroups G_1 and G_2, using subsets of the basis Q, and are given as follows:

- the G_1-G_1-biset B_1 has in the basis Q the wreath recursion
$$a = \langle\!\langle a^{-1}, a\rangle\!\rangle(1,2)\,,\quad r = \langle\!\langle a, r\rangle\!\rangle\,,\quad d = \langle\!\langle d, 1\rangle\!\rangle(1,2)\,,$$
and is isomorphic to the biset of the rational map z^2-2;
- the G_2-G_2-biset B_2 has in the basis $\{\ell_2\}$ the wreath recursion
$$b = \langle\!\langle c^{r^n}\rangle\!\rangle\,,\quad c = \langle\!\langle b^{r^{n-1}}\rangle\!\rangle\,,\quad r = \langle\!\langle r\rangle\!\rangle\,,$$
and is isomorphic to the biset of the rational map z^{-1} marked at $\{0,1,\infty\}$;
- the G_1-G_2-biset B_3 has in the basis $\{\ell_1\}$ the wreath recursion
$$b = \langle\!\langle a\rangle\!\rangle\,,\quad c = \langle\!\langle 1\rangle\!\rangle\,,\quad r = \langle\!\langle a\rangle\!\rangle\,;$$
- the bisets E_2 is the identity $\mathbb{Z}$-$\mathbb{Z}$-biset in the basis $\{\ell_1\}$, and the biset E_3 is the G_1-$\mathbb{Z}$-biset given in the basis $\{\ell_2\}$ by the wreath recursion $r = \langle\!\langle a\rangle\!\rangle$; these edge bisets embed naturally in their respective source and target vertex bisets.

Recalling from §7.2 the notation $\mathfrak{T}$ for the biprincipal biset of T, we have $\mathfrak{B}(f_n)\cong\mathfrak{T}^{\otimes n}\otimes\mathfrak{B}(f_0)$. Note that, if one changes B_2's basis to $\{\ell_2' := r^n\cdot\ell_2\}$, one obtains a simpler wreath recursion
$$b = \langle\!\langle c\rangle\!\rangle\,,\quad c = \langle\!\langle cbc^{-1}\rangle\!\rangle\,,\quad r = \langle\!\langle r\rangle\!\rangle\,;$$
that does not depend on n, but then one has to specify the embeddings of the edge $\mathbb{Z}$-$\mathbb{Z}$-biset E_2 in B_1 and B_2 respectively: the basis element ℓ_2 of E_2 maps to the basis element ℓ_2 in B_1, but to $r^{-n}\times$ the basis element ℓ_2' in B_2.

The maps f_n all admit the same algebraic realization on a singly noded complex stable curve, as $f, i\colon \mathbb{Y}\rightrightarrows\mathbb{X}$. By convention, we identify the post-critical points with the elementary loops in G representing them, so that the post-critical set is now $\{a,b,c,d\}$. We only give the map f, in red; we chose the coördinates on the spheres in $\mathbb{Y}$ so that i is either the identity map or the constant map on each component, so that it suffices to label, on $\mathbb{Y}$, the i-preimages of the post-critical set. The mapping i on spheres is, anyways, the same as the map λ in (7.2). We indicate inside the spheres the coördinates that we chose so as to make the maps rational; they are usually unimportant in those spheres of $\mathbb{Y}$ which get blown down, and which are drawn shaded. We also attempt to give the correct geometry to the spheres by indicating the angles at the post-critical points and their f-preimages:



7.4. The mating of $z^2 - 1$ with itself

This example appears in [28, §1.3.2]. Consider first the polynomial $g(z) = z^2 - 1$, with post-critical set $P_g = \{0, -1, \infty\}$. Choose as in §7.3 a base-point $* \in \mathbb{R}_+$ close to ∞, and a basis $\{\ell_1, \ell_2\}$ consisting of a short path ℓ_1 from $\sqrt{*+1}$ to $*$ and an upper half-circle from $-\sqrt{*+1}$ to $*$. Write $H = \langle a, b, t \mid tba\rangle$ for the fundamental group of $\widehat{\mathbb{C}} \setminus P_g$, with a, b, t elementary loops around $-1, 0, \infty$ respectively (i.e. a, b, t follow the straight lines from $*$ to the respective point). The presentation of the biset ${}_H B(g)_H$ is

$$a = \langle\!\langle a^{-1}, ba\rangle\!\rangle(1,2), \quad b = \langle\!\langle a, 1\rangle\!\rangle, \quad t = \langle\!\langle t, 1\rangle\!\rangle(1,2).$$

It may also be obtained by Algorithm V.1 starting from the external angle $1/3$ of the map $z^2 - 1$.

Let now the branched covering f be the *mating* of g with itself. Topologically, it is the following map. Take two copies of $\mathbb{C}$, and compactify each with a circle $\{\infty e^{i\theta} \mid \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$. Glue these two closed disks by identifying $\infty e^{2i\pi\theta}$ on the first with $\infty e^{-i\theta}$ on the second. Let g act on each copy, and note that they agree with the map $\infty e^{i\theta} \mapsto \infty e^{2i\theta}$ on the circle at infinity. In effect, we have decomposed S^2 in its upper and lower hemispheres and let g act independently on both.

A presentation of $B(f)$ may easily be computed. Consider a copy $\overline{H} = \langle \bar a, \bar b, \bar t \mid \bar t\bar b\bar a\rangle$, and the group

$$G = H *_{\langle t\rangle = \langle \bar t^{-1}\rangle} \bar H = \langle a, b, \bar a, \bar b \mid ba\bar b\bar a\rangle\,.$$

This is a sphere tree of groups with two vertices and a single edge corresponding to the fundamental group of the circle at infinity. The biset $B(f)$ is the fundamental biset of the following tree of bisets: it has two vertices each carrying the biset $B(g)$; we write $\bar B(g)$ for the biset of the second vertex to distinguish it from the first. An edge connects these vertices, carrying the biset $B(z^2)$ which is $\langle t\rangle$ as a set with actions $t^i \cdot t^j \cdot t^k = t^{2i+j+k}$. The inclusions of $B(z^2)$ in $B(g)$ are as follows: choosing for each of the bisets $B(z^2), B(g), \bar B(g)$ the same basis $\{\ell_1, \ell_2\}$, the maps are

$$\begin{cases} B(z^2) & \to B(g)\\ \ell_1 & \mapsto \ell_1\\ \ell_2 & \mapsto \ell_2\end{cases} \qquad \text{and} \qquad \begin{cases} B(z^2) & \to B(g)\\ \ell_1 & \mapsto \ell_1\\ \ell_2 & \mapsto t\cdot\ell_2\,.\end{cases}$$

We obtain in this manner the following presentation for ${}_G B(f)_G$:

$$a = \langle\!\langle a^{-1}, ba\rangle\!\rangle(1,2)\,,\quad b = \langle\!\langle a, 1\rangle\!\rangle\,,\quad \bar a = \langle\!\langle \bar b\bar a, \bar a^{-1}\rangle\!\rangle(1,2)\,,\quad \bar b = \langle\!\langle 1, \bar a\rangle\!\rangle\,.$$

We naturally have an invariant multicurve $\{(ba)^G\}$, since $ba = \langle\!\langle 1, ba\rangle\!\rangle(1,2)$; and its Thurston matrix is $(1/2)$. There is, however, another invariant multicurve

$$\Gamma = \{x^G\} \text{ with } x = \bar a a\,,$$

since $x = \langle\!\langle 1, x^{-\bar a}\rangle\!\rangle$; and its Thurston matrix is (1), so it is a Levy obstruction.

We have $G = G_1 *_{\mathbb Z} G_2$, with $G_1 = \langle a, \bar a, x^{-1} \mid \bar a a x^{-1}\rangle$ and $G_2 = \langle b^a, \bar b, x \mid b^a\bar b x\rangle$, glued along $\langle x\rangle = \mathbb Z$. The sphere tree of groups therefore consists of a single segment.

We change the basis of $B(f)$ to $Q = \{\ell_1, \bar a\ell_2\}$ so as to make more visible the decomposition of $B(f)$ as a tree of bisets; indeed in that basis $x = \langle\!\langle 1, x^{-1}\rangle\!\rangle$. The presentation of $B(f)$ becomes, on the generating set $\{a, \bar a, b^a, \bar b\}$,

$$a = \langle\!\langle x^{-1}, xb^a\rangle\!\rangle(1,2)\,,\quad \bar a = \langle\!\langle \bar b, 1\rangle\!\rangle(1,2)\,,\quad b^a = \langle\!\langle 1, a^{x^{-1}}\rangle\!\rangle\,,\quad \bar b = \langle\!\langle 1, \bar a\rangle\!\rangle\,.$$

Again we do not need to subdivide the tree of groups barycentrically. We note that, in the new basis of $B(f)$, the wreath recursion restricts to maps $G_1 \to G_2^2 \rtimes Q\wr$ and $G_2 \to 1 \times G_1$. Therefore, the decomposition of $B(f)$ as a sphere tree of bisets has two vertices B_1, B_2 joined by an edge E_2, and such that ρ sends B_i to G_i while λ sends B_i to G_{3-i}, and an additional trivial vertex B_3 above G_2 joined by an edge E_3 to B_1:

$$\begin{tikzcd} & & B_3 \arrow[dd, squiggly, bend left] \\ B_1 \arrow[urr, dash, "E_3"] \arrow[rr, dash, "E_2"] \arrow[d, "\rho"'] \arrow[drr, squiggly, "\lambda"] & & B_2 \arrow[d, "\rho"] \arrow[dll, squiggly, "\lambda"'] \\ G_1 \arrow[rr, dash, "\langle x \rangle"'] & & G_2. \end{tikzcd} \tag{7.3}$$

The vertex bisets are again obtained by restricting $B(f)$ while using subsets of the basis Q, and are given as follows:

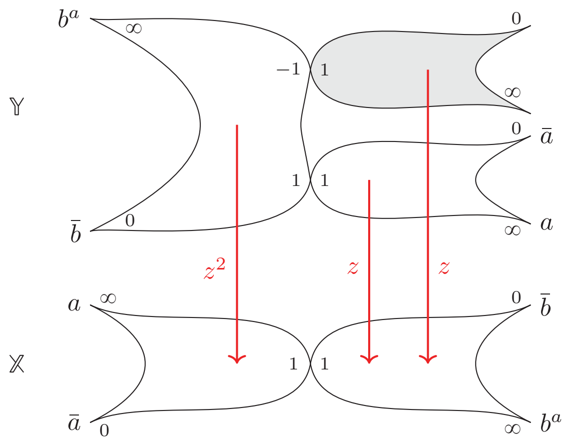
- the G_2-G_1-biset B_1 has in the basis Q the wreath recursion
$$a = \langle\!\langle x^{-1}, xb^a \rangle\!\rangle(1,2)\,, \quad \bar{a} = \langle\!\langle \bar{b}, 1 \rangle\!\rangle(1,2)\,, \quad x^{-1} = \langle\!\langle 1, x \rangle\!\rangle\,;$$
- the G_1-G_2-biset B_2 has in the basis $\{\bar{a}\ell_2\}$ the wreath recursion
$$b^a = \langle\!\langle a^{x^{-1}} \rangle\!\rangle\,, \quad \bar{b} = \langle\!\langle \bar{a} \rangle\!\rangle\,, \quad x = \langle\!\langle x^{-1} \rangle\!\rangle\,;$$
- the G_2-G_2-biset B_3 is trivial in the basis $\{\ell_1\}$: it has the wreath recursion $b^a = \bar{b} = x = \langle\!\langle 1 \rangle\!\rangle$. It corresponds to a sphere that gets shrunk under i.
- the bisets E_2 and E_3 are the identity $\mathbb{Z}$-$\mathbb{Z}$-bisets in the bases $\{\bar{a}\ell_2\}$ and $\{\ell_1\}$ respectively, and embed naturally in their respective source and target vertex bisets.

To obtain the pieces of the decomposition, we consider the first return map of $\lambda^{-1}\rho$ to G_1 via its biset $B_2 \otimes_{G_2} B_1$. Its wreath recursion is

$$a = \langle\!\langle x, ax^{-1} \rangle\!\rangle(1,2)\,, \quad \bar{a} = \langle\!\langle \bar{a}, 1 \rangle\!\rangle(1,2)\,, \quad x^{-1} = \langle\!\langle 1, x^{-1} \rangle\!\rangle$$

which is isomorphic to $B(z^2)$, with x^{-1} representing a loop around the fixed point 1.

The algebraic realization takes place on a singly noded complex stable curve, as $f, i \colon \mathbb{Y} \rightrightarrows \mathbb{X}$. We keep the convention of identifying the post-critical points with the elementary loops in G representing them, giving the map f in red and forcing i to be either the identity or the constant map on each component:



7.5. Maps doubly covered by diagonal torus endomorphisms

We consider next the endomorphism $(\begin{smallmatrix} m & 0 \\ 0 & n \end{smallmatrix})$ of the torus $\mathbb{R}^2/\mathbb{Z}^2$, of degree mn, and its projection to a map $f_{m,n}\colon S^2 \circlearrowleft$ via the Weierstraß function $\wp$ of the square lattice. The example $m = 3, n = 2$ is treated in [28, §1.3.3]. Without loss of generality, we restrict ourselves to $m \geqslant n$.

The critical points of $\wp$ are at $\frac{1}{2}\mathbb{Z}^2$, so $f_{m,n}$ has $\{\wp(0), \wp(\frac{1}{2}), \wp(\frac{i}{2}), \wp(\frac{1+i}{2})\}$ as post-critical set. If $m = n$, then f is a rational map — a *flexible Lattès map*, see [25] and §III.3 — and its pullback map $\sigma_{f_{m,n}}$ on Teichmüller space is the identity. In the general case, the pullback map associated to the map $f = \wp \circ (\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}) \circ \wp^{-1}$ is $\sigma_f(z) = (pz + r)/(qz + s)$, if one identifies Teichmüller space $\mathscr{T}_{\{a,b,c,d\}}$ with the upper half plane.

Let us write $a = \wp(0), b = \wp(\frac{1}{2}), c = \wp(\frac{i}{2}), d = \wp(\frac{1+i}{2})$; the post-critical graph depends on the parity of m and n, but in all cases a is a fixed point, b maps to a or b, etc. The map $f_{m,n}$ may be given by considering the rectangle $X = [0, 2] \times [0, 1] \subset \mathbb{R}^2$, with sides identified under $(0, y) \sim (2, y)$ and $(1 - x, 0) \sim (1 + x, 0)$ and $(1 - x, 2) \sim (1 + x, 2)$. This is topologically a sphere, and metrically a "pillowcase". Consider next the rectangle $Y = [0, 2m] \times [0, n]$ and the maps $f, i \rightrightarrows Y \to X$ given by $i(x, y) = (x/m, y/n)$ and $f(x, y) = (x, y)$ on $[0, 2] \times [0, 1]$, extended by reflections in the lines $y \in \mathbb{Z}$ and $x \in 2\mathbb{Z}$. The picture for $m = 3, n = 2$ is given in Figure 7.1.

Since all post-critical points are orbispace points of order 2, the sphere group of $f_{m,n}$ is

$$G = \langle a, b, c, d \mid dcba, a^2, b^2, c^2, d^2\rangle;$$

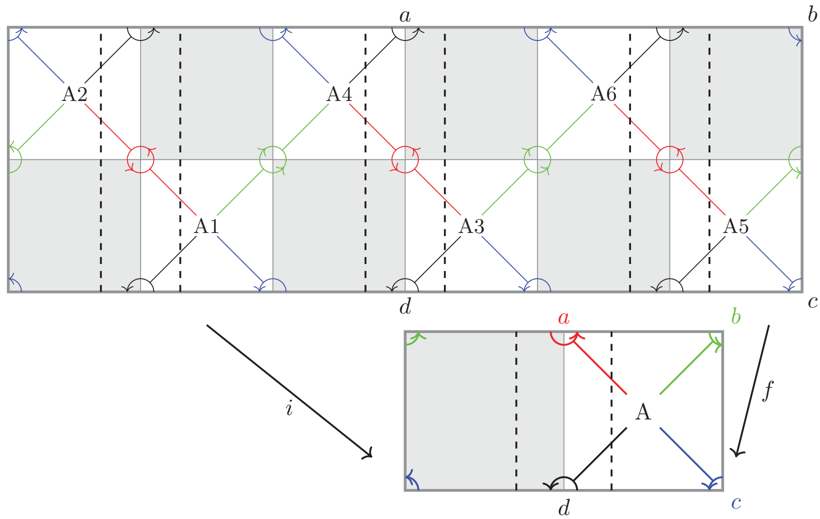


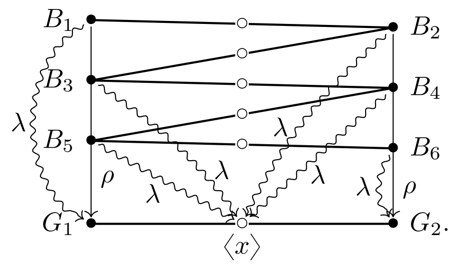
Figure 7.1. The map doubly covered by the torus endomorphism $m = 3, n = 2$

its subgroup $H = \langle ba, ad\rangle$ has index 2 and $H \cong \mathbb{Z}^2$. The biset $B(f_{m,n})$ can easily be computed from this picture, but the answer is not very illuminating. Let $\tilde f_{m,n}$ be the self-map of $\mathbb{R}^2/\mathbb{Z}^2$ defined by $(x, y) \mapsto (mx, ny)$; then the biset $B(\tilde f_{m,n})$ admits the following simple description as an H-H-biset: identify H with $\mathbb{Z}^2$. As a set, $B(\tilde f_{m,n}) = \mathbb{Z}^2$. The left and right actions of $\mathbb{Z}^2$ are given by $v \cdot \beta \cdot w = \left(\begin{smallmatrix} m & 0 \\ 0 & n\end{smallmatrix}\right)v + \beta + w$. From the degree-2 branched cover $\mathbb{R}^2/\mathbb{Z}^2 \to S^2$ we deduce that $B(\tilde f_{m,n})$ is a subbiset of $B(f_{m,n})$ of index 2, namely $B(f_{m,n})$ is, qua left H-set, the disjoint union of two copies of $B(\tilde f_{m,n})$.

We turn to the decomposition of $f_{m,n}$. Set $x = ad$; then the multicurve $\{x^G\}$ is invariant. It has m preimages mapping by degree n to itself, so its Thurston matrix is (m/n) and it is an obstruction.

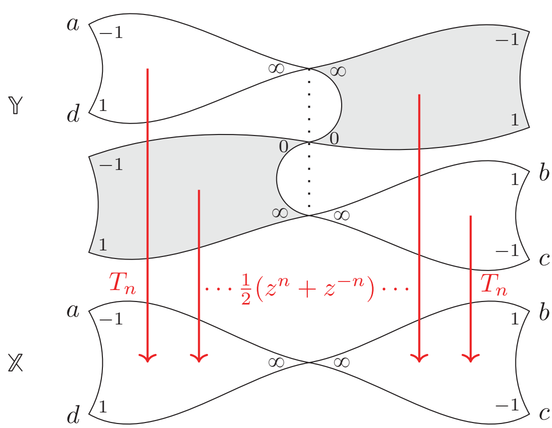
The sphere tree of groups decomposition of G associated with $\{x^G\}$ is $G = G_1 *_{\langle x\rangle} G_2$ with $G_1 = \langle a, d, x^{-1} \mid adx^{-1}\rangle$ and $G_2 = \langle b, c, x \mid bcx\rangle$, and the tree of bisets decomposition of $B(f_{m,n})$ has $m + 1$ vertices arranged in a chain. Note that we needed, in this case, to consider the barycentric subdivision of the tree of groups with two vertices and one edge, because the map λ sends some vertices to annuli. Here is the graph for $m = 5$; for even

m, both endpoints of the tree of bisets would map to G_1 by ρ:



(7.4)

We give directly the complex stable curve, which is singly noded. Denote by T_n the Chebyshev polynomial $T_n(z) = \cos(n \arccos z)$. We keep the convention of identifying the post-critical points with the elementary loops in G representing them, giving the map f in red and forcing i to be either the identity or the constant map on each component — and, in the latter case, indicating the blown-down spheres in shade. We consider m odd; if m is even, then the first and last spheres both map to the left sphere by i:



7.6. The formal mating $5/12 \uplus 5/12$

Douady and Hubbard, in their article [17], consider the formal mating with itself of the (obstructed) topological polynomial with lamination angle 5/12. This example is important because it is a Thurston map with six curves, four of which forming a chain, such that various subsets of these six curves define an annular obstruction. Because curves in an obstruction cannot intersect, this means that there is, in general, no such thing as a "maximal annular obstruction".

We start by describing the presentation of the map $f_{5/12}$, using this time the language of laminations; see [8]. An identical recursion arises if one starts with the angle 7/12. In that picture, the generators lie at angles 5/12, 5/6, 2/3 and 1/3. With the generators t, g_1, g_2, g_3, g_4 corresponding to the angles $0, 1/3, 5/12, 2/3, 5/6$ in increasing order, the biset $B(f_{5/12})$ has presentation

$$t = \langle\langle t, 1\rangle\rangle(1,2)\,, \quad g_1 = \langle\langle 1, g_3\rangle\rangle\,, \quad g_2 = \langle\langle g_1^{-1}g_2^{-1}g_3^{-1}, g_3g_2g_1\rangle\rangle(1,2)\,,$$
$$g_3 = \langle\langle g_1, g_4\rangle\rangle\,, \quad g_4 = \langle\langle g_2, 1\rangle\rangle\,.$$

The fundamental group is $H = \langle t, g_1, g_2, g_3, g_4 \mid g_4g_3g_2g_1t\rangle$. The biset $B(f_{5/12})$ admits a Levy obstruction $\{g_0^H\}$ with $g_0 = g_3^{g_2}g_1$, since $g_0 = \langle\langle g_4^{t^{-1}}, g_0^{g_2^{-1}}\rangle\rangle$. It is canonical. This Levy cycle comes from the external rays with angles 1/3 and 2/3 landing together. If one considers the subgroup $H_0 = \langle t, g_0, g_2, g_4 \mid g_4g_2g_0t\rangle$ of H, consisting of paths that do not cross a fixed arc between the punctures g_1 and g_3, one obtains a realizable biset with presentation

$$t = \langle\langle t, 1\rangle\rangle(1,2)\,, \quad g_0 = \langle\langle g_4^{t^{-1}}, g_0^{g_2^{-1}}\rangle\rangle\,,$$
$$g_2 = \langle\langle g_0^{-1}g_2^{-1}, g_2g_0\rangle\rangle(1,2)\,, \quad g_4 = \langle\langle g_2, 1\rangle\rangle\,,$$

which is the biset of a polynomial $\cong z^2 - 1.54369$. In effect, passing to the subgroup H_0 amounts to considering only curves in $(S^2, P_{f_{5/12}})$ that do not cross the curve g_0^H.

To compute the mating f of $f_{5/12}$ with itself, we consider as in §7.4 the group

$$G = G_{5/12} *_{\langle t\rangle} G_{5/12} = \langle g_1, \dots, g_4, h_1, \dots, h_4 \mid g_4g_3g_2g_1h_4h_3h_2h_1\rangle.$$

Its generators are lollipops around two copies $\cdot \Rightarrow x_1 \to x_2 \to x_3 \leftrightarrow x_4$ and $\cdot \Rightarrow y_1 \to y_2 \to y_3 \leftrightarrow y_4$ of the post-critical set of $f_{5/12}$. Let us write $t = h_4h_3h_2h_1 = (g_4g_3g_2g_1)^{-1}$. The presentation of $B(f)$ is then

$$g_1 = \langle\langle 1, g_3\rangle\rangle, \quad g_2 = \langle\langle tg_4, g_4^{-1}t^{-1}\rangle\rangle(1,2), \quad g_3 = \langle\langle g_1, g_4\rangle\rangle, \quad g_4 = \langle\langle g_2, 1\rangle\rangle,$$
$$h_1 = \langle\langle h_3, 1\rangle\rangle, \quad h_2 = \langle\langle h_4^{-1}t^{-1}, th_4\rangle\rangle(1,2), \quad h_3 = \langle\langle h_4, h_1\rangle\rangle, \quad h_4 = \langle\langle 1, h_2\rangle\rangle.$$

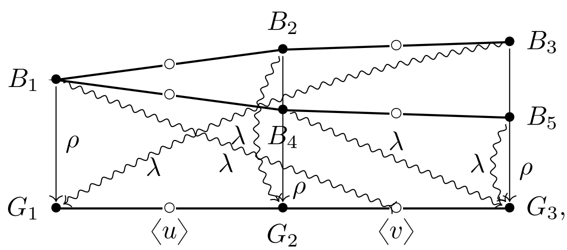
There are now many annular obstructions: setting as before $g_0 = g_3^{g_2}g_1$ and $h_0 = h_3^{h_2}h_1$, the multicurves $\{g_0^G\}$ and $\{h_0^G\}$ are both invariant with matrix (1); however, setting $r = g_3h_1$ and $s = g_1h_3$, we also have $r = \langle\langle s, g_4\rangle\rangle$ and $s = \langle\langle h_4, r\rangle\rangle$ so $\{r^G, s^G\}$ is a Levy multicurve. The four curves g_0, r, h_0, s intersect each other cyclically, so none of these are part of the canonical obstruction, see [17, p. 34]. In fact, the canonical obstruction is

$$\{u^G, v^G\} \text{ with } u = h_2 g_2, v = g_4^{t^{-1}} h_4\,,$$

since $u = \ll v^{-1}, v^{t g_4} \gg$ and $v = \ll 1, u^{h_2} \gg$. Its Thurston matrix is $\left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)$. We get a sphere decomposition

$$G = G_1 *_{\langle u\rangle} G_2 *_{\langle v\rangle} G_3 \text{ with } \begin{cases} G_1 = \langle g_2, h_2, u^{-1} \mid h_2 g_2 u^{-1}\rangle\,, \\ G_2 = \langle g_1, g_3^{g_2}, h_1^{g_2}, h_3, u, v \mid v h_3 u h_1^{g_2} g_3^{g_2} g_1\rangle\,, \\ G_3 = \langle g_4^{t^{-1}}, h_4, v^{-1} \mid g_4^{t^{-1}} h_4 v^{-1}\rangle\,. \end{cases}$$

The corresponding sphere tree of bisets decomposition is



(7.5)

with the following vertex bisets:

- the $\langle v\rangle$-G_1-biset B_1 has, in basis $\{\ell_1, g_1^{-1}g_2^{-1}g_3^{-1}\ell_2\}$, the wreath recursion

$$g_2 = \ll 1, 1 \gg (1,2)\,, \qquad h_2 = \ll v, v^{-1} \gg (1,2)\,, \qquad u^{-1} = \ll v, v^{-1} \gg\,;$$

- the G_2-G_2-biset B_2 has, in basis $\{g_1^{-1}g_2^{-1}g_3^{-1}\ell_2\}$, the wreath recursion

$$\begin{aligned} g_1 &= \ll (g_3^{g_2})^{g_1} \gg\,, & g_3^{g_2} &= \ll g_1 \gg\,, & u &= \ll v \gg\,, \\ h_1^{g_2} &= \ll h_3 \gg\,, & h_3 &= \ll (h_1^{g_2})^{g_3^{g_2} g_1} \gg\,, & v &= \ll u^{g_3^{g_2} g_1} \gg\,; \end{aligned}$$

- the G_1-G_3-biset B_3 has, in basis $\{\ell_2\}$, the wreath recursion

$$g_4^{t^{-1}} = \ll g_2 \gg\,, \qquad h_4 = \ll h_2 \gg\,, \qquad v = \ll u^{g_2} \gg\,;$$

- the G_3-G_2-biset B_4 has, in basis $\{\ell_1\}$, the wreath recursion

$$\begin{aligned} g_1 &= \ll 1 \gg\,, & g_3^{g_2} &= \ll g_4^{t^{-1}} \gg\,, & u &= \ll v^{-1} \gg\,, \\ h_1^{g_2} &= \ll 1 \gg\,, & h_3 &= \ll h_4 \gg\,, & v &= \ll 1 \gg\,; \end{aligned}$$

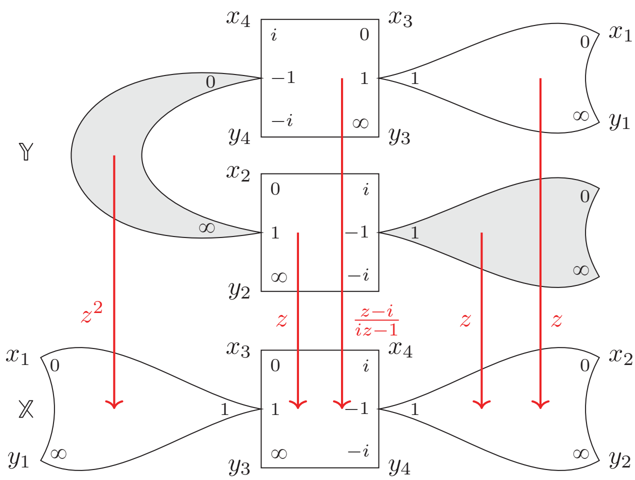
- the G_3-G_3-biset B_5 has, in basis $\{\ell_1\}$, the wreath recursion $g_4^{t^{-1}} = h_4 = v = \ll 1 \gg$. It corresponds to a sphere being contracted to a point.

The only return maps of $B(f)$ are the trivial biset B_5 and the degree-1 biset B_2. This last biset is the biset of an outer automorphism φ of order two: the wreath recursion of $B_2 \otimes B_2$ is conjugation by $g_0 = g_3^{g_2} g_1$. Indeed $\varphi(g_0) = g_0$ and the action of φ is

$$g_3^{g_2} \mapsto g_1 \mapsto (g_3^{g_2})^{g_0}, \quad u \mapsto v \mapsto u^{g_0}, \quad h_1^{g_2} \mapsto h_3 \mapsto (h_1^{g_2})^{g_0}.$$

On top of $\varphi(g_0) = g_0$, we also have $\varphi(h_0^{G_2}) = h_0^{G_2}$ and $\varphi(r^{g_2 G_2}) = s^{G_2}$ and $\varphi(s^{G_2}) = r^{g_2 G_2}$; so the simple closed curves g_0, r, h_0, s can be homotoped into periodic curves in the sphere G_2.

We are ready to give the complex stable curve on which f may be realized:



Note that the coördinates on the central small sphere in $\mathbb{X}$ are not uniquely determined; rather, choose any Möbius transformation μ that is an involution and that does not fix 0, 1 or ∞. Then, once the left and right small spheres are labelled by $0, 1, \infty$ as above, the central small sphere is labelled by $0, 1, \infty, \mu(0), \mu(1), \mu(\infty)$ and the covering f is given, on the top central small sphere of $\mathbb{Y}$, by μ.

7.7. Blowing up an arc

In [28, §1.3.4], Kevin Pilgrim describes a self-covering of the sphere, obtained from the $z \mapsto 2z$ map on the torus by rotating and blowing up an edge. It is a degree-5 map f, and Pilgrim asks whether it can be realized as a complex map (from the construction, it is clear that f is expanding, so it

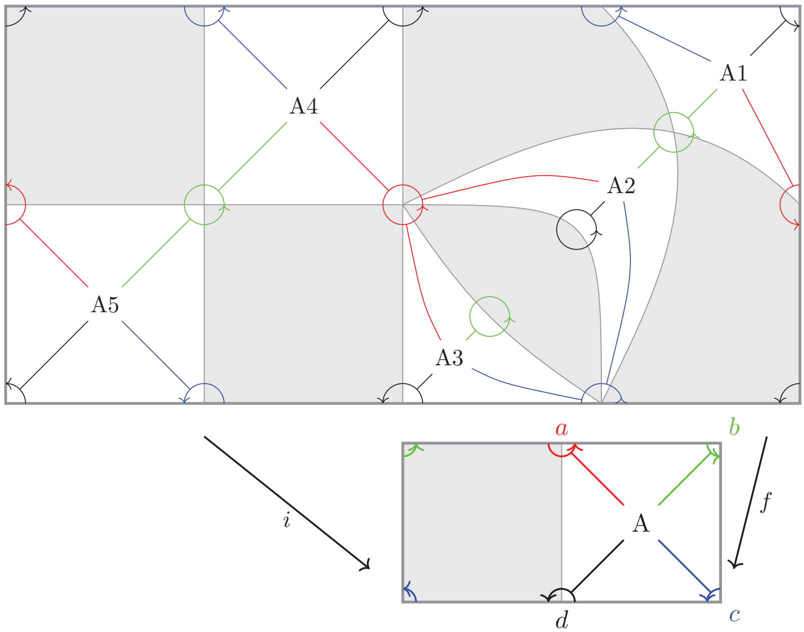


Figure 7.2. Pilgrim's "blowing up an arc" subdivision rule

cannot have any Levy obstruction, see Theorem C). We start by describing the map similarly to the examples in §7.5:

The map may be group-theoretically presented as follows — see Figure 7.2. Consider $G = \langle a, b, c, d \mid dcba\rangle$ generated by lollipops around the punctures. In the basis $\{\ell_1, \dots, \ell_5\}$ consisting of straight paths from $i(\mathrm{A}1), \dots, i(\mathrm{A}5)$ to the basepoint A, the biset $B(f)$ is presented as

$$\begin{aligned} a &= \langle\!\langle c^{-1}, 1, 1, 1, c\rangle\!\rangle(1,5)(2,4,3)\,,\\ b &= \langle\!\langle 1, 1, 1, d, d^{-1}\rangle\!\rangle(1,2)(4,5)\,,\\ c &= \langle\!\langle a, 1, 1, a^{-1}, 1\rangle\!\rangle(1,4)(2,3,5)\,,\\ d &= \langle\!\langle b, 1, d, a, c\rangle\!\rangle\,. \end{aligned}$$

Consider $x = ac$. One then computes directly

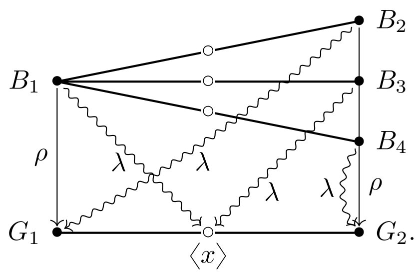
$$x = \langle\!\langle c^{-1}, cx^{-1}, 1, 1, x^{c^{-1}}\rangle\!\rangle(1,2)(4,5)\,,$$

so $\{x^G\}$ is an annular obstruction, with Thurston matrix $(\frac{1}{2} + \frac{1}{2})$. This already answers Pilgrim's question in the negative. However, let us study this example further, and decompose $B(f)$ as a sphere tree of bisets.

The tree of groups decomposition is $G = G_1 *_{\mathbb{Z}} G_2$, with $G_1 = \langle a, c, x^{-1} \mid acx^{-1}\rangle$ and $G_2 = \langle x, b, d^c \mid xd^cb\rangle$. To compute the sphere tree of bisets, we first change the basis of $B(f)$ to $Q := \{\ell_1, c\ell_2, c\ell_3, c\ell_4, c\ell_5\}$ so as to simplify the presentation of x, and obtain

$$\begin{aligned} a &= \langle\!\langle 1,1,1,1,1\rangle\!\rangle(1,5)(2,4,3)\,,\\ b &= \langle\!\langle c, c^{-1}, 1, d^c, d^{-c}\rangle\!\rangle(1,2)(4,5)\,,\\ c &= \langle\!\langle x, 1, 1, x^{-1}, 1\rangle\!\rangle(1,4)(2,3,5)\,,\\ d^c &= \langle\!\langle a, c, 1, b^x, d^c\rangle\!\rangle\,,\\ x &= \langle\!\langle 1, x^{-1}, 1, 1, x\rangle\!\rangle(1,2)(4,5)\,. \end{aligned}$$

The multicurve $\{x^G\}$ has 3 preimages, corresponding to the cycles $(1,2)(3)(4,5)$ of the permutation associated with x; so the sphere tree of bisets into which $B(f)$ decomposes has 4 vertex bisets. They are arranged as follows:



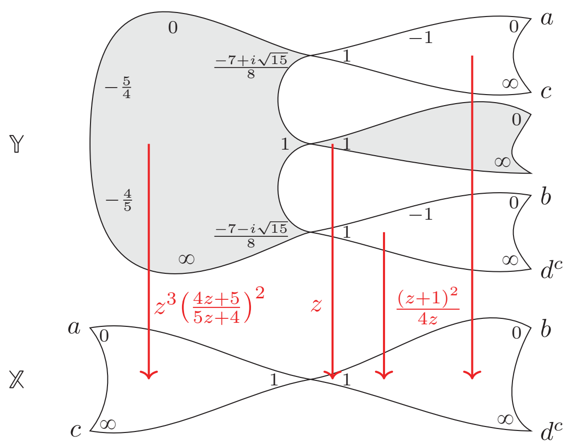
(7.6)

The vertex bisets are again obtained by restricting $B(f)$ while using subsets of the basis Q, and are given as follows:

- the $\langle x\rangle$-G_1-biset B_1 has in the basis Q the wreath recursion
$$\begin{aligned} a &= \langle\!\langle 1,1,1,1,1\rangle\!\rangle(1,5)(2,4,3)\,,\\ c &= \langle\!\langle x,1,1,x^{-1},1\rangle\!\rangle(1,4)(2,3,5)\,,\\ x^{-1} &= \langle\!\langle x,1,1,x^{-1},1\rangle\!\rangle(1,2)(4,5)\,; \end{aligned}$$
- the G_1-G_2-biset B_2 has in the subbasis $\{\ell_1, c\ell_2\}$ of Q the wreath recursion
$$x = \langle\!\langle 1, x^{-1}\rangle\!\rangle(1,2)\,,\quad b = \langle\!\langle c, c^{-1}\rangle\!\rangle(1,2)\,,\quad d^c = \langle\!\langle a, c\rangle\!\rangle\,;$$
- the $\langle x\rangle$-G_2-biset B_3 has in the subbasis $\{c\ell_3\}$ of Q the trivial wreath recursion $x = b = d^c = \langle\!\langle 1\rangle\!\rangle$; it corresponds to a sphere that gets blown down to a point on the annulus;
- the G_2-G_2-biset B_4 has in the subbasis $\{c\ell_4, c\ell_5\}$ of Q the wreath recursion
$$x = \langle\!\langle 1, x\rangle\!\rangle(1,2)\,,\quad b = \langle\!\langle d^c, d^{-c}\rangle\!\rangle(1,2)\,,\quad d^c = \langle\!\langle b^x, d^c\rangle\!\rangle\,;$$

it is isomorphic to the biset of the map $z^2 - 2$.

The biset B_1 is the biset of the rational map $z^3(4z+5)^2/(5z+4)^2$. The algebraic realization of f takes place on a singly noded complex stable curve. Keeping the same conventions, it is



7.8. Shishikura and Tan Lei's example

Tan Lei and Shishikura consider in [35] a mating of two polynomials of degree 3, and show that it is obstructed, but does not admit a Levy cycle (it is known that rational maps of degree 2 are obstructed if and only if they admit a Levy cycle). This example was further studied in [14].

Their example may be described as follows. Consider the polynomial

$$f(z) = z^3 + c\,,\ \text{with } (c^3+c)^3 + c = 0 \text{ and } c \approx -0.264425 + 1.26049i\,.$$

Its post-critical orbit is $w \Rrightarrow u \to v \to w$, and the angles landing at the critical point u are $\{11, 24, 37\}/39$. Consider then the polynomial

$$g(z) = (a-1)(3z^2 - 2z^3) + 1\,,\ \text{with } g(a) = 0 \text{ and } a \approx -0.42654\,.$$

Its post-critical orbit is $z \Rightarrow x \Rightarrow y \to z$, the angles landing at z are $\{21, 47\}/78$ and those landing at x are $\{11, 63\}/78$. Consider finally the mating h of f and g.

We choose as usual a basepoint $*$ close to the equator of h and write

$$G = \pi_1(S^2 \backslash P_h, *) = \langle u, v, w, x, y, z \mid uwvxyz \rangle$$

for standard generators of G consisting of lollipops along external rays of f and g respectively. For convenience, we write $t = uwv = (xyz)^{-1}$ for the loop along the equator. In the basis $\{\ell_1,\ell_2,\ell_3\}$ of $B(h)$ consisting of positively oriented paths along the equator, we obtain $t=uwv=(xyz)^{-1}=\langle\!\langle 1,1,t\rangle\!\rangle(1,2,3)$ and

$$\begin{aligned} u &= \langle\!\langle v^{-1},u^{-1},t\rangle\!\rangle(1,2,3)\,, & x &= \langle\!\langle 1,yz,y^{-1}\rangle\!\rangle(2,3)\,,\\ v &= \langle\!\langle 1,1,u\rangle\!\rangle\,, & y &= \langle\!\langle t^{-1},1,tx\rangle\!\rangle(1,3)\,,\\ w &= \langle\!\langle 1,v,1\rangle\!\rangle\,, & z &= \langle\!\langle 1,y,1\rangle\!\rangle\,. \end{aligned}$$

Consider now the multicurve $\{r^G,s^G\}$ with $r=vy$ and $s=u^w x^{v^{-1}}$. We have

$$r=\langle\!\langle t^{-1},1,ts^v\rangle\!\rangle(1,3)\,,\qquad s=\langle\!\langle s^{-v}t^{-1},r^{-v},t\rangle\!\rangle(1,3)\,,$$

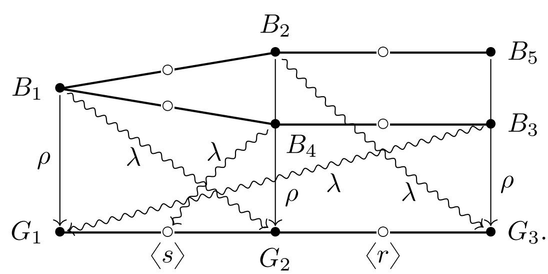
so $\{r^G,s^G\}$ is an annular obstruction with transition matrix $\left(\begin{smallmatrix}0&1\\ \frac12&\frac12\end{smallmatrix}\right)$. The corresponding decomposition of G has three vertices, and is

$$G=G_1*_{\langle s\rangle}G_2*_{\langle r\rangle}G_3 \text{ with } \begin{cases} G_1 &= \langle u^w,x^{v^{-1}},s^{-1}\mid u^w x^{v^{-1}}s^{-1}\rangle,\\ G_2 &= \langle w,z,r,s\mid zwsr\rangle,\\ G_3 &= \langle v,y,r^{-1}\mid vyr^{-1}\rangle.\end{cases}$$

In basis $Q\coloneqq\{v\ell_1,v\ell_2,(uw)^{-1}\ell_3\}$ the presentation of $B(h)$ becomes

$$\begin{aligned} u^w &= \langle\!\langle 1,w,1\rangle\!\rangle(1,2,3)\,, & x^{v^{-1}} &= \langle\!\langle 1,s^{-1},w^{-1}r^{-1}\rangle\!\rangle(2,3)\,,\\ r &= \langle\!\langle 1,1,s\rangle\!\rangle(1,3)\,, & s &= \langle\!\langle s^{-1},r^{-1},1\rangle\!\rangle(1,3)\,,\\ w &= \langle\!\langle 1,v,1\rangle\!\rangle\,, & z &= \langle\!\langle 1,y^{v^{-1}},1\rangle\!\rangle\,,\\ v &= \langle\!\langle 1,1,u^w\rangle\!\rangle\,, & y &= \langle\!\langle 1,1,x^{v^{-1}}\rangle\!\rangle(1,3)\,. \end{aligned}$$

From this presentation, just by looking at which of G_1,G_2,G_3 the entries belong to, we get the sphere tree of bisets decomposition



(7.7)

The vertex bisets are as usual obtained by restricting $B(h)$ while using subsets of the basis Q, and are given as follows:

- the G_2-G_1-biset B_1 has in the basis Q the wreath recursion

$$u^w=\langle\!\langle 1,w,1\rangle\!\rangle(1,2,3),\qquad x^{v^{-1}}=\langle\!\langle 1,s^{-1},w^{-1}r^{-1}\rangle\!\rangle(2,3),$$
$$s^{-1}=\langle\!\langle 1,r,s\rangle\!\rangle(1,3)\,;$$

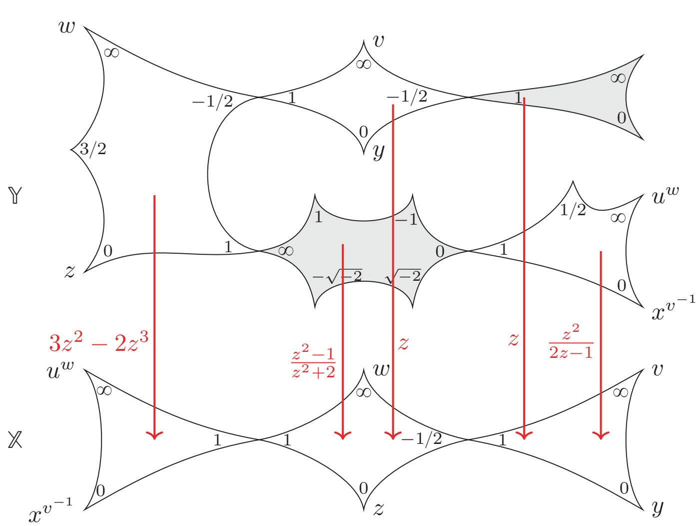
in appropriate coördinates, it is the biset of the map $3z^2 - 2z^3$;

- the G_3-G_2-biset B_2 has in the subbasis $\{\ell_2\}$ of Q the wreath recursion
$$w = \langle\!\langle v\rangle\!\rangle, \quad z = \langle\!\langle y^{v^{-1}}\rangle\!\rangle, \qquad r = \langle\!\langle 1\rangle\!\rangle, \quad s = \langle\!\langle r^{-1}\rangle\!\rangle;$$
- the G_1-G_3-biset B_3 has in the subbasis $\{\ell_1, \ell_3\}$ of Q the wreath recursion
$$v = \langle\!\langle 1, u^w\rangle\!\rangle, \quad y = \langle\!\langle 1, x^{v^{-1}}\rangle\!\rangle(1,2), \quad r^{-1} = \langle\!\langle s^{-1}, 1\rangle\!\rangle(1,2);$$
- the $\langle s\rangle$-G_2-biset B_4 has in the subbasis $\{\ell_1, \ell_3\}$ of Q the wreath recursion
$$w = \langle\!\langle 1, 1\rangle\!\rangle, \quad z = \langle\!\langle 1, 1\rangle\!\rangle, \quad r = \langle\!\langle 1, s\rangle\!\rangle(1,2), \quad s = \langle\!\langle s^{-1}, 1\rangle\!\rangle(1,2);$$
- the 1-G_3 biset B_5 is trivial on the subbasis $\{\ell_2\}$, and corresponds to a sphere that gets blown down to a point.

The only small cycle in the sphere tree of bisets is the G_1-G_1-biset $C := B_3 \otimes_{G_3} B_2 \otimes_{G_2} B_1$ and its two cyclic permutations. A presentation for C, in the basis $\{\ell_1\ell_2\ell_1, \ell_3\ell_2\ell_1, \ell_1\ell_2\ell_2, \ell_3\ell_2\ell_2, \ell_1\ell_2\ell_3, \ell_3\ell_2\ell_3\}$, is

$$\begin{aligned}
u^w &= \langle\!\langle 1,1,1,u^w,1,1\rangle\!\rangle(1,3,5)(2,4,6),\\
x^{v^{-1}} &= \langle\!\langle 1,1,1,u^w x^{v^{-1}},1,u^{-w}\rangle\!\rangle(3,6,4,5),\\
s^{-1} &= \langle\!\langle 1,1,1,1,s^{-1},1\rangle\!\rangle(1,5,2,6).
\end{aligned}$$

A direct calculation shows that it is isomorphic to $B((3z^2-2)^2/(3z^4-4z^6))$; this will also follow from the algebraic realization of h, which is as follows:



We remark that, since h does not admit any Levy obstruction, it is isotopic to an expanding map for a path metric on (S^2, P_h). It admits, therefore, a *Julia set*, defined for example as the accumulation set of iterated preimages of a generic point. On the other hand, $(3z^2 - 2z^3) \circ z \circ \frac{z^2}{2z-1}$ is a rational map, so also admits a Julia set. Chéritat investigates this example in [14] by comparing these Julia sets.

On the above noded sphere model, the Julia set of h is, on the first sphere, the Julia set of $(3z^2 - 2z^3) \circ z \circ \frac{z^2}{2z-1}$; on the other two spheres, it is the Julia set of its cyclic permutations.

Chéritat also asks, in [14, Question 2], to prove that the map of noded spheres obtained from pinching the canonical obstruction in h is indeed (in a different normalization) the map above. This follows from [31], and also follows immediately from computing the bisets in the decomposition, as we have done.

7.9. A Thurston map with infinitely generated centralizer

We conclude this survey with an example that shows that centralizers of Thurston maps can be sometimes quite complicated, and in particular not finitely generated (whence our notion of "sub-computable"). The example has degree 6 and 7 marked points. Many generalizations are possible, but we content ourselves with its direct description, see Figure 7.3.

The seven marked points $A = \{x_1, \dots, x_7\}$ of the Thurston map f are separated by two curves s, t as follows: x_3, x_4 lie on an f-fixed sphere S_0 separated by s from a sphere S_1 containing x_2, x_5 and on which f acts as z^2; and S_1 is separated by t from an f-fixed sphere S_2 containing x_1, x_6, x_7. The canonical obstruction of f is $\{s, t\}$. There are two other preimages of S_1, mapping by z^2 to S_1 and embedded in annuli about s and t respectively, and two other preimages of S_2 mapping by degree 2 to S_2, respectively with critical values $\{x_1, x_6\}$ and embedded in an annulus about s, and with critical values $\{x_1, x_7\}$ and embedded in an annulus about t.

The Thurston matrix of the multicurve $\mathscr{C} := \{s, t\}$ is $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 3 \end{smallmatrix}\right)$. Note that every curve in S_2 is a Levy cycle, however the return map on S_2 is the identity and the canonical obstruction does not contain the Levy cycles in degree-1 pieces.

The centralizer of f preserves the canonical obstruction, so is a subgroup of $\mathbf{Mod}(S^2, A, \mathscr{C}) \cong \mathbf{Mod}(S_0) \times \mathbf{Mod}(S_1) \times \mathbf{Mod}(S_2) \times \mathbb{Z}^{\mathscr{C}}$. Furthermore,

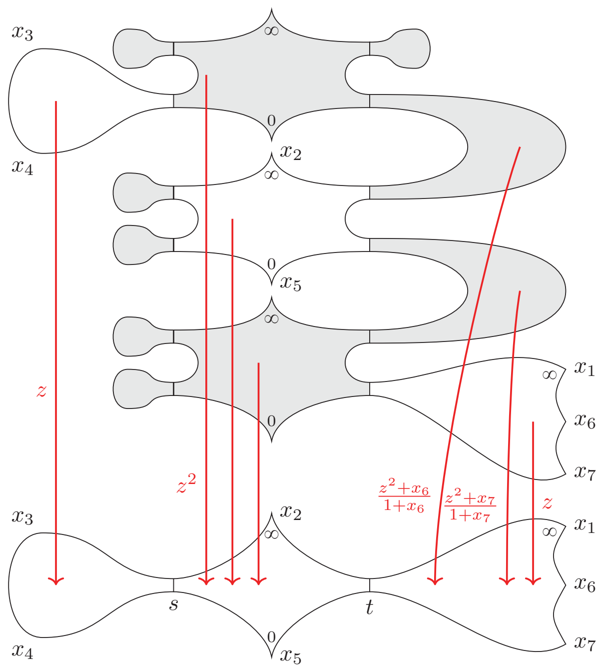


Figure 7.3. A Thurston map with infinitely generated centralizer

$\mathbf{Mod}(S_0) = 1$ because it contains only three marked points, and the projection of $Z(f)$ into $\mathbf{Mod}(S_1)$ is trivial because the restriction of f to S_1 is a rational self-map. Therefore, $Z(f)$ is a subgroup of $\mathbb{Z}^{\mathscr{C}} \times \mathbf{Mod}(S_2)$.

To compute it, we write down a presentation of $B(f)$, and compute some relations in its mapping class biset. We set

$$G = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \mid x_1x_2x_3x_4x_5x_6x_7 \rangle,$$

write $s = x_3x_4$ and $t = x_2x_3x_4x_5$, and in a basis $\{\ell_1, \dots, \ell_7\}$ we compute the presentation

$$\begin{aligned}
x_1 &= \langle\!\langle 1, s, s^{-1}, t, t^{-1}, x_1 \rangle\!\rangle (2,3)(4,5)\,,\\
x_2 &= \langle\!\langle 1, 1, s^{-1}, x_2 s, t^{-1}, t \rangle\!\rangle (1,2)(3,4)(5,6)\,,\\
x_3 &= \langle\!\langle x_3, 1, 1, 1, 1, 1 \rangle\!\rangle\,,\\
x_4 &= \langle\!\langle x_4, 1, 1, 1, 1, 1 \rangle\!\rangle\,,\\
x_5 &= \langle\!\langle 1, 1, x_5, 1, 1, 1 \rangle\!\rangle (1,2)(3,4)(5,6)\,,\\
x_6 &= \langle\!\langle 1, 1, 1, 1, 1, x_6 \rangle\!\rangle (2,3)\,,\\
x_7 &= \langle\!\langle 1, 1, 1, 1, 1, x_7 \rangle\!\rangle (4,5)\,,
\end{aligned} \tag{7.8}$$

giving $s = \llangle s,1,1,1,1,1\rrangle$ and $t = \llangle 1,s,s^{-1},t,t^{-1},t\rrangle$. We write σ,τ,α,β for Dehn twists about s,t,x_1x_6 and x_6x_7 respectively; their actions on G are given respectively by

$$\begin{aligned}
\sigma:&\quad x_3\mapsto x_3^s,\ x_4\mapsto x_4^s\,,\\
\tau:&\quad x_2\mapsto x_2^t,\ x_3\mapsto x_3^t,\ x_4\mapsto x_4^t,\ x_5\mapsto x_5^t\,,\\
\alpha:&\quad x_1\mapsto x_1^{tx_6t^{-1}},\ x_6\mapsto x_6^{t^{-1}x_1tx_6}\,,\\
\beta:&\quad x_6\mapsto x_6^{x_6x_7},\ x_7\mapsto x_7^{x_6x_7}\,,
\end{aligned}$$

all other generators being fixed. Naturally $[\sigma,\alpha]=[\tau,\alpha]=[\sigma,\beta]=[\tau,\beta]=1$ while $\langle\alpha,\beta\rangle$ is a free group of rank 2. We then compute

$$\begin{aligned}
B(f)\cdot\sigma &\cong \sigma\cdot B(f)\,, & B(f)\cdot\tau &\cong \sigma^2\tau^3\cdot B(f)\,,\\
B(f)\cdot\alpha &\cong \alpha\sigma^2\cdot B(f)\,, & B(f)\cdot\beta &\cong \beta\cdot B(f)\,.
\end{aligned}$$

For the second equality, the recursion of $\sigma^{-2}\tau^{-3}\cdot B(f)\cdot\tau$ in basis $\{s^2t^3\ell_1, st^3\ell_2, st^3\ell_3, t^2\ell_4, t^2\ell_5, \ell_6\}$ coincides with (7.8), while for the third equality, the recursion of $\sigma^{-2}\alpha^{-1}\cdot B(f)\cdot\alpha$ in basis $\{s^2\ell_1, s^2\ell_2, \ell_3,\dots,\ell_6\}$ coincides with (7.8).

Consider the homomorphism $\phi\colon\langle\alpha,\beta\rangle\to\mathbb{Z}$ which counts the total exponent in α of a word; it is the quotient by the normal closure of β. Then, for $w\in\langle\alpha,\beta\rangle$, the element $w\sigma^m\tau^n$ belongs to the centralizer of f if and only if $(m,n)=(m+2n+\phi(w),3n)$, if and only if $n=0$ and $w\in\ker(\phi)$. Therefore,

$$Z(f)=\langle\sigma\rangle\times\ker(\phi)=\langle\sigma\rangle\times\langle\beta,\beta^\alpha,\beta^{\alpha\beta},\dots\rangle\cong\mathbb{Z}\times F_\infty\,.$$

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