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On divergent fractional Laplace equations

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Abstract. — We consider the divergent fractional Laplace operator presented in [5] and we prove three types of results.

Firstly, we show that any given function can be locally shadowed by a solution of a divergent fractional Laplace equation which is also prescribed in a neighborhood of infinity.

Secondly, we take into account the Dirichlet problem for the divergent fractional Laplace equation, proving the existence of a solution and characterizing its multiplicity.

Finally, we take into account the case of nonlinear equations, obtaining a new approximation results.

These results maintain their interest also in the case of functions for which the fractional Laplacian can be defined in the usual sense.

Résumé. — Nous considérons le Laplacien fractionnaire divergent introduit dans [5] et démontrons trois types de résultats.

Premièrement, nous montrons que toute fonction donnée peut être approchée localement par une solution d'une équation de Laplace fractionnaire divergente, dont les valeurs sont de plus prescrites au voisinage de l'infini.

Deuxièmement, nous démontrons l'existence de solutions au problème de Dirichlet pour le Laplacien fractionnaire divergent, et caractérisons leur multiplicité.

Enfin, nous obtenons des résultats d'approximation dans le cadre d'équations non linéaires, résultats qui sont nouveaux même lorsque le Laplacien fractionnaire peut être défini au sens usuel.

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1. Introduction

Given $u : \mathbb{R}^n \to \mathbb{R}$ and $s \in (0,1)$, to define the fractional Laplacian of u,

$$(-\Delta)^s u(x) := \lim_{\rho \searrow 0} \int_{\mathbb{R}^n \setminus B_\rho(x)} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, \mathrm{d}y, \tag{1.1}$$

one typically needs two main requisites on the function u:

- u has to be sufficiently smooth in the vicinity of x, for instance $u \in C^\gamma(B_\delta(x))$ for some $\delta > 0$ and $\gamma > 2s$,
- u needs to have a controlled growth at infinity, for instance

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} \, \mathrm{d}x < +\infty. \tag{1.2}$$

Nevertheless, in [5] we have recently introduced a new notion of "divergent" fractional Laplacian, which can be used even when condition (1.2) is violated. This notion takes into account the case of functions with polynomial growth, for which the classical definition in (1.1) makes no sense, and it recovers the classical definition for functions with controlled growth such as in (1.2).

The notion of divergent fractional Laplacian possesses several interesting features and technical advantages, including suitable Schauder estimates in which the full smooth Hölder norm of the solution is controlled by a suitable seminorm of the nonlinearity. Moreover, compared to (1.1), the notion of divergent fractional Laplacian is conceptually closer to the classical case in the sense that it requires a sufficient degree of regularity of the function u at a given point, without global conditions (up to a mild control at infinity of polynomial type), thus attempting to make the necessary requests as close as possible to the case of the classical Laplacian.

In this article, we consider the setting of the divergent Laplacian and we obtain the following results:

- an approximation result with solutions of divergent Laplacian equations: we will show that these solutions can locally shadow any prescribed function, maintaining also a complete prescription at infinity,
- a characterization of the Dirichlet problem: we will show that the (possibly inhomogeneous) Dirichlet problem is solvable and we determine the multiplicity of the solutions,
- an approximation result with solutions of nonlinear divergent Laplacian equations, up to a small error also in the forcing term.

To state these results in detail, we now recall the precise framework for the divergent fractional Laplacian. Given $k \in \mathbb{N}$, we consider the space of functions

$$\mathcal{U}_k := \left\{ u : \mathbb{R}^n \to \mathbb{R}, \quad \begin{array}{c} \text{s.t. } u \text{ is continuous in } B_1 \text{ and} \\ \displaystyle\int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s+k}} \, \mathrm{d}x < +\infty \end{array} \right\}.$$

Then (see [5, Definition 1.1]) we use the notation

$$\chi_R(x) := \begin{cases} 1 & \text{if } x \in B_R, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_R, \end{cases}$$

and we say that

$$(-\Delta)^s u \stackrel{k}{=} f \qquad \text{in } B_1 \tag{1.3}$$

if there exist a family of polynomials P_R, which have degree at most $k-1$, and functions $f_R : B_1 \to \mathbb{R}$ such that $(-\Delta)^s u = f_R + P_R$ in B_1 in the viscosity sense, with

$$\lim_{R \to +\infty} f_R(x) = f(x)$$

for any $x \in B_1$.

Interestingly, one can also think that the right hand side of equation (1.3) is not just a function, but an equivalence class of functions modulo polynomials, since one can freely add to f a polynomial of degree k when $s \in \left(0, \frac{1}{2}\right]$ and of degree $k+1$ when $s \in \left(\frac{1}{2}, 1\right)$ (see [5, Theorem 1.5]).

The first result that we provide in this setting states that every given function can be modified in an arbitrarily small way in B_1, remaining unchanged in a large ball, in such a way to become s-harmonic with respect to the divergent fractional Laplacian.

Theorem 1.1 (All divergent functions are locally s-harmonic up to a small error). — *Let k, $m \in \mathbb{N}$ and $u : \mathbb{R}^n \to \mathbb{R}$ be such that $u \in C^m(\overline{B_1})$ and*

$$\int_{B_1^c} \frac{|u(x)|}{|x|^{n+2s+k}} \, \mathrm{d}x < +\infty.$$

Then, for any $\varepsilon > 0$ there exist u_ε and $R_\varepsilon > 1$ such that

$$(-\Delta)^s u_\varepsilon \stackrel{k}{=} 0 \quad \text{in } B_1, \tag{1.4}$$

$$\|u_\varepsilon - u\|_{C^m(B_1)} \leqslant \varepsilon \tag{1.5}$$

$$\text{and} \qquad u_\varepsilon = u \quad \text{in } B^c_{R_\varepsilon}. \tag{1.6}$$

A graphical sketch of Theorem 1.1 is given in Figure 1.1 (notice the possible wild oscillations of u_ε in $B_{R_\varepsilon} \setminus B_1$).

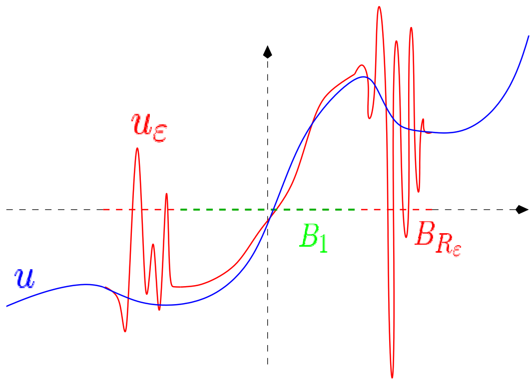

Figure 1.1. The approximation result in Theorem 1.1.

Remark 1.2. — When $k = 0$ and $u = 0$ outside B_2, Theorem 1.1 reduces to the main result of [4]. Interestingly, in the case considered here, one can preserve the values of the given function u at infinity and, if the growth of u at infinity is "too fast" for the classical fractional Laplacian to be defined, then the result still carries on, in the divergent fractional Laplace setting.

Remark 1.3. — We observe that Theorem 1.1 does not hold under the additional assumption that

$$|u_\varepsilon(x)| \leqslant P(x) \quad \text{for all } x \in \mathbb{R}^n, \tag{1.7}$$

for a given polynomial P (that is, one cannot replace a growth assumption at infinity with a pointwise bound). Indeed, under assumption (1.7), we have that

$$\int_{\mathbb{R}^n} \frac{|u_\varepsilon(x)|}{1+|x|^{n+2s+d}}\,\mathrm{d}x \leqslant \int_{\mathbb{R}^n} \frac{|P(x)|}{1+|x|^{n+2s+d}}\,\mathrm{d}x =: J < +\infty,$$

being $d \in \mathbb{N}$ the degree of the polynomial P. As a consequence of this and (1.4), we deduce from [5, Theorem 1.3] that for any $\gamma > 0$ such that γ and $\gamma + 2s$ are not integer,

$$\|u_\varepsilon\|_{C^{\gamma+2s}}(B_{1/2}) \leqslant C\,J,$$

for some C depending only on J, n, s, γ and d. In particular, if $\gamma + 2s \geqslant m$, we would have from (1.5) that

$$\begin{aligned} \varepsilon \geqslant \|u_\varepsilon - u\|_{C^m(B_1)} &\geqslant \|u_\varepsilon - u\|_{C^m(B_{1/2})} \\ &\geqslant \|u\|_{C^m(B_{1/2})} - \|u_\varepsilon\|_{C^m(B_{1/2})} \geqslant \|u\|_{C^m(B_{1/2})} - C\,J. \end{aligned}$$

This set of inequalities would be violated for $\varepsilon \in (0,1)$ by any function u satisfying

$$\|u\|_{C^m(B_{1/2})} \geqslant C\,J + 1. \tag{1.8}$$

That is, solutions with a large C^m-norm (more specifically with a norm as in (1.8)) cannot be approximated arbitrarily well by s-harmonic functions (not even "modulo polynomials") that satisfy a polynomial bound as in (1.7).

Interestingly, this remark is independent from R_ε in (1.6) (hence, it is not possible to arbitrarily improve the approximation results if we require an additional polynomial bound, even if we drop the request that the approximating function is compactly supported).

Theorem 1.1 is also related to some recent results in [1, 2, 3, 6, 8, 10] (see [11] for an elementary exposition in the case of the fractional Laplacian in dimension 1).

Next result focuses on the Dirichlet problem for divergent fractional Laplacians. We show that, given an external datum and a forcing term, the Dirichlet problem has a solution. Differently from the classical case, when $k \neq 0$ such solution is not unique, and we determine the dimension of the multiplicity space.

Theorem 1.4 (Solvability of the Dirichlet problem for divergent fractional Laplacians). — *Let $k \in \mathbb{N}$ and $u_0 : B_1^c \to \mathbb{R}$ be such that*

$$\int_{B_1^c} \frac{|u_0(x)|}{|x|^{n+2s+k}}\,\mathrm{d}x < +\infty.$$

Let f be continuous in B_1. Then, there exists a function $u \in \mathcal{U}_k$ such that

$$\begin{cases} (-\Delta)^s u \stackrel{k}{=} f & \text{in } B_1, \\ u = u_0 & \text{in } B_1^c. \end{cases} \tag{1.9}$$

Also, the space of solutions of (1.9) *has dimension N_k, with*

$$N_k := \sum_{j=0}^{k-1} \binom{j+n-1}{n-1}. \tag{1.10}$$

With the aid of Theorems 1.1 and 1.4, we can also consider the case of nonlinear equations, namely the case in which the right hand side depends also on the solution (as well as on its derivatives, since the result that we provide is general enough to comprise such a case too).

In this setting, we establish that any prescribed function satisfies any prescribed nonlinear (and possibly divergent) fractional Laplace equation, up to an arbitrarily small error, once we are allowed to make arbitrarily small modifications of the given function in a given region, preserving its values at infinity. The precise result that we have is the following one:

Theorem 1.5 (All divergent functions almost solve nonlinear equations). *Let k, $m \in \mathbb{N}$ and $u : \mathbb{R}^n \to \mathbb{R}$ be such that $u \in C^{2m}(\overline{B_1})$ and*

$$\int_{B_1^c} \frac{|u(x)|}{|x|^{n+2s+k}}\,\mathrm{d}x < +\infty.$$

Let

$$N(m) := n + \sum_{j=0}^{m} n^j$$

and let $F \in C^m(\mathbb{R}^{N(m)})$.

Then, for any $\varepsilon > 0$ there exist u_ε, $\eta_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ and $R_\varepsilon > 1$ such that

$$(-\Delta)^s u_\varepsilon(x) \stackrel{k}{=} F\big(x, u_\varepsilon(x), \nabla u_\varepsilon(x), \dots, D^m u_\varepsilon(x)\big) + \eta_\varepsilon(x) \qquad \text{for all } x \in B_1, \tag{1.11}$$

$$\|\eta_\varepsilon\|_{L^\infty(B_1)} \leqslant \varepsilon, \tag{1.12}$$

$$\|u_\varepsilon - u\|_{C^m(B_1)} \leqslant \varepsilon \tag{1.13}$$

$$\text{and} \quad u_\varepsilon = u \quad \text{in } B^c_{R_\varepsilon}. \tag{1.14}$$

Remark 1.6. — We think that it is a very interesting *open problem* to determine whether the statement in Theorem 1.5 holds true also with $\eta_\varepsilon := 0$. This would give that any given function can be locally approximated arbitrarily well by functions which solve exactly (and not only approximatively) a nonlinear equation.

Remark 1.7. — All the results presented here maintain their own interest even in the case $k = 0$: in this case, the definition of divergent fractional Laplacian boils down to the usual fractional Laplacian (see [5, Corollary 3.8]).

The rest of this article is organized as follows. In Section 2 we give the proof of Theorem 1.1, in Section 3 we deal with the proof of Theorem 1.4, and in Section 4 we focus on Theorem 1.5.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we first present an observation on the decay of the divergent fractional Laplacians for functions that vanish on a large ball:

Lemma 2.1. — *Let $k \in \mathbb{N}$ and $R > 3$. Let $u : \mathbb{R}^n \to \mathbb{R}$ be such that $u = 0$ in B_R and*

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s+k}}\,\mathrm{d}x < +\infty. \tag{2.1}$$

Then, there exists $f : B_1 \to \mathbb{R}$ such that $(-\Delta)^s u \overset{k}{=} f$ in B_1 and for which the following statement holds true: for any $\varepsilon > 0$ and any $m \in \mathbb{N}$, there exists $\overline{R}_\varepsilon > 3$ such that if $R \geqslant \overline{R}_\varepsilon$ then

$$\|f\|_{C^m(B_1)} \leqslant \varepsilon. \tag{2.2}$$

Proof. — From Remark 3.5 in [5], we can write that $(-\Delta)^s u \overset{k}{=} f$ in B_1, with

$$\begin{aligned} f(x) &= f_u(x) \\ &:= \int_{B_2} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y + \int_{B_2^c} \frac{u(x)}{|x-y|^{n+2s}}\,\mathrm{d}y + \int_{B_2^c} \frac{u(y)\,\psi(x,y)}{|y|^{n+2s+k}}\,\mathrm{d}y \\ &= \int_{B_R^c} \frac{u(y)\,\psi(x,y)}{|y|^{n+2s+k}}\,\mathrm{d}y, \end{aligned}$$

for some function ψ satisfying, for any $j \in \mathbb{N}$,

$$\sup_{x\in B_1,\, y\in B_2^c} |D_x^j \psi(x,y)| \leqslant C_j,$$

for some $C_j > 0$. In particular, for any $x \in B_1$,

$$|D^j f(x)| \leqslant \int_{B_R^c} \frac{|u(y)|\,|D_x^j\psi(x,y)|}{|y|^{n+2s+k}}\,\mathrm{d}y \leqslant C_j \int_{B_R^c} \frac{|u(y)|}{|y|^{n+2s+k}}\,\mathrm{d}y,$$

so the desired claim in (2.2) follows from (2.1). □

With this, we complete the proof of Theorem 1.1 in the following way.

Proof of Theorem 1.1. — From [4, Theorem 1.1] we know that there exist a function v_ε and $\rho_\varepsilon > 1$ such that

$$(-\Delta)^s v_\varepsilon = 0 \quad \text{in } B_1, \tag{2.3}$$

$$\|v_\varepsilon - u\|_{C^m(B_1)} \leqslant \varepsilon \tag{2.4}$$

$$\text{and} \quad v_\varepsilon = 0 \quad \text{in } B_{\rho_\varepsilon}^c. \tag{2.5}$$

For any $R > 3$, we also set $\widetilde{u}_R := (1-\chi_R)\,u$. Notice that

$$\widetilde{u}_R = u \quad \text{in } B_R^c. \tag{2.6}$$

In addition,

$$\widetilde{u}_R = 0 \quad \text{in } B_R, \tag{2.7}$$

so, in view of Lemma 2.1, there exist a function f_ε and $\overline{R}_\varepsilon > 3$ such that

$$(-\Delta)^s \widetilde{u}_{\bar{R}_\varepsilon} \overset{k}{=} f_\varepsilon \quad \text{in } B_2, \tag{2.8}$$

$$\text{and} \quad \|f_\varepsilon\|_{C^m(B_2)} \leqslant \varepsilon. \tag{2.9}$$

Now we consider the standard solution of the Dirichlet problem

$$\begin{cases} (-\Delta)^s w_\varepsilon = f_\varepsilon & \text{in } B_2, \\ w_\varepsilon = 0 & \text{in } B_2^c. \end{cases} \tag{2.10}$$

From [9, Proposition 1.1], we have that

$$\|w_\varepsilon\|_{C^s(\mathbb{R}^n)} \leqslant C\,\|f_\varepsilon\|_{L^\infty(B_2)}, \tag{2.11}$$

for some $C > 0$.

Now we take $\gamma := m - s$. Notice that $\gamma \notin \mathbb{N}$ and $\gamma + 2s = m + s \notin \mathbb{N}$. Then, by Schauder estimates (see e.g. [5, Theorem 1.3], applied here with $k := 0$), and exploiting (2.9) and (2.11), possibly renaming $C > 0$ line after line, we obtain that

$$\begin{aligned} \|w_\varepsilon\|_{C^m(B_1)} \leqslant \|w_\varepsilon\|_{C^{\gamma+2s}(B_1)} &\leqslant C\left([f_\varepsilon]_{C^\gamma(B_2)} + \int_{B_1^c} \frac{|w_\varepsilon(y)|}{|y|^{n+2s}}\,\mathrm{d}y\right) \\ &\leqslant C\left(\|f_\varepsilon\|_{C^m(B_2)} + \|w_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\right) \leqslant C\,\|f_\varepsilon\|_{C^m(B_2)} \leqslant C\varepsilon. \end{aligned} \tag{2.12}$$

Now we define

$$u_\varepsilon := v_\varepsilon + \widetilde{u}_{\bar{R}_\varepsilon} - w_\varepsilon.$$

Using (2.3), (2.10) and the consistency result in [5, Corollary 3.8], we see that

$$(-\Delta)^s v_\varepsilon \overset{0}{=} 0 \quad \text{and} \quad (-\Delta)^s w_\varepsilon \overset{0}{=} f_\varepsilon \quad \text{in } B_1.$$

Thus, the consistency result in [5, formula (1.7)] implies that

$$(-\Delta)^s v_\varepsilon \overset{k}{=} 0 \quad \text{and} \quad (-\Delta)^s w_\varepsilon \overset{k}{=} f_\varepsilon \quad \text{in } B_1.$$

Consequently, from (2.8), we deduce that $(-\Delta)^s u_\varepsilon \overset{k}{=} 0 + f_\varepsilon - f_\varepsilon$ in B_1, and this establishes (1.4).

Furthermore, from (2.4), (2.7) and (2.12), we see that

$$\|u_\varepsilon - u\|_{C^m(B_1)} \leqslant \|v_\varepsilon - u\|_{C^m(B_1)} + \|\widetilde{u}_{\bar{R}_\varepsilon}\|_{C^m(B_1)} + \|w_\varepsilon\|_{C^m(B_1)} \leqslant \varepsilon + 0 + C\varepsilon.$$

This proves (1.5) (up to renaming ε).

Now we take $R_\varepsilon := \rho_\varepsilon + \overline{R}_\varepsilon$. From (2.5), (2.6) and (2.10), we have that, in $B^c_{R_\varepsilon}$, it holds that $u_\varepsilon = 0 + u - 0$, which establishes (1.6), as desired. □

3. Proof of Theorem 1.4

First, we prove the existence result in Theorem 1.4. To this aim, we let u_0 and f be as in the statement of Theorem 1.4 and we define

$$u_1 := \chi_{B_2^c}\, u_0 \qquad \text{and} \qquad u_2 := \chi_{B_2 \setminus B_1}\, u_0.$$

We stress that u_1 is smooth in B_1 and u_2 is supported in B_2.

From [5, Remark 3.5], we can write $(-\Delta)^s u_1 \stackrel{k}{=} f_{u_1}$ in B_1, for a suitable function f_{u_1}.

Now we set $\widetilde{f} := f - f_{u_1}$ and we consider the solution of the standard problem

$$\begin{cases} (-\Delta)^s \widetilde{u} = \widetilde{f} & \text{in } B_1, \\ \widetilde{u} = u_2 & \text{in } B_1^c. \end{cases}$$

Hence, the consistency result in [5, Corollary 3.8 and formula (1.7)] give that

$$\begin{cases} (-\Delta)^s \widetilde{u} \stackrel{k}{=} \widetilde{f} & \text{in } B_1, \\ \widetilde{u} = u_2 & \text{in } B_1^c. \end{cases}$$

Then, we define $u := u_1 + \widetilde{u}$ and we see that $(-\Delta)^s u \stackrel{k}{=} f_{u_1} + \widetilde{f} = f$ in B_1. Moreover, in B_1^c it holds that $u = u_1 + u_2 = u_0$, namely u is a solution of (1.9). This establishes the existence result in Theorem 1.4.

Now, we prove the uniqueness claim in Theorem 1.4. For this, we observe that for any polynomial P of degree at most $k-1$ there exists a unique solution u_P of the standard problem

$$\begin{cases} (-\Delta)^s u_P = P & \text{in } B_1, \\ u_P = 0 & \text{in } B_1^c. \end{cases} \tag{3.1}$$

That is, in view of the consistency result in [5, Corollary 3.8], we have that $(-\Delta)^s u_P \stackrel{0}{=} P$ in B_1. Accordingly, from [5, formula (1.7)], we get that $(-\Delta)^s u_P \stackrel{k}{=} P$ in B_1. Then, by [5, formula (1.8)], it follows that u_P is a solution of

$$\begin{cases} (-\Delta)^s u_P \stackrel{k}{=} 0 & \text{in } B_1, \\ u_P = 0 & \text{in } B_1^c. \end{cases}$$

This means that if u is a solution of (1.9), then so is $u + u_P$.

Conversely, if u and v are two solutions, then $w := v - u$ satisfies

$$\begin{cases} (-\Delta)^s w \stackrel{k}{=} 0 & \text{in } B_1, \\ w = 0 & \text{in } B_1^c. \end{cases}$$

This and the consistency result in [5, Lemma 3.9] (used here with $j := 0$) give that $(-\Delta)^s w \stackrel{0}{=} P$ in B_1, for some polynomial P of degree at most $k-1$. Hence, using the consistency result in [5, Corollary 3.8], we can write

$$\begin{cases} (-\Delta)^s w = P & \text{in } B_1, \\ w = 0 & \text{in } B_1^c. \end{cases}$$

From the uniqueness of the solution of the standard problem in (3.1), we conclude that $w = u_P$, and so $v = u + u_P$.

These observations yield that the space of solutions of (1.9) is isomorphic to the space of polynomials P with degree less than or equal to $k-1$, which in turn has dimension N_k, as given in (1.10) (see e.g. [7]).

4. Proof of Theorem 1.5

We can extend u such that $u \in C^{2m}(B_{1+h})$, for some $h \in (0,1)$. Then, for all $x \in B_{1+h}$, we define $f(x) := F\big(x, u(x), \nabla u(x), \dots, D^m u(x)\big)$. Then, $f \in C^m(B_{1+h})$ and we can exploit Theorem 1.4 and obtain a function $v \in \mathcal{U}_k$ such that

$$\begin{cases} (-\Delta)^s v \overset{k}{=} f & \text{in } B_{1+h}, \\ v = 0 & \text{in } B^c_{1+h}. \end{cases}$$

By Theorem 1.3 in [5], we have that $v \in C^m(B_1)$. Hence, we can set $w := u - v \in C^m(B_1)$ and make use of Theorem 1.1 to find w_ε and $R_\varepsilon > 2$ such that

$$\begin{aligned} &(-\Delta)^s w_\varepsilon \overset{k}{=} 0 \quad \text{in } B_1, \\ &\|w_\varepsilon - w\|_{C^m(B_1)} \leqslant \varepsilon \\ \text{and} \quad &w_\varepsilon = w \quad \text{in } B^c_{R_\varepsilon}. \end{aligned}$$

Now, we define $u_\varepsilon := v + w_\varepsilon$. We observe that

$$\begin{aligned} (-\Delta)^s u_\varepsilon(x) &\overset{k}{=} (-\Delta)^s v(x) + (-\Delta)^s w_\varepsilon(x) \\ &\overset{k}{=} f(x) = F\big(x, u(x), \nabla u(x), \dots, D^m u(x)\big) \end{aligned}$$

for all $x \in B_1$.

This gives that (1.11) is satisfied with

$$\begin{aligned} \eta_\varepsilon(x) := &F\big(x, u(x), \nabla u(x), \dots, D^m u(x)\big) \\ &- F\big(x, u_\varepsilon(x), \nabla u_\varepsilon(x), \dots, D^m u_\varepsilon(x)\big). \end{aligned} \tag{4.1}$$

Moreover, in $B^c_{R_\varepsilon}$,

$$u_\varepsilon = v + w_\varepsilon = v + w = u,$$

and this proves (1.14).

Furthermore,

$$\|u_\varepsilon - u\|_{C^m(B_1)} = \|v + w_\varepsilon - u\|_{C^m(B_1)} = \|w_\varepsilon - w\|_{C^m(B_1)} \leqslant \varepsilon,$$

which establishes (1.13).

Then, we take

$$S := 2 + \sum_{j=0}^{m} \|D^j u\|_{L^\infty(B_1)}$$

and we denote by L the Lipschitz norm of F in $[-S, S]^{N(m)}$. Thus, employing (1.13) and (4.1), for all $x \in B_1$ we have that

$$\begin{aligned}
&|\eta_\varepsilon(x)| \\
&= \Big|F\big(x, u(x), \nabla u(x), \ldots, D^m u(x)\big) - F\big(x, u_\varepsilon(x), \nabla u_\varepsilon(x), \ldots, D^m u_\varepsilon(x)\big)\Big| \\
&\leqslant L \sum_{j=0}^{m} |D^j u(x) - D^j u_\varepsilon(x)| \leqslant Lm \, \|u_\varepsilon - u\|_{C^m(B_1)} \leqslant Lm\varepsilon,
\end{aligned}$$

and this gives (1.12), up to renaming ε.

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