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Weighted local Weyl laws for elliptic operators [(*)]

Alejandro Rivera [(1)]

Abstract. — Let A be an elliptic pseudo-differential operator of order m on a closed manifold $\mathcal{X}$ of dimension $n > 0$, self-ajdoint with respect to some positive smooth density $_{\mathcal{X}}$. Then, the spectrum of A is made up of a sequence of eigenvalues $(\lambda_k)_{k\geqslant 1}$ whose corresponding (orthogonal) eigenfunctions $(e_k)_{k\geqslant 1}$ are C^∞. Fix $s \in \mathbb{R}$ and define the following integral kernel on $\mathcal{X}$

$$K_L^s(x,y) = \sum_{0<\lambda_k\leqslant L} \lambda_k^{-s} e_k(x)\overline{e_k(y)}\,.$$

We derive asymptotic formulae near the diagonal for the kernels $K_L^s(x,y)$ when $L \to +\infty$ with fixed s. For $s = 0$, K_L^0 is the kernel of the spectral projector of A on the energy levels $]0, L]$, studied by Hörmander in [11]. In the present work we build on Hörmander's result to study the kernels K_L^s for $s \in \mathbb{R}$ fixed. If $s < \frac{n}{m}$, uniformly in $x \in \mathcal{X}$, $K_L^s(x,x) \asymp L^{-s+n/m}$ and, at distance $L^{-1/m}$ around the diagonal, the rescaled leading term behaves like the Fourier transform of an explicit function of the symbol of A. If $s = \frac{n}{m}$, under some explicit generic condition on the principal symbol of A, which holds if A is a differential operator, the integral kernel has a logarithmic divergence near the diagonal smoothed at scale $L^{-1/m}$, so that on the diagonal it is pointwise of order $\ln(L)$. Our results also hold when A is an elliptic differential operator on a compact open subset of $\mathbb{R}^n$ and Dirichlet boundary conditions are imposed on the e_k.

1. Introduction

1.1. Context and presentation of the results

The purpose of the present work is to compute pointwise asymptotics of the integral kernels of certain operators defined by functional calculus from

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either elliptic self-adjoint pseudo-differential operators on a closed manifold or on a compact manifold with boundary, with Dirichlet boundary conditions. Stating the results in full generality requires some vocabulary from microlocal analysis and some additional definitions. For this reason, we start by stating our results in the simpler case of elliptic self-adjoint differential operators on a closed manifold. The general case is presented in Section 2. This, of course, leads to some redundancy between different statements which we accept for the sake of accessibility and transparency of the main results.

Let $\mathcal{X}$ be a smooth compact manifold without boundary, of positive dimension $n > 0$ and equipped with a smooth positive density $\mathrm{d}\mu_{\mathcal{X}}$. Let A be an elliptic differential operator on $\mathcal{X}$ of positive order m. By this we mean that in any local coordinate system $x = (x_1, \dots, x_n)$ on $\mathcal{X}$ defined on $U \subset \mathbb{R}^n$, A, acts on $C_c^\infty(U)$ as

$$\sum_{0 \leqslant |\alpha| \leqslant m} a_\alpha(x)(-i\partial)^\alpha$$

where $\alpha \in \mathbb{N}^n$ and $a_\alpha \in C^\infty(\mathbb{R}^n)$ and for each $\xi \in \mathbb{R}^n \setminus \{0\}$, we have

$$\sigma_A(x,\xi) := \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha > 0\,.$$

The function σ_A is called the *principal symbol* of A in these coordinates. It is well known (and easy to check) that the principal symbol of A defines a smooth function on the complement of the zero section of $T^*\mathcal{X}$ independent of the choice of coordinates. We assume that A is symmetric with respect to the L^2-scalar product on $(\mathcal{X}, \mathrm{d}\mu_{\mathcal{X}})$. Then one can show (see Section 2.1) that A has a unique self-adjoint extension whose spectrum is made up of a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of real eigenvalues diverging to $+\infty$ with smooth L^2-normalized eigenfunctions $(e_k)_{k\in\mathbb{N}}$ forming a Hilbert basis for $L^2(\mathcal{X}, \mathrm{d}\mu_{\mathcal{X}})$. For each $L \geqslant 0$, let Π_L be the L^2 orthogonal projector on the space spanned by the eigenfunctions e_k such that $0 < \lambda_k \leqslant L$. Since this space is finite-dimensional, Π_L has a smooth integral kernel $E_L \in C^\infty(\mathcal{X} \times \mathcal{X})$. More explicitely,

$$\forall\,(x,y) \in \mathcal{X} \times \mathcal{X},\ E_L(x,y) = \sum_{0<\lambda_k\leqslant L} e_k(x)\overline{e_k(y)}\,.$$

In [11], Hörmander studied the behavior of this kernel[1] on a neighborhood of the diagonal as $L \to +\infty$. Integrating E_L over the diagonal he

[1] Actually, Hörmander considered eigenvalues $\lambda_k \leqslant L$ without requiring $\lambda_k > 0$, but this only adds a bounded error term to the estimates.

recovered the following estimate, also known as *Weyl's law*:

$$\mathrm{Card}\{k \in \mathbb{N} \mid \lambda_k \leqslant L\} \sim \frac{1}{(2\pi)^n} \int_{\mathcal{X}} \int_{\sigma_A(x,\xi)\leqslant 1} \mathrm{d}x\mathrm{d}\xi \times L^{\frac{n}{m}}$$

Hörmander's result is stronger than the above estimates in two respects. First because the error term obtained is smaller than the ones known before and is sharp in all generality. Secondly, the result actually provides local information concerning the behavior of the kernel E_L near the diagonal, which is why is sometimes called the *local Weyl law*. We will state this theorem in Section 2.2 (see Theorem 2.4).

In recent years, Hörmander's local Weyl law has received a lot of attention because E_L turns out to be the covariance of a certain Gaussian field on $\mathcal{X}$ defined as a random linear combination of eigenfunctions of A. Following the early work [1], more recently, several authors have studied the average length of the zero set of these functions, the average number of connected components with given topologies, as well as concentration in probability of these quantities (see for instance [4, 6, 7, 10, 15, 18, 19, 20, 24, 30]). Indeed, if $(Z_k)_{k\geqslant 0}$ is a sequence of i.i.d. real random variables with law $\mathcal{N}(0,1)$, then, the one parameter family $(f_L)_{L\geqslant 0}$ of Gaussian fields defined at each $x \in \mathcal{X}$ as

$$f_L(x) = \sum_{0<\lambda_k\leqslant L} Z_k e_k(x)$$

is such that for each $x, y \in \mathcal{X}$, $\mathbb{E}[f_L(x)f_L(y)] = E_L(x,y)$. In particular, Hörmander's pointwise Weyl law implies that

$$\mathrm{Var}(f_L(x)) = E_L(x,x) = \frac{1}{(2\pi)^n} \int_{\sigma_A(x,\xi)\leqslant 1} \widetilde{\mathrm{d}_x\mu(\xi)} \times L^{\frac{n}{m}} + O(L^{\frac{n-1}{m}})$$

where $\widetilde{\mathrm{d}_x\mu_{\mathcal{X}}}(\xi)$ is the density induced[(2)] on $T_x^*\mathcal{X}$ by $\mathrm{d}\mu_{\mathcal{X}}$. In particular, $\mathrm{d}\mu_{\mathcal{X}}(x)\widetilde{\mathrm{d}_x\mu_{\mathcal{X}}}(\xi)$ equals $\mathrm{d}x\mathrm{d}\xi$, the Lebesgue measure on $\mathbb{R}^{2n}$. In [22] we studied a natural variation of this random linear combination of eigenfunctions in dimension $n = 2$ and observed a very different asymptotic behavior of the covariance function. More precisely, in the case where A is the Laplacian on a closed surface, we studied the random fields

$$g_L = \sum_{0<\lambda_k\leqslant L} \lambda_k^{-1/2} Z_k e_k\,.$$

(2) More precisely, if $(x_1, \ldots, x_n)$ are local coordinates on $\mathcal{X}$ such that $\mathrm{d}\mu_{\mathcal{X}}(x) = g(x)\mathrm{d}x$ in these coordinates, $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ defines a basis of $T_x^*\mathcal{X}$ for each $x = (x_1, \ldots, x_n)$ and we define $\widetilde{\mathrm{d}_x\mu_{\mathcal{X}}}(\xi)$ as $\frac{1}{g(x)}$ times the Lebesgue measure defined by declaring this basis to be orthonormal. This definition is invariant by measure preserving coordinate changes so $\widetilde{\mathrm{d}_x\mu_{\mathcal{X}}}(\xi)$ is indeed induced by $\mathrm{d}\mu_{\mathcal{X}}$ independently of the choice of coordinates.

Unlike for the previous case, we found that for any $x \in \mathcal{X}$,

$$\operatorname{Var}(g_L(x)) = \frac{1}{4\pi} \ln(L) + O(1)\,.$$

Following this work, we are interested in studying more general random linear combinations of these eigenfunctions. To this end, it is essential to gather some information about the corresponding covariance function. The purpose of this article is to provide an asymptotic for these kernels similar to the one we have for E_L. For each $s \in \mathbb{R}$ we consider the kernel

$$K_L^s(x,y) = \sum_{0<\lambda_k\leqslant L} \lambda_k^{-s} e_k(x)\overline{e_k(y)}\,.$$

These kernels converge in distribution to the integral kernels of A^{-s} as $L \to +\infty$ but diverge on the diagonal for small or negative values of s. The pointwise behavior of the limiting kernel on the diagonal, which is well defined for large values of s, has been studied for instance in [25] and [26]. In [25], the author proved that, as a function of s, the limit admitted a meromorphic extension to the whole complex plane. We focus instead on a fixed s for which the kernel diverges and study its pointwise divergence near the diagonal. We call these results *weighted local Weyl laws* by analogy with E_L (which is just K_L^0) because of the weights λ_k^{-s} on the terms of the sum defining K_L^s. As we shall see, the kernels K_L^s experience a sudden change in their asymptotic behavior between the phases $s < \frac{n}{m}$ and $s = \frac{n}{m}$. All our results will be local so we take the liberty of omitting the composition with the chart when writing functions on $\mathcal{X}$ in local coordinates. Our first result provides information when $s < \frac{n}{m}$.

Theorem 1.1 (Kernel asymptotics when $s < \frac{n}{m}$). — *Recall that A is an elliptic differential operator on $\mathcal{X}$ of order $m > 0$. Assume that $s < \frac{n}{m}$. Fix $x_0 \in \mathcal{X}$ and consider local coordinates $x = (x_1, \dots, x_n)$ for $\mathcal{X}$ centered at x_0 and defined on an open subset $U \subset \mathbb{R}^n$ such that $\mathrm{d}\mu_{\mathcal{X}}$ agrees with the Lebesgue measure in these coordinates*(3) *. Then, there exists $V \subset U$ an open neighborhood of 0 such that, in these coordinates, for each $\alpha, \beta \in \mathbb{N}^n$, we have the following estimates.*

(3) Such coordinates always exist by [17].

(1) *Uniformly for $L \geqslant 1$, $x \in V$ and $X, Y \in \mathbb{R}^n$ such that $x + L^{-1/m}X$, $x + L^{-1/m}Y \in V$*

$$L^{s-(n+|\alpha|+|\beta|)/m}\ \partial_x^\alpha \partial_y^\beta K_L^s\left(x + L^{-1/m}X, x + L^{-1/m}Y\right) = \frac{1}{(2\pi)^n} \int_{\sigma_A(x,\xi)\leqslant 1} e^{i\langle \xi, X-Y\rangle} \frac{(i\xi)^\alpha(-i\xi)^\beta}{\sigma_A(x,\xi)^s} \mathrm{d}\xi + O\Big(L^{s-(n+|\alpha|+|\beta|)/m} + L^{-1/m}\ln{(L)^\eta}\Big)$$

where $\eta = 1$ if $s = (n + |\alpha| + |\beta| - 1)/m$ and 0 otherwise.

(2) *Let $\varepsilon > 0$. Then, uniformly for $x, y \in V$ such that $|x - y| > \varepsilon$ and $L \geqslant 1$,*

$$L^{s-(n+|\alpha|+|\beta|)/m}\ \partial_x^\alpha \partial_y^\beta K_L^s(x, y) = O\Big(L^{s-(n+|\alpha|+|\beta|)/m} + L^{-1/m}\ln{(L)^\eta}\Big)$$

where $\eta = 1$ if $s = (n + |\alpha| + |\beta| - 1)/m$ and 0 otherwise.

Here $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\langle\,\cdot\,,\cdot\,\rangle$ is the Euclidean scalar product.

Note that the case where $s = 0$ and $\alpha = \beta = 0$ is Theorem 5.1 of [11] (see Theorem 2.4 and the discussion below for more details about this case). Let us say a few words about the two error terms appearing in both points one and two of the theorem. The first term, $L^{s-(n+|\alpha|+|\beta|)/m}$, corresponds to the $O(1)$ error in $\partial_x^\alpha \partial_y^\beta K_L^s(x, y)$, coming from the low eigenvalues, which we never try to control. In the second term, $L^{-1/m}\ln(L)^\eta$, the factor $L^{-1/m}$ is analogous (and directly linked to) Hörmander's error term (see Theorem 2.4 below), while the $\ln(L)^\eta$ appears in an integration by parts made to deal with the λ_k^{-s} powers in the expansion. Depending on the values of s, n, m and $d = |\alpha|+|\beta|$ either one of the two error terms could dominate. More precisely, Hörmander's error $L^{-1/m}\ln(L)^\eta$ always dominates the low frequency error $L^{s-(n+|\alpha|+|\beta|)/m}$, except when $\alpha = \beta = 0$ and $(n-1)/m < s < n/m$, in which case it is the low frequency term that dominates.

We prove Theorem 1.1 at the end of Section 2. Before stating the second result, we introduce the following notation. Firstly, for each $x \in \mathcal{X}$, the density $\mathrm{d}_x\mu_{\mathcal{X}}$ on $T_x\mathcal{X}$ defines a canonical dual density $\widetilde{\mathrm{d}_x\mu_{\mathcal{X}}}$ on $T_x^*\mathcal{X}$. For each $x \in \mathcal{X}$, let

$$S_x^A = \{\xi \in T_x^*\mathcal{X} \,|\, \sigma_A(x,\xi) = 1\}\,.$$

Since σ_A is m-homogeneous, S_x^A is a smooth compact hypersurface of $T_x^*\mathcal{X}$ strictly star-shaped[(4)] around the origin and the map

$$\begin{aligned} F : S_x^A \times \left]0,+\infty\right[&\longrightarrow T_x^*\mathcal{X} \setminus \{0\} \\ (\omega, t) &\longmapsto t\omega \end{aligned}$$

is a diffeomorphism. If $\mathrm{d}t$ is the Lebesgue measure on $]0,+\infty[$, we define a smooth density $\mathrm{d}_x\nu$ on S_x^A by

$$\mathrm{d}_x\nu(\omega)t^{n-1}\mathrm{d}t = F^*\widetilde{\mathrm{d}_x\mu_{\mathcal{X}}}(\xi)\,.$$

In particular, for each $u \in C_c^\infty(T_x^*\mathcal{X})$,

$$\int_{T_x^*\mathcal{X}} u(\xi)\widetilde{\mathrm{d}_x\mu_{\mathcal{X}}}(\xi) = \int_0^{+\infty}\int_{S_x^A} u(t\xi)\mathrm{d}_x\nu(\xi)t^{n-1}\mathrm{d}t\,. \tag{1.1}$$

This implies in turn that

$$\nu_x(S_x^A) := \int_{S_x^A} \mathrm{d}_x\nu = n\int_{\{\sigma_A(x,\xi)\leqslant 1\}} \widetilde{\mathrm{d}_x\mu_{\mathcal{X}}}(\xi)\,. \tag{1.2}$$

Our second result deals with the case where $s = \frac{n}{m}$. While Theorem 1.1 proves that K_L^s grows at rate $L^{n/m-s}$ for $s < n/m$, and that the main term depends continuously on s, the following result shows that this is not true for $s = n/m$. Indeed, while the first point is analogous to the results of Theorem 1.1, the second point is quite different (and requires additional tools).

Theorem 1.2 (Kernel asymptotics when $s = \frac{n}{m}$). — *Recall that A is an elliptic differential operator on $\mathcal{X}$ of order $m > 0$. Assume that $s = \frac{n}{m}$. Fix $x_0 \in \mathcal{X}$ and consider local coordinates $x = (x_1, \dots, x_n)$ for $\mathcal{X}$ centered at x_0 and defined on an open subset $U \subset \mathbb{R}^n$ such that $\mathrm{d}\mu_{\mathcal{X}}$ agrees with the Lebesgue measure in these coordinates. Then, there exists an open neighborhood $V \subset U$ of 0 such that, for each $\alpha, \beta \in \mathbb{N}^n$, the following holds.*

(1) • *Assume that $(\alpha,\beta) \neq (0,0)$. In these coordinates, uniformly for $L \geqslant 1$, $x \in V$ and $X, Y \in \mathbb{R}^n$ such that $x + L^{-1/m}X, x + L^{-1/m}Y \in V$*

$$\begin{aligned} L^{-(|\alpha|+|\beta|)/m}\, &\partial_x^\alpha\partial_y^\beta K_L^{n/m}\Big(x + L^{-1/m}X, x + L^{-1/m}Y\Big) \\ &= \frac{1}{(2\pi)^n}\int_{\sigma_A(x,\xi)\leqslant 1} e^{i\langle\xi, X-Y\rangle}\frac{(i\xi)^\alpha(-i\xi)^\beta}{\sigma_A(x,\xi)^{n/m}}\mathrm{d}\xi \\ &\qquad + O\Big(L^{-1/m}\ln(L)^\eta\Big)\,. \end{aligned}$$

where $\eta = 1$ if $1 = |\alpha| + |\beta|$ and 0 otherwise.

(4) More precisely, for each $\xi \in T_x^*\mathcal{X} \setminus \{0\}$, the ray $\{t\xi \,|\, t > 0\}$ intersects S_x^A exactly at $\sigma_A(x,\xi)^{-1/m}\xi$ and does so transversally.

- *Assume* $(\alpha, \beta) \neq (0,0)$ *and let* $\varepsilon > 0$. *Then, uniformly for* $x, y \in V$ *such that* $|x-y| > \varepsilon$,
$$L^{-(|\alpha|+|\beta|)/m}\ \partial_x^\alpha \partial_y^\beta K_L^{n/m}(x,y) = O\Big(L^{-1/m}\ln(L)^\eta\Big)$$
where $\eta = 1$ *if* $1 = |\alpha| + |\beta|$ *and* 0 *otherwise.*

(2)
- *Uniformly for* $x, y \in V$ *and* $L \geqslant 1$, *in these coordinates,*
$$K_L^{n/m}(x,y) = g_A(x,y)\left[\ln\big(L^{1/m}\big) - \ln_+\big(L^{1/m}|x-y|\big)\right] + O(1)$$
where
$$g_A(x,y) = \frac{n}{2(2\pi)^n} \times \left(|\{\sigma_A(x,\xi) \leqslant 1\}| + |\{\sigma_A(y,\xi) \leqslant 1\}|\right)$$
where $\ln_+(t) = \ln(t) \vee 0$ *and where* $|\{\sigma_A(x,\xi) \leqslant 1\}| = \int_{\sigma_A(x,\xi)\leqslant 1} \mathrm{d}\xi$.
- *There exists a symmetric bounded function* $Q : U \times U \to \mathbb{R}$ *such that, uniformly for* $\kappa \geqslant 1$, $L \geqslant 1$ *and* $x, y \in V$ *such that* $|x-y| \geqslant \kappa L^{-1/m}$, *in these coordinates,*
$$K_L^{n/m}(x,y) = -g_A(x,y)\ln(|x-y|) + Q(x,y) + O\Big(\kappa^{-1/k}\Big)$$
where, if $n = 1$ *then* $k = 1$ *and if* $n \geqslant 2$ *then* $k = m$.

Here $|\alpha| = \alpha_1 + \cdots + \alpha_n$ *and* $\langle\,\cdot\,,\cdot\,\rangle$ *is the Euclidean scalar product.*

This theorem (especially the second point) generalizes Theorem 3 of [22], which proved the second point in the case where $s = 1$, $\mathcal{X}$ was a closed surface (so $n = 2$) with a Riemannian metric and A was the associated Laplacian (so $m = 2$). The main challenge in the extension comes from the need to apply a generalized stationary phase formula on the level sets of the symbol. In [22], this is Proposition 23, where the traditional stationary phase formula applies directly. This general setting requires tools from singularity theory that are deployed in Section 7. The second point of Theorem 1.2 will follow from Theorem 2.8 below. As is apparent, in Figure 3.1, the proof of this result is more complex than that of the others. We prove Theorem 1.2 at the end of Section 2.

Corollary 1.3. — *The Schwartz kernel* $K \in \mathcal{D}'(\mathcal{X} \times \mathcal{X})$ *of* $A^{-n/m}$ *belongs to* $L^1(\mathcal{X} \times \mathcal{X})$. *Moreover, for each smooth distance function* $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ *on* $\mathcal{X}$ *there exists a bounded symmetric function* $Q_{A,d} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, *smooth on the complement of the diagonal, such that, for any distinct* $x, y \in \mathcal{X}$,
$$K(x,y) = -g_A(x,y)\ln(d(x,y)) + Q_{A,d}(x,y)\,.$$

We prove Corollary 1.3 at the end of Section 2.

1.2. An important example: the Laplacian

As explained above, this work is motivated by recent interest in the kernel K_L^0 as the covariance function of a Gaussian field. In further work, we wish to study certain Gaussian fields arising naturally in geometry and statistical mechanics with covariance K_L^s. One such field is the Gaussian Free Field, which is a central object in statistical mechanics today. In Corollaries 1.4 and 1.5 we detail our main results in this special case.

Let $(\mathcal{X}, g)$ be a closed Riemannian manifold of dimension $n \geqslant 2$. Let $\Delta = -\operatorname{div} \circ \nabla$ be the Laplace operator on $\mathcal{X}$ and let $|\mathrm{d}V_g|$ be the Riemannian volume density on $\mathcal{X}$. Then, Δ is an elliptic differential operator with principal symbol $\sigma(x,\xi) = g_x^{-1}(\xi,\xi)$ where g_x^{-1} is the scalar product induced on $T_x^*\mathcal{X}$ by g_x. Moreover, Δ is symmetric with respect to the L^2-scalar product induced by the density $|\mathrm{d}V_g|$ on $\mathcal{X}$. Let $(\lambda_k)_{k\in\mathbb{N}}$ be the sequence of eigenvalues of Δ (counted with multiplicity) and arranged in increasing order. Let $(e_k)_{k\in\mathbb{N}}$ be a Hilbert basis of $L^2(\mathcal{X}, |\mathrm{d}V_g|)$ made up of real valued functions, such that for each $k \in \mathbb{N}$, $\Delta e_k = \lambda_k e_k$. For each $L > 0$, each $s > 0$ and each $(x,y) \in \mathcal{X} \times \mathcal{X}$, let

$$K_L^s(x,y) = \sum_{0<\lambda_k\leqslant L} \lambda_k^{-s} e_k(x) e_k(y)\,.$$

Moreover, consider $(Z_k)_{k\in\mathbb{N}}$ a family of independent standard normals and for each $s \in \mathbb{R}$ and $L > 0$ define

$$f_L^s(x) = \sum_{0<\lambda_k\leqslant L} \lambda_k^{-s/2} Z_k e_k(x)\,.$$

Then, f_L^s is an a.s. smooth, centered Gaussian field on $\mathcal{X}$ with covariance K_L^s. In the case $s = 1$, K_L^1 converges in distribution as $L \to +\infty$ to Green's function on $\mathcal{X}$ which is the (generalized) covariance function for the Gaussian Free Field (see for instance [27]). Equivalently, as $L \to +\infty$, f_L^s converges a.s. in the space of distributions to the Gaussian Free Field. We have the following results. In the case where $s < n/2$, K_L^s converges at scale $L^{-1/2}$ to a non-trivial function after rescaling by a polynomial factor.

Corollary 1.4 (The Laplacian: $s < n/2$). — *Assume that $s < n/2$. Fix $x_0 \in \mathcal{X}$ and consider local coordinates $x = (x_1, \dots, x_n)$ for $\mathcal{X}$ centered at x_0 such that $|\mathrm{d}V_g|$ agrees with the Lebesgue measure in these coordinates. Then, there exists $V \subset U$ an open neighborhood of 0 such that, in these coordinates, for each $\alpha, \beta \in \mathbb{N}^n$, we have the following estimates.*

(1) *In these coordinates, uniformly for $L \geqslant 1$, $x \in V$ and $X, Y \in \mathbb{R}^n$ such that $x + L^{-1/m}X, x + L^{-1/m}Y \in V$*

$$L^{s-(n+|\alpha|+|\beta|)/2}\ \partial_x^\alpha \partial_y^\beta K_L^s \left(x + L^{-1/2}X, x + L^{-1/2}Y\right) = \frac{1}{(2\pi)^n} \int_{|\xi|_x^2 \leqslant 1} e^{i\langle \xi, X-Y\rangle} \frac{(i\xi)^\alpha (-i\xi)^\beta}{|\xi|_x^{2s}} \mathrm{d}\xi + O\Big(L^{s-(n+|\alpha|+|\beta|)/2} + L^{-1/2} \ln (L)^\eta\Big)$$

where $\eta = 1$ if $s = (n + |\alpha| + |\beta| - 1)/2$ and 0 otherwise. Here $|\xi|_x^2 = g_x^{-1}(\xi, \xi)$.

(2) *Let $\varepsilon > 0$. Then, uniformly for $x, y \in V$ such that $|x - y| > \varepsilon$ and for $L \geqslant 1$,*

$$\partial_x^\alpha \partial_y^\beta K_L^s\,(x, y) = O\Big(1 + L^{(n+|\alpha|+|\beta|-2s-1)/2} \ln (L)^\eta\Big)$$

where $\eta = 1$ if $s = (n + |\alpha| + |\beta| - 1)/2$ and 0 otherwise.

Proof. — This follows directly from Theorem 1.1 with $m = 2$, $s < n/2$, $A = \Delta$ and $\sigma_A(x, \xi) = |\xi|_x^2$. □

A direct consequence of Corollary 1.4 is that, for $s < n/2$, in the same coordinates as in the corollary, for each $x \in V$, the random field

$$X \mapsto L^{-s/2} f_L^s(x + L^{-1/2}X)$$

converges in distribution as $L \to +\infty$ to a smooth stationary Gaussian field on $\mathbb{R}^n$ with covariance

$$(X, Y) \mapsto \frac{1}{(2\pi)^n} \int_{|\xi|_x^2 \leqslant 1} e^{i\langle \xi, X-Y\rangle} \frac{\mathrm{d}\xi}{|\xi|^{2s}}$$

at a rate uniform in x. On the other hand, if $s = n/2$, although the derivatives of K_L^s also have non-trivial local limits at scale $L^{-1/2}$, K_L^s itself converges pointwise to a distribution with a logarithmic singularity on the diagonal. Note that when $s = 1$, the first part of the second point of Corollary 1.5 below yields Theorem 3 of [22].

Corollary 1.5 (The Laplacian: $s = n/2$). — *Assume that $s = n/2$. Fix $x_0 \in \mathcal{X}$ and consider local coordinates $x = (x_1, x_2)$ for $\mathcal{X}$ centered at x_0 defined on an open subset $U \subset \mathbb{R}^2$ such that $|\mathrm{d}V_g|$ agrees with the Lebesgue measure in these coordinates. Then, there exists an open neighborhood $V \subset U$ of 0 such that, for each $\alpha, \beta \in \mathbb{N}^n$, the following holds.*

(1) • *In these coordinates, uniformly for* $L \geqslant 1$, $x \in V$ *and* $X, Y \in \mathbb{R}^n$ *such that* $x + L^{-1/m}X, x + L^{-1/m}Y \in V$

$$L^{-(|\alpha|+|\beta|)/2}\ \partial_x^\alpha \partial_y^\beta K_L^{n/2}\left(x + L^{-1/2}X, x + L^{-1/2}Y\right)$$
$$= \frac{1}{(2\pi)^n}\int_{|\xi|_x^2 \leqslant 1} e^{i\langle \xi, X-Y\rangle}\frac{(i\xi)^\alpha(-i\xi)^\beta}{|\xi|_x^n}\mathrm{d}\xi$$
$$+ O\Big(L^{s-(n+|\alpha|+|\beta|)/2} + L^{-1/2}\ln(L)^\eta\Big)$$

where $\eta = 1$ *if* $1 = |\alpha| + |\beta|$ *and* 0 *otherwise. Here* $|\xi|_x^2 = g_x^{-1}(\xi, \xi)$ *and* $\mathrm{d}\xi$ *is the Lebesgue measure.*

• *Let* $\varepsilon > 0$*. Then, uniformly for* $x, y \in V$ *such that* $|x - y| > \varepsilon$ *and for* $L \geqslant 1$,

$$\partial_x^\alpha \partial_y^\beta K_L^s(x, y) = O\Big(1 + L^{(|\alpha|+|\beta|-1)/2}\ln(L)^\eta\Big)$$

where $\eta = 1$ *if* $1 = |\alpha| + |\beta|$ *and* 0 *otherwise.*

(2) • *Uniformly for* $x, y \in V$ *and* $L \geqslant 1$*, in these coordinates,*

$$K_L^{n/2}(x, y) = \frac{\left|S^{n-1}\right|}{(2\pi)^n}\left[\ln\left(L^{1/2}\right) - \ln_+\left(L^{1/2}|x-y|\right)\right] + O(1)$$

where $\ln_+(t) = \ln(t) \vee 0$.

• *There exists a symmetric bounded function* $Q : U \times U \to \mathbb{R}$ *such that, uniformly for* $\kappa \geqslant 1$, $L \geqslant 1$ *and* $x, y \in V$ *such that* $|x - y| \geqslant \kappa L^{-1/2}$*, in these coordinates,*

$$K_L^{n/2}(x, y) = \frac{\left|S^{n-1}\right|}{(2\pi)^n}\ln(|x-y|) + Q(x, y) + O\Big(\kappa^{-1/2}\Big).$$

Proof. — This follows directly from Theorem 1.2 with $m = 2$, $s = n/2$, $A = \Delta$ and $\sigma_A(x, \xi) = |\xi|_x^2$. We simply use that if B_n is the n-dimensional euclidean ball, $n|B_n| = |S^{n-1}|$. □

In the case $s = n/2$, Corollary 1.5 implies that the family $(f_L^s)_{L>0}$ defines a family of log-correlated fields. More precisely, the pointwise variance of f_L^s is equivalent to $C_n \ln(L^{1/2})$ as $L \to +\infty$ where $C_n = |S^{n-1}|/(2\pi)^n$, and the correlations decay logarithmically until they reach order $O(1)$. This behavior was observed in [22] for $n = 2$ and is consistent with the behavior of the two-dimensional discrete Gaussian Free Field, whose covariance is the Green's function for the 2D simple random walk (see for instance [5, Chapter 8]).

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2. Statement of the main results

In this section, we present the main objects of study and state our results in full generality. In Section 2.1 we present the general framework of the article. In Section 2.2 we state Hörmander's local Weyl law. In Section 2.3 we state the generalizations of the local Weyl law proved in this paper. We finish off by deducing Theorems 1.1 and 1.2 as well as Corollary 1.3.

2.1. General setting

In this article, we consider simultaneously two different elliptic eigenvalue problems. Since our arguments hold indifferently for the two cases, we present them in this section using the same notations. The first case is a closed eigenvalue problem. In this case we will follow [11, 13, 14]. In the second case, we consider a Dirichlet eigenvalue problem, for which our main references will be [16] and [29].

Setting 1. — In this setting we follow [13] and [14]. Here $\mathcal{X}$ is a compact manifold without boundary. We will consider symbols and classical symbols on open subsets of $\mathbb{R}^n$ defined as in Definitions 18.1.1 and 18.1.5 of [13] or Definition 2.2 below. To any such symbol we associate a pseudo-differential operator by left quantization as in Chapter XVIII of [13]. Pseudo-differential operators on $\mathcal{X}$ are defined, as in [13], to be operators on $\mathcal{X}$ which, when read in local charts, are pseudo-differential operators modulo operators with smooth kernel (see [13, Definition 18.1.20]).

We consider an elliptic pseudo-differential operator A of positive order m acting on $C^\infty(\mathcal{X})$. We assume A is symmetric for the L^2-scalar product on $(\mathcal{X}, \mathrm{d}\mu_{\mathcal{X}})$. This implies that the principal symbol σ_A of A is a real valued positive C^∞ m-homogeneous function on the complement of the zero section of $T^*\mathcal{X}$. Under these assumptions, A has a unique self-adjoint extension in $L^2(\mathcal{X}, \mathrm{d}\mu_{\mathcal{X}})$ whose spectrum forms a discrete sequence $(\lambda_k)_{k\in\mathbb{N}}$ of real numbers diverging to $+\infty$ and whose corresponding eigenfunctions e_k are of class C^∞ (see for instance of [14, Section 29.1]). For each $L > 0$, set

$$\forall\, x, y \in \mathcal{X}, E_L(x,y) = \sum_{0<\lambda_j\leqslant L} e_j(x)\overline{e_j(y)}\,.$$

Finally, we assume that the symbol of A is a classical symbol (see Definition 2.2 below).

Setting 2. — In this setting, we follow [16]. Here $\mathcal{X}$ is a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{X}$ and $\mathrm{d}\mu_{\mathcal{X}}$ is the Lebesgue measure. We consider the Dirichlet eigenvalue problem

$$\begin{aligned} Au &= \lambda u \quad \text{on } \mathcal{X}\,; \\ u &= 0 \qquad \text{on } \partial\mathcal{X} \end{aligned}$$

where A is an elliptic differential operator of even order $m \geqslant 1$ with principal symbol σ_A and $\lambda \in \mathbb{C}$. We assume that the following conditions (from [16, Chapter 2, Section 1.4]) are satisfied

- For each $u, v \in C_c^\infty(\mathcal{X})$, $\int u(x)\overline{Av(x)}\mathrm{d}\mu_{\mathcal{X}}(x) = \int Au(x)\overline{v(x)}\mathrm{d}\mu_{\mathcal{X}}(x)$. In other words, A is symmetric in $L^2(\mathcal{X})$.
- The operator A is properly elliptic in $\overline{\mathcal{X}}$ in the sense of Definition 1.2 of [16, Chapter 2, Section 1.4].

As explained in the appendix (see Proposition B.1 in Appendix B), there exists a sequence $(\lambda_k)_k$ of real numbers going to infinity and a sequence $(e_k)_k$ of smooth functions such that for each k, $\int_{\mathcal{X}} |e_k(x)|^2 \mathrm{d}\mu_{\mathcal{X}}(x) = 1$, such that the linear span of the e_k is dense in $L^2(\mathcal{X}, \mathrm{d}\mu_{\mathcal{X}})$ and such that e_k solves the aforementioned eigenvalue problem with $\lambda = \lambda_k$. For each $L > 0$, set

$$\forall\, x, y \in \overline{\mathcal{X}}, E_L(x,y) = \sum_{0<\lambda_j\leqslant L} e_j(x)\overline{e_j(y)}\,.$$

Remark 2.1. — For simplicity of reference we have chosen to restrict Setting 2 to bounded open subsets of $\mathbb{R}^n$. We find it very likely that similar results hold for manifolds with boundary. In any case, our results will hold as long as one can find an analog of Lemma A.1, which is the only input we really use. In particular, [29], which we use as a reference below, works on manifolds with boundary, but cites as a reference, Chapter 2 of [16], in which $\mathcal{X}$ is an open subset of $\mathbb{R}^n$.

2.2. Hörmander's local Weyl law

We begin by stating Hörmander's local Weyl law, for which we need the following definitions. First we define a family of symbol classes

Definition 2.2.

- *Fix $m \in \mathbb{R}$. Let $U \subset \mathbb{R}^p$ be an open subset and let $\sigma \in C^\infty(U \times \mathbb{R}^n)$. We say that σ is a* symbol of order m *and write $\sigma \in S^m(U; \mathbb{R}^n)$ (or just $\sigma \in S^m$ when no confusion is possible), if, for each compact* *subset $K \subset U$ and each $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^p$, there exists $C_{K,\alpha,\beta}$ such that*

$$\sup_{x\in K,\xi\in\mathbb{R}^n} |\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)| \leqslant C_{K,\alpha,\beta}(1+|\xi|)^{m-|\alpha|}\,.$$

- *If $\sigma(x,\xi)$ is a polynomial of degree m in ξ with coefficients varying smoothly with x, then $\sigma \in S^m$.*
- *If there exists $R < +\infty$ such that $|\xi| > R$ implies that $\sigma(x,\xi)$ is m-homogeneous in ξ, then $\sigma \in S^m$.*
- *Let $\sigma \in S^m(U;\mathbb{R}^n)$. We say that σ is a classical symbol if there exist symbols $(\sigma_k)_{k\in\mathbb{N}}$ such that for each $k \in \mathbb{N}$, there exists $R_k < +\infty$ such that $\sigma_k(x,\xi)$ is $(m-k)$-homogeneous in ξ for $|\xi| \geqslant R_k$. and such that for each $N \in \mathbb{N}$,*

$$\sigma - \sum_{k=0}^{N} \sigma_k \in S^{m-N-1}(U;\mathbb{R}^n)\,.$$

We define the principal symbol of the classical symbol σ as the function on $U \times (\mathbb{R}^n \setminus \{0\})$ $(x,\xi) \mapsto \lim_{t\to+\infty} t^{-m}\sigma(x,t\xi)$.

We then define a class of phase functions[5] as follows.

Definition 2.3. — *Given an open subset $U \subset \mathbb{R}^n$, we will say that a function $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$ is a* proper phase function *if it satisfies the following conditions.*

1. *The function ψ is a symbol of order one in its third variable.*
2. *For each $(x,y,\xi) \in U \times U \times \mathbb{R}^n$, $\langle x-y,\xi\rangle = 0$ implies that $\psi(x,y,\xi) = 0$.*
3. *For each $x \in U$ and $\xi \in \mathbb{R}^n$, $\partial_x\psi(x,y,\xi)|_{y=x} = \xi$.*
4. *There exists $\psi_\infty \in C^\infty(U \times U \times \mathbb{R}^n)$ satisfying all of the above properties and 1-homogeneous in ξ such that*

$$t^{-1}\psi(x,y,t\xi) \xrightarrow[t\to+\infty]{} \psi_\infty(x,y,\xi)$$

where the convergence takes place in $C^\infty(U \times U \times \mathbb{R}^n)$.

An important example of proper phase function to have in mind is the phase function $\psi(x,y,\xi) = \langle x-y,\xi\rangle$. Hörmander's local Weyl law may be stated as follows. Let us consider an operator A from either of the two settings presented in Section 2.1. In Setting 2 we also fix a family of boundary operators $(B_j)_j$ satisfying the assumptions required therein. In both settings

[5] This definition is inspired by Definition 2.3 of [11]. However, our notion of proper phase function is more restrictive than Hörmander's. Namely, Hörmander does not require condition 4.

we deduce the existence of a sequence $(\lambda_k)_k$ of (real) eigenvalues, diverging to $+\infty$ and $(e_k)_k$ a sequence of smooth, L^2-normalized eigenfunctions (either of A or of the boundary value problem) associated to them. We then define, in either setting

$$E_L(x,y) = \sum_{0<\lambda_j\leqslant L} e_j(x)\overline{e_j(y)}\,.$$

Recall that σ_A is the principal symbol of A, which we assumed to be positive homogeneous of order $m>0$ in the second variable.

Theorem 2.4 (Local Weyl law [11, Theorem 5.1] for $P=\mathrm{Id}$). — *Fix a point in $\mathcal{X}$ and consider local coordinates $(x_1,\dots,x_n)$ around it. Suppose further that the density $\mathrm{d}\mu_{\mathcal{X}}$ agrees with the Lebesgue measure in these coordinates. Let σ_A be the principal symbol of A in these coordinates. Then, there exists an open neighborhood U of $0\in\mathbb{R}^n$, a proper phase function $\psi\in C^\infty(U\times U\times\mathbb{R}^n)$ such that the following holds. Let P be a differential operator of order d with constant coefficients acting on $C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ with principal symbol $\sigma_P : T^*\mathbb{R}^{2n}\to\mathbb{C}$ (a homogeneous polynomial of degree d in the momentum variables). Then, there exists a constant $C<+\infty$, such that, in these coordinates, for each $x,y\in U$ and $L>0$,*

$$\left|PE_L(x,y) - \frac{1}{(2\pi)^n}\int_{\sigma_A(y,\xi)\leqslant L} e^{i\psi(x,y,\xi)}\sigma_P(\partial_{x,y}\psi(x,y,\xi))\mathrm{d}\xi\right| \leqslant C(1+L)^{(n+d-1)/m}\,.$$

Here $\partial_{x,y}\psi(x,y,\xi)\in T^*_{(x,y)}(U\times U)\simeq\mathbb{R}^{2n}$ denotes the derivative of ψ with respect to the variables (x,y). Moreover, for each neighborhood $W\subset U\times U$ of the diagonal there exists $C>0$ such that in local coordinates, for each $(x,y)\in(U\times U)\setminus W$ and $L>0$,

$$\big|PE_L(x,y)\big| \leqslant C(1+L)^{(n+d-1)/m}\,.$$

Finally, there exist a symbol[6] *$\sigma\in S^1$ and a constant $R<+\infty$ such that $\sigma=\sigma_A^{1/m}$ outside of a compact set and for each $x,y\in U$ and $\xi\in\mathbb{R}^n$ such that $|\xi|\geqslant R$,*

$$\sigma(x,\partial_x\psi(x,y,\xi)) = \sigma(y,\xi)\,. \tag{2.1}$$

Here $\partial_{x,y}\psi$ denotes the partial derivative of ψ with respect to the couple (x,y).

(6) If $\sigma_A^{1/m}$ is smooth then we can actually take $\sigma=\sigma_A^{1/m}$.

Remark 2.5.

- Recall that in Setting 2 (the Dirichlet case), the notation $\mathcal{X}$ denotes an open subset of $\mathbb{R}^n$. Therefore, Theorem 2.4 is an estimate of PE_L away from the boundary $\partial\mathcal{X}$.
- Since σ_A is m-homogeneous, by Definition 2.3, Theorem 2.4 implies that

$$PE_L(x,x) = \frac{1}{(2\pi)^n}\int_{\sigma_A(x,\xi)\leqslant 1} \sigma_P(\xi,-\xi)\mathrm{d}\xi \times L^{(n+d)/m}(1+O(L^{-1/m}))\,.$$

  In particular, on the diagonal, the main term grows like $L^{(n+d)/m}$. Moreover, the change of variables $\xi = L^{1/m}\eta$ reveals that, taking $x = x_0 + L^{-1/m}X$ and $y = x_0 + L^{-1/m}Y$ the integral expression converges – after rescaling by $L^{-(n+d)/m}$ – to a smooth function in (X,Y) and x.
- The approximation for $PE_L(x,y)$ given in Theorem 2.4 may seem asymmetric with respect to x and y because of the integration domain $\{\xi \in \mathbb{R}^n : \sigma_A(x,\xi) \leqslant L\}$. However, unless $|x-y| = O(L^{1/m})$, the main term is no longer greater than the error term. On the other hand, if $|x-y| = O(L^{1/m})$, replacing $\sigma_A(y,\xi)$ by $\sigma_A(x,\xi)$ in the integration domain yields a negligible error.
- The asymptotic provided by Theorem 2.4 is coordinate dependent since the notion of proper phase function is not invariant.
- Given $\sigma \in S^1$, equation (2.1), satisfied by ψ is called the *eikonal equation*. As explained in [11, Section 3], it has a unique solution satisfying the properties required in Definition 2.3. The statement about the eikonal equation is not usually stated as part of the local Weyl law but the function ψ provided by the theorem does satisfy this property and it will be useful in our proofs.
- The case where $P = \mathrm{Id}$ and was proved by Hörmander in [11]. The case where $x = y$ and $\mathcal{X}$ is a closed manifold was treated in [23] with some restrictions on P. Finally, Gayet and Welschinger extended this result to a general P (see [7, Theorem 2.3]) on a closed manifold. While in their statement, $x = y$, their proof yields the off-diagonal case with only minor modifications.
- Hörmander manages to lift the compactness assumption using results on the local nature of the propagator $e^{itA^{1/m}}$, in the case of $P = \mathrm{Id}$. We believe that similar arguments would work for general P but focus on the compact case for simplicity. Note however that the only input from Hörmander's work used here is Theorem 2.4. Obtaining such a result in the non-compact case with the appropriate functional analysis setting would immediately extend our results.

- One recent result closely related to this theorem is Canzani and Hanin's asymptotics for the monochromatic spectral projector of the Laplacian under some dynamical assumption on the geodesic flow (see [2] and [3]).

For the convenience of the reader, in Appendix A we provide a proof of the full result relying on the wave kernel asymptotics provided in [11].

2.3. Weighted local Weyl laws

In the present article, we generalize Theorem 2.4 in the following way. Consider A and P as in Theorem 2.4 and take U, and ψ as provided by this theorem. Recall that in the coordinates on U, the measure $\widetilde{\mathrm{d}\mu_{\mathcal{X}}(x)\mathrm{d}_x\mu_{\mathcal{X}}}(\xi)$ agrees with the Lebesgue measure $\mathrm{d}x\mathrm{d}\xi$ on $\mathbb{R}^{2n}$. In particular, Theorems 2.6 and 2.8 below hold in both settings presented in Section 2.1. We stress that in Setting 2, however, $\mathcal{X}$ denotes an bounded open subset of $\mathbb{R}^n$, so the results only hold away from the boundary $\partial\mathcal{X}$.

We derive an asymptotic expression for the kernel of the operator $(A\Pi_L)^z$, as $L \to +\infty$ for z in a certain half-plane of $\mathbb{C}$. More precisely, we always assume that $n + d + mz_1 \geqslant 0$. This generalized Hörmander's result which corresponds to $z = 0$.

Interestingly, the behavior will differ between the case $n + d + mz_1 > 0$, which we call *subcritical*, and the case where $n + d + mz = 0$ which we call *critical*. We do not cover the case $\{n + d + mz_1 = 0,\ z_2 \neq 0\}$. In particular, this distinction, and the fact that it depends on d, means that a kernel wich corresponds to the critical case for $d = 0$, is subcritical for $d > 0$. In particular, this kernel and its derivatives have very different behaviors when $L \to +\infty$.

Theorem 2.6 (Generalized Weyl law: subcritical case). — *Fix $z = z_1 + iz_2 \in \mathbb{C}$. Let $f \in C^\infty(\mathbb{R})$ such that $f(t) = t^z$ for t large enough. Let K_L be the Schwartz kernel of $\Pi_L f(A)$. Suppose that $n + d + mz_1 > 0$. For each $x, y \in U$ and $L \geqslant 1$, let*

$$R_L(x,y) = L^{-z_1-(n+d)/m}\Bigg[PK_L(x,y) \\ - \frac{1}{(2\pi)^n}\int_{\sigma_A(y,\xi)\leqslant 1} e^{i\langle\xi,x-y\rangle L^{1/m}}\sigma_A(y,\xi)^z\sigma_P(\xi,-\xi)\mathrm{d}\xi\Bigg].$$

Then, for each open neighborhood V of $0 \in U$ such that $\overline{V} \subset U$ is compact, the following holds.

(1) *Uniformly for $L \geqslant 1$, $x \in V$ and $X, Y \in \mathbb{R}^n$ such that $x + L^{-1/m}X$, $x + L^{-1/m}Y \in V$*

$$R_L\Big(x + L^{-1/m}X, x + L^{-1/m}Y\Big) = O\Big(L^{-z_1-(n+d)/m} + L^{-1/m}\ln(L)^\eta\Big)$$

where $\eta = 1$ if $n + d + mz = 1$ and 0 otherwise.

(2) *Let $W \subset U \times U$ be a neighborhood of the diagonal. Uniformly for $L \geqslant 1$ and $(x, y) \in V \times V \setminus W$,*

$$PK_L(x, y) = O\Big(L^{-z_1-(n+d)/m} + L^{z_1+(n+d-1)/m}\ln(L)^\eta\Big)$$

where $\eta = 1$ if $n + d + mz = 1$ and 0 otherwise.

The two error terms appearing in both points of the theorem are analogous to those in Theorem 1.1. Below said theorem, we discuss the interpretation and relative size of these error terms. We prove Theorem 2.6 in Section 5. As we will see below, Theorem 1.1 and the first assertion of Theorem 1.2 are both direct consequences of Theorem 2.6. Before stating Theorem 2.8 below, we must introduce some more terminology. One key ingredient of the proof will be the decay of certain oscillatory integrals depending on the level sets of σ_A. To observe this behavior we must impose certain condition on σ_A. This is the object of Definition 2.7.

Definition 2.7 (Admissible homogeneous symbols).

- *Fix $n \in \mathbb{N}$, $n \geqslant 1$ and $m \in \,]0; +\infty[$. For each $U \subset \mathbb{R}^n$, let $S_h^m(U) \subset C^\infty(U \times (\mathbb{R}^n \setminus \{0\}))$ be the set of smooth functions m-homogeneous in the second variable. We write $S_{h,+}^m(U)$ for the set of positive valued functions in $S_h^m(U)$. The map*

$$S_h^m(U) \to C^\infty(U \times S^{n-1}),$$

restricting the second variable to the unit sphere, is a bijection. We endow $S_h^m(U)$ with the topology induced by the Whitney topology on $C^\infty(U \times S^{n-1})$ (see Definition 3.1 of Chapter II of [9]). Although elements of S_h^m are not smooth at $U \times \{0\}$ we call them homogeneous symbols since they coincide with symbols outside of a small neighborhood of $U \times \{0\}$.

- *Fix $m \in \mathbb{R}$, $m > 0$ and $k_0 \in \mathbb{N}$, $k_0 \geqslant 2$. We say that a function $\sigma \in S_{h,+}^m(U)$ is k_0-admissible if*

$$\forall\,(x, \xi) \in U \times (\mathbb{R}^n \setminus \{0\}),\ \exists\, k \in \{2, \dots, k_0\},$$
$$\sigma(x, \xi)^{k-1}\partial_\xi^k \sigma(x, \xi) \neq \frac{m(m-1)\dots(m-k+1)}{m^k}(\partial_\xi \sigma(x, \xi))^{\otimes k}. \quad (2.2)$$

This condition is invariant if we see σ as a function on $T^*\mathcal{X}$ because coordinate changes act linearly on the fibers of $T^*\mathcal{X}$. It is stable and generic for k_0 large enough, as explained in Proposition 7.6.

Theorem 2.8 (Generalized Weyl law: critical case). — *We use the same notations as in Theorem 2.6. Suppose that $n+d+mz=0$ and that either $n=1$ or σ_A is k_0-admissible for some $k_0 \geqslant 2$. For each $x, y \in U$ let*

$$Y_P(y) = \int_{S_y^A} \sigma_P\left(\xi, -\xi\right) \mathrm{d}_y\nu(\xi)\,.$$

Then, there exists $V \subset U$ an open neighborhood of 0 such that the following holds.

(1) *Uniformly for $(x,y) \in V \times V$ and $L \geqslant 1$,*

$$PK_L(x,y) = \frac{1}{(2\pi)^n} Y_P(y)\Big[\ln\big(L^{1/m}\big) - \ln_+\big(L^{1/m}|x-y|\big)\Big] + O(1)\,.$$

(2) *There exists $Q \in L^\infty(V \times V)$ such that, uniformly for $\kappa \geqslant 1$, $L \geqslant 1$ and $(x,y) \in V \times V$ such that $|x-y| \geqslant \kappa L^{-1/m}$,*

$$PK_L(x,y) = -\frac{1}{(2\pi)^n} Y_P(y) \ln(|x-y|) + Q(x,y) + O\Big(\kappa^{-1/k_0}\Big)\,.$$

Here, if $n=1$ we set $k_0=1$.

We prove Theorem 2.8 in Section 6. As we will see below, the second point of Theorem 1.2 follows directly from this theorem.

Remark 2.9.

- The critical case, $n+d+mz=0$ depends not only on the order m of the operator A or the power z, but also on the order d of P. In particular, if z is critical for A with $P=\mathrm{Id}$, it becomes sub-critical for any choice of differential operator P of positive order.
- The quantity $Y_P(y)$ is smooth in y by definition, but it is obviously not symmetric in x and y, even for $P=\mathrm{Id}$. This may seem counter-intuitive. Notice however that, in the main term of the expression of PK_L given in Theorem 2.8, replacing $Y_P(y)$ by $Y_P(x)$ only yields an error of order $O(1)$. In the first point, this does not change the estimate. In the second point, replacing $Y_P(y)$ by $Y_P(x)$ would simply amount to adding $\frac{1}{2\pi}\left(Y_P(y) - Y_P(x)\right)\ln(|x-y|)$ to $Q(x,y)$, which is bounded.
- From (1.1) we get the following alternate expression for Y_P:

$$Y_P(y) = (n+d)\int_{\{\sigma_A(y,\xi)\leqslant 1\}} \sigma_P(\xi,-\xi)\mathrm{d}\xi\,.$$

Here we are assuming, as in the rest of the subsection, that $\widetilde{\mathrm{d}_x \mu_{\mathcal{X}}}(\xi)$ agrees with the Lebesgue measure $\mathrm{d}\xi$ on $\mathbb{R}^n$.

Remark 2.10. — The admissibility condition on the symbol of A may appear to be unfamiliar. However, in practice, it is often satisfied. Here are two important examples of families of admissible homogeneous symbols:

- If $n \geqslant 2$ and the level sets $S_x^A = \{\xi \in \mathbb{R}^n : \sigma_A(x,\xi) = 1\}$ are strictly convex, $\partial_\xi^2 \sigma_A$ is positive when restricted to their tangent spaces. Therefore, it cannot be a multiple of $(\partial_\xi \sigma)^{\otimes 2}$ so Theorem 2.8 applies with $k_0 = 2$.
- If σ_A is a positive homogeneous polynomial of degree $m \in \mathbb{N}$ in ξ, $m \geqslant 1$, then it is m-admissible. Indeed, otherwise, taking $k = k_0 = m$, we would have, for some $(x,\xi) \in U \times (\mathbb{R}^n \setminus \{0\})$, $\sigma_A(x,\xi)^{m-1} \partial_\xi^m \sigma_A(x,\xi) = 0$. But since $\sigma_A(x,\xi) > 0$ we have $\partial_\xi^m \sigma_A(x,\xi) = 0$ which implies that all the coefficients of $\sigma_A(x,\cdot\,)$ vanish. This contradicts the fact that $\sigma_A(x,\xi) > 0$. In particular, Theorem 2.8 applies for all differential operators, and in particular to the situation of Theorem 1.2.

As we shall see in Section 7, admissibility is equivalent to the fact that there is no linear map $\mathbb{R}^n \to \mathbb{R}$ whose restriction to S_x^A has a critical point of infinite order. For instance, it rules out the case where S_x^A contains an open subset of an affine hyperplane.

In addition to the two examples of the last remark, we prove the following theorem.

Theorem 2.11 (Admissible homogeneous symbols are generic). — *Fix $n \in \mathbb{N}$, $n \geqslant 2$ and let $U \subset \mathbb{R}^n$ be an open subset. There exists $k_0 = k_0(n) \in \mathbb{N}$ such that for each $m > 0$, the set of elements of $S^m_{h,+}(U)$ that are k_0-admissible (see Definition 2.7), is open and dense in $S^m_{h,+}(U)$.*

Theorem 2.11 follows immediately from Proposition 7.6, which is proved Section 7.2. The integer k_0 is explicit (see Proposition 7.6).

Finally, though we do not use this in the proof of Theorems 1.1 and 1.2, we prove the following result, which might be useful in further applications.

Theorem 2.12 (Super-critical case). — *We use the same notations as in Theorem 2.6. Suppose that $n + d + mz_1 < 0$. Then, there exists and a function $K_\infty \in C^d(U \times U)$ such that the following holds. For each compact subset $\Omega \subset U \times U$, uniformly for $(x,y) \in \Omega$,*

$$PK_L(x,y) = PK_\infty(x,y) + O\Big(L^{z_1 + (n+d)/m}\Big).$$

Remark 2.13. — In Theorems 2.6, 2.8 and 2.12, the setting provided in Section 2.1 only comes into play through Theorem 2.4. Therefore, if one could weaken the hypotheses for this theorem, one would automatically extend Theorems 2.6 and 2.8 as a corollary. In particular, since Hörmander proves Theorem 2.4 for $P = \mathrm{Id}$ without any compactness assumption or boundary condition, both of these results remain valid in this case.

Let us check that Theorems 2.6 and 2.8 imply the results presented in the introduction.

Proof of Theorem 1.1. — Both results follow from Theorem 2.6 applied to the first setting of Section 2.1 with $z = -s$ by taking $P = \partial_x^\alpha \partial_y^\beta$. In this case, the order of P is $d = |\alpha| + |\beta|$ and we have, for any $\xi \in \mathbb{R}^n$,

$$\sigma_P(\xi, -\xi) = (i\xi)^\alpha (-i\xi)^\beta \,. \qquad \square$$

Proof of Theorem 1.2. — Set $z = -s = -n/m$. For the first part, set $P = \partial_x^\alpha \partial_y^\beta$ near 0 and proceed as in the proof of Theorem 1.1. Indeed, since $(\alpha, \beta) \neq (0,0)$, we have $n + d + mz_1 = |\alpha| + |\beta| > 0$. For the second part, since by Remark 2.10 σ_A is m-admissible, and since $n + d + mz_1 = 0$, we apply Theorem 2.8 instead. In our case, $P = \mathrm{Id}$ so for each $x, y \in U$, $Y_P(y) = \nu_y\left(S_y^A\right)$ so

$$K_L^s(x,y) = \frac{1}{(2\pi)^n} \nu_y\left(S_y^A\right)\left[\ln\!\left(L^{1/m}\right) - \ln_+\!\left(L^{1/m}|x-y|\right)\right] + O(1)\,.$$

By (1.2), $\nu_y(S_y^A) = n|\{\sigma_A(y,\xi) \leqslant 1\}|$. But since $K_L^s(x,y) = \overline{K_L^s(y,x)}$, we may replace $\nu_y(S_y^A) = n|\{\sigma_A(y,\xi) \leqslant 1\}|$ in the above expression by $\frac{n|\{\sigma_A(x,\xi)\leqslant 1\}| + n|\{\sigma_A(y,\xi)\leqslant 1\}|}{2}$ as announced[(7)]. $\square$

Proof of Corollary 1.3. — We use the notations of Theorem 2.8. First of all, by definition, as $L \to +\infty$, $K_L^s \to K$ in distribution. Moreover, by Theorem 1.2, any point in $\mathcal{X}$ has a neighborhood V such that the sequence $(K_L^s)_{L \geqslant 1}$ is uniformly bounded on $V \times V$ by a locally integrable function and converge pointwise towards $-g_A(x,y)\ln(|x-y|) + Q(x,y)$ where $Q \in L^\infty(V \times V)$ on the complement of the diagonal in $V \times V$. In particular, they converge in distribution to this function. This implies that when restricted to $C^\infty(V \times V)$,

$$K(x,y) = -g_A(x,y)\ln(|x-y|) + Q(x,y)\,.$$

Now, given any smooth distance d on $\mathcal{X}$, for each x, y distinct,

$$\ln(|x-y|) = \ln(d(x,y)) + \ln\!\left(\frac{|x-y|}{d(x,y)}\right)$$

[(7)] As explained in Remark 2.9, replacing the former by the latter in the expression of $K_L^s(x,y)$ creates a change of order $O(1)$.

and the second term is bounded so, on $V \times V$,

$$K(x,y) = -g_A(x,y)\ln(d(x,y)) + Q_A(x,y)$$

for some $Q_A \in L^\infty(V \times V)$. But K is the integral kernel of A^s which is a self-adjoint pseudo-differential operator (see [26] or [14, Proposition 29.1.9]). In particular, it is smooth and symmetric outside the diagonal (see for instance [13, Theorem 18.1.16]). Hence, Q_A must also be symmetric and smooth outside the diagonal. □

Remark 2.14. — Theorems 2.6 and 2.8 hold in both Setting 1 and Setting 2 of Section 2.1. Since the proofs of Theorems 1.1 and 1.2 and Corollary 1.3 are purely local once one admits Theorems 2.6 and 2.8, these results also hold in Setting 2.

3. Heuristics and proof outline

In this section we provide a heuristic justification for Theorems 2.6 and 2.8 and an outline of the skeleton of the proof. At the end of this section, we also provide a proof map to highlight the dependencies between intermediate results leading to the proofs of Theorems 2.6, 2.8 and 2.12, see Figure 3.1.

3.1. Heuristics

In order to get a sense of the kind of calculations we will carry out in the rest of the article, let us present a simple example, with few non-rigorous steps in order to shorten the argument. We assume that $\mathcal{X}$ is a closed Riemannian manifold and that A denotes the associated Laplacian(8). Then, A is indeed elliptic of order $m = 2$ and self-adjoint with respect to the Riemannian volume density $\mathrm{d}\mu$. Moreover the symbol of A is $\sigma_A(x,\xi) = \|\xi\|_x^2$ where $\|\cdot\|_x$ is the norm induced by the Riemannian metric on $T_x^*\mathcal{X}$. Thus, in orthonormal coordinates $S_x^A = S^{n-1}$. Finally, we take $P = \mathrm{Id}$. Now, if $s \leqslant \frac{n}{2}$,

$$\begin{aligned} K_L^s(x,y) &= \int_1^L \lambda^{-s} E'_\lambda(x,y)\mathrm{d}\lambda + O(1) \\ &= L^{-s}E_L(x,y) + \int_1^L s\lambda^{-s-1}E_\lambda(x,y)\mathrm{d}\lambda + O(1)\,. \quad (3.1) \end{aligned}$$

(8) Here we use the convention that $\Delta = -\operatorname{div}\circ\nabla$ so that the operator is non-negative.

Here, we artificially cut-off the first eigenvalues since, in the sum defining K_L^s, they contribute by a term independent of L. By Theorem 2.4,

$$E_L(x,y) = \frac{1}{(2\pi)^n}\int_{|\xi|^2\leqslant L} e^{i\psi(x,y,\xi)}\mathrm{d}\xi + O(L^{(n-1)/2}) \tag{3.2}$$

where ψ is a phase function. To simplify the computations[9], we take

$$\psi(x,y,\xi) = \langle x-y,\xi\rangle\,. \tag{3.3}$$

Replacing E_λ in Equation (3.1) the expression given by Equations (3.2) and (3.3) we get

$$\begin{aligned} K_L^s(x,y) = \frac{1}{(2\pi)^n}\Bigg[L^{-s}\int_{\|\xi\|^2\leqslant L} e^{i\langle x-y,\xi\rangle}\mathrm{d}\xi \\ + \int_1^L s\lambda^{-s-1}\int_{\|\xi\|^2\leqslant\lambda} e^{i\langle x-y,\xi\rangle}\mathrm{d}\xi\mathrm{d}\lambda\Bigg] + O(L^{(n-1)/2-s})\,. \end{aligned}$$

Integrating by parts in λ, the boundary term $\lambda = L$ cancels out the first term. The case $\lambda = 1$ is independent of L and can be absorbed in an $O(1)$ error. We obtain

$$K_L^s(x,y) = \frac{1}{(2\pi)^n}\int_1^L (1/2)\lambda^{(n-2)/2-s}J(|x-y|\lambda^{1/2})\mathrm{d}\lambda + O(1+L^{(n-1)/2-s})$$

$$\text{where } J(t) = \int_{S^{n-1}} e^{it\omega_1}\mathrm{d}\omega\,.$$

Here the radius of the ball $\{\|\xi\|^2\leqslant\lambda\}$ is $\lambda^{1/2}$ which accounts for a factor of $(1/2)\lambda^{-1/2}$ in the resulting expression. Note that $J(t)=(t)^{-(n-2)/2}J_{(n-2)/2}(t)$ where J_ν is the ν-th Bessel function of the first kind (see e.g. [8, p. 198]). Making the change of variables $\eta = \lambda^{1/2}$ yields

$$K_L^s(x,y) = \frac{1}{(2\pi)^n}\int_1^{L^{1/2}} \eta^{n-1-2s}J(|x-y|\eta)\mathrm{d}\eta + O(1+L^{(n-1)/2-s})\,. \tag{3.4}$$

Here we see the two error terms from Theorem 1.1 appear. The 1 comes from the low frequency region while the $L^{(n-1)/2-s}$ comes from the high frequency cut-off. At this point we distinguish between $s<\frac{n}{2}$ and $s=\frac{n}{2}$. In the former case, assume that $|x-y| = hL^{-1/2}$ for some fixed h. This new information on s implies that the integrand in (3.4) is integrable at zero so

(9) As explained in the following section, it is not obvious at all that this approximation is valid. Suffice to say for now that our estimates are interesting near the diagonal, around which ψ shares many properties with the function $\langle x-y,\xi\rangle$ by Definition 2.3.

we can extend the integral up to an $O(1)$ error. We obtain

$$\begin{aligned}
K^s_L(x,y) &= \frac{1}{(2\pi)^n}\int_0^{L^{1/2}} \eta^{n-1-2s} J(hL^{-1/2}\eta)\mathrm{d}\eta + O(1+L^{(n-1)/2-s})\\
&= \frac{1}{(2\pi)^n}\int_0^{1} \widetilde{\eta}^{n-1-2s} J(|h|\widetilde{\eta})\mathrm{d}\widetilde{\eta} \times L^{n/2-s} + O(1+L^{(n-1)/2-s})\\
&= \frac{1}{(2\pi)^n}\int_{\|\xi\|^2\leqslant 1} e^{i\langle h,\xi\rangle}\mathrm{d}\xi \times L^{n/2-s} + O(1+L^{(n-1)/2-s})\,.
\end{aligned}$$

This is the conclusion of Theorem 1.1 for $A=\Delta$ and $P=\mathrm{Id}$.

Assume now that $s=\frac{n}{2}$ and $x\neq y$. Starting off from Equation (3.4) and writing $\zeta = |x-y|\eta$,

$$K^s_L(x,y) \sim \frac{1}{(2\pi)^n}\int_{|x-y|}^{|x-y|L^{1/2}} \zeta^{-1} J(\zeta)\mathrm{d}\zeta\,. \tag{3.5}$$

Note that, by the stationary phase method (or standard Bessel function estimates),

$$J(t) = O(t^{-(n-1)/2}) \tag{3.6}$$

for $t\geqslant 1$ and $J(0)=|S^{n-1}|$. Therefore, in dimensions $n\geqslant 2$, the integrand is L^1 away from zero, and equivalent to $|S^{n-1}|\zeta^{-1}$ at 0. This crucial observation basically allows us to replace expression (3.5) by the following ansatz, allowing for $O(1)$ errors:

$$\begin{aligned}
K^s_L(x,y) &= \frac{|S^{n-1}|}{(2\pi)^n}\int_{|x-y|}^{|x-y|L^{1/2}} \mathbb{1}\big[\zeta\leqslant 1\big]\frac{\mathrm{d}\zeta}{\zeta} + O(1)\\
&= \frac{|S^{n-1}|}{(2\pi)^n}\Big[\ln\big(L^{1/2}\big) - \ln_+\big(L^{1/2}|x-y|\big)\Big] + O(1)\,.
\end{aligned}$$

This is the essential statement of Theorem 1.2. The case $n=1$ is similar in spirit but requires a trick to replace the stationary phase method.

3.2. Proof strategy

There are two main obstacles to carry out the above calculation rigorously in the general case and Sections 4 and 7 are devoted to dealing with them. The first is to justify Equation (3.3). This is the role of Lemmas 4.1, 4.2 and 4.3 that roughly state that ψ behaves like $\langle x-y,\xi\rangle$. The second difficulty is to obtain an analog of Equation (3.6) when S^{n-1} is replaced by $S^A_x = \{\xi\,,\,\sigma_A(x,\xi)=1\}$ for a general symbol σ_A. Indeed, in this case, the standard stationary phase method need not apply and we must use more general results on oscillatory integrals. This requires the assumption that

σ_A be admissible (see Definition 2.7). To make this point more precise, let us introduce some notation.

As in the previous section, we fix once and for all a point in $\mathcal{X}$ and consider a local chart centered at this point defined on $U \subset \mathbb{R}^n$ given by Theorem 2.4. We also take P with principal symbol σ_P, $W \subset U \times U$ and $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$ as in this theorem.

The following quantity will be central in our proofs. For any $t > 0$, $x, y \in U$ and $\xi \in \mathbb{R}^n$ let

$$H_P(x, y, \xi, t) = e^{i\psi(x,y,t\xi)}\sigma_P\left(t^{-1}\partial_{x,y}\psi(x, y, t\xi)\right) \tag{3.7}$$

and

$$J_A(x, y, t) = \int_{S_y^A} H_P(x, y, \xi, t)\mathrm{d}_y\nu(\xi). \tag{3.8}$$

The function $J_A(x, y, t)$ specializes to the function J of the previous subsection when $\sigma_P = 1$, $\psi(x, y, \xi) = \langle x - y, \xi\rangle$ and $S_y^A = S^{n-1}$. Since σ_P is d-homogeneous, H_P satisfies the following equation. For any $s, t > 0$, $x, y \in U$ and $\xi \in \mathbb{R}^n$,

$$H_P(x, y, s\xi, t) = s^d H_P(x, y, \xi, st). \tag{3.9}$$

We will prove the following proposition.

Proposition 3.1 (Decay of J_A). — *Assume that σ_A is k_0-admissible. Then, there exists $V \subset U$ an open neighborhood of 0 and $C < +\infty$ such that, uniformly for distinct $x, y \in V$ and $t > 0$*

$$|J_A(x, y, t)| \leqslant C(t|x - y|)^{-\frac{1}{k_0}}.$$

The proof of Proposition 3.1 is divided into two steps. First, we will prove that the admissibility condition on σ_A implies a property governing the decay of certain oscillatory integrals over the level sets of σ_A that we define below in Definition 3.2. Next, we prove that this property implies the required behavior of J_A. More precisely, we introduce the following terminology.

Definition 3.2 (Deformations of height functions and non-degenerate level sets). — *Let $\varepsilon > 0$, $m > 0$, $E \subset \mathbb{R}^p$ a neighborhood of 0 and let $U \subset \mathbb{R}^n$ be an open subset. Let $\sigma \in C^\infty(U \times \mathbb{R}^n \setminus \{0\})$ be homogeneous of degree m in the second variable. For each $x \in U$ let $S_x^A = \{\xi \in \mathbb{R}^n \mid \sigma(x, \xi) = 1\}$ and $\mathrm{d}_x\mu$ be the area measure on S_x^A. Let $S^*U = \{(x, \xi) \in U \times \mathbb{R}^n \mid \xi \in S_x^A\}$.*

(1) *Given a compact subset $\Omega \subset U \times (\mathbb{R}^n \setminus \{0\})$ let $X = \{(x, \tau, \xi) \in \Omega \times \mathbb{R}^n \mid \xi \in S_x^A\}$. We call a* deformation of the height function for σ over Ω *any family $(f_\eta)_{\eta\in E}$ of continuous, real-valued functions on X, smooth in the third variable ξ, with the following properties:*
   - *for each $(x, \tau, \xi) \in \Omega \times \mathbb{R}^n$ such that $\xi \in S_x^A$, $f_0(x, \tau, \xi) = \langle\tau, \xi\rangle$*

- *for each $\alpha \in \mathbb{N}^n$, the map $\eta \mapsto \partial_\xi^\alpha f_\eta$ is continuous for the topology of uniform convergence on compact sets.*

(2) *We say that σ has* ε-non-degenerate level sets *if, for any compact subset Ω of $U \times (\mathbb{R}^n \setminus \{0\})$ and any deformation of the height function $(f_\eta)_\eta$ for σ over Ω there exists $V \subset \mathbb{R}^p$ a neighborhood of 0 depending only on Ω (and ε) such that for each $\gamma \in C^\infty(S^*U)$ and each continuous family of smooth functions on $(u_\eta)_\eta \in (C^\infty(\mathbb{R}^n))^E$, there exists $C < +\infty$ such that for each $\eta \in V$, each $(x, \tau) \in \Omega$ and each $\lambda > 0$,*

$$\left| \int_{S_x^A} e^{i\lambda f_\eta(x,\tau,\xi)} u_\eta(\xi)\gamma(x,\xi) \mathrm{d}_x\nu(\xi) \right| \leqslant C\lambda^{-\varepsilon}. \tag{3.10}$$

We say that σ has non-degenerate level sets *if it has ε-non-degenerate level sets for some $\varepsilon > 0$.*

(3) *Let $\varepsilon > 0$. We say that a homogeneous symbol on a manifold has non-degenerate (resp. ε-non-degenerate) level sets if it has this property when written in any local coordinate system.*

Remark 3.3.

- In Definition 3.2, Equation (3.10) will be used to bound $J_A(x, y, t)$ for the proof of Proposition 3.1.
- Since, by Definition 2.3, as in Section 3.1, we want to take "$\psi(x, y, \xi) = \langle x - y, \xi \rangle$", we will see the former as a deformation of the latter. In other words, ψ will be realized as a deformation of the height function.
- The notion of ε-non-degenerate level sets expresses the "non-flatness" of these level sets. Indeed, for instance, in the worse case scenario, if S_x^A contains an open subset of an affine hyperplane, say of the form $\{x_1 = h\}$ for some fixed $h \in \mathbb{R}$, and is otherwise strictly convex, the following integral does not decay to zero as $\lambda \to +\infty$

$$\int_{S_x^A} e^{i\lambda\xi_1} \mathrm{d}_x\nu(\xi).$$

Proposition 3.1 will then be a consequence of the following results. Recall the definition of admissible homogeneous symbols (Definition 2.7). On the one hand, we will prove:

Proposition 3.4 (Admissible homogeneous symbols have non-degenerate level sets). — *Fix $n, k_0 \in \mathbb{N}$, $n \geqslant 1$, $k_0 \geqslant 2$. Let $U \subset \mathbb{R}^n$ be an open subset and let $\sigma \in S_{h,+}^m(U)$. If σ is k_0-admissible, then it has $\frac{1}{k_0}$-non-degenerate level sets.*

The proof of this proposition, which is presented in Section 7.1, is entirely independent of the rest of the present text and uses different techniques. It is followed by Section 7.2, in which we prove that the admissibility condition is generic in a suitable sense.

On the other hand, in Section 4.2, we will prove the following result.

Lemma 3.5. — *Fix $\varepsilon > 0$. Suppose that the principal symbol σ_A has ε-non-degenerate level sets (see Definition 3.2). Then, there exists $V \subset U$ an open neighborhood of 0 and $C < +\infty$ such that, uniformly for distinct $x, y \in V$ and $t > 0$*

$$|J_A(x,y,t)| \leqslant C(t|x-y|)^{-\varepsilon}\,.$$

Remark 3.6.

- Lemma 3.5 corresponds to Proposition 23 of [22] for $\varepsilon = \frac{1}{2}$ although, in that setting, the non-degeneracy condition was always satisfied.
- In the one dimensional case, Lemma 3.5 is replaced by Lemma 4.5.
- The proof of Lemma 3.5 relies on a control of $t^{-1}\partial_{x,y}\psi(x,y,t\xi)$ uniform in $t \in \,]0,+\infty[$. These estimates are carried out in Section 4.1. We deal with the region $t \ll 1$ by hand. The region $t \gg 1$ comes by assumption in ψ from the fourth point of Definition 2.3.

After proving all of these results, we carry out the calculation sketched in Section 3.1 in Sections 5 and 6 as we will now explain in more detail. We therefore suggest that the reader have Section 3.1 in mind for what follows. The integration by parts is of course valid in a general setting. This allows us to obtain an expression like Equation (3.1) where the map λ^{-s} is replaced by $f(\lambda)$ for some adequate function f. More explicitely, in Section 5, we derive the following result. We start by introducing a suitable function $f : \,]0,+\infty[\, \mapsto \mathbb{C}$ and studying the asymptotics of the following kernel:

$$K_L^f : (x,y) \longmapsto \sum_{0<\lambda_k\leqslant L} f(\lambda_k) e_k(x)\overline{e_k}(y)\,.$$

This is again a smooth function. Since all of our results are local, we fix once and for all a point in $\mathcal{X}$ and consider $x = (x_1, \dots, x_n)$ the local coordinate system in which this point has coordinates $x = (0, \dots, 0)$, provided by Theorem 2.4 and defined on an open neighborhood U of 0 in $\mathbb{R}^n$. Recall that in Theorem 2.4, W was a neighborhood of the diagonal. With some abuse of notation, we will write W for this neighborhood read in the present chart so that $W \subset U \times U$.

Proposition 3.7 (Integral expression for PK_L^f). — *Take $f : \mathbb{R} \to \mathbb{C}$, continuously differentiable, with support in $]0, +\infty[$. Then, in local coordinates, uniformly for each $x, y \in U$, for each $L > 0$,*

$$PK_L^f(x,y) = \frac{1}{(2\pi)^n} \int_{\sigma_A(y,\xi) \leqslant L} e^{i\psi(x,y,\xi)} \sigma_P(\partial_{x,y}\psi(x,y,\xi)) f(\sigma_A(y,\xi)) \mathrm{d}\xi \\ + O\Big(f(L) L^{(n+d-1)/m}\Big) + O\left(\int_0^L f'(\lambda) \lambda^{(n+d-1)/m} \mathrm{d}\lambda\right).$$

In addition, if $W \subset U \times U$ is a neighborhood of the diagonal, then, uniformly for any $(x,y) \in (U \times U) \setminus W$, for each $L \geqslant 1$,

$$PK_L^f(x,y) = O\Big(f(L) L^{(n+d-1)/m}\Big) + O\left(\int_0^L f'(\lambda) \lambda^{(n+d-1)/m} \mathrm{d}\lambda\right).$$

Finally, the constants implied by the O's do not depend on f.

We prove Proposition 3.7 in Section 5. Then, we consider the case where f is of the form $f(t) = \chi(t) t^z$ where $z = z_1 + iz_2 \in \mathbb{C}$ and χ is some smooth function with support in $]0, +\infty[$ equal to 1 for t large enough. In Section 5, we prove Theorem 2.12 using only a crude estimate from Theorem 2.4, and we also deduce Theorem 2.6 from Proposition 3.7 and results from Section 4. Next, in Section 6 we prove Theorem 2.8 using again Proposition 3.7 but also Proposition 3.1. We end this section with a diagram detailing the dependencies between different results involved in the proofs of Theorems 2.6, 2.8 and 2.12.

4. Preliminary results

As before, in this section we fix once and for all a point in $\mathcal{X}$ and consider a local chart centered at this point defined on $U \subset \mathbb{R}^n$ given by Theorem 2.4. We also take P with principal symbol σ_P, $W \subset U \times U$ and $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$ as in this theorem. The object of this section is to estimate the behavior of the phase ψ near the diagonal and to prove Lemma 4.3.

4.1. Basic properties of the phase ψ

The phase ψ from Theorem 2.4 will frequently appear in the calculations below. We begin by deducing a list of properties of ψ from those given in Definition 2.3. We gather these properties in Lemma 4.1. It is easy to check that all these properties are satisfied by the function $\psi(x,y,\xi) = \langle x - y, \xi \rangle$.

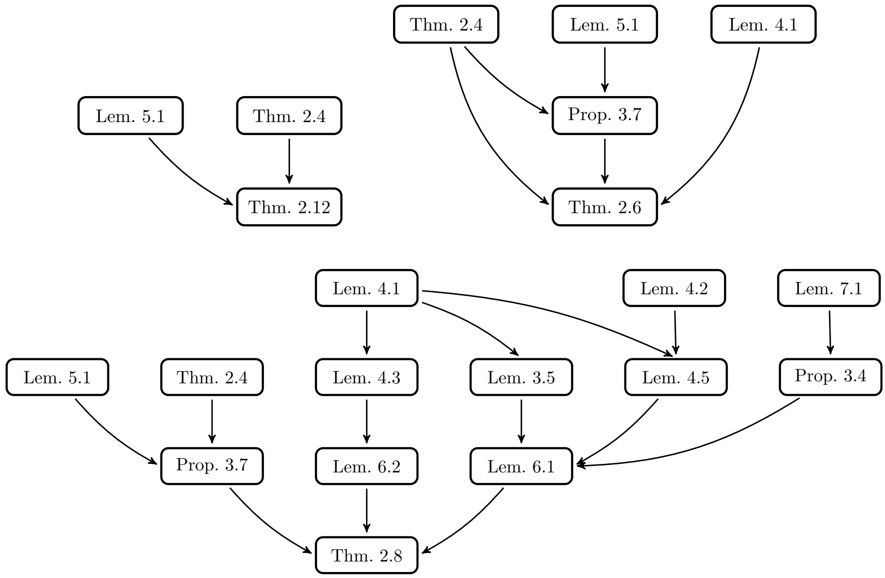

Figure 3.1. A map of the proofs of Theorems 2.6, 2.8 and 2.12. The result at the origin of each arrow is used in the proof of the result at its target.

Next, we present an additional lemma, Lemma 4.2, for the case $n = 1$. Finally, we use Lemma 4.1 to deduce some properties of the function H_P defined in Equation (3.7). For each $x, y \in U$ and $\xi \in \mathbb{R}^n$, let

$$\psi_0(x,y,\xi) = \partial_\xi \psi(x,y,0)\xi = \sum_{j=1}^{m} \partial_{\xi_j}\psi(x,y,0)\xi_j \,. \tag{4.1}$$

Lemma 4.1. — *Let $U \subset \mathbb{R}^n$ and let $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$ be a proper phase function. For each $t > 0$, let $\psi_t = t^{-1}\psi(\,\cdot\,,\cdot\,,t\,\cdot)$. Then,*

(1) *For each $x, y \in U$ and each $t > 0$, $\psi_t(x,y,0) = 0$.*
(2) *For each $x \in U$, each $t > 0$ and each $\xi \in \mathbb{R}^n$, $\psi_t(x,x,\xi) = 0$.*
(3) *For each $x \in U$, each $t > 0$ and each $\xi \in \mathbb{R}^n$, $\partial_{x,y}\psi_t(x,x,\xi) = (\xi, -\xi)$.*
(4) *The sequence $(\psi_t)_{t>0}$ converges in $C^\infty(U \times U \times \mathbb{R}^n)$ as $t \to 0$ to the function ψ_0 defined in (4.1). In other words, for each compact subset $\Omega \subset U$, each $R < +\infty$ and each $\alpha, \beta, \gamma \in \mathbb{R}^n$,*

$$\lim_{t\to 0} \sup_{x,y\in\Omega,\ |\xi|\leqslant R} \left|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi_t(x,y,\xi) - \partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi_0(x,y,\xi)\right| = 0.$$

(5) *The sequence $(\psi_t)_{t\geqslant 0}$ is bounded in $C^\infty(U \times U \times \mathbb{R}^n)$.*

Proof of Lemma 4.1. — Let $t > 0$, $x, y \in U$ and $\xi \in \mathbb{R}^n$. Then, $\langle x - y, 0\rangle = \langle x - x, t\xi\rangle = 0$ so $\psi_t(x, x, \xi) = \psi_t(x, y, 0) = 0$ by the second point of Definition 2.3. This proves the first two points of Lemma 4.1. By point 3 of Definition 2.3, for each $x \in U$ and $\xi \in \mathbb{R}^n$, $\partial_x \psi(x, x, \xi) = \xi$, so $\partial_x \psi_t(x, x, \xi) = t^{-1}(t\xi) = \xi$. Next, by differentiating the following equality

$$\psi_t(x + sv, x + sv, \xi) = 0$$

with respect to $s \in \mathbb{R}$, at $s = 0$, where $x \in U$, $\xi \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, we get

$$\partial_x \psi_t(x, x, \xi) + \partial_y \psi_t(x, x, \xi) = 0.$$

This proves the third point of Lemma 4.1.

To prove the fourth point, first, fix $\beta, \gamma \in \mathbb{N}^n$ and let $\Omega \subset U$ be a compact subset and $R < +\infty$. Then, for each $x, y \in \Omega$ and $\xi \in \mathbb{R}^n$ such that $|\xi| \leqslant R$,

$$\partial_x^\beta \partial_y^\gamma \psi_t(x, y, \xi) = t^{-1} \partial_x^\beta \partial_y^\gamma \psi(x, y, t\xi).$$

By the first point, of Lemma 4.1, $\partial_x^\beta \partial_y^\gamma \psi(x, y, 0) = 0$. We apply Taylor's formula to $t \mapsto \partial_x^\beta \partial_y^\gamma \psi(x, y, t\xi)$ uniformly for $t \leqslant 1$, $x, y \in \Omega$ and $\xi \in \mathbb{R}^n$ such that $|\xi| \leqslant R$ and get

$$t^{-1} \partial_x^\beta \partial_y^\gamma \psi(x, y, t\xi) = 0 + \partial_x^\beta \partial_y^\gamma (\partial_\xi \psi_0(x, y, \xi)) + O(t)\,.$$

In particular, as $t \to 0$, $\partial_x^\beta \partial_y^\gamma \psi_t \to \partial_x^\beta \partial_y^\gamma \psi_0$ uniformly for $x, y \in \Omega$ and $\xi \in \mathbb{R}^n$, $|\xi| \leqslant R$. Next, fix $\alpha \in \mathbb{N}$ and suppose $|\alpha| \geqslant 1$. Then, for each $x, y \in K$, $\xi \in \mathbb{R}^n$, $|\xi| \leqslant R$ and $t > 0$,

$$\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi_t(x, y, \xi) = t^{|\alpha|-1} \partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi(x, y, t\xi).$$

If $|\alpha| = 1$, as $t \to 0$ the right hand side converges uniformly to $\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi(x, y, 0) = \partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma (\partial_\xi \psi_0(x, y, \xi))$. On the other hand, if $|\alpha| > 1$, as $t \to 0$ it converges uniformly to $0 = \partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma (\partial_\xi \psi_0(x, y, \xi))$. This proves the fourth point of Lemma 4.1. Lastly, the family $(\psi_t)_{t>0}$ is obviously continuous into $C^\infty(U \times U \times \mathbb{R}^n)$ for $t > 0$. By the fourth point of Lemma 4.1 we may extend it by continuity to $t = 0$. On the other hand, by the fifth point of Definition 2.3, it also converges as $t \to \infty$. In particular, the family $(\psi_t)_{t \geqslant 0}$ is uniformly bounded in $C^\infty(U \times U \times \mathbb{R}^n)$. This proves the fifth point of Lemma 4.1. □

We use the following lemma to prove Lemma 4.5 below, which is the analog of Proposition 3.1 we use in dimension $n = 1$. It is the only place where we use the fact that ψ satisfies the eikonal equation (2.1).

Lemma 4.2. — *Assume that $n = 1$. For each segment $I \subset U$ there exists $c \in \,]0, +\infty[$ such that for each $x, y \in I$ and $\xi \in \mathbb{R}$, $\frac{1}{c}|x - y| \leqslant |\partial_\xi \psi(x, y, \xi)| \leqslant c|x - y|$ and $|\partial_\xi^2 \psi(x, y, \xi)| \leqslant c|x - y|(1 + |\xi|)^{-1}$.*

Proof of Lemma 4.2. — Let us fix $I \subset U$ a compact interval. Since the symbol σ_A is m-homogeneous and $\dim(\mathcal{X}) = 1$ there exists a positive function $\varrho \in C^\infty(U)$ such that $\sigma_A(x,\xi) = \varrho(x)^m|\xi|^m$ for $\xi \neq 0$ and $x \in U$. By construction of ψ there exist $C_1 < +\infty$ and symbols $\tau \in S^0(U \times \mathbb{R})$ and $\sigma \in S^1(U \times \mathbb{R})$ such that $\sigma(x,\xi) = \varrho(x)|\xi| + \tau(x,\xi)$ for $|\xi| \geqslant C_1$ and $x \in U$ and such that

$$\forall\, \xi \in \mathbb{R} \setminus [-C_1, C_1],\ \forall\, x,y \in U,\ \sigma(x, \partial_x\psi(x,y,\xi)) = \sigma(y,\xi).$$

Since $\tau \in S^0$ and since ϱ, being positive and continuous, is bounded from below on I, there exists $C_2 \in [\max(C_1,1), +\infty[$ such that for any $x \in I$ and $\xi \in \mathbb{R}$ such that $|\xi| \geqslant C_2$,

$$\frac{1}{2}\varrho(x)|\xi| \leqslant \sigma(x,\xi) \leqslant 2\varrho(x)|\xi|\,;$$
$$C_2^{-1} \leqslant \operatorname{sign}(\xi)\partial_\xi\sigma(x,\xi) \leqslant C_2.$$

Let $(\sigma^{-1})(x,\cdot\,)$ be the inverse of $\sigma(x,\cdot\,) : [C_2,+\infty[\, \to [\sigma(x,C_2)+\infty[$. Let us fix $x_0 \in I$. Then, for any $x \in I$,

$$\partial_x\psi(x,x_0,\xi) = (\sigma^{-1})(x, \sigma(x_0,\xi)). \tag{4.2}$$

Differentiating this Equation with respect to ξ we obtain the following expression for $\partial_\xi\partial_x\psi$.

$$\partial_\xi\partial_x\psi(x,x_0,\xi) = \partial_\xi(\sigma^{-1})(x,\sigma(x_0,\xi))\partial_\xi\sigma(x_0,\xi)\,.$$

Now, by definition of σ^{-1}, we have, for $x \in I$ and $\xi \in \mathbb{R}$ such that $\xi \geqslant C_3 = \max_{y\in I}\sigma(y,C_2)$,

$$\partial_\xi(\sigma^{-1})(x,\xi) = \big(\partial_\xi\sigma(x,\sigma^{-1}(x,\xi))\big)^{-1} = \big(\varrho(x) + \partial_\xi\tau(x,\sigma^{-1}(x,\xi))\big)^{-1}\ ,$$

where $\varrho(x)$ is bounded on I from above and below by positive constants and $\partial_\xi\tau(x,\sigma^{-1}(x,\xi))$ is $O(|\sigma^{-1}(x,\xi)|^{-1})$ uniformly for $x \in I$. Since

$$\sigma^{-1}(x,\xi) \xrightarrow[\xi\to+\infty]{} +\infty$$

then there exists $C_4 > 0$ such that for any $x \in I$ and any $\xi \geqslant C_4 \geqslant \max(C_3, C_2)$,

$$C_4^{-1} \leqslant \partial_\xi(\sigma^{-1})(x,\xi) \leqslant C_4. \tag{4.3}$$

Therefore,

$$C_2^{-1}C_4^{-1} \leqslant \partial_x\partial_\xi\psi(x,x_0,\xi) \leqslant C_2C_4\,.$$

Recall that, by the first point of Lemma 4.1, $\psi(x,x,\xi) = 0$ for any $x \in U$ and any $\xi \in \mathbb{R}$. Thus, for any $x \in I$, $\xi \geqslant C_4$,

$$|\partial_\xi\psi(x,x_0,\xi)| = \left|\int_{x_0}^{x} \partial_\xi\partial_x\psi(y,x_0,\xi)\mathrm{d}y\right| \in \big[C_5^{-1}|x-x_0|, C_5|x-x_0|\big]$$

where $C_5 = C_2C_4$ is independent of the choice of x_0. The case where $\xi < 0$ is symmetric and this proves the first identity announced in the lemma. For the second identity, we start by differentiating Equation (4.2) with respect to ξ to obtain

$$\partial_\xi^2\partial_x\psi(x,x_0,\xi)=\partial_\xi^2(\sigma^{-1})(x,\sigma(x_0,\xi))(\partial_\xi\sigma(x_0,\xi))^2 \\ +\partial_\xi(\sigma^{-1})(x,\sigma(x_0,\xi))\sigma_\xi^2(x_0,\xi)\,. \quad (4.4)$$

To deal with the second term of the right hand side, observe that, since σ is a symbol of order one and by Equation (4.3), there exists a constant $C_6<+\infty$ such that for any $x,x_0\in I$ and any $\xi\in\mathbb{R}$,

$$\left|\partial_\xi(\sigma^{-1})(x,\sigma(x_0,\xi))\sigma_\xi^2(x_0,\xi)\right|\leqslant C_6(1+|\xi|)^{-1}. \quad (4.5)$$

For the first term we proceed as follows. By definition of σ^{-1}, we have, for any $x\in I$ and $\xi\geqslant C_3$,

$$\partial_\xi^2\sigma(x,\sigma^{-1}(x,\xi))(\partial_\xi(\sigma^{-1})(x,\xi))^2+\partial_\xi\sigma(x,\sigma^{-1}(x,\xi))\partial_\xi^2(\sigma^{-1})(x,\xi)=0.$$

By Equation (4.3), since σ is a symbol of order one and since $\partial_\xi\sigma$ is bounded from below on $[C_2,+\infty[$, there exists $C_7<+\infty$ such that for each $x,x_0\in I$ and $\xi\geqslant C_3$,

$$\left|\partial_\xi^2(\sigma^{-1})(x,\sigma(x_0,\xi))(\partial_\xi\sigma(x_0,\xi))^2\right|\leqslant C_7(1+|\xi|)^{-1}\,. \quad (4.6)$$

We use Equations (4.5) and (4.6) on the right hand side of Equation (4.4) and get, for each $x,x_0\in I$ and $\xi\geqslant C_3$,

$$\left|\partial_x\partial_\xi^2\psi(x,x_0,\xi)\right|\leqslant(C_6+C_7)(1+|\xi|)^{-1}\,.$$

As before, since for all $x\in I$ and $\xi\in\mathbb{R}$, $\psi(x,x,\xi)=0$, we have

$$\left|\partial_\xi^2\psi(x,x_0,\xi)\right|\leqslant\int_{x_0}^{x}\left|\partial_x\partial_\xi^2\psi(y,x_0,\xi)\right|\mathrm{d}y\leqslant C_8|x-x_0|(1+|\xi|)^{-1}$$

where $C_8=C_6+C_7$. The case $\xi<0$ is symmetric. □

From Lemma 4.1, we deduce the following properties of the function H_P defined in Equation (3.7).

Lemma 4.3. — *The function H_P satisfies the following properties.*

(1) *The function $t\mapsto H_P(\,\cdot\,,\cdot\,,\cdot\,,t)$ extends continuously to $t=0$ as a function from $\mathbb{R}_+$ to $C^\infty(U\times U\times\mathbb{R}^n)$ and*

$$H_P(x,y,\xi,0)=\sigma_P\left(\partial_{x,y}\psi_0(x,y,\xi)\right)$$

where ψ_0 is defined as in Equation (4.1)*.*

(2) *Uniformly for $t\geqslant 0$ and x,y in compact subsets of U and $\xi\in\mathbb{R}^n$,*

$$H_P(x,y,\xi,t)-H_P(x,y,\xi,0)=O\big(t|x-y||\xi|^{d+1}\big)\,.$$

Note that the assertions are both easy to check for the prototype $H_P(x,y,t)=e^{it\langle x-y,\xi\rangle}\sigma_P(\xi,-\xi)$.

Remark 4.4. — Lemma 4.3 implies that the function $t \mapsto J_A(\,\cdot\,,\cdot\,,t)$ extends continuously to $t=0$ as a function from $\mathbb{R}_+$ to $C^\infty(U\times U)$ and

$$J_A(x,y,0)=\int_{S_y^A}\sigma_P(\partial_{x,y}\psi_0(x,y,\xi))\mathrm{d}_y\nu(\xi). \tag{4.7}$$

Proof. — The first statement follows from the fourth point of Lemma 4.1. For the second statement, by Equation (3.9), we may therefore restrict our attention to the case where $\xi\in S_y^A$. Next, we observe that by the second point of Lemma 4.1, $H_P(y,y,\xi,t)=H_P(y,y,\xi,0)$. The function H_P is clearly C^1 with respect to its first variable so that $|H_P(x,y,\xi,t)-H_P(x,y,\xi,0)|$ is no greater than

$$|x-y|\sup_{s\in[0,1]}\left|\partial_x H_P(sx+(1-s)y,y,\xi,t)-\partial_x H_P(sx+(1-s)y,y,\xi,0)\right|.$$

Let us fix $\Omega\subset U$ a compact set. Then by Taylor's inequality, there exists $C_1<+\infty$ such that for each $x,y\in\Omega$, $\xi\in S_y^A$ and each $t>0$,

$$|\partial_{x,y}\psi(x,y,t\xi)-\partial_{x,y}\psi(x,y,0)-\partial_{x,y}\psi_0(x,y,\xi)t|\leqslant C_1t^2\,.$$

By the first point of Lemma 4.1, $\partial_{x,y}\psi(x,y,0)=0$, so that

$$\partial_{x,y}\psi_t(x,y,\xi)=\partial_{x,y}\psi_0(x,y,\xi)+O(t) \tag{4.8}$$

uniformly in $x,y\in\Omega$ and $\xi\in S_y^A$. On the other hand by the fifth point of Lemma 4.1, $(\psi_t)_{t>0}$ is bounded in C^∞. In particular, there exists a constant $C_2<+\infty$ such that for each $t>0$, each $x,y\in K$ and each $\xi\in S_y^A$, $|\psi_t(x,y,\xi)|\leqslant C_2$. In other words

$$\psi(x,y,t\xi)=O(t) \tag{4.9}$$

uniformly in $x,y\in\Omega$ and $\xi\in S_y^A$. Applying estimates (4.8) and (4.9) to each occurrence of ψ in H_P, we see that uniformly for x,y in compact subsets of U and $\xi\in S_y^A$,

$$\partial_x H_P(x,y,\xi,t)=\partial_x H_P(x,y,\xi,0)+O(t)\,,$$

which completes the proof. □

4.2. Decay of J_A: Proof of Lemma 3.5 and its analogue in dimension one

In this subsection, we use the results of the previous subsection to prove Lemma 3.5. We will use this lemma in the proof of the multi-dimensional case of Theorem 2.8 (see Section 6). In the one dimensional case, we will use Lemma 4.5 presented below.

Proof of Lemma 3.5. — To prove this lemma, we interpret J_A as an oscillatory integral whose phase is a deformation of $(\omega,\tau)\mapsto\langle\omega,\tau\rangle$. First, fix $\Omega\subset U$ a compact neighborhood of 0. Let $r_0>0$ be such that $\Omega_0=\{x\in\mathbb{R}^n \mid \exists\ y\in\Omega,\ |y-x|\leqslant r_0\}\subset U$. By the fourth point of Lemma 4.1 the family $(\psi_t)_{t>0}$ extends by continuity to $t=0$ in C^∞. For each $t\geqslant 0$, $y\in U$, $0<r\leqslant r_0$ and $\xi,\tau\in\mathbb{R}^n$ such that $|\tau|\leqslant 1$, let

$$f_{t,r}(y,\xi,\tau)=r^{-1}\psi_t((y+r\tau),y,\xi).$$

Let $\alpha\in\mathbb{N}^n$. The Taylor expansion of $\partial_\xi^\alpha\psi_t(y+r\tau,y,\xi)$ along r yields, for each $y\in\Omega$, $|\tau|\leqslant 1$, $0<r\leqslant r_0$, $t\geqslant 0$ and $\xi\in S_y^A$,

$$\left|\partial_\xi^\alpha\psi_t(y+r\tau,y,\xi)-\partial_\xi^\alpha\langle\xi,\tau\rangle\right|\leqslant\frac{1}{2}C_1 r$$

where

$$C_1=\sup\left\{|\partial_x\partial_\xi\psi_s(w',w,\xi)|\,\middle|\,w\in\Omega,\ w'\in\Omega_0,\ \xi\in S_w^A,\ s\geqslant 0\right\}.$$

The constant C_1 is finite by the fifth point of Lemma 4.1. In particular,

$$\lim_{r\to 0}f_{t,r}(y,\xi,\tau)=\langle\xi,\tau\rangle$$

smoothly in ξ, uniformly in $t\geqslant 0$, $y\in\Omega$ and $\tau\in\mathbb{R}^n$ such that $|\tau|\leqslant 1$. In particular, we have proved first that $f_{t,r}(y,\xi,\tau)\xrightarrow[t,r\to 0]{}\langle\xi,\tau\rangle$ in this same topology, and second that for each $\alpha\in\mathbb{N}^n$, the map $(t,r)\to\partial_\xi^\alpha f_{t,r}$ is continuous at $(t,0)$ for any $t\geqslant 0$ for the topology of uniform convergence. Since this map is obviously continuous as long as $r>0$ we have proved that the family $(f_{t,r})_{t,r}$ is a deformation of the height function in the sense of Definition 3.2. Now let $x\in U$ be such that $0<r:=|x-y|\leqslant r_0$ and let $\tau=\frac{x-y}{|x-y|}$. Then $|\tau|=1$ and

$$\psi(x,y,t\xi)=t|x-y|f_{t,|x-y|}(y,\xi,\tau).$$

Moreover, by the fifth point of Lemma 4.1, the function

$$\xi\mapsto\sigma_P(\partial_{x,y}\psi_t(x,y,\xi))$$

is bounded in $C^\infty(\mathbb{R}^n)$ uniformly for $x,y\in\Omega$ and $t\geqslant 1$. Hence, the fact that the function σ_A has ε-non-degenerate level sets (see Definition 3.2) implies the existence an open neighborhood $V\subset U$ of 0 and a constant $C>0$ such that, uniformly for $x,y\in V$ and $t>0$,

$$\left|\int_{S_y^A}e^{i\psi(x,y,t\xi)}\sigma_P(\partial_{x,y}\psi_t(x,y,\xi))\mathrm{d}_y\nu(\xi)\right|\leqslant C(t|x-y|)^{-\varepsilon}.$$

Here we took $\lambda=t|x-y|$ in Equation (3.10). □

In dimension $n=1$, the symbol will never have non-degenerate level sets (in fact they will be discrete). Instead of Lemma 3.5 we will use the following result.

Lemma 4.5. — *Assume that $n = 1$. For each compact interval $I \subset U$, there exists $C < +\infty$ such that for each $0 < a \leqslant b$, each $\eta \in \{-1, +1\}$ and each $x, y \in I$*

$$\left|\int_{\eta a}^{\eta b} e^{i\psi(x,y,|x-y|^{-1}\eta)}\sigma_P(|x-y|\partial_{x,y}\psi(x,y,|x-y|^{-1}\eta)\sigma_A(y,\xi)^{-(d+1)/m}\mathrm{d}\eta\right| \leqslant Ca^{-1}. \quad (4.10)$$

Proof of Lemma 4.5. — Let $I \subset U$ be a compact interval. First of all, since σ_A is homogeneous of degree m and $n = 1$, there exists a positive function $\varrho \in C^\infty(U)$ such that $\sigma_A(x, \eta) = \varrho(x)|\eta|^m$. Thus, we may replace $\sigma_A(x, \eta)$ by $|\eta|^m$ in Equation (4.10). Observe that for each $t, \lambda > 0$, $x, y \in U$ and $\eta \in \mathbb{R}$,

$$\psi_t(x, y, \lambda\eta) = \lambda\psi_{\lambda t}(x, y, \eta).$$

This Equation, combined with the fifth point of Lemma 4.1 implies that there exists $C < +\infty$ such that for each $x, y \in I$, each $t > 0$ and $\eta \in \mathbb{R}$

$$|\partial_{x,y}\psi_t(x, y, \eta)| \leqslant C|\eta| \text{ and } |\partial_{x,y}\partial_\xi\psi_t(x, y, \eta)| \leqslant C.$$

Since moreover σ_P is homogeneous of degree d, we have, uniformly for $x, y \in I$ and for non-zero $\eta \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}&\sigma_P(|x-y|\partial_{x,y}\psi(x,y,|x-y|^{-1}\eta))|\eta|^{-d-1}\\ &\qquad = \sigma_P(\partial_{x,y}\psi_{|x-y|^{-1}}(x,y,\eta))|\eta|^{-d-1} = O(|\eta|^{-1})\\ &\partial_\eta[\sigma_P(|x-y|\partial_{x,y}\psi(x,y,|x-y|^{-1}\eta))|\eta|^{-d-1}]\\ &\qquad = \partial_\eta[\sigma_P(\partial_{x,y}\psi_{|x-y|^{-1}}(x,y,\eta))|\eta|^{-d-1}] = O(|\eta|^{-2}).\end{aligned}$$

In addition, again uniformly for $x, y \in I$ and non-zero $\eta \in \mathbb{R} \setminus \{0\}$, by Lemma 4.2, $\partial_\eta^2[\psi(x, y, |x-y|^{-1}\eta)] = O(|\eta|^{-1})$ and $\partial_\eta[\psi(x, y, |x-y|^{-1}\eta)]$ is bounded from above and below by a positive constant. Now, setting momentarily $u(\eta) := \psi(x, y, |x-y|^{-1}\eta)$ and $v(\eta) = \sigma_P(|x-y|\partial_{x,y}\psi(x, y, |x-y|^{-1}\eta))|\eta|^{-d-1}$, we have, for any $a, b > 0$ such that $a \leqslant b$,

$$\int_a^b e^{iu(\eta)}v(\eta)\mathrm{d}\eta = \left[\frac{1}{i}e^{iu(\eta)}\frac{v(\eta)}{u'(\eta)}\right]_{\eta=a}^b - \int_a^b \frac{1}{i}e^{iu(\eta)}\left(\frac{v'(\eta)}{u'(\eta)} - \frac{v(\eta)u''(\eta)}{u'(\eta)^2}\right)\mathrm{d}\eta.$$

The preceding observations show that, uniformly for $x, y \in I$, $0 < a \leqslant b$ and $\eta \in [a, b]$, we have $\frac{v(a)}{u'(a)} = O(a^{-1})$, $\frac{v(b)}{u'(b)} = O(b^{-1})$, $\frac{v'(\eta)}{u'(\eta)} = O(\eta^{-2})$ and $\frac{v(\eta)u''(\eta)}{u'(\eta)^2} = O(\eta^{-2})$. Consequently, there exists $C < +\infty$ such that for any $x, y \in \Omega$ and any $0 < a \leqslant b$,

$$\left|\int_a^b e^{i\psi(x,y,|x-y|^{-1}\eta)}\sigma_P(|x-y|\partial_{x,y}\psi(x,y,|x-y|^{-1}\eta))|\eta|^{-d-1}\mathrm{d}\eta\right| \leqslant Ca^{-1}.$$

The proof for $\int_{-b}^{-a}$ is identical. $\square$

5. The off-critical case: Proof of Theorem 2.12, Proposition 3.7 and Theorem 2.6

In this section, we prove Theorem 2.12, Proposition 3.7 and Theorem 2.6. We use only Theorem 2.4 and Lemma 4.1.

Let $f : \mathbb{R} \to \mathbb{C}$, continuously differentiable, with support in $]0, +\infty[$. For each $L \geqslant 1$, let K_L^f be the integral kernel of $\Pi_L f(A)$. Later in this section we will consider the case $f(z) = t^z$ for t large enough. We begin by linking K_L^f with E_L.

Lemma 5.1. — *For any $L \in \mathbb{R}$,*

$$K_L^f = f(L)E_L - \int_0^L f'(\lambda)E_\lambda \mathrm{d}\lambda.$$

This lemma generalizes Proposition 21 of [22].

Proof. — The functions $L \mapsto E_L$ and $L \mapsto K_L^f$ are locally constant and define distributions on $\mathbb{R}$ with values in $C^\infty(\mathcal{X} \times \mathcal{X})$. We denote by $'$ the weak derivative with respect to L of these kernels. For all $L > 0$, and $x, y \in \mathcal{X}$

$$E_L(x,y) = \sum_{\lambda_k \leqslant L} e_k(x)\overline{e_k(y)}; \quad K_L^f = \sum_{\lambda_k \leqslant L} f(\lambda_k)e_k(x)\overline{e_k(y)}\,,$$

so that

$$\begin{aligned} K_L^f(x,y)' &= \sum_{k \in \mathbb{N}} \delta_{\lambda_k}(L) f(\lambda_k) e_k(x)\overline{e_k(y)} \\ &= f(L) \sum_{k \in \mathbb{N}} \delta_{\lambda_k}(L) e_k(x)\overline{e_k(y)} = f(L)E'_L \end{aligned}$$

and

$$K_L^f = \int_0^L f(\lambda)E'_\lambda \mathrm{d}\lambda.$$

By integration by parts,

$$K_L^f = f(L)E_L - f(0)E_0 - \int_0^L f'(\lambda)E_\lambda \mathrm{d}\lambda = f(L)E_L - \int_0^L f'(\lambda)E_\lambda \mathrm{d}\lambda$$

since $f(0) = 0$. □

We can now prove both Theorem 2.12 and Proposition 3.7 using Theorem 2.4. We start with Theorem 2.12.

Proof of Theorem 2.12. — Throughout the proof we fix $z = z_1 + iz_2 \in \mathbb{C}$ such that $(n + d)/m < z_1$ and $f : \mathbb{R} \to \mathbb{C}$ continuously differentiable such that $f(t) = t^z$ for t large enough. As in the statement of the theorem, we omit the superscript in K_L^f. Let $L > 0$. Then, by Lemma 5.1, we have, for each $t \geqslant L$, if $f : \mathbb{R} \to \mathbb{C}$ is continuously differentiable,

$$K_{L+t} = f(L+t)E_{L+t} - \int_0^{L+t} f'(\lambda)E_\lambda \mathrm{d}\lambda\,.$$

If we apply the operator P, then, for all large enough values of $L > 0$ and all $t \geqslant 0$,

$$PK_{L+t} - PK_L = (L+t)^z PE_{L+t} - L^z PE_L - \int_L^{L+t} z\lambda^{z-1} PE_\lambda \mathrm{d}\lambda\,.$$

By Theorem 2.4, we have, uniformly for $(x,y) \in U \times U$ and $t \geqslant 0$,

$$(L+t)^z PE_{L+t}(x,y) = O\Big(L^{z_1+(n+d)/m}\Big)$$

and

$$\int_L^{L+t} z\lambda^{z-1} E_\lambda \mathrm{d}\lambda = O\bigg(\int_L^{+\infty} \lambda^{-1+z_1+(n+d)/m} \mathrm{d}\lambda\bigg) = O\Big(L^{z_1+(n+d)/m}\Big).$$

In particular, uniformly for $(x,y) \in U \times U$ and $t \geqslant 0$,

$$PK_{L+t}(x,y) - PK_L(x,y) = O\Big(L^{z_1+(n+d)/m}\Big).$$

Since, $(n+d)/m < z_1$, this last estimate implies that the sequence $(PK_L)_{L>0}$ is a Cauchy sequence in $C^0(U \times U)$. Therefore, it converges uniformly on compact subsets of $U \times U$ to some function $K_\infty^P \in C^0(U \times U)$. Since this is actually true for any differential operator of order at most d (indeed, if $d' \leqslant d$, we still have $z_1 + (n+d')/m < 0$), all the derivatives of K_L, of order up to d, converge uniformly on compact sets. But this means that the limit K_∞ of $(K_L)_{L>0}$ is actually of class C^d and that the limits of the respective derivatives converge to the derivatives of the limit. In particular,$K_\infty^P = PK_\infty$. $\square$

We now move on to Proposition 3.7.

Proof of Proposition 3.7. — By Theorem 2.4, uniformly for $x, y \in U$ and $L \geqslant 1$,

$$\begin{aligned} PE_L(x,y) &= \frac{1}{(2\pi)^n} \int_{\sigma_A(y,\xi)\leqslant L} e^{i\psi(x,y,\xi)} \sigma_P(\partial_{x,y}\psi(x,y,\xi))\mathrm{d}\xi + O\Big(L^{(n+d-1)/m}\Big) \\ &= \frac{1}{(2\pi)^n} \int_0^{L^{1/m}} J_A(x,y,t) t^{n+d-1} \mathrm{d}t + O\Big(L^{(n+d-1)/m}\Big). \end{aligned}$$

In the second equality we used the definition of $\mathrm{d}\nu$ (see (1.1)) and J_A (see (3.8)) as well as the fact that σ_P is d-homogeneous. Consequently, uniformly for any $x,y\in U$ and $L\geqslant 1$,

$$-\int_0^L f'(\lambda)PE_\lambda(x,y)\mathrm{d}\lambda = -\frac{1}{(2\pi)^n}\int_0^L f'(\lambda)\int_0^{\lambda^{1/m}} J_A(x,y,t)t^{n+d-1}\mathrm{d}t\mathrm{d}\lambda \\ + O\left(\int_{-\infty}^L f'(\lambda)\lambda^{(n+d-1)/m}\mathrm{d}\lambda\right).$$

Integrating by parts along λ the first term in the right hand side, we get

$$-f(L)PE_L(x,y)+\frac{1}{(2\pi)^n}\int_0^L f(\lambda)\frac{1}{m}\lambda^{\frac{1}{m}-1}J_A(x,y,\lambda^{1/m})\lambda^{(n+d-1)/m}\mathrm{d}\lambda \\ + O\Big(f(L)L^{(n+d-1)/m}\Big).$$

Setting $u=\lambda^{1/m}$ we get

$$\begin{aligned}\int_0^L f(\lambda)\frac{1}{m}\lambda^{\frac{1}{m}-1}J_A(x,y,\lambda^{1/m})\lambda^{(n+d-1)/m}\mathrm{d}\lambda \\ &= \int_0^{L^{1/m}} f(u^m)J_A(x,y,u)u^{n+d-1}\mathrm{d}u \\ &= \int_{\sigma_A(y,\xi)\leqslant L} e^{i\psi(x,y,\xi)}f(\sigma_A(y,\xi))\sigma_P(\partial_{x,y}\psi(x,y,\xi))\mathrm{d}\xi.\end{aligned}$$

By Lemma 5.1,

$$PK_L^f = f(L)PE_L - \int_0^L f'(\lambda)PE_\lambda\mathrm{d}\lambda.$$

Replacing the integral term by the expression derived above, we see that the $f(L)PE_L$ terms cancel out, leaving the equation from the first result of Proposition 3.7. For the case where $(x,y)\in U\times U\setminus W$, $W\subset U\times U$ a neighborhood of the diagonal, we just apply the corresponding estimate from Theorem 2.4 and proceed accordingly. □

For the proof of Theorem 2.6, From now on, we assume that $f(t)=t^z$ for large enough $t>0$ and omit the superscript f in K_L^f.

Proof of Theorem 2.6. — Throughout the proof, we let $\eta=1$ if $n+d+mz=1$ and 0 otherwise and set $g(L)=1+L^{(n+d-1)/m+z_1}\ln(L)^\eta$. We also fix a compact subset $\Omega\subset U$ which is a neighborhood of 0. Firstly, changing f on a compact set affects PK_L by adding a linear combination of smooth functions (independent of L). On the other hand, it changes the right-hand side of the equation from Proposition 3.7 by an $O(1)$ term. Thus, we may assume that $f(t)=t^z\mathbb{1}\big[t\geqslant 1\big]$. By Proposition 3.7, uniformly for

$x \in \Omega$, $L \geqslant 1$ and $X, Y \in \mathbb{R}^n$ such that $x + L^{-1/m}X, x + L^{-1/m}Y \in \Omega$, $PK_L\left(x + L^{-1/m}X, x + L^{-1/m}Y\right)$ equals

$$\frac{1}{(2\pi)^n}\int_{1\leqslant\sigma_A(x+L^{-1/m}Y,\xi)\leqslant L}\sigma_A(x+L^{-1/m}Y,\xi)^z e^{i\psi(x+L^{-1/m}X,x+L^{-1/m}Y,\xi)} \\ \times\sigma_P\left(\partial_{x,y}\psi\left(x+L^{-1/m}X,x+L^{-1/m}Y,\xi\right)\right)\mathrm{d}\xi + O(g(L)). \quad (5.1)$$

We need to check that replacing each occurrence of $x + L^{-1/m}Y$ or $x + L^{-1/m}X$ by w in the integrand will produce an error of order $O(g(L))$. More precisely, we make the following claim.

Claim 5.2. — *Uniformly for $x \in \Omega$, $\xi \in \mathbb{R}^n \setminus \{0\}$, $L \geqslant 1$ and $X, Y \in \mathbb{R}^n$ such that $x+L^{-1/m}X, x+L^{-1/m}Y \in U$ such that $1 \leqslant \sigma_A(x+L^{-1/m}Y,\xi) \leqslant L$, the quantity*

$$\sigma_A(x+L^{-1/m}Y,\xi)^z e^{i\psi(x+L^{-1/m}X,x+L^{-1/m}Y,\xi)} \\ \times\sigma_P\left(\partial_{x,y}\psi\left(x+L^{-1/m}X,x+L^{-1/m}Y,\xi\right)\right) \quad (5.2)$$

equals

$$e^{iL^{-1/m}\langle\xi,X-Y\rangle}\sigma_A(x,\xi)^z\sigma_P(\xi,-\xi) + O\Big(|\xi|^{mz_1+d}L^{-1/m}\Big). \quad (5.3)$$

The claim follows by Taylor expansion in the x, y variables and keeping track of the homogeneity in ξ. The only subtlety lies in the linearization of the phase.

Proof of Claim 5.2. — Throughout the proof we fix $x \in \Omega$, $L \geqslant 1$, $X, Y \in \mathbb{R}^n$ such that $x + L^{-1/m}X, x + L^{-1/m}Y \in \Omega$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ such that $\sigma_A(x + L^{-1/m}Y, \xi) \leqslant L$. Unless otherwise stated, all the O estimates will be uniform with respect to these parameters. First of all, since σ_A is a positive m-homogeneous symbol in its second variable, σ_P is homogeneous of order d and $\partial_{x,y}\psi$ is a symbol of order 1 in its third variable, applying Taylor's inequality with respect to the L-dependent variables everywhere except the exponential in the quantity (5.2) shows that it equals

$$\sigma_A(x,\xi)^z e^{i\psi(x+L^{-1/m}X,x+L^{-1/m}Y,\xi)}\sigma_P\left(\partial_{x,y}\psi\left(x,x,\xi\right)\right) \\ + O\Big(|\xi|^{mz_1+d}L^{-1/m}\Big). \quad (5.4)$$

Here the $|\xi|^{mz_1}$ appears regardless of the sign of z_1 because σ_A is positive homogeneous. Since ψ is a symbol of order one in ξ and $|\xi| = O\big(L^{1/m}\big)$,

$$\psi(x+L^{-1/m}X,x+L^{-1/m}Y,\xi) \\ = \psi(x,x,\xi) + \partial_x\psi(x,x,\xi)L^{-1/m}x + \partial_y\psi(x,x,\xi)L^{-1/m}y + O\big(L^{-1/m}\big).$$

By points two and three of Lemma 4.1 we get

$$\psi(x+L^{-1/m}X, x+L^{-1/m}Y, \xi) = L^{-1/m}\langle X-Y, \xi\rangle + O\big(L^{-1/m}\big).$$

Using this estimate in the exponential, together with the fact the rest of the integrand is $O\big(|\xi|^{mz_1+d}\big)$ we obtain that the quantity (5.4) equals

$$e^{iL^{-1/m}\langle \xi, X-Y\rangle}\sigma_A(x,\xi)^z\sigma_P(\xi,-\xi) + O\Big(|\xi|^{mz_1+d}L^{-1/m}\Big)$$

which is exactly (5.3). □

By Claim 5.2 and Equation (5.1) $PK_L(x+L^{-1/m}X, x+L^{-1/m}Y)$ equals

$$\begin{aligned} PK_L&\left(x+L^{-1/m}X, x+L^{-1/m}Y\right) \\ &= \frac{1}{(2\pi)^n}\int_{1\leqslant\sigma_A(x+L^{-1/m}Y,\xi)\leqslant L} e^{iL^{-1/m}\langle\xi,X-Y\rangle}\sigma_A(x,\xi)^z\sigma_P(\xi,-\xi)\mathrm{d}\xi \\ &\qquad + O\left(L^{-1/m}\int_{1\leqslant\sigma_A(x+L^{-1/m}Y,\xi)\leqslant L}|\xi|^{mz_1+d}\mathrm{d}\xi\right). \end{aligned} \tag{5.5}$$

But since $mz_1+d+n>0$ and σ_A is m-homogeneous, the remainder is $O\big(L^{z_1+(n+d-1)/m}\big) = O(g(L))$. For each $x\in U$, $Y\in\mathbb{R}^n$ and each $L\geqslant 1$ let $\Delta(x,Y,L)$ be the symmetric difference of the sets $\{\xi\in\mathbb{R}^n \mid 1\leqslant\sigma_A(x,\xi)\leqslant L\}$ and $\{\xi\in\mathbb{R}^n \mid 1\leqslant\sigma_A(x+L^{-1/m}Y,\xi)\leqslant L\}$, whenever $x+L^{-1/m}Y\in\Omega$. Since σ_A is positive m-homogeneous in ξ and smooth in x, there exists $0<C<+\infty$ such that for each $L\geqslant 1$, $x\in\Omega$ and $Y\in\mathbb{R}^n$ such that $x+L^{-1/m}Y\in\Omega$, $\mathrm{Vol}(\Delta(x,Y,L))\leqslant CL^{(n-1)/m}$ and for each $\xi\in\Delta(x,Y,L)$, $C^{-1}L^{1/m}\leqslant|\xi|\leqslant CL^{1/m}$. Consequently, in Equation (5.5) we can replace the integration domain by $\{\xi\in\mathbb{R}^n \mid 1\leqslant\sigma_A(x,\xi)\leqslant L\}$ and produce an error of order $O\big(L^{z_1+(n+d-1)/m}\big) = O(g(L))$ uniformly for $x\in\Omega$, $L\geqslant 1$ and $Y\in\mathbb{R}^n$ such that $x+L^{-1/m}Y\in\Omega$. In other words,

$$\begin{aligned} PK_L&\left(x+L^{-1/m}X, x+L^{-1/m}Y\right) \\ &= \frac{1}{(2\pi)^n}\int_{1\leqslant\sigma_A(x,\xi)\leqslant L} e^{iL^{-1/m}\langle\xi,X-Y\rangle}\sigma_A(x,\xi)^z\sigma_P(\xi,-\xi)\mathrm{d}\xi + O(g(L)). \end{aligned}$$

Moreover, since $mz_1+d+n>0$ and the integrand scales like $|\xi|^{mz_1+d}$ near 0, adding the region $\sigma_A(x,\xi)\leqslant 1$ to the integration domain creates a bounded error. Following this by the change of variable $\xi = L^{1/m}\zeta$ shows that uniformly for $x\in\Omega$, $L\geqslant 1$ and $X,Y\in\mathbb{R}^n$ such that $x+L^{-1/m}X$,

$x + L^{-1/m}Y \in \Omega$,

$$PK_L\left(x + L^{-1/m}X, x + L^{-1/m}Y\right) = \frac{1}{(2\pi)^n}\int_{\sigma_A(x,\zeta)\leqslant 1} e^{i\langle\zeta, X-Y\rangle}\sigma_A(x,\zeta)^z\sigma_P(\zeta,-\zeta)\mathrm{d}\zeta L^{z+(n+d)/m} + O(g(L))\,.$$

This proves the first statement of the theorem for $V = \mathring{\Omega}$. To prove the second statement, observe that by Lemma 5.1, uniformly for $L \geqslant 1$ and $x, y \in \Omega$,

$$\begin{aligned} PK_L(x,y) &= f(L)PE_L - \int_0^L f'(\lambda)PE_\lambda(x,y)\mathrm{d}\lambda \\ &= L^z PE_L(x,y) - \int_1^L \lambda^{z-1}PE_\lambda(x,y)\mathrm{d}\lambda + O(1) \end{aligned}$$

Next, fix $W \subset V \times V$ a neighborhood of the diagonal. By Theorem 2.4, there exists $C' > 0$ such that for any $(x,y) \in (V \times V)\setminus W$ and any $L \geqslant 1$, $|PE_L(x,y)| \leqslant C'L^{(n+d-1)/m}$, which implies

$$|PK_L(x,y)| \leqslant C'\left(L^{z_1+(n+d-1)/m} + \int_1^L \lambda^{z_1-1+(n+d-1)/m}\mathrm{d}\lambda\right) = O(g(L))\,.$$

This proves the second statement of Theorem 2.6. □

6. The critical case: Proof of Theorem 2.8

In this section we prove Theorem 2.8. We use the admissibility condition through Proposition 3.4. Suppose that $n + d + mz = 0$, so that $z = -\frac{d+n}{m}$. By Proposition 3.7, uniformly for $x, y \in U$,

$$PK_L(x,y) = \frac{1}{(2\pi)^n}\int_{\sigma_A(y,\xi)\leqslant L} e^{i\psi(x,y,\xi)}\sigma_P(\partial_{x,y}\psi(x,y,\xi))f(\sigma_A(y,\xi))\mathrm{d}\xi + O\big(L^{-1/m}\big).$$

Let $C < +\infty$ be such that $f(t) = t^z$ for $t > C$. Then,

$$\begin{aligned} PK_L(x,y) &= \frac{1}{(2\pi)^n}\int_{C\leqslant\sigma_A(y,\xi)\leqslant L} e^{i\psi(x,y,\xi)}\sigma_P(\partial_{x,y}\psi(x,y,\xi))\sigma_A(y,\xi)^{-(d+n)/m}\mathrm{d}\xi \\ &\qquad + Q_1(x,y) + O\big(L^{-1/m}\big) \end{aligned}$$

where

$$Q_1(x,y) = \frac{1}{(2\pi)^n}\int_{\sigma_A(y,\xi)\leqslant C} e^{i\psi(x,y,\xi)}\sigma_P(\partial_{x,y}\psi(x,y,\xi))f(\sigma_A(y,\xi))\mathrm{d}\xi\,.$$

We will split the first integral term in the last expression of PK_L as follows. For any $x, y \in U$, let

$$\begin{aligned} I_L(x,y) = \frac{1}{(2\pi)^n} \int_{C \leqslant \sigma_A(y,\xi) \leqslant L} & \mathbb{1}\left[\sigma_A(y,\xi)|x-y|^m \geqslant 1\right] e^{i\psi(x,y,\xi)} \\ & \times \sigma_P(\partial_{x,y}\psi(x,y,\xi))\sigma_A(y,\xi)^{-(d+n)/m} \mathrm{d}\xi \\ II_L(x,y) = \frac{1}{(2\pi)^n} \int_{C \leqslant \sigma_A(y,\xi) \leqslant L} & \mathbb{1}\left[\sigma_A(y,\xi)|x-y|^m < 1\right] e^{i\psi(x,y,\xi)} \\ & \times \sigma_P(\partial_{x,y}\psi(x,y,\xi))\sigma_A(y,\xi)^{-(d+n)/m} \mathrm{d}\xi. \end{aligned}$$

The integral I_L represents the off-diagonal contributions while the II_L represents the near diagonal contributions, where the split occurs at $|x-y| \asymp \sigma_A(y,\xi)^{-1/m}$. Then, uniformly for $x, y \in U$,

$$PK_L(x,y) = I_L(x,y) + II_L(x,y) + Q_1(x,y) + O\big(L^{-1/m}\big). \tag{6.1}$$

Theorem 2.8 is an easy consequence of the following two lemmas.

Lemma 6.1. — *Let $k_0 \in \mathbb{N}$, $k_0 \geqslant 2$. Suppose that either $n = 1$ or σ_A is $\frac{1}{k_0}$-admissible. There exist an open neighborhood $V \subset U$ of $0 \in \mathbb{R}^n$, a function $Q_2 \in L^\infty(V \times V)$ and a constant $C < +\infty$ such that for any $x, y \in V$ and $L \geqslant 1$,*

$$\Big| I_L(x,y) - Q_2(x,y) \Big| \leqslant C \min\Big(L^{-1/k_0 m}|x-y|^{-1/k_0}, 1\Big).$$

In dimension one, we prove the lemma using Lemma 4.5 while in the case of admissible σ_A we use Proposition 3.1. This proof is the only place where we use these results.

Lemma 6.2. — *For any open neighborhood $V \subset U$ of 0 such that $\overline{V} \subset U$ is compact, there exists a constant $C < +\infty$ such that for all $x, y \in V$ and $L \geqslant 1$,*

$$\left| II_L(x,y) - \frac{1}{(2\pi)^n} Y_P(y) \Big[\ln\big(L^{1/m}\big) - \ln_+\big(L^{1/m}|x-y|\big)\Big] \right| \leqslant C\,.$$

Moreover $II_L(x,y)$ is independent of L as long as $L \geqslant 1$ and $L|x-y|^m \geqslant 1$.

Let us first prove that these lemmas imply Theorem 2.8.

Proof of Theorem 2.8. — Let V be the intersection of the V's appearing in Lemmas 6.1 and 6.2. Firstly, Lemma 6.1 implies that $I_L(x,y)$ is uniformly bounded for $x, y \in V$ and $L \geqslant 1$. Secondly, Lemma 6.2 implies that, uniformly for $x, y \in V$ and $L \geqslant 1$,

$$II_L(x,y) = \frac{1}{(2\pi)^n} Y_P(y) \Big[\ln\big(L^{1/m}\big) - \ln_+\big(L^{1/m}|x-y|\big)\Big] + O(1)\,.$$

Plugging these two estimates in Equation (6.1) we get the first point of Theorem 2.8. For the second point, we begin by observing that by Lemma 6.2, there exists a bounded function $Q_3 \in L^\infty(V \times V)$ such that for each $L \geqslant |x-y|^{-m}$,

$$II_L(x,y) = -\frac{1}{(2\pi)^n} Y_P(y) \ln(|x-y|) + Q_3(x,y)\,.$$

Moreover, if $L \geqslant |x-y|^{-m}$ then $L^{-1/k_0}|x-y|^{-1/k_0 m} \leqslant 1$ so by Lemma 6.1, uniformly for any such x, y and L,

$$I_L(x,y) = Q_2(x,y) + O\Big(L^{-1/k_0}|x-y|^{-1/k_0 m}\Big)\,.$$

Applying these two estimates to Equation (6.1) we deduce that, uniformly for $x, y \in V$ and $L \geqslant 1$ such that $|x-y| \geqslant L^{-1/m}$,

$$PK_L(x,y) = -\frac{1}{(2\pi)^n} Y_P(y) \ln(|x-y|) + Q(x,y) + O\Big(L^{-1/k_0}|x-y|^{-1/k_0 m}\Big)$$

where $Q = Q_1 + Q_2 + Q_3 \in L^\infty(V \times V)$. This proves the estimate in the second point of Theorem 2.8. $\square$

Proof of Lemma 6.1. — Suppose first that $\mathcal{X}$ has dimension $n = 1$ and fix $\Omega \subset U$ a compact neighborhood of 0. For $x \neq y$, setting $\eta = |x-y|\xi$, the integral $I_L(x,y)$ equals

$$\int_{a(x,y)}^{b(x,y,L)} e^{i\psi(x,y,|x-y|^{-1}\eta)} \sigma_P\left(|x-y|\partial_{x,y}\psi\left(x,y,|x-y|^{-1}\eta\right)\right) \\ \times \sigma_A(y,\eta)^{-(d+1)/m} \mathrm{d}\eta.$$

where $a(x,y)$ and $b(x,y,L)$ are the positive numbers defined by

$$\sigma_A(y, a(x,y)) = \max(C|x-y|^m, 1)$$
$$\text{and} \quad \sigma_A(y, b(x,y,L)) = \max(|x-y|^m L, 1).$$

Since σ_A is elliptic positive homogeneous of degree $m > 0$ there exists $C_1 > 0$ such that for each $x, y \in \Omega$ and $L \geqslant 1$,

$$b(x,y,L) \geqslant C_1 \min\big(|x-y| L^{-1/m}\big).$$

By Lemma 4.5, $I_L(x,y)$ converges to some limit $Q_2(x,y)$ as $L \to +\infty$ in such a way that the remainder term is $O\big(\min\left(|x-y|^{-1}L^{-1/m}, 1\right)\big)$. The case where $x = y$ follows by continuity and we have proved the lemma in the one-dimensional case with $V = \mathring{\Omega}$.

Suppose now that $n \geqslant 2$ and σ_A is $\frac{1}{k_0}$-admissible for some integer $k_0 \geqslant 2$. By Equations (1.1) and (3.8), for any $L \geqslant 1$ and $x, y \in U$,

$$\begin{aligned} I_L(x,y) &= \frac{1}{(2\pi)^n} \int_{C^{1/m}}^{L^{1/m}} \mathbb{1}\left[|x-y|t \geqslant 1\right] J_A(x,y,t) \frac{\mathrm{d}t}{t} \\ &= \frac{1}{(2\pi)^n} \int_{C^{1/m}|x-y|}^{L^{1/m}|x-y|} \mathbb{1}\left[s \geqslant 1\right] J_A\left(x,y,|x-y|^{-1}s\right) \frac{\mathrm{d}s}{s} \end{aligned}$$

By Proposition 3.1, there exist an open neighborhood $V \subset U$ of 0 and a constant $C_3 > 0$ such that, uniformly for distinct $x, y \in V$ and $t > 0$, $|J_A(x,y,t)| \leqslant C_3 \left(|x-y|t\right)^{-1/k_0}$. Therefore, for each $x, y \in V$ and $L > 0$,

$$\begin{aligned} &\left|(2\pi)^n I_L(x,y) - \int_{C^{1/m}|x-y|}^{+\infty} \mathbb{1}\left[s \geqslant 1\right] J_A\left(x,y,|x-y|^{-1}s\right) \frac{\mathrm{d}s}{s}\right| \\ &\leqslant C_3 \int_{\max(|x-y|L^{\frac{1}{m}},1)}^{+\infty} s^{-1-1/k_0} \mathrm{d}s = \frac{C_3}{k_0} \min\Big(1, L^{-1/k_0 m}|x-y|^{-1/k_0}\Big)\,. \end{aligned}$$

By continuity, this stays true for $x = y$. This proves the lemma for σ_A admissible with

$$Q_2(x,y) = \int_{C^{1/m}|x-y|}^{+\infty} \mathbb{1}\left[s \geqslant 1\right] J_A\left(x,y,|x-y|^{-1}s\right) \frac{\mathrm{d}s}{s}\,. \qquad \square$$

Proof of Lemma 6.2. — Let $\Omega \subset U$ be a compact neighborhood of the origin. The indicator function in the integral defining $II_L(x,y)$ changes the integration domain from $C \leqslant \sigma_A(y,\xi) \leqslant L$ to $C \leqslant \sigma_A(y,\xi) \leqslant |x-y|^{-1/m}$ as long as $L|x-y|^m > 1$. Hence, $II_L(x,y)$ is independent of L whenever $L|x-y|^m > 1$. For each $y \in U$ and each $0 \leqslant r_1 \leqslant r_2$, we set

$$\mathcal{A}_y(r_1,r_2) = \{\xi \in \mathbb{R}^n \mid r_1 \leqslant \sigma_A(y,\xi) \leqslant r_2\}\,.$$

Recall that

$$\begin{aligned} II_L(x,y) = \frac{1}{(2\pi)^n} \int_{\mathcal{A}_y(C,L)} & \mathbb{1}\left[\sigma_A(y,\xi)|x-y|^m < 1\right] e^{i\psi(x,y,\xi)} \\ & \times \sigma_P(\partial_{x,y}\psi(x,y,\xi))\sigma_A(y,\xi)^{-(n+d)/m} \mathrm{d}\xi\,. \end{aligned}$$

By Equation (3.7), the integrand equals

$$\mathbb{1}\left[\sigma_A(y,\xi)|x-y|^m < 1\right] H_P\Big(x,y,\sigma_A(y,\xi)^{-1/m}\xi, \sigma_A(y,\xi)^{1/m}\Big)\, \sigma_A(y,\xi)^{-n/m}\,.$$

Since σ_A is positive homogeneous of degree m, $\sigma_A(y,\xi)^{-1/m}\xi$ is uniformly bounded for $y \in \Omega$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. By the second point of Lemma 4.3, uniformly for $x, y \in \Omega$ and $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned} H_P\Big(x, y, \sigma_A(y,\xi)^{-1/m}\xi, \sigma_A(y,\xi)^{1/m}\Big) \\ = H_P\Big(x, y, \sigma_A(y,\xi)^{-1/m}\xi, 0\Big) + O\Big(|x-y|\sigma_A(y,\xi)^{1/m}\Big)\,. \end{aligned}$$

Again by m-homogeneity and positivity, $|x-y|\mathbb{1}\left[\sigma_A(y,\xi)|x-y|^m < 1\right] \times \sigma_A(y,\xi)^{(1-n)/m}$ is uniformly integrable in ξ for $x, y \in \Omega$ so

$$\begin{aligned} II_L(x,y) = \frac{1}{(2\pi)^n}\int_{\mathcal{A}_y(C,L)} \mathbb{1}\left[\sigma_A(y,\xi)|x-y|^m < 1\right] \\ \times H_P\Big(x, y, \sigma_A(y,\xi)^{-1/m}\xi, 0\Big)\, \sigma_A(y,\eta)^{-n/m}\mathrm{d}\xi + O(1)\,. \end{aligned}$$

Fix two distinct points $x, y \in U$. The change of variables $\eta = |x-y|\xi$ in the integral yields

$$\int_{|x-y|\mathcal{A}_y(C,L)} \mathbb{1}\left[\sigma_A(y,\eta) < 1\right] H_P(x, y, |x-y|\eta, 0)\sigma_A(y,\eta)^{-n/m}\mathrm{d}\eta$$

which, by definition of J_A (see Equation (3.8)), equals

$$J_A(x,y,0)\int_{C^{1/m}|x-y|}^{L^{1/m}|x-y|} \mathbb{1}\left[|x-y|s < 1\right]\frac{\mathrm{d}s}{s}\,.$$

Observe that for any $0 < a \leqslant b$,

$$\int_a^b \mathbb{1}\left[t < 1\right]\frac{\mathrm{d}t}{t} = \ln(b) - \ln_+(b) - \ln(a) + \ln_+(a)$$

where $\ln_+(s) = \max(\ln(s), 0)$. In our setting, uniformly for distinct $x, y \in \Omega$,

$$\begin{aligned} \int_{C^{1/m}}^{L^{1/m}} \mathbb{1}\left[|x-y|s < 1\right]\frac{\mathrm{d}s}{s} &= \int_{C^{1/m}|x-y|}^{L^{1/m}|x-y|} \mathbb{1}\left[t < 1\right]\frac{\mathrm{d}t}{t} \\ &= \ln\big(L^{1/m}\big) - \ln_+\big(L^{1/m}|x-y|\big) + O(1)\,. \end{aligned}$$

Hence, uniformly for any $(x,y) \in \Omega \times \Omega$ and $L \geqslant 1$,

$$II_L(x,y) = \frac{1}{(2\pi)^n}J_A(x,y,0)\left[\ln\big(L^{1/m}\big) - \ln_+\big(L^{1/m}|x-y|\big)\right] + O(1).$$

Now, by Equation (4.7) and the second point of Definition 2.3, followed by (1.1),

$$\begin{aligned} J_A(x,y,0) &= \int_{S_y^A} \sigma_P(\partial_{x,y}\psi(y,y,\xi))\mathrm{d}_y\nu(\xi) + O(|x-y|) \\ &= \int_{S_y^A} \sigma_P(\xi,-\xi)\mathrm{d}_y\nu(\xi) + O(|x-y|) = Y_P(y) + O(|x-y|)\,. \end{aligned}$$

Since the quantity

$$|x-y|\left[\ln\left(L^{1/m}\right)-\ln_{+}\left(L^{1/m}|x-y|\right)\right]$$

is uniformly bounded in $L>0$ and $x,y\in\Omega$, we deduce that

$$II_L(x,y)=\frac{1}{(2\pi)^n}Y_P(y)\left[\ln\left(L^{1/m}\right)-\ln_{+}\left(L^{1/m}|x-y|\right)\right]+O(1)$$

so the lemma is proved with $V=\mathring{\Omega}$. $\square$

7. Admissible homogeneous symbols

In this section, we deal with results concerning admissible homogeneous symbols (see Definition 2.7). These results are useful in the proofs of Theorems 2.11 and 2.8. More precisely, in Section 7.1 we prove Proposition 3.4 which says that admissible homogeneous symbols have non-degenerate level sets and is used in the proof of Theorem 2.8. Then, in Proposition 7.6 of Section 7.2 we prove that admissibility is both stable and generic in a suitable topology. Theorem 2.11 follows directly from Proposition 7.6.

Throughout this section we will use the following notation. Let $U\subset\mathbb{R}^p$ and $V\subset\mathbb{R}^q$ be two open subsets and $f\in C^\infty(U\times V)$. For each $k\in\mathbb{N}$ and $(x,y)\in U\times V$, we denote by $\mathrm{d}^k f(x,y)$ the k-th differential of f at (x,y), which is a symmetric k-linear form on $\mathbb{R}^{p+q}$. Moreover, we will denote by $\mathrm{d}_x^k f(x,y)$ the k-th differential of $f(\,\cdot\,,y)$ at (x,y) and define $\mathrm{d}_y f^k f(x,y)$ likewise. We nevertheless retain the notation $\partial_x f$ (resp. $\partial_y f$) if $p=1$ (resp. $q=1$).

We will also use Euler's identity for homogeneous functions. Namely, if $f:\mathbb{R}^p\setminus\{0\}\to\mathbb{R}$ is m-homogeneous, then, for each $x\in\mathbb{R}^p$, $\mathrm{d}f(x)(x)=mf(x)$. We will actually apply this identity as follows. For each $k\in\mathbb{N}$, $\mathrm{d}^k f(x)$ is k-homogeneous so

$$\mathrm{d}^{k+1}f(x)(x,\dots)=(m-k)\mathrm{d}^k f(x)\,. \tag{7.1}$$

7.1. Proof of Proposition 3.4

The object of this subsection is to prove Proposition 3.4. To prove this result, we will use partitions of unity and local charts to carry the integral onto $\mathbb{R}^n$ and then apply the following lemma, which we prove later in the section.

Lemma 7.1. — *Let $n \in \mathbb{N}$, $n \geqslant 1$. Let $U \subset \mathbb{R}^n$ be an open neighborhood of* 0 *and $(f_\eta)_{\eta \in E}$ be a continuous family of smooth functions on U indexed by $E \subset \mathbb{R}^p$, an open neighborhood of* 0. *Fix $k \geqslant 1$ and assume that $\mathrm{d}^k f_0(0) \neq 0$. Then, there exist $E' \subset E$ and $U' \subset U$ two open neighborhoods of the origin in $\mathbb{R}^p$ and $\mathbb{R}^n$ respectively, such that for each $u \in C_c^\infty(U')$ there exists $C(u) < +\infty$ such that for each $\lambda > 0$ and each $\eta \in E'$,*

$$\left| \int_{U'} e^{i\lambda f_\eta(x)} u(x) \mathrm{d}x \right| \leqslant C(u) \lambda^{-\frac{1}{k}} \,.$$

Moreover, $C(u)$ depends continuously on u in the $C_c^\infty(U')$ topology.

We now begin the proof of Proposition 3.4.

Proof of Proposition 3.4. — Take Ω, γ, $(f_\eta)_\eta$ and $(u_\eta)_\eta$ as in Definition 3.2. Recall that $\mathrm{d}_x\nu$ is the measure on S_x^A defined in (1.1). By using partitions of unity on $\mathbb{R}^n$, we may fix $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ and assume that the functions u_η are supported near ξ_0. Let $\xi_1, \dots, \xi_{n-1} \in \mathbb{R}^n$ be such that $(\xi_0, \xi_1, \dots, \xi_{n-1})$ forms a basis for $\mathbb{R}^n$. For any $x \in U$, let

$$\begin{aligned}\beta_x : (t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1} \longmapsto \sigma(x, \xi_0 + t_1\xi_1 + \dots + t_{n-1}\xi_{n-1})^{-\frac{1}{m}} \\ \times (\xi_0 + t_1\xi_1 + \dots + t_{n-1}\xi_{n-1}) \in S_x^A \,.\end{aligned}$$

The map β_x defines a local coordinate system at $\sigma(x,\xi_0)^{-\frac{1}{m}}\xi_0 \in S_x^A$. Moreover, the map $x \mapsto \beta_x \in C^\infty(\mathbb{R}^{n-1})$ is continuous. The density $g_x = \frac{\beta_x^*(\gamma(x,\cdot)\mathrm{d}_x\mu)}{\mathrm{d}t} \in C^\infty(\mathbb{R}^{n-1})$ also depends continously on $x \in U$ in $C^\infty(\mathbb{R}^{n-1})$. Now, for any $\lambda > 0$, $\eta \in E$ and $(x,\tau) \in \Omega$, if u_η is supported close enough to ξ_0,

$$\int_{S_x^A} e^{i\lambda f_\eta(x,\tau,\xi)} u_\eta(\xi)\gamma(x,\xi)\mathrm{d}_x\nu(\xi) = \int_{\mathbb{R}^{n-1}} e^{i\lambda f_\eta(x,\tau,\beta_x(t))} u_\eta(\beta_x(t)) g_x(t) \mathrm{d}t \,.$$

We now set $\widetilde{E} = U \times \mathbb{R}^n \times E$, for any $\widetilde{\eta} = (x,\tau,\eta) \in \widetilde{E}$, $\widetilde{f}_{\tilde{\eta}} = f_\eta(x,\tau,\beta_x(\,\cdot\,)) \in C^\infty(\mathbb{R}^{n-1})$ and $\widetilde{u}_{\tilde{\eta}} = u_\eta(\beta_x(\,\cdot\,))g_x \in C^\infty(\mathbb{R}^{n-1})$. By compactness, it is enough to fix $(x_0,\tau_0) \in \Omega$ and prove estimate (3.10) for $\widetilde{\eta} = (x,\tau,\eta)$ close enough to $\widetilde{\eta}_0 = (x_0,\tau_0,0)$. Also, without loss of generality, we may assume $x_0 = 0$. Our task is therefore to find $C > 0$ such that for each $\widetilde{\eta}$ close enough to $\widetilde{\eta}_0$ and each $\lambda > 0$,

$$\left| \int_{\mathbb{R}^{n-1}} e^{i\lambda \tilde{f}_{\tilde{\eta}}(t)} \widetilde{u}_{\tilde{\eta}}(t) \mathrm{d}t \right| \leqslant C\lambda^{-\frac{1}{k_0}} \,.$$

We wish to apply Lemma 7.1. The estimate is obvious for $\lambda \leqslant 1$ while, for $\lambda \geqslant 1$, replacing k_0 by some smaller integer would improve the estimate. Thus, we need only to check that there exists $k \in \{1, \dots, k_0\}$ such that

$$\mathrm{d}^k \widetilde{f}_{\tilde{\eta}_0}(0) \neq 0 \,. \tag{7.2}$$

Let $g = \widetilde{f}_{\tilde{\eta}_0}$. Since $f_0(x, \tau_0, \xi) = \langle \tau_0, \xi \rangle$, we have, for all $t \in \mathbb{R}^{n-1}$,

$$g(t) = (\langle \tau_0, \xi_1 \rangle t_1 + \cdots + \langle \tau_0, \xi_{n-1} \rangle t_{n-1} + \langle \tau_0, \xi_0 \rangle) \\ \times \sigma(0, \xi_0 + t_1 \xi_1 + \cdots + t_{n-1} \xi_{n-1})^{-\frac{1}{m}}.$$

We proceed by contradiction and assume that $\mathrm{d}^j g(0) = 0$ for each $j \in \{1, \dots, k\}$. To understand how this condition affects σ we use the following claim which we prove at the end.

Claim 7.2. — *Let $U \subset \mathbb{R}^p$ be an open neighborhood of 0 and $f \in C^\infty(U)$ be positive valued. Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $k \in \mathbb{N}$ such that $k \geqslant 1$. Assume that there exist $b \in \mathbb{R}$ and $\tau \in \mathbb{R}^p$ such that $(\tau, b) \neq (0, 0)$ such that, writing $h : x \in \mathbb{R}^n \mapsto \langle \tau, x \rangle + b \in \mathbb{R}$ we have, for each $j \in \{1, \dots, k\}$,*

$$\mathrm{d}^j [h f^\alpha](0) = 0. \tag{7.3}$$

Then,

$$f(0)^{k-1} \mathrm{d}^k f(0) = (\alpha + 1)(2\alpha + 1) \dots ((k-1)\alpha + 1)(\mathrm{d} f(0))^{\otimes k}. \tag{7.4}$$

We wish to use this claim with $\alpha = -\frac{1}{m}$, $h(t) = \langle \tau_0, \xi_1 \rangle t_1 + \cdots + \langle \tau_0, \xi_{n-1} \rangle t_{n-1} + \langle \tau_0, \xi_0 \rangle$ and $f(t) = \sigma(0, \xi_0 + t_1 \xi_1 + \cdots + t_{n-1} \xi_{n-1})$. In order to apply it, the only thing to check is that h is not identically 0. But $h = 0$ would imply that $\langle \tau_0, \xi_0 \rangle = \cdots = \langle \tau_0, \xi_{n-1} \rangle = 0$. This cannot happen since $\tau_0 \neq 0$. Hence, by Claim 7.2 we have the following equality between (symmetric) k-forms on the hyperplane H spanned by $(\xi_1, \dots, \xi_{n-1})$,

$$\sigma(0, \xi_0)^{k-1} \mathrm{d}_\xi^k \sigma(0, \xi_0)|_{H^k} = C(m, k)(\mathrm{d}_\xi \sigma(0, \xi_0))^{\otimes k}|_{H^k} \tag{7.5}$$

where

$$C(m, k) = \left(-\frac{1}{m} + 1\right) \cdots \left(-\frac{k-1}{m} + 1\right) = \frac{m(m-1)\dots(m-k+1)}{m^k}.$$

Next, we make the following claim, which we prove at the end.

Claim 7.3. — *Let m be a positive real number and let $f \in C^\infty(\mathbb{R}^p \setminus \{0\})$ be a real-valued m-homogeneous function. Then, for each $x \in \mathbb{R}^p \setminus \{0\}$, each hyperplane $H \subset \mathbb{R}^p$ not containing x and each $k_0 \geqslant 2$,*

$$\forall\, k \in \{2, \dots, k_0\}, \\ f(x)^{k-1} \mathrm{d}^k f(x) = \frac{m(m-1)\dots(m-k+1)}{m^k} (\mathrm{d} f(x))^{\otimes k} \tag{7.6}$$

is equivalent to

$$\forall\, k \in \{2, \dots, k_0\}, \\ f(x)^{k-1} \mathrm{d}^k f(x)\big|_H = \frac{m(m-1)\dots(m-k+1)}{m^k} (\mathrm{d} f(x))^{\otimes k}\big|_H. \tag{7.7}$$

This claim implies that σ actually satisfies Equation (7.5) on the whole of $T_\xi^*\mathbb{R}^n \simeq \mathbb{R}^n$. By the assumption on σ, this equation cannot be satisfied for all $k \leqslant k_0$. Hence, $\mathrm{d}^k g(0)$ cannot vanish for each $k \in \{1,\dots,k_0\}$ (recall that $g(t) = \widetilde{f}_{\tilde{\eta}_0}(t) = f_0(x_0,\tau_0,\beta_{x_0}(t)))$. In particular, there exists $k \in \{1,\dots,k_0\}$ for which $\widetilde{f}_{\tilde{\eta}}$ satisfies Equation (7.2). Hence, Lemma 7.1 applies for this k and we are done. All that remains is to prove Claims 7.2 and 7.3.

Proof of Claim 7.2. — Let f, τ, b, α, h and k be as in the statement of the claim. Let $g(x) = h(x)^{-\frac{1}{\alpha}}$. First of all, by Equation (7.3) with $j=1$,

$$f(0)\tau = -\alpha b \mathrm{d}f(0)$$

In particular, since $(\tau, b) \neq 0$ and $f(0) > 0$, we actually have $b \neq 0$. Thus, the function $g : x \mapsto h(x)^{-\frac{1}{\alpha}}$ is well defined and positive near the origin. Moreover, $hg^\alpha = 1$ so all of its derivatives vanish. Consequently, for each $j \in \{1,\dots,k\}$, $\mathrm{d}^j(f^\alpha g^{-\alpha})(0) = 0$ which in turn gives, for each $j \in \{1,\dots,k\}$, $\mathrm{d}^j(fg^{-1})(0) = 0$ (here we use the fact that $fg^{-1} = (f^\alpha g^{-\alpha})^{\frac{1}{\alpha}}$ which is well defined near 0). In particular, the Taylor expansions of f and g coincide to the k-th order up to a multiplicative constant. By homogeneity of Equation (7.4) we may assume that they agree up to order k. But

$$\begin{aligned}\mathrm{d}^k g(0) &= \prod_{j=0}^{k-1}\left(-\frac{1}{\alpha} - j\right) \times b^{-\frac{1}{\alpha}-k}\tau^{\otimes k} \\ &= \left(b^{-\frac{1}{\alpha}}\right)^{1-k}(\alpha+1)(2\alpha+1)\dots((k-1)\alpha+1)\left(-\alpha b^{\frac{1}{\alpha}+1}\right)^{-k}\tau^{\otimes k}\end{aligned}$$

and $g(0) = b^{-\frac{1}{\alpha}}$ and $\mathrm{d}g(0) = (-\alpha b^{\frac{1}{\alpha}+1})^{-1}\tau$. Thus,

$$g(0)^{k-1}\mathrm{d}^k g(0) = (\alpha+1)(2\alpha+1)\dots((k-1)\alpha+1)(\mathrm{d}g(0))^{\otimes k}.$$

Since f agrees with g up to order k, f satisfies Equation (7.4). ☐

Proof of Claim 7.3. — Equation (7.6) implies (7.7) by restriction to H. Let us assume (7.7) and prove the converse. Since $x \notin H$, $\mathbb{R}x \oplus H$ generate $\mathbb{R}^p$. By multilinearity, it is enough to prove (7.6) when the k forms are evaluated on families of the form $(x,\dots,x,y_1,\dots,y_h)$ where $y_1,\dots,y_h \in H$ and $h \in \{1,\dots,k\}$. Now, since f is homogeneous, by Euler's Equation (see (7.1)), for any $h \leqslant k$, and for any $y_1,\dots,y_h \in H$,

$$\mathrm{d}^k f(x)(x,\dots,x,y_1,\dots,y_h) = \underbrace{(m-h)\dots(m-k+1)}_{1 \text{ if } k=h}\mathrm{d}^h f(x)(y_1,\dots,y_h)$$

and

$$(\mathrm{d}f(x))^{\otimes k}(x,\dots,x,y_1,\dots,y_h) = m^{k-h}f^{k-h}(x)(\mathrm{d}f(x))^{\otimes h}(y_1,\dots,y_h).$$

Applying (7.7) to compare the right hand sides of each line we get Equation (7.6). ☐

This concludes the proof of Proposition 3.4 □

The proof of Lemma 7.1 will combine two results from singularity theory and oscillatory integral asymptotics which we state now.

The following theorem is a corollary of the Malgrange preparation theorem presented in [12]. We give a slightly different formulation and add the continuity with respect to smooth perturbations, which actually follows from Hörmander's original proof.

THEOREM 7.4 ([12, Theorem 7.5.13]). — *Let $U \subset \mathbb{R}\times\mathbb{R}^n$ (resp. $E \subset \mathbb{R}^p$) be an open neighborhood of $0 \in \mathbb{R}\times\mathbb{R}^n$ (resp. $0 \in \mathbb{R}^p$) and $(f_\eta)_{\eta\in E}$ be a continuous family of smooth functions on U. We denote by (t,x) the elements of U. Let $k \in \mathbb{N}$, $k \geqslant 2$. Assume that for each $\eta \in E$ and $j \in \{0,\dots,k-1\}$*

$$\partial_t^j f_\eta(0,0) = 0$$

and that $\partial_t^k f_\eta(0,0) > 0$. Then, there exist $W \subset \mathbb{R}\times\mathbb{R}^n$ (resp. $V \subset \mathbb{R}^n$) a neighbohood of $0 \in \mathbb{R}\times\mathbb{R}^n$ (resp. $0 \in \mathbb{R}^n$) with $U' \subset \mathbb{R}\times V$ such that for each $\eta \in E$, there exist $\phi_\eta \in C^\infty(W)$ as well as $a_\eta^1,\dots,a_\eta^{k-1} \in C^\infty(V)$, satisfying, for any $\eta \in E$, $(t,x) \in W$,

$$\begin{aligned} \phi_\eta(0,0) &= 0\,,\\ \partial_t\phi_\eta(0,0) &> 0\,,\\ a_\eta^1(0) = \cdots = a_\eta^{k-1}(0) &= 0\,,\\ \textit{and}\quad f_\eta(\phi_\eta(t,x),x) &= t^k + \sum_{j=0}^{k-1} a_\eta^j(x)t^j\,. \end{aligned}$$

Moreover, one can choose these functions such that the maps $\eta \mapsto \phi_\eta$ and $\eta \mapsto a_\eta^j$ are continuous into C^∞.

The Lemma 7.5 is a direct consequence of Van der Corput's lemma.

LEMMA 7.5 ([28, Chapter 8, Section 1.2]). — *Let $k \in \mathbb{N}$, $k \geqslant 2$. There exist $\delta = \delta(k) > 0$, $V \subset \mathbb{R}^k$ an open neighborhood of 0 such that for all $u \in C_c^\infty(]-\delta,\delta[)$ there exists $C(u) < +\infty$ such that for all $\lambda > 0$ and $(a_0,\dots,a_{k-1}) \in V$,*

$$\left|\int_{\mathbb{R}} e^{i\lambda(t^k + a_{k-1}t^{k-1}+\cdots+a_0)}u(t)\mathrm{d}t\right| \leqslant C(u)\lambda^{-\frac{1}{k}}\,.$$

Moreover $C(u)$ depends continuously on $u \in C_c^\infty(]-\delta,\delta[)$.

Proof of Lemma 7.1. — Let $(f_\eta)_\eta$ and $k \geqslant 1$ be as in the statement of the lemma. Assume that $\mathrm{d}^k f_0(0) \neq 0$. Then, by the multilinear polarization formula (see [21, Chapter 3, Section 2]), there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that $\mathrm{d}^k f_0(0)(v,v,\dots,v) \neq 0$. Without loss of generality, we may assume that $v = e_n := (0, \dots, 0, 1)$. We write $x = (\widetilde{x}, x_n) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. Let $u \in C_c^\infty(U)$ be such that $u(\widetilde{x}, x_n) \neq 0$ implies that $\|\widetilde{x}\|_\infty \leqslant 1$. Then, for each $\eta \in E$ and $\lambda > 0$,

$$\left| \int_U e^{i\lambda f_\eta(x)} u(x) \mathrm{d}x \right| \leqslant \max_{\tilde{x} \in \mathbb{R}^{n-1}} \left| \int_{\mathbb{R}} e^{i\lambda f_\eta(\tilde{x}, x_n)} u(\widetilde{x}, x_n) \mathrm{d}x_n \right| .$$

This way by replacing η by $(\eta, \widetilde{x})$ and f_η by $f_\eta(\widetilde{x}, \cdot\,)$ we have reduced the problem to the one dimensional case. From now on, we assume that $n = 1$, so that U is an open subset of $\mathbb{R}$.

For each $\eta \in E$, each $x \in U$, and $q = (q_0, \dots, q_{k-1}) \in \mathbb{R}^k$, let

$$\begin{aligned} g_\eta(x, q) = f_\eta(x) - f_\eta(0) - f_\eta'(0)x - \dots - \frac{1}{(k-1)!} f_\eta^{(k-1)}(0) x^{k-1} \\ + q_0 + q_1 x + \dots + q_{k-1} x^{k-1} . \end{aligned}$$

We will first prove the desired bound where we replace f_η by $g_\eta(\,\cdot\,, q)$, uniformly for q close enough to 0 and then deduce the result for f_η itself as a phase.

The map $\eta \mapsto g_\eta$ is continuous from E to $C^\infty(U \times \mathbb{R}^k)$. Moreover, for each $\eta \in E$ close enough to 0 we have

$$\forall\, j \in \{0, \dots, k-1\}, \quad \partial_x^j g_\eta(0,0) = 0 \quad \partial_x^k g_\eta(0,0) \neq 0.$$

Replacing f_η by $-f_\eta$ does not change the estimate since it amounts to complex conjugation of the integrand. With this in mind, we may assume that $\partial_x^k g_\eta(0,0) > 0$. By Theorem 7.4, there exist $W \subset \mathbb{R} \times \mathbb{R}^k$ and continuous families of smooth functions $(a_\eta^1)_\eta \dots, (a_\eta^{k-1})_\eta$, as well as $(\phi_\eta)_\eta$ defined respectively in a neighborhood of $0 \in \mathbb{R}^k$ and a neighborhood of $(0,0)$ in $U \times \mathbb{R}^k$ such that for each $\eta \in E$ and (x, q) close enough to 0 and $(0,0)$ respectively,

$$\begin{aligned} \phi_\eta(0,0) = 0 \,, \\ \partial_t \phi_\eta(0,0) > 0 \,, \\ a_\eta^1(0) = \dots = a_\eta^{k-1}(0) = 0 \,, \\ \text{and } g_\eta(\phi_\eta(x,q), q) = x^k + \sum_0^{k-1} a_\eta^j(q) x^j \,. \end{aligned}$$

Hence, if $u \in C_c^\infty(\mathbb{R})$ is supported close enough to 0, we have, for all $\eta \in E$ close enough to 0 and all $q \in \mathbb{R}^k$,

$$\begin{aligned} &\int_{\mathbb{R}} e^{i\lambda g_\eta(y,q)} u(y) \mathrm{d}y \\ &\qquad = \int_{\mathbb{R}} e^{i\lambda(x^k + a_\eta^{k-1}(q)x^{k-1} + \dots + a_\eta^0(q))} u(\phi_\eta(x,q)) \left(\phi_\eta^{-1}(\,\cdot\,, q)\right)'(x) \mathrm{d}x \,. \end{aligned}$$

By Lemma 7.5, there exist $W_1 \subset \mathbb{R}^k \times E$ a neighborhood of 0 such that $\delta > 0$ such that for each $(q, \eta) \in W_1$, for each $v \in C_c^\infty(]-\delta, \delta[)$, there exists $C'(v) < +\infty$ such that for each $\lambda > 0$ and each (q, η) close enough to $(0, 0)$,

$$\left| \int_{\mathbb{R}} e^{i\lambda(x^k + a_\eta^{k-1}(q)x^{k-1} + \cdots + a_\eta^0(q))} v(x) \mathrm{d}x \right| \leqslant C'(v)\lambda^{-\frac{1}{k}} .$$

Moreover, Lemma 7.5 specifies that the map $v \in C_c^\infty(]-\delta, \delta[) \mapsto C'(v) \in \mathbb{R}$ is continuous. By continuity, there exist $\varepsilon > 0$ and $W_2 \subset W_1$ a compact neighborhood of 0 such that for any $(q, \eta) \in W_2$ and any $x \in \mathbb{R}$ with $|x| \geqslant \delta/2$, $|\phi_\eta(x, q)| \geqslant \varepsilon$. In particular, the map $(q, \eta, u) \in W_2 \times C_c^\infty(]-\varepsilon, \varepsilon[) \mapsto u(\phi_\eta(\,\cdot\,, q)) \left(\phi_\eta(\,\cdot\,, q)^{-1}\right)' \in C_c^\infty(]-\delta, \delta[)$ is well defined and continuous. Consequently, so is the map

$$\begin{aligned} W_2 \times C_c^\infty(]-\varepsilon, \varepsilon[) &\longrightarrow \mathbb{R} \\ (q, \eta, u) &\longmapsto C_{q,\eta}(u) = C'\left(u(\phi_\eta(\,\cdot\,, q)) \left(\phi_\eta(\,\cdot\,, q)^{-1}\right)'\right). \end{aligned}$$

By compactness, $C(u) = \sup_{(q,\eta)\in W_2} C_{q,\eta}(u)$ is finite and continuous in u. We have proved that for any $(q, \eta) \in W_2$, any $\lambda > 0$ and any $u \in C_c^\infty(]-\varepsilon, \varepsilon[)$,

$$\left| e^{i\lambda g_\eta(y,q)} u(y) \mathrm{d}y \right| \leqslant C(u)\lambda^{-\frac{1}{k}} .$$

To obtain the corresponding estimate with f_η instead of $g_\eta(\,\cdot\,, q)$, we make the following two observations. First, for each $\eta \in E$, and $x \in U$,

$$g_\eta(x, f_\eta(0), \ldots, f_\eta^{(k-1)}(0)) = f_\eta(x) .$$

Second, since $f_0(0) = \cdots = f_0^{(k-1)}(0) = 0$, there exists $E' \subset E$ a neighborhood of 0 such that for each $\eta \in E'$, $(f_\eta(0), \ldots, f_\eta^{(k-1)}(0), \eta) \in W_2$. Thus, for each $\eta \in E'$ each $u \in C_c^\infty(]-\varepsilon, \varepsilon[)$ and each $\lambda > 0$,

$$\left| e^{i\lambda f_\eta(y)} u(y) \mathrm{d}y \right| \leqslant C(u)\lambda^{-\frac{1}{k}}$$

and the proof is over. □

7.2. Genericity and stability of the non-degeneracy condition

The goal of this subsection is to prove Proposition 7.6 below, which says roughly that admissible homogeneous symbols are stable and generic. To give a precise meaning to this statement, use the topology we defined on the space of homogeneous symbols $S_h^m(U)$ (see Definition 2.7). We have the following proposition.

Proposition 7.6. — *For all $n \in \mathbb{N}$, $n \geqslant 2$ we define $k_0 = k_0(n) \in \mathbb{N}$ as follows. We set $k_0(2) = 5$, $k_0(3) = 3$, $k_0(4) = 3$ and $\forall\, n \geqslant 5$, $k_0(n) = 2$. Fix $n \geqslant 2$ and $m > 0$. Let $U \subset \mathbb{R}^n$ be an open subset. Then, the set of $\sigma \in S^m_{h,+}(U)$ such that for each $(x,\xi) \in U \times (\mathbb{R}^n \setminus \{0\})$ there exists $j \in \{2,\dots,k_0\}$ such that*

$$\sigma^{j-1}(x,\xi)\mathrm{d}^j_\xi\sigma(x,\xi) \neq \frac{m(m-1)\dots(m-j+1)}{m^j}\left(\mathrm{d}_\xi\sigma(x,\xi)\right)^{\otimes j} \tag{7.8}$$

is open and dense in $S^m_{h,+}(U)$.

To prove this proposition, we will apply Thom's transversality theorem (see [9, Theorem 4.9 of Chapter II]) to a well chosen submanifold of the jet bundle of $U \times S^{n-1}$ whose codimension grows with the degree of admissibility we consider. Lemmas 7.7, 7.8, 7.9, 7.10 and 7.11 below are devoted to the construction of this manifold. The proof of Proposition 7.6 is presented only after these are stated and proved. Throughout the rest of the section we fix $n \in \mathbb{N}$, $n \geqslant 2$, $U \subset \mathbb{R}^n$ an open subset and $m \in \mathbb{R}$, $m > 0$. We start by introducing some notation.

Notation.

(1) For each $j,p \in \mathbb{N}$, $p \geqslant 1$, let Sym^j_p be the space of symmetric j-linear forms over $\mathbb{R}^p$. This is a vector space of dimension $\binom{p+j-1}{j}$. We adopt the convention that $\mathrm{Sym}^0_p = \mathbb{R}$.
(2) Let Z be a smooth manifold. For each $k \geqslant 0$ we denote by $\mathcal{J}^k(Z)$ the k-th jet space of mappings from X to $\mathbb{R}$, that is, the space $J^k(Z,\mathbb{R})$ introduced in [9, Definition 2.1 of Chapter II]. For any $p \in \mathbb{N}$ and any open subset $V \subset \mathbb{R}^p$, the space $\mathcal{J}^k(V)$ is canonically isomorphic to $V \times \bigoplus_{j=0}^k \mathrm{Sym}^j_p$. We will denote its elements by (ξ,ω) where $\xi \in V$ and $\omega = (\omega_0,\dots,\omega_k) \in \bigoplus_{j=0}^k \mathrm{Sym}^j_p$.
(3) Let Z be a smooth manifold and $k \in \mathbb{N}$. For each $f \in C^\infty(Z)$, we write $j^k f$ for the section of $\mathcal{J}^k(Z)$ whose value at each point is the k-jet of f at this point (see [9, paragraph below Definition 2.1 of Chapter II]). At a point $x \in Z$, the k-jet $j^k f(x)$ is essentially the Taylor expansion of order k of f at x. In fact, if $Z = \mathbb{R}^p$, $j^k f(x) = (x, f(x), \mathrm{d}f(x), \dots, \mathrm{d}^k f(x))$.

Since the jet bundle $\mathcal{J}^k(\mathbb{R}^n \setminus \{0\})$ is quite explicit, we will make most of our contructions inside it and them "push them down" onto the sphere. In the following lemma, we build the map we need to "push down" our constructions.

Lemma 7.7. — *Let $\iota : S^{n-1} \to \mathbb{R}^n$ be the canonical injection. Then, there exists a bundle morphism*

$$\rho : \iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\}) \to \mathcal{J}^k(S^{n-1})$$

such that the following diagram commutes:

$$\begin{tikzcd} C^\infty(\mathbb{R}^n \setminus \{0\}) \arrow[r, "\iota^*"] \arrow[d, "\iota^*(j^k\cdot)"'] & C^\infty(S^{n-1}) \arrow[d, "j^k"] \\ \iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\}) \arrow[r, "\rho"] & \mathcal{J}^k(S^{n-1}). \end{tikzcd}$$

Here the top arrow is the restriction map while the left arrow is the restriction of the k-jet to the sphere.

Lemma 7.7 states that it is equivalent to restrict a function to the sphere and consider its k-jet or to consider its k-jet and restrict it to the tensor powers of the tangent space of the sphere.

Proof. — We construct ρ by defining its action on each fiber. Let $\xi \in S^{n-1}$ and let (V, ϕ) be a chart $\phi : V \to \mathbb{R}^{n-1}$ of S^{n-1} near ξ. Then, for each $f \in C^\infty(\mathbb{R}^n)$, the k-th order Taylor expansion of $f \circ \phi^{-1}$ at ξ depends only on the k-th order Taylor expansion of f at ξ and the dependence is linear. This defines a linear map $\rho|_\xi : \iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\})|_\xi \to \mathcal{J}^k(S^{n-1})|_\xi$. The corresponding fiberwise map ρ is clearly smooth and defines a morphism of smooth vector bundles. Moreover, by construction, for each $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and each $\xi \in S^{n-1}$, $\rho|_\xi \left(j^k f(\xi)\right) = j^k(f \circ \iota)(\xi)$ so the diagram does indeed commute. □

Notation. — For each $k \in \mathbb{N}$, each $\xi \in \mathbb{R}^n$ and each $\omega = (\omega_0, \dots, \omega_k) \in \bigoplus_{j=0}^k \mathrm{Sym}_n^j$ we introduce the following notation. For each $j \in \{0, \dots, k\}$, $\omega_j|_{\xi^\perp}$ is the restriction of ω_j to the orthogonal of ξ in $\mathbb{R}^n$. Moreover, we set $\omega|_{\xi^\perp} = (\omega_0|_{\xi^\perp}, \dots, \omega_k|_{\xi^\perp})$.

In the following lemma, we check that the set of jets of homogeneous maps is a smooth submanifold of $\iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\})$ and give an explicit description of it. Moreover, we show that the "push down" map ρ maps it diffeomorphically on the space $\mathcal{J}^k(S^{n-1})$.

Lemma 7.8. — *Fix $k \in \mathbb{N}$. Let H_m^k be the subset of $\iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\})$ of jets of m-homogeneous symbols. Then,*

(1) *The set H_m^k is characterized by the following equations:*

$$H_m^k = \bigcap_{j=0}^{k-1} \left\{ (\xi, \omega) \in \iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\}) \middle| \omega_{j+1}(\xi, \dots) = (m-j)\omega_j \right\}.$$

(2) *The set* H^k_m *is a submanifold of* $\iota^*\mathcal{J}^k\left(U\times\mathbb{R}^n\setminus\{0\}\right)$ *of the same dimension as* $\mathcal{J}^k(S^{n-1})$.
(3) *The map* $\rho|_{H^k_m}: H^k_m\to\mathcal{J}^k\left(U\times S^{n-1}\right)$ *is a diffeomorphism.*

Proof. — We set

$$\widetilde{H^k_m}=\bigcap_{j=0}^{k-1}\left\{(\xi,\omega)\in\iota^*\mathcal{J}^k(\mathbb{R}^n\setminus\{0\})\middle|\,\omega_{j+1}(\xi,\dots)=(m-j)\omega_j\right\}.$$

Firstly, each m-homogeneous $f\in C^\infty(\mathbb{R}^n\setminus\{0\})$, satisfies Euler's equation (see (7.1)). Next, notice that if f is m-homogeneous, then, for each $j\in\{1,\dots,k\}$, $\xi\mapsto \mathrm{d}^jf(\xi)$ is homogeneous of order $m-j$ so that for each $\xi\in\mathbb{R}^n\setminus\{0\}$, $\mathrm{d}^{j+1}f(\xi)(\xi,\dots)=(m-j)\mathrm{d}^jf(\xi)$. Therefore, for each $\xi\in\mathbb{R}^n\setminus\{0\}$, $j^kf(\xi)\in\widetilde{H^k_m}$. We have shown that $H^k_m\subset\widetilde{H^k_m}$. Next, notice that for each $f\in C^\infty(S^{n-1})$, the m-homogeneous function $\xi\mapsto|\xi|^mf\left(\frac{\xi}{|\xi|}\right)$ restricts back to f on S^{n-1}. Therefore, we have $\mathcal{J}^k(S^{n-1})=\rho(H^k_m)\subset\rho(\widetilde{H^k_m})\subset\mathcal{J}^k(S^{n-1})$. So we have

$$\rho(H^k_m)=\rho\left(\widetilde{H^k_m}\right)=\mathcal{J}^k(S^{n-1}).\tag{7.9}$$

Given this equation, in order to prove the lemma, it is enough to prove points (2) and (3) with H^k_m replaced by $\widetilde{H^k_m}$, which we call (2′) and (3′) respectively. Indeed, point (3′) will imply that $\rho|_{\widetilde{H^k_m}}$ is one-to-one so by Equation (7.9), we will have $H^k_m=\widetilde{H^k_m}$ which is point (1). Moreover, since we will have already proved points (2) and (3) for $\widetilde{H^k_m}$ we will have them for H^k_m. Let us start by proving (2′). For each $j\in\{0,\dots,k-1\}$ set

$$F^j_m:(\xi,\omega)\mapsto\omega_{j+1}(\xi,\dots)-(m-j)\omega_j$$

so that $\widetilde{H^k_m}=\cap_{j=0}^{k-1}\left(F^j_m\right)^{-1}(0)$. Let us prove that the map

$$F_m=\left(F^0_m,\dots,F^{k-1}_m\right):\iota^*\mathcal{J}^k(\mathbb{R}^n\setminus\{0\})\to\bigoplus_{j=0}^{k-1}\mathrm{Sym}^j_n$$

is a submersion. Fix $(\xi,\omega)\in\iota^*\mathcal{J}^k(\mathbb{R}^n\setminus\{0\})$. Let $(\eta_0,\dots,\eta_{k-1})\in\bigoplus_{j=0}^{k-1}\mathrm{Sym}^j_n\simeq T_{F_m(\xi,\omega)}\bigoplus_{j=0}^{k-1}\mathrm{Sym}^j_n$. Then, for each $j\in\{0,\dots,k-1\}$,

$$\mathrm{d}_{\omega_{j+1}}F^j_m(\xi,\omega)\left(|\xi|^{-2}\langle\xi,\cdot\rangle\otimes\eta_j\right)=\eta_j\,.$$

In particular, $\mathrm{d}_{(\xi,\omega)}F_m$ is surjective. Therefore $\widetilde{H^k_m}$ is a submanifold of $\iota^*\mathcal{J}^k(\mathbb{R}^n\setminus\{0\})$ of codimension

$$\mathrm{codim}_{\iota^*\mathcal{J}^k(\mathbb{R}^n\setminus\{0\})}\left(\widetilde{H^k_m}\right)=\sum_{j=0}^{k-1}\dim(\mathrm{Sym}^j_n)=\sum_{j=0}^{k-1}\binom{n+j-1}{j}.$$

Indeed, recall that $\dim(\mathrm{Sym}_n^j) = \binom{n+j-1}{j}$. Using this identity, we also have:

$$\dim\big(\iota^*\mathcal{J}^k(\mathbb{R}^n \setminus \{0\})\big) = (n-1) + \sum_{j=0}^{k} \binom{n+j-1}{j};$$
$$\dim\big(\mathcal{J}^k(S^{n-1})\big) = (n-1) + \sum_{j=0}^{k} \binom{n+j-2}{j}.$$

Therefore, firstly $\dim(\widetilde{H_m^k}) = (n-1) + \binom{n+k-1}{k}$ and secondly

$$\dim\Big(\widetilde{H_m^k}\Big) - \dim\big(\mathcal{J}^k(S^{n-1})\big) = \binom{n+k-1}{k} - \sum_{j=0}^{k} \binom{n+j-2}{j} = 0\,. \quad (7.10)$$

In the last equality we use a well known binomial formula which is easily checked by induction on k. The conclusion here is that $\widetilde{H_m^k}$ has the same dimension as $\mathcal{J}^k(S^{n-1})$ so we have proved $(2')$. To prove $(3')$ observe that ρ is linear on each fiber of $\iota^*\mathcal{J}^k(\mathbb{R}^n \setminus \{0\})$ so that its derivative $d\rho$ is constant on each fiber. Moreover, it is equivariant with respect to the automorphisms of the base space S^{n-1} so its derivative must have the same rank on different fibers. Since ρ is surjective (see Equation (7.9)) $\mathrm{d}\rho$ must be of maximal rank. This proves that ρ is a local diffeomorphism. But since it is a morphism of vector bundles, it must be a diffeomorphism, which is the claim of $(3')$. This concludes the proof of the lemma. □

In the following lemma, we build a submanifold of H_m^k that describes the condition of non-admissibility and compute its codimension.

Lemma 7.9. — *For each $k \in \mathbb{N}$, $k \geqslant 2$, define*

$$Y_m^k = \bigcap_{j=2}^{k} \left\{ (\xi,\omega) \in \iota^*\mathcal{J}^k(\mathbb{R}^n \setminus \{0\}) \;\middle|\; \begin{array}{l} \omega_0 > 0,\ \omega_0^{j-1}\omega_j|_{\xi^\perp} \\ \quad = \frac{m(m-1)\dots(m-j+1)}{m^j}\left(\omega_1|_{\xi^\perp}\right)^{\otimes j} \end{array} \right\}.$$

Then, $Y_m^k \cap H_m^k$ is a closed submanifold of H_m^k of codimension $\sum_{j=2}^{k} \binom{n+j-2}{j}$.

Proof. — For each $j \in \{0,\dots,k-1\}$, each $l \in \{2,\dots,k\}$ and each $(\xi,\omega) \in \iota^*\mathcal{J}^k(\mathbb{R}^n \setminus \{0\})$, let, as before, $F_m^j(\xi,\omega) = \omega_{j+1}(\xi,\dots) - (m-j)\omega_j \in \mathrm{Sym}_n^{j-1}$. Moreover, let $\mathrm{Sym}_n^l|_{\xi^\perp}$ be the set of symmetric l-linear forms acting on the orthogonal of ξ in $\mathbb{R}^n$ and let $G_m^l(\xi,\omega) = \omega_l|_{\xi^\perp} - \frac{m(m-1)\dots(m-l+1)}{m^l}\left(\omega_1|_{\xi^\perp}\right)^{\otimes l} \in \mathrm{Sym}_n^l|_{\xi^\perp}$. Then, $Y_m^k \cap H_m^k$ is the intersection of the zero sets of the functions F_m^j and G_m^l for $j \in \{0,\dots,k-1\}$ and $l \in \{2,\dots,k\}$. In particular, it is closed. Note first that $\partial_{\omega_0} F_m^0 = m \neq 0 \in \mathrm{Hom}\left(\mathrm{Sym}_n^0, \mathrm{Sym}_n^0\right) \simeq \mathbb{R}$. In particular this map is invertible. We will now prove that for each $l \in \{2,\dots,k\}$, the map $(\mathrm{d}_{\omega_l} F_m^{l-1}, \mathrm{d}_{\omega_l} G_m^l)$ is of maximal rank on Y_m^k. For any $(\xi,\omega) \in Y_m^k$ and any $l \in \{2,\dots,k\}$, $(\mathrm{d}_{\omega_l} F_m^{l-1}(\xi,\omega), \mathrm{d}_{\omega_l} G_m^l(\xi,\omega))$ acts as follows.

$$\begin{aligned} \mathrm{Sym}_n^l &\longrightarrow \mathrm{Sym}_n^{l-1} \bigoplus \mathrm{Sym}_n^l |_{\xi^\perp} \\ \eta_l &\longmapsto (\eta_l(\xi,\dots), \omega_0^{l-1}\eta_l|_{\xi^\perp})\,. \end{aligned}$$

But this map is invertible. To see this, let $pr_{\xi^\perp}^* : \mathrm{Sym}_n^l|_{\xi^\perp} \to \mathrm{Sym}_n^l$ be the pull-back map by the orthogonal projection onto the orthogonal of ξ. Also, recall that on Y_m^k, we have $\omega_0 > 0$. Then, the inverse of $(\mathrm{d}_{\omega_l} F_m^{l-1}(\xi,\omega), \mathrm{d}_{\omega_l} G_m^l(\xi,\omega))$ is

$$\begin{aligned} \mathrm{Sym}_n^{l-1} \bigoplus \mathrm{Sym}_n^l |_{\xi^\perp} &\longrightarrow \mathrm{Sym}_n^l \\ (\eta_{l-1}, \eta|_\perp) &\longmapsto |\xi|^{-2}\langle \xi, \cdot\rangle \otimes \eta_{l-1} + \omega_0^{1-l} pr_{\xi^\perp}^* \eta_\perp\,. \end{aligned}$$

All in all, we have shown so far that $\partial_{\omega_0} F_m^0$ is surjective and that for each $l \in \{2,\dots,k\}$, $(\mathrm{d}_{\omega_l} F_m^{l-1}, \mathrm{d}_{\omega_l} G_m^l)$ is of maximal rank. Therefore, $Y_m^k \cap H_m^k$ is a submanifold of H_m^k of codimension

$$\begin{aligned} &\mathrm{codim}_{H_m^k}(Y_m^k \cap H_m^k) \\ &\qquad = \mathrm{codim}_{\iota^*\mathcal{J}^k(\mathbb{R}^n\setminus\{0\})}(Y_m^k \cap H_m^k) - \mathrm{codim}_{\iota^*\mathcal{J}^k(\mathbb{R}^n\setminus\{0\})}(H_m^k) \\ &\qquad = 1 + \sum_{l=2}^k \binom{n+l-1}{l} - \sum_{j=0}^{k-1}\binom{n+j-1}{j} \\ &\qquad = \binom{n+k-1}{k} - \binom{n+1-1}{1} \\ &\qquad = \sum_{j=2}^k \binom{n+j-2}{j} \end{aligned}$$

where in the last line we use the same binomial identity as in (7.10). ☐

So far we have neglected the U coordinate in the product $U \times S^{n-1}$. To take this coordinate into account, in the following lemma, we introduce a submersion $pr_2 : \mathcal{J}^k\left(U \times S^{n-1}\right) \to \mathcal{J}^k(S^{n-1})$ by which we will pull back the submanifold $\rho\left(Y_m^k\right)$.

Lemma 7.10. — *Let $k \in \mathbb{N}$. Let $\pi : U \times S^{n-1} \to S^{n-1}$ be the map $(x,\xi) \mapsto \xi$. Also, for each $x \in U$, let $\iota_x : S^{n-1} \to U \times S^{n-1}$ be the map $\xi \mapsto (x,\xi)$. Then, there exists a surjective vector bundle morphism $pr_2 : \mathcal{J}^k\left(U \times S^{n-1}\right) \to \pi^*\mathcal{J}^k(S^{n-1})$ such that for each $x \in U$, the following* *diagram commutes:*

$$\begin{tikzcd}
C^\infty\left(U\times S^{n-1}\right) \arrow[rr, "\iota_x^*"] \arrow[d, "j^k"] & & C^\infty(S^{n-1}) \arrow[d, "j^k"] \\
\mathcal{J}^k\left(U\times S^{n-1}\right) \arrow[r, "pr_2"] & \pi^*\mathcal{J}^k(S^{n-1}) \arrow[r, "\iota_x^*"] & \mathcal{J}^k(S^{n-1}).
\end{tikzcd}$$

In particular, pr_2 is a submersion.

Proof. — Given $f\in C^\infty\left(U\times S^{n-1}\right)$ and $x\in U$, the k-jet of $f(x,\cdot)$ at $\xi\in S^{n-1}$ depends only on the k-jet of f at (x,ξ). This allows us to define a map $pr_2|_{(x,\xi)}:\mathcal{J}^k\left(U\times S^{n-1}\right)|_{(x,\xi)}\to\pi^*\mathcal{J}^k(S^{n-1})_{(x,\xi)}$. This defines a bundle morphism $pr_2:\mathcal{J}^k\left(U\times S^{n-1}\right)\to\pi^*\mathcal{J}^k\left(\right)$. The fact that the diagram commutes follows by construction. Finally, since the composition of the top and right arrows : $j^k\circ\iota_x^*$ is onto, so is the composition of the left and bottom arrows. But this implies that the composition of bottom arrows is onto. Since $\pi^*\mathcal{J}^k(S^{n-1})$ and $\mathcal{J}^k(S^{n-1})$ have the same rank, then pr_2 must also be onto. In particular, it defines a submersion from the manifold $\mathcal{J}^k\left(U\times S^{n-1}\right)$ to the manifold $\pi^*\mathcal{J}^k(S^{n-1})$. $\square$

In this last lemma, we check that the previous construction does indeed characterize non-admissibility of a symbol by the intersection of the k-jet with the submanifold constructed in Lemma 7.9 and "pushed down" by ρ.

Lemma 7.11. — *Let $k\in\mathbb{N}$, $k\geqslant 2$. Let $\sigma\in S^m_{h,+}(U)$. Then, the two following statements are equivalent:*

- *There exists $(x,\xi)\in U\times(\mathbb{R}^n\setminus\{0\})$ such that for each $j\in\{2,\dots,k\}$*

$$\sigma^{j-1}(x,\xi)\mathrm{d}^j_\xi\sigma(x,\xi)=\frac{m(m-1)\dots(m-j+1)}{m^j}\left(\mathrm{d}_\xi\sigma(x,\xi)\right)^{\otimes j}\tag{7.11}$$

- *The image $pr_2\circ j^k\left(\sigma|_{U\times S^{n-1}}\right)\left(U\times S^{n-1}\right)$ intersects $\rho\left(Y^k_m\right)$.*

Proof. — Firstly, Equation (7.11) is homogeneous in ξ so there exists a pair $(x,\xi)\in U\times(\mathbb{R}^n\setminus\{0\})$ satisfying it if and only if there exists such a pair in $U\times S^{n-1}$. Now, since σ is m-homogeneous, for each $x\in U$, $j^k(\sigma(x,\cdot))(S^{n-1})\subset H^k_m$. Therefore, $(x,\xi)\in U\times S^{n-1}$ satisfy Equation (7.11) if and only if $j^k(\sigma(x,\cdot))(\xi)\in Y^k_m\cap H^k_m$ (here we use that the symbols are positive, as well as m-homogeneous). Since, moreover, by Lemma 7.9, $\rho|_{H^k_m}$ is bijective, this is equivalent to $\rho\circ j^k(\sigma(x,\cdot))(\xi)\in\rho(Y^k_m)$. But, by Lemmas 7.7 and 7.10, $\rho\circ j^k(\sigma(x,\cdot))=j^k(\sigma(x,\cdot)|_{S^{n-1}})=pr_2\circ j^k\left(\sigma|_{U\times S^{n-1}}\right)(x,\cdot)$. To conclude, we have proved that for any $(x,\xi)\in U\times S^{n-1}$, (x,ξ) satisfies Equation (7.11) if and only if $pr_2\circ j^k\left(\sigma|_{S^{n-1}}\right)(x,\xi)\in\rho\left(Y^k_m\right)$. This concludes the proof of the lemma. $\square$

We are now ready to prove Proposition 7.6.

Proof of Proposition 7.6. — Firstly, by Lemma 7.11, Equation (7.8) has solutions in $U \times (\mathbb{R}^n \setminus \{0\})$ if and only if $j^k\left(\sigma|_{U\times S^{n-1}}\right)(U \times S^{n-1}) \cap pr_2^{-1}\left(\rho(Y_m^k)\right) \neq \emptyset$. Now, by Lemmas 7.8 and 7.9, $\rho\left(Y_m^k\right)$ is a closed submanifold of $\mathcal{J}^k(S^{n-1})$ of codimension $\sum_{j=2}^k \binom{n+j-2}{j}$. Since moreover, by Lemma 7.10, pr_2 is a submersion, $Z_m^k = pr_2^{-1}\left(\rho(Y_m^k)\right)$ has the same codimension in $\mathcal{J}^k\left(U \times S^{n-1}\right)$. At this point, we apply Thom's transversality theorem ([9, Corollary 4.10 of Chapter II]). This theorem states that the set of functions $f \in C^\infty\left(U \times S^{n-1}\right)$ such that $j^k(f)(U\times S^{n-1})$ is transverse[(10)] to Z_m^k is open and dense. But $j^k(f)\left(U \times S^{n-1}\right)$ has dimension at most $2n-1$ so if k is such that

$$2n - 1 < \sum_{j=2}^{k} \binom{n+j-2}{j} \tag{7.12}$$

then such a transverse intersection must be empty. Inequality (7.12) is satisfied for instance for $n = 2$ and $k = 5$, for $n \in \{3, 4\}$ and $k = 3$ and for $n \geqslant 5$ and $k = 2$. This ends the proof of the proposition. $\square$

Appendix A. C^∞ asymptotics for E_L: Proof of Theorem 2.4

In this section, we prove Theorem 2.4 by following closely the approach used in [11] and in [7]. As explained above, [7] contains all the essential arguments for Theorem 2.4 despite the focus on the case where $x = y$ and $\mathcal{X}$ is closed. In this section we merely wish to confirm this by revisiting the proof. We consider A, σ_A and E_L indifferently as in any of the two settings presented in Section 2.1.

A.1. Preliminaries

The following lemma summarizes the results proved in [11, Section 4] for the closed manifold setting. For the Dirichlet boundary value problem, this was proved in [29, Section 3]. We introduce the following notation. For each $T > 0$, set $\mathsf{e}_T = E_{T^m}$. In [11], Hörmander studies the local spectral density $\frac{\mathrm{d}}{\mathrm{d}T}\mathsf{e}_T(x, y)$ via it's Fourier transform in T, $U(t) = \mathcal{F}_T[\mathsf{e}_T](t)$, which is the solution to the problem

$$(i\partial_t - A^{1/m})U(t) = 0$$

(10) See [9, Definition 4.1 of Chapter II] for a definition and general presentation of transversality.

with initial value $U(0) = \mathrm{Id}$. The operator $U(t)$ is estimated using the WKB method leading to the results recorded in Lemma A.1 below. To estimate e_T, Hörmander takes the inverse Fourier transform and integrates in T. We will follow a similar procedure in the following subsections, but we will keep track of the derivatives of the kernel.

LEMMA A.1 (see [11, Section 4] in Setting 1, and [29, Section 3] in[11] Setting 2). — *Firstly, the spectral function* $\mathsf{e}_T(x,y)$ *defines a tempered distribution of the* T *variable with values in* $C^\infty(\mathcal{X}\times\mathcal{X})$. *In addition, for each set of local coordinates in which* $\mathrm{d}\mu_{\mathcal{X}}$ *coincides with the Lebesgue measure on* $\mathbb{R}^n$, *there is an open neighborhood* U *of* $0\in\mathbb{R}^n$ *such that there exist* $\varepsilon>0$, *a proper phase function* $\psi\in C^\infty(U\times U\times\mathbb{R}^n)$, *a symbol* $\sigma\in S^1(U,\mathbb{R}^n)$, *a function* $k\in C^\infty(U\times U\times\,]-\varepsilon,\varepsilon[)$ *and a symbol* $q\in S^0(U\times\,]-\varepsilon,\varepsilon[\,\times U,\mathbb{R}^n)$, *for which*

$$\mathcal{F}_T[\mathsf{e}'_T(x,y)](t)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}q(x,t,y,\xi)e^{i(\psi(x,y,\xi)-t\sigma(y,\xi))}\mathrm{d}\xi+k(x,y,t).$$

Here $\mathcal{F}_T$ *(resp.* $'$*) denotes the Fourier transform (resp. the derivative) with respect to the variable* T, *in the sense of temperate distributions, and the integral is to be understood in the sense of Fourier integral operators (see* [11, Theorem 2.4]*). We have*

(1) *The function* ψ *satisfies the Equation*

$$\forall\, x,y\in U,\ \xi\in\mathbb{R}^n,\ \sigma(x,\partial_x\psi(x,y,\xi))=\sigma(y,\xi).$$

(2) *For each* $t\in\,]-\varepsilon,\varepsilon[$ *and* $\xi\in\mathbb{R}^n$, *the function* $q(\,\cdot\,,t,\cdot\,,\xi)$ *has compact support in* $U\times U$ *uniformly in* (t,ξ) *and* $q(x,0,y,\xi)-1$ *is a symbol of order* -1 *as long as* x,y *belong to some open neighborhood* U_0 *of* 0 *in* U.

(3) $\sigma-\sigma_A^{\frac{1}{m}}$ *has compact support.*

We will also need the following classical lemma. Here and below, $\mathcal{S}(\mathbb{R})$ will denote the space of Schwartz functions.

LEMMA A.2. — *For each* $\varepsilon>0$ *there is a function* $\rho\in\mathcal{S}(\mathbb{R})$ *such that* $\mathcal{F}(\rho)$ *has compact support contained in* $]-\varepsilon,\varepsilon[$, $\rho>0$ *and* $\mathcal{F}(\rho)(0)=1$.

Proof. — Choose $f\in\mathcal{S}(\mathbb{R})$ whose Fourier transform has support in $]-\frac{\varepsilon}{2},\frac{\varepsilon}{2}[$ and $h(t)=e^{-\frac{1}{2}t^2}$. Then it is easy to see that one can choose a constant $Z>0$ so that $\rho=\frac{1}{Z}f^2*f^2*h$ satisfies the required properties. □

[11] To avoid any confusion, we stress once more that in Setting 2, $\mathcal{X}$ denotes a bounded open subset of $\mathbb{R}^n$. We do not obtain any results on the boundary of the domain in the present work.

Before we proceed, let us fix U, ψ, q, k and ρ as in Lemmas A.1 and A.2 (with the same ε) as well as a constant coefficient differential operator P on $\mathbb{R}^n \times \mathbb{R}^n$ of order d with principal symbol σ_P which is a homogeneous polynomial in $2n$ variables. Let $\mathsf{e}_{T,P} = P\mathsf{e}_T$. In order to estimate this $\mathsf{e}_{T,P}$, we will first convolve it with ρ in order to estimate it using Lemma A.1. Then, we will compare $\mathsf{e}_{T,P}$ to its convolution with ρ which we denote, somewhat liberally, by

$$\rho * \mathsf{e}_{T,P} = \int_{\mathbb{R}} \rho(\lambda)\mathsf{e}_{T-\lambda,P}\mathrm{d}\lambda\,.$$

Thus, ρ is a function of the frequency variable T. Its Fourier transform will be a function of the time variable t. The starting point of our calculations will be Equation (A.1) below, which follows from Lemma A.1. In this equation and later on in this section, the notation $\mathcal{F}_t^{-1}$ will denote the inverse Fourier transform acting on the variable t.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}&(\rho * \mathsf{e}_{\lambda,P}(x,y))|_{\lambda=T}\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathcal{F}_t^{-1}\Big[\mathcal{F}(\rho)(t)P\Big(q(x,t,y,\xi)e^{i(\psi(x,t,y,\xi)-t\sigma(y,\xi))}\Big)\Big](T)\mathrm{d}\xi\\ &\qquad\qquad + \mathcal{F}_t^{-1}\big[\mathcal{F}(\rho)(t)Pk(x,t,y)\big](T). \quad \text{(A.1)}\end{aligned}$$

A.2. Estimating the convolved kernel

In this section we provide the following expression for $\rho*\mathsf{e}_{T,P}$ in the local coordinates chosen in Lemma A.1.

Lemma A.3. — *There is an open set $V \subset U$ containing 0 such that, as $T \to \infty$ and uniformly for $(x,y) \in V \times V$,*

$$\rho * \mathsf{e}_{T,P}(x,y) = \frac{1}{(2\pi)^n}\int_{\sigma(y,\xi)\leqslant T} \sigma_P(\partial_{x,y}\psi(x,y,\xi))e^{i\psi(x,y,\xi)}\mathrm{d}\xi + O(T^{n+d-1})\,.$$

In order to do so we use the three lemmas stated below, whose proofs are given at the end of the section. To begin with, we use the information of Lemma A.1 to give a first expression for $\rho * \mathsf{e}_{T,P}$.

Lemma A.4. — *The quantity*

$$\begin{aligned}&\rho * \mathsf{e}_{T,P}(x,y)\\ &- \int_{-\infty}^{T} \frac{1}{(2\pi)^n}\int_{T_y^*M} \mathcal{F}_t^{-1}\Big[\mathcal{F}(\rho)P\Big(q(x,t,y,\xi)e^{i(\psi(x,y,t,\xi)-t\sigma(y,\xi))}\Big)\Big](\lambda)\mathrm{d}\xi\mathrm{d}\lambda\end{aligned}$$

is bounded uniformly for $(x,y) \in U \times U$.

Here and below $\mathcal{F}$ is the Fourier transform and the occasional subscript indicates the variable on which the transform is taken. Let us now investigate the effect of the differential operator P on the right hand side of this expression. By the Leibniz rule, there is a finite family of symbols $(\sigma_j)_{0\leqslant j\leqslant d} \in C^\infty(U \times]-\varepsilon,\varepsilon[\times U,\mathbb{R}^n)^{d+1}$ such that for each j, σ_j is homogeneous of degree j, such that

$$P\Big[q(x,t,y,\xi)e^{i(\psi(x,y,\xi)-t\sigma(y,\xi))}\Big] = \left[\sum_{j=0}^{d}\sigma_j(x,t,y,\xi)\right]e^{i(\psi(x,y,\xi)-t\sigma(y,\xi))}$$

and such that

$$\sigma_d(x,t,y,\xi) = q(x,t,y,\xi)\sigma_P(\partial_{x,y}(\psi(x,y,\xi)-t\sigma(y,\xi)))\,.$$

Now, for each j, let

$$R_j(x,y,T,\xi) = \frac{1}{(2\pi)^{n+1}}\int_{\mathbb{R}}\mathcal{F}(\rho)(t)\sigma_j(x,t,y,\xi)e^{itT}\mathrm{d}t$$

and

$$S_j(x,y,T) = \int_{-\infty}^{T}\int_{\mathbb{R}^n}R_j(x,y,\lambda-\sigma(y,\xi),\xi)e^{i\psi(x,y,\xi)}\mathrm{d}\xi\mathrm{d}\lambda\,.$$

Then,

$$\int_{-\infty}^{T}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathcal{F}_t^{-1}\Big[\mathcal{F}(\rho)P\Big(q(x,t,y,\xi)e^{i(\psi(x,y,\xi)-t\sigma(y,\xi))}\Big)\Big](\lambda)\mathrm{d}\xi\mathrm{d}\lambda \\ = \sum_{j=0}^{d}S_j(x,y,T)\,.$$

Each S_j will grow at an order corresponding to the degree of the associated symbol. This is shown in the following lemma.

Lemma A.5. — *There is an open set $V \subset U$ containing 0 such that, as $T\to\infty$ and uniformly for $(x,y)\in V\times V$,*

$$S_j(x,y,T) = \frac{1}{(2\pi)^n}\int_{\sigma(y,\xi)\leqslant T}\sigma_j(x,0,y,\xi)e^{i\psi(x,y,\xi)}\mathrm{d}\xi + O(T^{n+j-1})\,.$$

Similarly since $q(x,0,y,\xi)-1\in S^{-1}(U_0\times U_0,\mathbb{R}^n)$, from a computation analogous to the proof of Lemma A.5 and left to the reader, replacing σ_d by

$$(q(x,0,y,\xi)-1)\sigma_P(\partial_{x,y}(\psi(x,y,\xi)-t\sigma(y,\xi)))\in S^{d-1}$$

one can remove q from the main term, which results in the following.

Lemma A.6. — *There is an open set $V \subset U$ containing 0 such that, as $T \to \infty$ and uniformly for $(x,y) \in V \times V$,*

$$S_d(x,y,T) = \frac{1}{(2\pi)^n} \int_{\sigma(y,\xi) \leqslant T} \sigma_P(\partial_{x,y}(\psi(x,y,\xi) - t\sigma(y,\xi)))e^{i\psi(x,y,\xi)} \mathrm{d}\xi + O(T^{n+d-1}) .$$

The juxtaposition of these results yields Lemma A.3.

Proof of Lemma A.4. — Since $k \in C^\infty(U \times U \times]-\varepsilon,\varepsilon[)$ and $\mathcal{F}(\rho)$ is supported in $]-\varepsilon,\varepsilon[$,

$$\mathcal{F}_t^{-1}\big[\mathcal{F}(\rho)(t)Pk(x,t,y)\big](T) \in \mathcal{S}(\mathbb{R}) .$$

Therefore, by Equation (A.1),

$$\rho * \mathsf{e}_{T,P}(x,y) - \int_{-\infty}^{T} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}_t^{-1}\Big[\mathcal{F}(\rho)P\Big(q(x,t,y,\xi)e^{i(\psi(x,y,\xi)-t\sigma(y,\xi))}\Big)\Big](\lambda)\mathrm{d}\xi\mathrm{d}\lambda$$

is bounded. □

Proof of Lemma A.5. — In this proof, all generic constants will be implicitly uniform with respect to $(x,y) \in V \times V$. Let us fix $y \in V$ and define the following three domains of integration.

$$\begin{aligned} D_1 &= \{(\lambda,\xi) \in \mathbb{R} \times \mathbb{R}^n \mid \lambda \leqslant T,\ \sigma(y,\xi) \leqslant T\} \\ D_2 &= \{(\lambda,\xi) \in \mathbb{R} \times \mathbb{R}^n \mid \lambda \leqslant T,\ \sigma(y,\xi) > T\} \\ D_3 &= \{(\lambda,\xi) \in \mathbb{R} \times \mathbb{R}^n \mid \lambda > T,\ \sigma(y,\xi) \leqslant T\} . \end{aligned}$$

Moreover, for $l = 1,2,3$, let $I_l = \int_{D_l} R_j(x,y,\lambda-\sigma(y,\xi),\xi)e^{i\psi(x,y,\xi)}\mathrm{d}\xi\mathrm{d}\lambda$. We will prove that I_2 and I_3 are $O(T^{n+j-1})$. The following calculation will then yield the desired identity. Here we use Fubini's theorem and the fact that $\mathcal{F}(\rho)(0) = \int_{\mathbb{R}} \rho(\lambda)\mathrm{d}\lambda = 1$.

$$\begin{aligned} S_j(x,y,T) &= I_1 + I_2 = I_1 + I_3 + O(T^{n+j-1}) \\ &= \int_{\sigma(y,\xi)\leqslant T} \left[\int_{\mathbb{R}} R_j(x,y,s,\xi)\mathrm{d}s\right] e^{i\psi(x,y,\xi)}\mathrm{d}\xi + O(T^{n+j-1}) \\ &= \frac{1}{(2\pi)^n} \int_{\sigma(y,\xi)\leqslant T} \sigma_j(x,0,y,\xi)e^{i\psi(x,y,\xi)}\mathrm{d}\xi + O(T^{n+j-1}). \end{aligned}$$

First of all, R_j is rapidly decreasing in the third variable and, since σ is elliptic of degree 1, bounded by $\sigma(y,\xi)^j$ with respect to the last variable, ξ. Therefore, for each $N > 0$ there is a constant $C > 0$ such that

$$|R_j(x,y,\lambda,\xi)| \leqslant \frac{C\sigma(y,\xi)^j}{(1+|\lambda|)^N} .$$

Since σ is elliptic of order 1, the hypersurface $T^{-1}\{\sigma(y,\xi) = T\} \subset \mathbb{R}^n$ converges smoothly for $T \to \infty$ uniformly in y to $S_y^A = \{\sigma_A(y,\xi) = 1\}$ and the volume of $\{\sigma(x,\xi) = \beta\} \subset \mathbb{R}^n$ is $O(\beta^{n-1})$. Taking $N = 2n + j + 1$, we deduce that

$$\begin{aligned}
|I_2| &\leqslant C \int_{-\infty}^{T} \int_{\sigma(y,\xi)>T} \frac{\sigma(y,\xi)^j}{(1+|\lambda-\sigma(y,\xi)|)^{2n+j+1}} \mathrm{d}\xi \mathrm{d}\lambda \\
&\leqslant C \int_{-\infty}^{T} \int_{T}^{+\infty} \frac{\beta^{n+j-1}}{(1+|\lambda-\beta|)^{2n+j+1}} \mathrm{d}\beta \mathrm{d}\lambda \\
&\leqslant C \int_{T}^{+\infty} \int_{-\infty}^{T-\beta} \frac{\beta^{n+j-1}}{(1+|s|)^{2n+j+1}} \mathrm{d}s \mathrm{d}\beta \\
&\leqslant C \int_{T}^{\infty} \frac{\beta^{n+j-1}}{(1+\beta-T)^{2n+j}} \mathrm{d}\beta \\
&\leqslant C \int_{0}^{+\infty} \frac{(\gamma+T)^{n+j-1}}{(1+\gamma)^{2n+j}} \mathrm{d}\gamma \\
&\leqslant C T^{n+j-1}.
\end{aligned}$$

Here we applied first the change of variables $s = \lambda - \beta$ and then $\gamma = \beta - T$. The case of I_3 is analogous and by a similar calculation we deduce that I_1 is well defined. □

A.3. Comparison of the kernel and its convolution

In this section we set about proving that $\mathsf{e}_{T,P}$ is close enough to its convolution with ρ. This is encapsulated in the following lemma.

Lemma A.7. — *There is an open set $V \subset U$ containing 0 such that, as $T \to \infty$ and uniformly for $(x,y) \in V \times V$,*

$$\rho * \mathsf{e}_{T,P}(x,y) - \mathsf{e}_{T,P}(x,y) = O(T^{n+d-1}).$$

As before, the proofs are relegated to the end of the section. In order to prove Lemma A.7 we first estimate the growth of the R_j as follows.

Lemma A.8. — *There is an open set $V \subset U$ containing 0 such that, as $T \to \infty$ and uniformly for $(x,y) \in V \times V$,*

$$\int_{\mathbb{R}^n} R_j(x,y,T-\sigma(y,\xi),\xi) e^{i\psi(x,y,\xi)} \mathrm{d}\xi = O(T^{n+j-1}).$$

This lemma follows from a computation analogous to the bound on I_2 and I_3 given in the proof of Lemma A.5 above and the details are left to the reader. It allows us to prove a second intermediate result from which we obtain Lemma A.7 directly.

Lemma A.9. — *There is an open set $V \subset U$ containing 0 such that, as $T \to \infty$ and uniformly for $(x,y) \in V \times V$,*

$$\mathsf{e}_{T+1,P}(x,y) - \mathsf{e}_{T,P}(x,y) = O(T^{n+d-1})\,.$$

Proof of Lemma A.9. — We begin with the case where $x = y$ and P is of the form $P_1 \otimes P_1$. For brevity we define

$$u(T) = \mathsf{e}_{T,P}(x,x) = \sum_{\lambda_k \leqslant T} |(P_1 e_k)(x)|^2\,.$$

Recall $\rho > 0$ so it stays greater than some constant $a > 0$ on the interval $[-1,0]$. Moreover u is increasing so by Equation (A.1) and Lemma A.8,

$$\begin{aligned}
0 \leqslant u(T+1) - u(T) &= \int_T^{T+1} u'(\lambda)\mathrm{d}\lambda \leqslant \frac{1}{a}\int_T^{T+1} \rho(T-\lambda)u'(\lambda)\mathrm{d}\lambda \\
&\leqslant \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}T}(\rho * u) \leqslant \frac{1}{a}\sum_{j=0}^{d}\int_{\mathbb{R}^n} R_j(x,x,T-\sigma(x,\xi))\mathrm{d}\xi + O(T^{n+d-1}) \\
&= O(T^{n+d-1}).
\end{aligned}$$

Now if P is of the form $P_1 \otimes P_2$, and for any x and y, let

$$X = (P_1 e_k(x))_{T<\lambda_k\leqslant T+1} \quad \text{and} \quad Y = (P_2 e_k(y))_{T<\lambda_k\leqslant T+1}$$

be two vectors in some $\mathbb{C}^q$ which we equip with the standard hermitian product "$\star$". Then, $\mathsf{e}_{T+1,P}(x,y) - \mathsf{e}_{T,P}(x,y) = X \star \overline{Y}$ so

$$|\mathsf{e}_{T+1,P}(x,y) - \mathsf{e}_{T,P}(x,y)|^2 \leqslant |X|^2|Y|^2 \leqslant CT^{2n+2d-2}$$

using first, Cauchy–Schwarz and second, the above estimate. In general P is a locally finite sum of operators of the form $P_1 \otimes P_2$. □

Proof of Lemma A.7. — First of all, according to Lemma A.9 there is a constant C such that for all $T \geqslant 0$ and λ,

$$|\mathsf{e}_{T+\lambda,P}(x,y) - \mathsf{e}_{T,P}(x,y)| \leqslant C(1+|\lambda|+T)^{n+d-1}(1+|\lambda|).$$

Consequently

$$\begin{aligned}
|(\rho * \mathsf{e}_{T,P}(x,y) - \mathsf{e}_{T,P}(x,y)| &\leqslant \left|\int \rho(\lambda)\mathsf{e}_{T+\lambda,P}(x,y)\mathrm{d}\lambda - \mathsf{e}_{T,P}(x,y)\right| \\
&\leqslant \int \rho(\lambda)\big|\mathsf{e}_{T+\lambda,P}(x,y) - \mathsf{e}_{T,P}(x,y)\big|\mathrm{d}\lambda \\
&\leqslant C\int \rho(\lambda)(1+|\lambda|+T)^{n+d-1}(1+|\lambda|)\mathrm{d}\lambda \\
&\leqslant C'T^{n+d-1}
\end{aligned}$$

for some $C' > 0$. Here we used that $\rho > 0$, ρ is rapidly decreasing and $\int_{\mathbb{R}} \rho(\lambda)\mathrm{d}\lambda = \mathcal{F}(\rho)(0) = 1$. □

A.4. Conclusion

Combining Lemmas A.3 and A.7 we obtain the following:

$$\mathsf{e}_{T,P}(x,y) = \frac{1}{(2\pi)^n} \int_{\sigma(y,\xi)\leqslant T} \sigma_P(\partial_{x,y}\psi(x,y,\xi)) e^{i\psi(x,y,\xi)} \mathrm{d}\xi + O(T^{n+d-1}) \,.$$

Since $\sigma - \sigma_A^{\frac{1}{m}}$ has compact support, replacing one by the other adds only a $O(1)$ term. Therefore,

$$\mathsf{e}_{T,P}(x,y) = \frac{1}{(2\pi)^n} \int_{\sigma_A(y,\xi)^{1/m}\leqslant T} \sigma_P(\partial_{x,y}\psi(x,y,\xi)) e^{i\psi(x,y,\xi)} \mathrm{d}\xi + O(T^{n+d-1}) \,.$$

This estimate is valid and uniform for $x, y \in V$. To conclude, notice that $\sigma_A(x,\xi)^{1/m} \leqslant T$ is equivalent to $\sigma_A(x,\xi) \leqslant T^m$. Since $\mathsf{e}_{T,P} = PE_{T^m}$, replacing T by $L^{1/m}$ in the last estimate we get

$$\begin{aligned} &E_{L,P}(x,y) \\ &\quad = \frac{1}{(2\pi)^n} \int_{\sigma_A(y,\xi)\leqslant L} \sigma_P(\partial_{x,y}\psi(x,y,\xi)) e^{i\psi(x,y,\xi)} \mathrm{d}\xi + O(L^{(n+d-1)/m}) \end{aligned}$$

as announced. Note that the only restriction on V is that it be chosen inside U_0 from Lemma A.1 (2). In particular, it may be chosen independently of P.

Appendix B. Existence and regularity of eigenfunctions in Setting 2

In the present section, we prove that, in Setting 2 of Section 2.1, the spectrum of A is indeed a discrete sequence of real numbers diverging to $+\infty$, whose associated eigenfunctions are smooth.

We will use the notations of Setting 2. Recall in particular that $\mathcal{X}$ is a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{X}$ and that $\mathrm{d}\mu_{\mathcal{X}}(x)$ is the Lebesgue measure $\mathrm{d}x$. We will also denote by $C^\infty(\overline{\mathcal{X}})$ the space of C^∞ functions on $\mathcal{X}$ whose derivatives all extend by continuity up to the boundary. For any $s \in \mathbb{N}$, we denote by $H^s(\mathcal{X})$ the L^2 Sobolev space of order s on $\mathcal{X}$ (see [16, Section 1.1, Chapter 1]). We will also denote by $H^s_0(\mathcal{X})$ the completion of $C^\infty_c(\mathcal{X})$ for the H^s topology and we will write $C^\infty_0(\mathcal{X})$ for the space of smooth functions on $\overline{\mathcal{X}}$ vanishing on $\partial\mathcal{X}$.

Proposition B.1. — *There exists a sequence of real numbers $(\lambda_j)_{j\in\mathbb{N}}$, going to infinity, as well as a Hilbert basis $(e_j)_{j\in\mathbb{N}}$ of $L^2(\mathcal{X})$ whose elements belong to $C_0^\infty(\mathcal{X})$, such that for each $j \in \mathbb{N}$, $Ae_j = \lambda_j e_j$.*

Proof. — Let $P = A + C$ for some large constant $C > 0$. Firstly, we observe that there exists a choice of $C \in]0, +\infty[$ such that if $u \in C^\infty(\overline{\mathcal{X}})$ satisfies $Pu = 0$ and $u|_{\partial\mathcal{X}} = 0$ then, by Theorem 9.2 of [16, Chapter 2], $u = 0$. Then, by Theorem 5.4 and Proposition 5.3 of [16, Chapter 2], P extends to a topological isomorphism $P_0 : H_0^m(\mathcal{X}) \to L^2(\mathcal{X})$. In particular, since $\mathcal{X}$ is compact, by Sobolev embeddings, P_0^{-1} defines a compact operator from $L^2(\mathcal{X})$ to $L^2(\mathcal{X})$. Therefore, P_0 has discrete spectrum, its eigenspaces are finite dimensional and its eigenfunctions belong to $H_0^m(\mathcal{X})$. Finally, applying Theorem 5.4 of [16, Chapter 2] once again, we deduce by induction that these eigenfunctions belong to $\cap_{s>0} H^s(\mathcal{X}) = C^\infty(\overline{\mathcal{X}}$ (using Sobolev embeddings once more). In particular, they belong to the domain of P. Since they also belong to $H_0^m(\mathcal{X})$, they must vanish on $\partial\mathcal{X}$. Applying Theorem 9.2 of [16, Chapter 2] a second time, we deduce that the eigenvalues of P are positive. In particular, these eigenfunctions form a Hilbert basis of $L^2(\mathcal{X})$ whose elements belong to $C_0^\infty(\mathcal{X})$. Moreover, the elements of this basis are also eigenfunctions of A and the corresponding eigenvalues, counted with multiplicity, form a sequence of real numbers diverging to $+\infty$ as announced. ☐

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