ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

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Tome XXXI, n° 2 (2022), p. 491–499.

https://doi.org/10.5802/afst.1700

Annales de la faculté des sciences de Toulouse Volume XXXI, n° 2, 2022
pp. 491-499

The divergence equation with L^∞ source [(*)]

Eduard Curcă [(1)]

Abstract. — A well-known fact is that there exists $g \in L^\infty(\mathbb{T}^2)$ with zero integral, such that the equation

$$\operatorname{div} f = g \qquad (*)$$

has no solution $f = (f_1, f_2) \in W^{1,\infty}(\mathbb{T}^2)$. This was proved by Preiss ([4]), using an involved geometric argument, and, independently, by McMullen ([2]), via Ornstein's non-inequality. We improve this result: roughly speaking, we prove that, there exists $g \in L^\infty$ for which $(*)$ has no solution such that $\partial_2 f_2 \in L^\infty$ and f is "slightly better" than L^1. Our proof relies on Riesz products in the spirit of the approach of Wojciechowski ([6]) for the study of $(*)$ with source $g \in L^1$. The proof we give is elementary, self-contained and completely avoids the use of Ornstein's non-inequality.

Résumé. — Notre point de départ est le résultat suivant de non existence : il existe $g \in L^\infty(\mathbb{T}^2)$, d'integrale nulle et telle que l'équation

$$\operatorname{div} f = g \qquad (*)$$

n'ait pas de solution $f = (f_1, f_2) \in W^{1,\infty}(\mathbb{T}^2)$. Ce résultat a été obtenu indépendamment par Preiss ([4]), en utilisant un argument géométrique délicat, et par McMullen ([2]), via la non-inégalité d'Ornstein. Nous améliorons substantiellement ce résultat, en montrant qu'en général $(*)$ n'a pas de solution satisfaisant $\partial_2 f_2 \in L^\infty$, avec f « un peu mieux » que L^1. Notre démonstration est basée sur les produits Riesz dans l'esprit de l'approche de Wojciechowski ([6]) pour l'étude de $(*)$ avec source $g \in L^1$. La démonstration est élémentaire et évite completement l'utilisation de la non-inégalité d'Ornstein.

[(*)] Reçu le 10 février 2020, accepté le 29 mai 2020.

Keywords: divergence equation, Riesz products, function spaces.

2020 *Mathematics Subject Classification:* 42B37, 42B05.

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The author was supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

Article proposé par Radu Ignat.

1. Introduction

In this paper, we improve the following result of Preiss ([4]) and McMullen ([2, Theorem 2.1]):

Theorem 1.1. — *There exists $g \in L^\infty(\mathbb{T}^2)$ with zero integral, such that there are no f_1, $f_2 \in W^{1,\infty}(\mathbb{T}^2)$ with*

$$g = \partial_1 f_1 + \partial_2 f_2.$$

The proof in [4] is "geometric", the one in [2] relies essentially on Ornstein's non-inequality ([3]).

Note that, in the above statement, the conditions on f_1, f_2 are isotropic, i.e., we require $\partial_l f_j \in L^\infty(\mathbb{T}^2)$ for all $l, j = 1, 2$. In what follows, we will prove that, under some mild regularity assumptions on f_1, f_2, the above requirements can be weakened to anisotropic conditions. Namely, it is enough to impose $\partial_2 f_2 \in L^\infty(\mathbb{T}^2)$. In order to state this more precisely, we introduce the following spaces of distributions.

Suppose $\lambda : \mathbb{N} \to (0, \infty)$ is a decreasing function such that $\lambda(k) \to 0$ when $k \to \infty$. To such a function we associate the Banach space of those distributions whose Fourier transform decays at the rate at least λ. More precisely, consider the space

$$S_\lambda(\mathbb{T}^2) := \left\{ f \in \mathcal{D}'(\mathbb{T}^2) \;\middle|\; \sup_{n \in \mathbb{Z}^2} \frac{|\widehat{f}(n)|}{\lambda(|n|)} < \infty \right\},$$

endowed with the norm given by

$$\|f\|_{S_\lambda} := \sup_{n \in \mathbb{Z}^2} \frac{|\widehat{f}(n)|}{\lambda(|n|)}, \quad f \in S_\lambda(\mathbb{T}^2).$$

To mention only few examples, we note that, for any $m \in \mathbb{N}^*$, $W^{m,1}(\mathbb{T}^2) \hookrightarrow S_\lambda(\mathbb{T}^2)$, with $\lambda(|n|) = 1/(1+|n|)^m$ and, if $s > 0$, the fractional Sobolev space $H^s(\mathbb{T}^2)$ is embedded in $S_\lambda(\mathbb{T}^2)$ for $\lambda(|n|) = 1/(1+|n|)^s$.

With this notation, we can formulate our result.

Theorem 1.2. — *Suppose $\lambda : \mathbb{N} \to (0, \infty)$ is decreasing to 0. There exists $g \in L^\infty(\mathbb{T}^2)$ such that there are no f_0, f_1, $f_2 \in S_\lambda(\mathbb{T}^2)$ with $\partial_2 f_2 \in L^\infty(\mathbb{T}^2)$ and*

$$g = f_0 + \partial_1 f_1 + \partial_2 f_2.$$

We can easily observe that Theorem 1.2 implies Theorem 1.1. Indeed, if f_1, $f_2 \in W^{1,\infty}(\mathbb{T}^2)$ then $\partial_2 f_2 \in L^\infty(\mathbb{T}^2)$ and, as we mentioned above, we have f_1, $f_2 \in S_\lambda(\mathbb{T}^2)$ for $\lambda(|n|) = 1/(1+|n|)$. Also, even the weaker regularity condition f_0, f_1, $f_2 \in H^\varepsilon(\mathbb{T}^2)$, $\partial_2 f_2 \in L^\infty(\mathbb{T}^2)$ ($\varepsilon > 0$, a small fixed number) rules out the existence of a solution. Intuitively, $f \in S_\lambda(\mathbb{T}^2)$, with λ slowly decaying, means that f is "slightly better" than L^1. The above result asserts that solutions with such regularity satisfying $\partial_2 f_2 \in L^\infty(\mathbb{T}^2)$ need not exist.

Finally, we discuss the most important aspect, which is the proof of Theorem 1.2. Our proof completely avoids the use of Ornstein's non-inequality. It is an adaptation of the Riesz products based proof, given by Wojciechowski in [6], of the fact that there exist L^1 functions which are not divergences of $W^{1,1}$ vector fields. We follow the general structure of his proof making the needed modifications in order to handle the L^∞ case. While the proof in [6] relies on a relatively difficult lemma ([6, Lemma 1]), in our case, the role of this lemma will be played by Lemma 2.1 below, which is elementary and easy. Another aspect of our proof is the presence of the function λ. This allows us to quantify the regularity that we impose to the solution and to improve the result described by Theorem 1.1. The approach based on Ornstein's non-inequality does not seem to be suited for obtaining this improvement.

We also mention that the proof of Theorem 1.2 given below is self-contained and elementary.

Acknowledgements

The author thanks Petru Mironescu for useful discussions and suggestions.

2. Proof of Theorem 1.2

Before starting the proof, we recall first the following well-known elementary fact (see [5, Lemma 6.3, p. 118]):

Lemma 2.1. — *Suppose $z_1, \dots, z_N$ are some complex numbers. Then, there exist $\sigma_1, \dots, \sigma_N \in \{0, 1\}$ such that*

$$\left| \sum_{k=1}^{N} \sigma_k z_k \right| \geqslant \frac{1}{\pi} \sum_{k=1}^{N} |z_k|.$$

Proof. — We follow [5]. View $z_1, \dots, z_N$ as vectors in $\mathbb{R}^2$. For a given $\theta \in [0, 2\pi]$, let $r_\theta := (\cos\theta, \sin\theta)$. If H_θ is the half-plane given by

$$H_\theta := \left\{ z \in \mathbb{R}^2 \,\middle|\, \langle z, r_\theta \rangle \geqslant 0 \right\},$$

we have

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=1,z_k\in H_\theta}^{N} z_k\right|\mathrm{d}\theta \geqslant \frac{1}{2\pi}\int_0^{2\pi}\sum_{j=1}^{N}\langle z_j, r_\theta\rangle^+\mathrm{d}\theta = \sum_{j=1}^{N}\frac{1}{2\pi}\int_0^{2\pi}\langle z_j, r_\theta\rangle^+\mathrm{d}\theta,$$

and we easily see that, for all j,

$$\frac{1}{2\pi}\int_0^{2\pi}\langle z_j, r_\theta\rangle^+\mathrm{d}\theta = |z_j|\frac{1}{2\pi}\int_0^{2\pi}(\cos\theta)^+\mathrm{d}\theta = \frac{1}{\pi}|z_j|.$$

Using the above inequality, we complete the proof of Lemma 2.1 via a mean value argument: there exists $\theta_0 \in [0, 2\pi]$ such that

$$\left|\sum_{k=1,z_k\in H_{\theta_0}}^{N} z_k\right| \geqslant \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=1,z_k\in H_\theta}^{N} z_k\right|\mathrm{d}\theta \geqslant \frac{1}{\pi}\sum_{j=1}^{N}|z_j|,$$

and one can choose $\sigma_k = 1$, if $z_k \in H_{\theta_0}$ and $\sigma_k = 0$, otherwise. □

We will also need few facts concerning the trigonometric polynomials.

Fix a finite sequence $(a_k)_{k=1,N}$ in $\mathbb{Z}^2$. For each finite sequence $(\alpha_1,\dots,\alpha_N)$ of complex numbers we have the following expansion rule for Riesz products:

$$\prod_{k=1}^{N}(1+\alpha_k\cos\langle t, a_k\rangle) = 1 + \sum_{k=1}^{N}\sum_{\substack{\varepsilon_1,\dots,\varepsilon_k\in\{-1,0,1\}\\ \varepsilon_k\neq 0}}\left(\prod_{\varepsilon_j\neq 0}\frac{\alpha_j}{2}\right)e^{i\langle t,\varepsilon_1 a_1+\cdots+\varepsilon_k a_k\rangle}. \quad (2.1)$$

(This can be easily proved by induction on N.)

Suppose, moreover, that $(a_k)_{k=1,N}$, with $a_k \in (\mathbb{Z}\setminus\{0\})^2$ is componentwise lacunary, i.e., there exists a constant $M>3$ such that $|a_{k+1}(1)|/|a_k(1)| > M$ and $|a_{k+1}(2)|/|a_k(2)| > M$ for all $1 \leqslant k \leqslant N-1$. Then, all the expressions $\varepsilon_1 a_1 + \cdots + \varepsilon_k a_k$ in the above formula are distinct and nonzero. Hence, if $\alpha_1, \dots, \alpha_N$ and $\beta_1, \dots, \beta_N$ are complex numbers, by using the above formula and the relation between convolution and the Fourier transform, we obtain

$$\prod_{k=1}^{N}(1+\alpha_k\cos\langle\,\cdot\,, a_k\rangle) * \prod_{k=1}^{N}(1+\beta_k\cos\langle\,\cdot\,, a_k\rangle) = \prod_{k=1}^{N}(1+\frac{\alpha_k\beta_k}{2}\cos\langle\,\cdot\,, a_k\rangle). \quad (2.2)$$

Indeed, using (2.1) we find that the coefficient of $e^{i\langle t,\varepsilon_1 a_1+\cdots+\varepsilon_k a_k\rangle}$ in the left hand side of (2.2) is

$$\left(\prod_{\varepsilon_j\neq 0}\frac{\alpha_j}{2}\right)\left(\prod_{\varepsilon_j\neq 0}\frac{\beta_j}{2}\right)=\prod_{\varepsilon_j\neq 0}\frac{(\alpha_j\beta_j/2)}{2}.$$

Since, all the expressions $\varepsilon_1 a_1+\cdots+\varepsilon_k a_k$ are distinct, the above identity permits us to recover the right hand side of (2.2) via (2.1).

We will also use the following standard algebraic identity which can be proved by induction on N:

$$\prod_{k=1}^{N}(1+c_k)=1+\sum_{k=1}^{N}c_k\prod_{j=1}^{k-1}(1+c_j) \tag{2.3}$$

for any complex numbers $c_1,\dots,c_N$.

Proof of Theorem 1.2. — Suppose that the assertion of Theorem 1.2 is false and fix a function λ as in the statement. Then, by the open mapping principle, there exists a constant $C>0$ such that for any $g\in L^\infty(\mathbb{T}^2)$ there exist distributions f_0, f_1, $f_2\in S_\lambda(\mathbb{T}^2)$, satisfying $g=f_0+\partial_1 f_1+\partial_2 f_2$, with the properties that $\partial_2 f_2\in L^\infty(\mathbb{T}^2)$ and

$$\|f_0\|_{S_\lambda}+\|f_1\|_{S_\lambda}+\|f_2\|_{S_\lambda}+\|\partial_2 f_2\|_{L^\infty}\leqslant C\|g\|_{L^\infty}. \tag{2.4}$$

Let N be a large positive integer such that $\ln N>25\pi C$ and consider the functions on $\mathbb{T}^2$

$$g_N(t):=\prod_{k=1}^{N}\left(1+\frac{i}{k}\cos\langle t,a_k\rangle\right)\quad\text{and}\quad G_N(t):=\prod_{k=1}^{N}(1+\cos\langle t,a_k\rangle),$$

where the finite sequence $(a_k)_{k=1,N}$ in $(\mathbb{N}^*)^2$ is defined below.

Using Lemma 2.1, applied to the sequence of complex numbers

$$z_k:=\frac{1}{k}\prod_{j=1}^{k-1}\left(1+\frac{i}{2j}\right)\quad\text{for } k=1,\dots,N,$$

(here and after the product over an empty set is by convention equal to 1), we can find a sequence $\sigma_1,\dots,\sigma_N\in\{0,1\}$ such that

$$\left|\sum_{k=1}^{N}\frac{\sigma_k}{k}\prod_{j=1}^{k-1}\left(1+\frac{i}{2j}\right)\right|\geqslant\frac{1}{\pi}\sum_{k=1}^{N}\frac{1}{k}\prod_{j=1}^{k-1}\left(1+\frac{1}{4j^2}\right)^{\frac{1}{2}}\geqslant\frac{1}{\pi}\sum_{k=1}^{N}\frac{1}{k}\geqslant\frac{1}{\pi}\ln N. \tag{2.5}$$

Now we impose the sequence $(a_k)_{k=1,N}$ to satisfy the following properties:

(i) $(a_k)_{k=1,N}$ is component-wise lacunary;

(ii) If $\sigma_k = 1$, then

$$\left|a_k(1) + \sum_{1\leqslant j\leqslant k-1} \varepsilon_j a_j(1)\right| \lambda\left(\left|a_k(2) + \sum_{1\leqslant j\leqslant k-1} \varepsilon_j a_j(2)\right|\right) < \frac{1}{4^N}$$
$$\text{for all } \varepsilon_1, \ldots, \varepsilon_{k-1} \in \{-1, 0, 1\};$$

(iii) If $\sigma_k = 0$, then

$$\left|a_k(2) + \sum_{1\leqslant j\leqslant k-1} \varepsilon_j a_j(2)\right| \lambda\left(\left|a_k(1) + \sum_{1\leqslant j\leqslant k-1} \varepsilon_j a_j(1)\right|\right) < \frac{1}{4^N}$$
$$\text{for all } \varepsilon_1, \ldots, \varepsilon_{k-1} \in \{-1, 0, 1\}.$$

(By convention the sum over an empty set is equal to 0.)

Such a sequence can be easily constructed by induction on k: if $a_1, \ldots, a_{k-1}$ are chosen, then we choose $a_k(2)$ much larger than $a_k(1)$, or $a_k(1)$ much larger than $a_k(2)$, depending on whether $\sigma_k = 1$ or $\sigma_k = 0$ respectively. Since λ is decreasing to 0, we can satisfy in this way the conditions (ii), respectively (iii). Also, the condition (i) can be easily satisfied.

We now return to the proof of Theorem 1.2. Note that

$$\|g_N\|_{L^\infty} = \prod_{k=1}^{N} \left(1 + \frac{1}{k^2}\right)^{\frac{1}{2}} \leqslant e^{\frac{\pi^2}{12}} < 3,$$
$$\text{and also } G_N \geqslant 0 \text{ and } \|G_N\|_{L^1} = 1. \quad (2.6)$$

(We can see that $\|G_N\|_{L^1} = 1$ by using (2.1).)

Using (2.2) and (2.3), we get

$$\begin{aligned} G_N * g_N(t) &= \prod_{k=1}^{N} \left(1 + \frac{i}{2k} \cos\langle t, a_k\rangle\right) \\ &= 1 + \sum_{k=1}^{N} \frac{i}{2k} \cos\langle t, a_k\rangle \prod_{j=1}^{k-1} \left(1 + \frac{i}{2j} \cos\langle t, a_j\rangle\right). \quad (2.7) \end{aligned}$$

Consider the sets

$$A := \bigcup_{\substack{k=1\\ \sigma_k=1}}^{N} \{\varepsilon_1 a_1 + \cdots + \varepsilon_k a_k \,|\, \varepsilon_1, \ldots, \varepsilon_k \in \{-1,0,1\},\ \varepsilon_k \neq 0\},$$

$$B := \bigcup_{\substack{k=1\\ \sigma_k=0}}^{N} \{\varepsilon_1 a_1 + \cdots + \varepsilon_k a_k \,|\, \varepsilon_1, \ldots, \varepsilon_k \in \{-1,0,1\},\ \varepsilon_k \neq 0\}.$$

Since the sequence $(a_k)_{k=1,N}$ is component-wise lacunary, we have $(\{0\}\times\mathbb{Z})\cap(A\cup B)=\varnothing$, $(\mathbb{Z}\times\{0\})\cap(A\cup B)=\varnothing$ and $A\cap B=\varnothing$, while clearly $|A\cup B|\leqslant 3^N$. In particular, $|A|\leqslant 3^N$, $|B|\leqslant 3^N$.

Using now (2.7), we have

$$P_A G_N * g_N(t) = \sum_{k=1}^{N} \frac{i\sigma_k}{2k} \cos\langle t, a_k\rangle \prod_{j=1}^{k-1}\left(1+\frac{i}{2j}\cos\langle t, a_j\rangle\right),$$

and from (2.5) we obtain

$$|P_A G_N * g_N(0)| = \left|\sum_{k=1}^{N} \frac{i\sigma_k}{2k} \prod_{j=1}^{k-1}\left(1+\frac{i}{2j}\right)\right| \geqslant \frac{1}{2\pi}\ln N, \tag{2.8}$$

where P_A is the linear operator on trigonometric polynomials, satisfying $P_A e^{i\langle t,n\rangle} = e^{i\langle t,n\rangle}$ if $n\in A$ and $P_A e^{i\langle t,n\rangle}=0$ otherwise.

On the other hand, according to our assumption and (2.6), we can find $f_0,\ f_1,\ f_2 \in S_\lambda(\mathbb{T}^2)$, satisfying $g_N = f_0 + \partial_1 f_1 + \partial_2 f_2$, with the properties that $\partial_2 f_2 \in L^\infty(\mathbb{T}^2)$ and

$$\|f_0\|_{S_\lambda} + \|f_1\|_{S_\lambda} + \|f_2\|_{S_\lambda} + \|\partial_2 f_2\|_{L^\infty} \leqslant 3C.$$

Let us note that

$$P_A G_N * g_N = P_A G_N * f_0 + P_A G_N * \partial_1 f_1 + P_A G_N * \partial_2 f_2. \tag{2.9}$$

We next estimate each term on the right hand side of (2.9).

For the second term, we have:

$$\begin{aligned}
&\|P_A G_N * \partial_1 f_1\|_{L^\infty}\\
&\qquad = \|G_N * P_A \partial_1 f_1\|_{L^\infty} \leqslant \|G_N\|_{L^1}\|P_A\partial_1 f_1\|_{L^\infty} = \|P_A\partial_1 f_1\|_{L^\infty}\\
&\qquad \leqslant |A| \max_{n\in A} |\widehat{\partial_1 f_1}(n)| = |A|\max_{n\in A}|n(1)||\widehat{f_1}(n)|\\
&\qquad \leqslant |A|\max_{n\in A}|n(1)|\lambda(|n|)\|f_1\|_{S_\lambda} \leqslant |A|\max_{n\in A}|n(1)|\lambda(|n(2)|)\|f_1\|_{S_\lambda}\\
&\qquad \leqslant 3^N 4^{-N} 3C < 3C,
\end{aligned}$$

where we have used (ii).

For the third term, we observe that, thanks to the identity $G_N = P_A G_N + P_B G_N + 1$, we have $P_A G_N * \partial_2 f_2 = G_N * \partial_2 f_2 - P_B G_N * \partial_2 f_2$. Hence, we can write:

$$\begin{aligned}
&\|P_A G_N * \partial_2 f_2\|_{L^\infty}\\
&= \|G_N * \partial_2 f_2 - P_B G_N * \partial_2 f_2\|_{L^\infty} \leqslant \|G_N * \partial_2 f_2\|_{L^\infty} + \|P_B G_N * \partial_2 f_2\|_{L^\infty}\\
&\leqslant \|G_N\|_{L^1}\|\partial_2 f_2\|_{L^\infty} + \|G_N\|_{L^1}\|P_B \partial_2 f_2\|_{L^\infty} = \|\partial_2 f_2\|_{L^\infty} + \|P_B \partial_2 f_2\|_{L^\infty}\\
&\leqslant 3C + |B| \max_{n\in B} |\widehat{\partial_2 f_2}(n)| = 3C + |B| \max_{n\in B} |n(2)||\widehat{f_2}(n)|\\
&\leqslant 3C + 3^N \max_{n\in B} |n(2)|\lambda(|n|)\|f_2\|_{S_\lambda} \leqslant 3C + 3^N \max_{n\in B} |n(2)|\lambda(|n(1)|)\|f_2\|_{S_\lambda}\\
&\leqslant 3C + 3^N 4^{-N} 3C < 6C,
\end{aligned}$$

where we have used (iii) to pass from the fourth to the fifth line.

Finally, the first term is easier to handle. We have:

$$\begin{aligned}
\|P_A G_N * f_0\|_{L^\infty} &= \|G_N * P_A f_0\|_{L^\infty} \leqslant \|P_A f_0\|_{L^\infty} \leqslant |A| \max_{n\in A} |\widehat{f_0}(n)|\\
&\leqslant |A| \max_{n\in A} \lambda(|n|)\|f_0\|_{S_\lambda} \leqslant |A| \max_{n\in A} |n(1)|\lambda(|n(2)|)\|f_0\|_{S_\lambda}\\
&\leqslant 3^N 4^{-N} 3C < 3C.
\end{aligned}$$

These estimates together with (2.9) give us

$$\|P_A G_N * g_N\|_{L^\infty} \leqslant 3C + 6C + 3C = 12C,$$

which contradicts (2.8), since $\ln N > 25\pi C$. □

Remarks. —

(1). — Similarly, a closer look to the proof in [6] gives the following analogue of Theorem 1.2 in the case of L^1.

Theorem 2.2. — *Suppose $\lambda : \mathbb{N} \to (0,\infty)$ is decreasing to 0. There exists $g \in L^1(\mathbb{T}^2)$ such that there are no f_0, f_1, $f_2 \in S_\lambda(\mathbb{T}^2)$ with $\partial_2 f_2 \in L^1(\mathbb{T}^2)$ and*

$$g = f_0 + \partial_1 f_1 + \partial_2 f_2.$$

(2). — The d-dimensional case, with $d \geqslant 3$, can be easily obtained from Theorem 1.2. More precisely, we have

Theorem 2.3. — *Let $d \geqslant 2$. Suppose $\lambda : \mathbb{N} \to (0,\infty)$ is decreasing to 0. There exists $g \in L^\infty(\mathbb{T}^d)$ such that there are no f_0, f_1, $f_2, \dots, f_d \in \mathcal{D}'(\mathbb{T}^d)$ with f_0, f_1, $f_2 \in S_\lambda(\mathbb{T}^d)$, $\partial_2 f_2 \in L^\infty(\mathbb{T}^d)$ and*

$$g = f_0 + \partial_1 f_1 + \partial_2 f_2 + \cdots + \partial_d f_d.$$

Indeed, consider a $g' \in C^\infty(\mathbb{T}^2)$ and $\psi \in C^\infty(\mathbb{T}^{d-2})$ such that $0 \leqslant \psi \leqslant 1$ and $\int_{\mathbb{T}^{d-2}} \psi = 1$. If the above result were not true, we could find $f_0, f_1, f_2, \ldots, f_d \in \mathcal{D}'(\mathbb{T}^d)$ such that

$$g' \otimes \psi = f_0 + \partial_1 f_1 + \partial_2 f_2 + \cdots + \partial_d f_d$$

and

$$\|f_0\|_{S_\lambda(\mathbb{T}^d)} + \|f_1\|_{S_\lambda(\mathbb{T}^d)} + \|f_2\|_{S_\lambda(\mathbb{T}^d)} + \|\partial_2 f_2\|_{L^\infty(\mathbb{T}^d)} \leqslant C\|g'\|_{L^\infty(\mathbb{T}^2)}.$$

Without loss of generality, we can suppose that $f_0, f_1, f_2, \ldots, f_d$ are smooth. Integrating this equation in the last $d-2$ coordinates, we reduce the problem to the 2-dimensional case: $g' = f'_0 + \partial_1 f'_1 + \partial_2 f'_2$ where

$$f'_j(t) := \int_{\mathbb{T}^{d-2}} f_j(t,\tau)\,\mathrm{d}\tau, \quad \text{for } j = 0, 1, 2,$$

satisfy

$$\|f'_0\|_{S_\lambda(\mathbb{T}^2)} + \|f'_1\|_{S_\lambda(\mathbb{T}^2)} + \|f'_2\|_{S_\lambda(\mathbb{T}^2)} + \|\partial_2 f'_2\|_{L^\infty(\mathbb{T}^2)} \leqslant C\|g'\|_{L^\infty(\mathbb{T}^2)}.$$

Here, we have used the fact that, for all $n' \in \mathbb{Z}^2$,

$$\left|\widehat{f'_j}(n')\right| = \left|\widehat{f_j}(n',0)\right| \leqslant \lambda(|(n',0)|)\|f_j\|_{S_\lambda(\mathbb{T}^d)} = \lambda(|n'|)\|f_j\|_{S_\lambda(\mathbb{T}^d)}.$$

(3). — Using Lemma 2.1, and adapting the technique in [1], we can obtain similar anisotropic Ornstein type inequalities adapted to the L^∞ case. We give below an example. For any $\varepsilon > 0$, there exists a trigonometric polynomial f on $\mathbb{T}^2$, depending on ε, such that

$$\varepsilon\|\partial_1^3\partial_2^2 f\|_{L^\infty} \geqslant \|\partial_1^4 f\|_{L^\infty} + \|\partial_1^2\partial_2^4 f\|_{L^\infty} + \|\partial_1\partial_2^6 f\|_{L^\infty} + \|\partial_2^8 f\|_{L^\infty}.$$

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