Annales Henri Lebesgue
5 (2022) 1151-1160

ANNALES
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BOREL FRACTIONAL COLORINGS OF SCHREIER GRAPHS

COLORIAGES FRACTIONNAIRES BORÉLIENS DE GRAPHES DE SCHREIER

Abstract. — Let Γ be a countable group and let G be the Schreier graph of the free part of the Bernoulli shift $\Gamma \curvearrowright 2^\Gamma$ (with respect to some finite subset $F \subseteq \Gamma$). We show that the Borel fractional chromatic number of G is equal to 1 over the measurable independence number of G. As a consequence, we asymptotically determine the Borel fractional chromatic number of G when Γ is the free group, answering a question of Meehan.

Résumé. — Soit Γ un groupe dénombrable. Considérons G le graphe de Schreier de la partie libre du décalage de Bernoulli $\Gamma \curvearrowright 2^\Gamma$ (par rapport à un ensemble fini $F \subseteq \Gamma$). Nous montrons que le nombre chromatique fractionnaire borélien de G est égal à 1 sur le nombre d'indépendance mesurable de G. Comme conséquence, nous déterminons l'asymptotique du nombre chromatique fractionnaire borélien de G lorsque Γ est le groupe libre, ce qui répond à une question de Meehan.

1. Definitions and results

All graphs in this paper are undirected and simple. Recall that for a graph G, a subset $I \subseteq V(G)$ is *G-independent* if no two vertices in I are adjacent in G. The *chromatic number* of G, denoted by $\chi(G)$, is the least $\ell \in \mathbb{N}$ such that there exist

Keywords: fractional coloring, Borel sets, Borel combinatorics, Schreier graph, group action, symbolic dynamics.
2020 *Mathematics Subject Classification:* 03E15, 05C15, 37B10.
DOI: `https://doi.org/10.5802/ahl.145`
(*) This research is partially supported by the NSF grant DMS-2045412.

G-independent sets $I_1, \ldots, I_\ell$ whose union is $V(G)$. (If no such ℓ exists, we set $\chi(G) \coloneqq \infty$.) The sequence $I_1, \ldots, I_\ell$ is called an *ℓ-coloring* of G, where we think of the vertices in I_i as being assigned the color i.

Fractional coloring is a well-studied relaxation of graph coloring. For an introduction to this topic, see the book [SU97] by Scheinerman and Ullman. Given $k \in \mathbb{N}$, the *k-fold chromatic number* of G, denoted by $\chi^k(G)$, is the least $\ell \in \mathbb{N}$ such that there are G-independent sets $I_1, \ldots, I_\ell$ which cover every vertex of G at least k times (such a sequence $I_1, \ldots, I_\ell$ is called a *k-fold ℓ-coloring*). Note that the sets $I_1, \ldots, I_\ell$ need not be distinct. In particular, if $I_1, \ldots, I_{\chi(G)}$ is a $\chi(G)$-coloring of G, then, by repeating each set k times, we obtain a k-fold $k\chi(G)$-coloring, which shows that

$$\chi^k(G) \leqslant k\chi(G) \quad \text{for all } k.$$

This inequality can be strict; for example, the 5-cycle C_5 satisfies $\chi(C_5) = 3$ but $\chi^2(C_5) = 5$. It is therefore natural to define the *fractional chromatic number* $\chi^*(G)$ of G by the formula

$$\chi^*(G) \coloneqq \inf_{k \geqslant 1} \frac{\chi^k(G)}{k}.$$

In this note we investigate fractional colorings from the standpoint of Borel combinatorics. For a general overview of Borel combinatorics, see the surveys [KM20] by Kechris and Marks and [Pik21] by Pikhurko. The study of fractional colorings in this setting was initiated by Meehan [Mee18]; see also [KM20, § 8.6]. We say that a graph G is *Borel* if $V(G)$ is a standard Borel space and $E(G)$ is a Borel subset of $V(G) \times V(G)$. The *Borel chromatic number* $\chi_{\mathrm{B}}(G)$ of G is the least $\ell \in \mathbb{N}$ such that there exist Borel G-independent sets $I_1, \ldots, I_\ell$ whose union is $V(G)$. The *Borel k-fold chromatic number* $\chi^k_{\mathrm{B}}(G)$ is defined analogously, and the *Borel fractional chromatic number* $\chi^*_{\mathrm{B}}(G)$ is

$$\chi^*_{\mathrm{B}}(G) \coloneqq \inf_{k \geqslant 1} \frac{\chi^k_{\mathrm{B}}(G)}{k}.$$

A particularly important class of Borel graphs are Schreier graphs of group actions. Let Γ be a countable group with identity element $\mathbf{1}$ and let $F \subseteq \Gamma$ be a finite subset. The *Cayley graph* $G(\Gamma, F)$ of Γ is the graph with vertex set Γ in which two distinct group elements γ, δ are adjacent if and only if $\gamma = \sigma\delta$ for some $\sigma \in F \cup F^{-1}$. This definition can be extended as follows. Let $\Gamma \curvearrowright X$ be a Borel action of Γ on a standard Borel space X. The action $\Gamma \curvearrowright X$ is *free* if

$$\gamma \cdot x \neq x \quad \text{for all } \ x \in X \quad \text{and} \quad \mathbf{1} \neq \gamma \in \Gamma.$$

The *Schreier graph* $G(X, F)$ of an action $\Gamma \curvearrowright X$ is the graph with vertex set X in which two distinct points x, $y \in X$ are adjacent if and only if $y = \sigma \cdot x$ for some $\sigma \in F \cup F^{-1}$. Note that the Cayley graph $G(\Gamma, F)$ is a special case of this construction corresponding to the left multiplication action $\Gamma \curvearrowright \Gamma$. More generally, when the action $\Gamma \curvearrowright X$ is free, $G(X, F)$ is obtained by putting a copy of the Cayley graph $G(\Gamma, F)$ onto each orbit.

A crucial example of a Borel action is the *(Bernoulli) shift* $\Gamma \curvearrowright 2^\Gamma$, given by the formula

$$(\gamma \cdot x)(\delta) \coloneqq x(\delta\gamma) \quad \text{for all} \quad x \colon \Gamma \to 2 \quad \text{and} \quad \gamma, \ \delta \in \Gamma.$$

We use β to denote the "coin flip" probability measure on 2^Γ, obtained as the product of countably many copies of the uniform probability measure on $2 = \{0, 1\}$. Note that β is invariant under the shift action. The *free part* of 2^Γ, denoted by $\mathrm{Free}(2^\Gamma)$, is the set of all points $x \in 2^\Gamma$ with trivial stabilizer. In other words, $\mathrm{Free}(2^\Gamma)$ is the largest subspace of 2^Γ on which the shift action is free. It is easy to see that the shift action $\Gamma \curvearrowright 2^\Gamma$ is free β-almost everywhere, i.e., $\beta(\mathrm{Free}(2^\Gamma)) = 1$.

Let G be a Borel graph and let μ be a probability (Borel) measure on $V(G)$. The *μ-independence number* of G is the quantity $\alpha_\mu(G) \coloneqq \sup_I \mu(I)$, where the supremum is taken over all μ-measurable G-independent subsets $I \subseteq V(G)$. Note that if $I_1, \ldots, I_\ell$ is a Borel k-fold ℓ-coloring of G, then

$$\ell\alpha_\mu(G) \geqslant \mu(I_1) + \cdots + \mu(I_\ell) \geqslant k,$$

which implies $\chi^*_{\mathrm{B}}(G) \geqslant 1/\alpha_\mu(G)$. Our main result is a matching upper bound for Schreier graphs:

Theorem 1.1. — *Let Γ be a countable group and let $F \subseteq \Gamma$ be a finite set. If $\Gamma \curvearrowright X$ is a free Borel action on a standard Borel space, then*

$$\chi^*_{\mathrm{B}}(G(X, F)) \leqslant \frac{1}{\alpha_\beta\left(G\left(\mathrm{Free}\left(2^\Gamma\right), F\right)\right)}. \tag{1.1}$$

In particular,

$$\chi^*_{\mathrm{B}}\left(G\left(\mathrm{Free}\left(2^\Gamma\right), F\right)\right) = \frac{1}{\alpha_\beta\left(G\left(\mathrm{Free}\left(2^\Gamma\right), F\right)\right)}. \tag{1.2}$$

While (1.2) is a special case of (1.1), it is possible to deduce (1.1) from (1.2) using a theorem of Seward and Tucker–Drob [STD16], which asserts that every free Borel action of Γ admits a Borel Γ-equivariant map to $\mathrm{Free}(2^\Gamma)$. Nevertheless, we will give a simple direct proof of (1.1) in § 2.

An interesting feature of Theorem 1.1 is that it establishes a precise relationship between a *Borel* parameter χ^*_{B} and a *measurable* parameter α_β. We find this somewhat surprising, since ignoring sets of measure 0 usually significantly reduces the difficulty of problems in Borel combinatorics. For instance, given a Borel graph G and a probability measure μ on $V(G)$, one can consider the *μ-measurable chromatic number* $\chi_\mu(G)$, i.e., the least $\ell \in \mathbb{N}$ such that there exist μ-measurable G-independent sets $I_1, \ldots, I_\ell$ whose union is $V(G)$. By definition, $\chi_\mu(G) \leqslant \chi_{\mathrm{B}}(G)$, and it is often the case that this inequality is strict—see [KM20, § 6] for a number of examples. By contrast, as an immediate consequence of Theorem 1.1 we obtain the opposite inequality $\chi^*_{\mathrm{B}}(G) \leqslant \chi_\beta(G)$, where G is the Schreier graph of the free part of the shift:

Corollary 1.2. — *Let Γ be a countable group and let $F \subseteq \Gamma$ be a finite set. Set $G \coloneqq G(\mathrm{Free}(2^\Gamma), F)$. Then $\chi^*_{\mathrm{B}}(G) \leqslant \chi_\beta(G)$.*

Proof. — Follows from Theorem 1.1 and the inequality $\alpha_\beta(G) \geqslant 1/\chi_\beta(G)$. □

As a concrete application of Theorem 1.1, consider the free group case. For $n \geqslant 1$, let $\mathbb{F}_n$ be the free group of rank n generated freely by elements $\sigma_1, \ldots, \sigma_n$ and let G_n denote the Schreier graph of the free part of the shift action $\mathbb{F}_n \curvearrowright 2^{\mathbb{F}_n}$ with respect to the set $\{\sigma_1, \ldots, \sigma_n\}$. Then every connected component of G_n is an (infinite)

$2n$-regular tree. In particular, the chromatic number of G_n is 2. On the other hand, Marks [Mar16] proved that $\chi_{\mathrm{B}}(G_n) = 2n+1$. Meehan inquired where between these two extremes the Borel fractional chromatic number of G_n lies:

Question 1.3 ([Mee18, Question 4.6.3]; see also [KM20, Problem 8.17]). — *What is the Borel fractional chromatic number of G_n? Is it always equal to 2?*

Using Theorem 1.1 together with some known results we asymptotically determine $\chi^*_{\mathrm{B}}(G_n)$ (and, in particular, give a negative answer to the second part of Question 1.3):

Corollary 1.4. — *For all $n \geqslant 1$, we have*

$$\chi^*_{\mathrm{B}}(G_n) \;=\; (2+o(1))\frac{n}{\log n},$$

where $o(1)$ denotes a function of n that approaches 0 as $n \to \infty$.

In other words, the Borel fractional chromatic number of G_n is less than its ordinary Borel chromatic number roughly by a factor of $\log n$. We present the derivation of Corollary 1.4 in § 3.

2. Proof of Theorem 1.1

We shall use the following theorem of Kechris, Solecki, and Todorcevic:

Theorem 2.1 (Kechris–Solecki–Todorcevic [KST99, Proposition 4.6]). — *If G is a Borel graph of finite maximum degree d, then $\chi_{\mathrm{B}}(G) \leqslant d+1$.*

Fix a countable group Γ and a finite subset $F \subseteq \Gamma$. Without loss of generality, we may assume that $\mathbf{1} \notin F$. Say that a set $I \subseteq 2^\Gamma$ is *independent* if $I \cap (\sigma \cdot I) = \varnothing$ for all $\sigma \in F$ (when $I \subseteq \mathrm{Free}(2^\Gamma)$, this exactly means that I is $G(\mathrm{Free}(2^\Gamma), F)$-independent). For brevity, let

$$\alpha_\beta \;\coloneqq\; \alpha_\beta\left(G\left(\mathrm{Free}\left(2^\Gamma\right), F\right)\right).$$

Lemma 2.2. — *For every $\alpha < \alpha_\beta$, there is a clopen independent set $I \subseteq 2^\Gamma$ such that $\beta(I) \geqslant \alpha$.*

Proof. — Let $J \subseteq \mathrm{Free}(2^\Gamma)$ be a β-measurable independent set with $\beta(J) > \alpha$. Since β is regular [Kec95, Theorem 17.10] and 2^Γ is zero-dimensional, there is a clopen set $C \subseteq 2^\Gamma$ with

$$\mu(J \vartriangle C) \;\leqslant\; \frac{\beta(J)-\alpha}{|F|+1}.$$

Set $I \coloneqq C \setminus \bigcup_{\sigma \in F}(\sigma \cdot C)$. By construction, I is clopen and independent. Moreover, if $x \in J \setminus I$, then either $x \in J \setminus C$ or $x \in (\sigma \cdot C) \setminus (\sigma \cdot J)$ for some $\sigma \in F$. Therefore,

$$\beta(I) \;\geqslant\; \beta(J) \;-\; (|F|+1)\beta(J \vartriangle C) \;\geqslant\; \alpha. \qquad \square$$

Let $\Gamma \curvearrowright X$ be a free Borel action on a standard Borel space. Fix an arbitrary clopen independent set $I \subseteq 2^\Gamma$. We will prove that $\chi^*_{\mathrm{B}}(G(X,F)) \leqslant 1/\beta(I)$, which yields Theorem 1.1 by Lemma 2.2. Since I is clopen, there exist finite sets $D \subseteq \Gamma$ and $\Phi \subseteq 2^D$ such that

$$I \;=\; \left\{x \in 2^\Gamma \;:\; x|_D \in \Phi\right\},$$

where $x|_D$ denotes the restriction of x to D. Note that

$$\beta(I) = \frac{|\Phi|}{2^{|D|}}.$$

Let $N \coloneqq |DD^{-1}|$ and consider the graph $H \coloneqq G(X, DD^{-1})$. Every vertex in H has precisely $N-1$ neighbors (we are subtracting 1 to account for the fact that a vertex is not adjacent to itself). By Theorem 2.1, this implies that $\chi_{\mathrm{B}}(H) \leqslant N$. In other words, we may fix a Borel function $f\colon X \to N$ such that $f(u) \neq f(v)$ whenever u, $v \in X$ are distinct points satisfying $v \in DD^{-1} \cdot u$. This implies that for each $x \in X$, the restriction of f to the set $D \cdot x$ is injective. Now, to each mapping $\varphi\colon N \to 2$, we associate a Borel Γ-equivariant function $\pi_\varphi\colon X \to 2^\Gamma$ as follows:

$$\pi_\varphi(x)(\gamma) \coloneqq (\varphi \circ f)(\gamma \cdot x) \quad \text{for all} \quad x \in X \quad \text{and} \quad \gamma \in \Gamma.$$

Let $I_\varphi \coloneqq \pi_\varphi^{-1}(I)$. Since π_φ is Γ-equivariant, I_φ is $G(X,F)$-independent. Consider any $x \in X$ and let

$$S_x \coloneqq \{f(\gamma \cdot x) : \gamma \in D\}.$$

By the choice of f, S_x is a subset of N of size $|D|$. Whether or not x is in I_φ is determined by the restriction of φ to S_x; furthermore, there are exactly $|\Phi|$ such restrictions that put x in I_φ. Thus, the number of mappings $\varphi\colon N \to 2$ for which $x \in I_\varphi$ is

$$|\Phi| 2^{N-|D|} = \beta(I) 2^N.$$

Since this holds for all $x \in X$, we conclude that the sets I_φ cover every point in X exactly $\beta(I)2^N$ times. Therefore, $\chi^*_{\mathrm{B}}(G(X,F)) \leqslant 1/\beta(I)$, as desired.

3. Proof of Corollary 1.4

Thanks to Theorem 1.1, in order to establish Corollary 1.4 it is enough to verify that

$$\alpha_\beta(G_n) = \left(\frac{1}{2} + o(1)\right) \frac{\log n}{n}.$$

There are a number of known constructions that witness the lower bound

$$\alpha_\beta(G_n) \geqslant \left(\frac{1}{2} + o(1)\right) \frac{\log n}{n};$$

see, e.g., [LW07] by Lauer and Wormald and [GG10] by Gamarnik and Goldberg. Moreover, by [Ber19, Corollary 1.2], even the inequality $\chi_\beta(G_n) \leqslant (2+o(1))n/\log n$ holds. For the upper bound

$$\alpha_\beta(G_n) \leqslant \left(\frac{1}{2} + o(1)\right) \frac{\log n}{n}, \tag{3.1}$$

we shall use a theorem of Rahman and Virág [RV17], which says that the largest density of a factor of i.i.d. independent set in the d-regular tree is at most $(1 + o(1)) \log d/d$. In the remainder of this section we describe their result and explain how it implies the desired upper bound on $\alpha_\beta(G_n)$.

Fix an integer $n \geqslant 1$. For our purposes, it will be somewhat more convenient to work on the space $[0,1]^{\mathbb{F}_n}$ instead of $2^{\mathbb{F}_n}$, where $[0,1]$ is the unit interval equipped with the usual Lebesgue probability measure. The product measure on $[0,1]^{\mathbb{F}_n}$ is denoted by λ. Let H_n be the Schreier graph of the shift action $\mathbb{F}_n \curvearrowright [0,1]^{\mathbb{F}_n}$ corresponding to the standard generating set of $\mathbb{F}_n$. We remark that, by a theorem of Abért and Weiss [AW13] (see also [KM20, Theorem 6.46]), $\alpha_\beta(G_n) = \alpha_\lambda(H_n)$, so it does not really matter whether we are working with G_n or H_n.

Set $d \coloneqq 2n$ and let $\mathbb{T}_d$ denote the Cayley graph of the free group $\mathbb{F}_n$ with respect to the standard generating set. In other words, $\mathbb{T}_d$ is an (infinite) d-regular tree. We view $\mathbb{T}_d$ as a *rooted* tree, whose root is the vertex $\mathbf{1}$, i.e., the identity element of $\mathbb{F}_n$. Let $\mathfrak{A}$ be the automorphism group of $\mathbb{T}_d$, i.e., the set of all bijections $\mathfrak{A}\colon \mathbb{F}_n \to \mathbb{F}_n$ that preserve the edges of $\mathbb{T}_d$, and let $\mathfrak{A}_\bullet \subseteq \mathfrak{A}$ be the subgroup comprising the root-preserving automorphisms, i.e., those $\mathfrak{A} \in \mathfrak{A}$ that map $\mathbf{1}$ to $\mathbf{1}$. The space $[0,1]^{\mathbb{F}_n}$ is equipped with a natural right action $[0,1]^{\mathbb{F}_n} \circlearrowright \mathfrak{A}$. Namely, for $\mathfrak{A} \in \mathfrak{A}$ and $x \in [0,1]^{\mathbb{F}_n}$, the result of acting by $\mathfrak{A}$ on x is the function $x \cdot \mathfrak{A}\colon \mathbb{F}_n \to [0,1]$ given by

$$(x \cdot \mathfrak{A})(\delta) \coloneqq x(\mathfrak{A}(\delta)) \quad \text{for all} \quad \delta \in \mathbb{F}_n.$$

For each $\gamma \in \mathbb{F}_n$, there is a corresponding automorphism $\mathfrak{A}_\gamma \in \mathfrak{A}$ sending every group element $\delta \in \mathbb{F}_n$ to $\delta\gamma$. The mapping $\mathbb{F}_n \to \mathfrak{A}\colon \gamma \mapsto \mathfrak{A}_\gamma$ is an antihomomorphism of groups, that is, we have

$$\mathfrak{A}_{\gamma\sigma} = \mathfrak{A}_\sigma \circ \mathfrak{A}_\gamma \quad \text{for all} \quad \gamma, \sigma \in \mathbb{F}_n,$$

where $\circ$ denotes composition. In particular, $\{\mathfrak{A}_\gamma : \gamma \in \mathbb{F}_n\}$ is a subgroup of $\mathfrak{A}$ isomorphic to $\mathbb{F}_n$. The right action $[0,1]^{\mathbb{F}_n} \circlearrowright \mathfrak{A}$ and the left action $\mathbb{F}_n \curvearrowright [0,1]^{\mathbb{F}_n}$ are related by the formula

$$x \cdot \mathfrak{A}_\gamma = \gamma \cdot x \quad \text{for all} \quad x \in [0,1]^{\mathbb{F}_n}.$$

A set $X \subseteq [0,1]^{\mathbb{F}_n}$ is called $\mathfrak{A}_\bullet$-*invariant* if $x \cdot \mathfrak{A} \in X$ for all $x \in X$ and $\mathfrak{A} \in \mathfrak{A}_\bullet$. The Rahman–Virág theorem can now be stated as follows:

Theorem 3.1 (Rahman–Virág [RV17, Theorem 2.1]). — *If $I \subseteq [0,1]^{\mathbb{F}_n}$ is an $\mathfrak{A}_\bullet$-invariant λ-measurable H_n-independent set, then*

$$\lambda(I) \leqslant (1 + o(1))\frac{\log d}{d} = \left(\frac{1}{2} + o(1)\right)\frac{\log n}{n}.$$

Theorem 3.1 is almost the result we want, except that we need an upper bound on the measure of *every* (not necessarily $\mathfrak{A}_\bullet$-invariant) λ-measurable H_n-independent set I. To remove the $\mathfrak{A}_\bullet$-invariance assumption, we use the following consequence of Theorem 3.1:

Corollary 3.2. — *There exists a Borel graph Q with a probability measure μ on $V(Q)$ such that:*

- *every connected component of Q is a d-regular tree; and*
- $\alpha_\mu(Q) \leqslant (1/2 + o(1)) \log n/n$.

Proof. — We use a construction that was studied by Conley, Kechris, and Tucker–Drob in [CKTD13]. Let Ω be the set of all points $x \in [0,1]^{\mathbb{F}_n}$ such that $x \cdot \mathfrak{A} \neq x$ for every non-identity automorphism $\mathfrak{A} \in \mathfrak{A}$. Let us make a couple observations about Ω. Notice that, by definition, the set Ω is invariant under the action $[0,1]^{\mathbb{F}_n} \circlearrowright \mathfrak{A}$; in particular, it is invariant under the shift action $\mathbb{F}_n \curvearrowright [0,1]^{\mathbb{F}_n}$. Furthermore, the induced action of $\mathbb{F}_n$ on Ω is free (indeed, even the action $\Omega \circlearrowleft \mathfrak{A}$ is free). Since every injective mapping $\mathbb{F}_n \to [0,1]$ belongs to Ω, we conclude that $\lambda(\Omega) = 1$. Now consider the quotient space $V := \Omega/\mathfrak{A}_\bullet$. As the group $\mathfrak{A}_\bullet$ is compact, the space V is standard Borel [CKTD13, paragraph preceding Lemma 7.8]. Let μ be the push-forward of λ under the quotient map $\Omega \to V$, and let Q be the graph with vertex set V in which two vertices $\boldsymbol{x}$, $\boldsymbol{y} \in V$ are adjacent if and only if there are representatives $x \in \boldsymbol{x}$ and $y \in \boldsymbol{y}$ that are adjacent in H_n. Conley, Kechris, and Tucker–Drob [CKTD13, Lemma 7.9] (see also [Tho20, Proposition 1.9]) showed that every connected component of Q is a d-regular tree. Furthermore, by construction, a set $I \subseteq V$ is Q-independent if and only if its preimage under the quotient map is H_n-independent. Since the quotient map establishes a one-to-one correspondence between subsets of V and $\mathfrak{A}_\bullet$-invariant subsets of Ω, Theorem 3.1 is equivalent to the assertion that $\alpha_\mu(Q) \leqslant (1/2 + o(1)) \log n/n$, as desired. □

In view of Corollary 3.2, the following lemma completes the proof of (3.1):

Lemma 3.3. — *Let Q be a Borel graph in which every connected component is a d-regular tree and let μ be a probability measure on $V(Q)$. Then $\alpha_\mu(Q) \geqslant \alpha_\beta(G_n)$.*

In the case when Q is the Schreier graph of a free measure-preserving action of $\mathbb{F}_n$, the conclusion of Lemma 3.3 follows from the Abért–Weiss theorem [AW13]. To handle the general case, we rely on a strengthening of a recent result of Tóth [Tót21] due to Grebík [Gre22], which, roughly, asserts that every d-regular Borel graph is "approximately" induced by an action of $\mathbb{F}_n$.

To state this result precisely, we introduce the following terminology. A *Borel partial action* $\boldsymbol{p}$ of $\mathbb{F}_n$ on a standard Borel space X, in symbols $\boldsymbol{p}\colon \mathbb{F}_n \curvearrowright^* X$, is a sequence of Borel partial injections $p_1, \ldots, p_n \colon X \dashrightarrow X$. Given a Borel graph Q, we say that a Borel partial action $\boldsymbol{p}\colon \mathbb{F}_n \curvearrowright^* V(Q)$ is a *partial Schreier decoration* of Q if $p_i(x)$ is adjacent to x for all $1 \leqslant i \leqslant n$ and $x \in \mathrm{dom}(p_i)$. If $\boldsymbol{p}$ is a partial Schreier decoration of a graph Q, then we let $C(Q, \boldsymbol{p})$ be the set of all vertices $x \in V(Q)$ such that x belongs to both the domain and the image of every p_i and the neighborhood of x in Q is equal to the set $\{p_1(x), \ldots, p_n(x), p_1^{-1}(x), \ldots, p_n^{-1}(x)\}$. A *Schreier decoration* of Q is a partial Schreier decoration $\boldsymbol{p}$ such that $C(Q, \boldsymbol{p}) = V(Q)$. It is easy to see that Q admits a Schreier decoration if and only if it is the Schreier graph of a Borel action of $\mathbb{F}_n$.

Now we can state Grebík's result:

Theorem 3.4 (Grebík [Gre22, Theorem 0.2(III)]). — *Let Q be a d-regular Borel graph and let μ be a probability measure on $V(Q)$. Then for every $\varepsilon > 0$, Q admits a partial Schreier decoration $\boldsymbol{p}$ such that $\mu(C(Q, \boldsymbol{p})) \geqslant 1 - \varepsilon$.*

With Theorem 3.4 in hand, we are ready to establish Lemma 3.3.

Proof of Lemma 3.3. — Recall that we denote the generators of $\mathbb{F}_n$ by $\sigma_1, \ldots, \sigma_n$. Let Q be a Borel graph in which every connected component is a d-regular tree and let μ be a probability measure on $V(Q)$. Thanks to Lemma 2.2, it suffices to show that $\alpha_\mu(Q) \geqslant \beta(I)$ for every clopen independent set $I \subseteq 2^{\mathbb{F}_n}$, where, as in § 2, we say that I is independent if $I \cap (\sigma_i \cdot I) = \varnothing$ for each $1 \leqslant i \leqslant n$.

Fix a clopen independent set $I \subseteq 2^{\mathbb{F}_n}$. Since I is clopen, we can write

$$I = \left\{x \in 2^{\mathbb{F}_n} : x|_D \in \Phi\right\},$$

where $D \subset \mathbb{F}_n$ and $\Phi \subseteq 2^D$ are finite sets. Furthermore, we may assume without loss of generality that $D = \{\gamma \in \mathbb{F}_n : |\gamma| \leqslant k\}$ for some $k \in \mathbb{N}$, where $|\gamma|$ denotes the word norm of γ. For a vertex $x \in V(Q)$, we let $N^k(x)$ be the set of all vertices that are joined to x by a path of length at most k. Since every connected component of Q is a d-regular tree, we have $|N^k(x)| = |D|$ for all $x \in V(Q)$. This allows us to define a probability measure μ_k on $V(Q)$ via

$$\mu_k(A) \coloneqq \int \frac{\left|A \cap N^k(x)\right|}{|D|} \,\mathrm{d}\mu(x) \quad \text{for all Borel } \ A \subseteq V(Q).$$

We have now prepared the ground for an application of Theorem 3.4. Fix $\varepsilon > 0$ and let $\boldsymbol{p}$ be a partial Schreier decoration of Q such that

$$\mu_k(C(Q, \boldsymbol{p})) \geqslant 1 - \frac{\varepsilon}{|D|},$$

which exists by Theorem 3.4. Let C_k be the set of all $x \in V(Q)$ such that $N^k(x) \subseteq C(Q, \boldsymbol{p})$. Then

$$\begin{aligned} 1 - \frac{\varepsilon}{|D|} &\leqslant \mu_k\left(C\left(Q, \boldsymbol{p}\right)\right) \\ &= \int \frac{\left|C\left(Q, \boldsymbol{p}\right) \cap N^k(x)\right|}{|D|} \,\mathrm{d}\mu(x) \leqslant \mu(C_k) + \left(1 - \frac{1}{|D|}\right)\left(1 - \mu\left(C_k\right)\right) \\ &= \frac{1}{|D|}\mu(C_k) + 1 - \frac{1}{|D|}, \end{aligned}$$

which implies that $\mu(C_k) \geqslant 1 - \varepsilon$. The importance of the set C_k lies in the fact that for each $x \in C_k$ and $\gamma \in D$, there is a natural way to define the notation $\gamma \cdot x$. Namely, we write γ as a reduced word:

$$\gamma = \sigma_{i_1}^{s_1} \cdots \sigma_{i_\ell}^{s_\ell},$$

where $0 \leqslant \ell \leqslant k$, each index i_j is between 1 and n, and each s_j is 1 or -1. Since $N^k(x) \subseteq C(Q, \boldsymbol{p})$, there is a unique sequence x_0, x_1, …, x_ℓ of vertices with

$$x_0 = x \quad \text{and} \quad x_j = p_{i_j}^{s_j}\left(x_{j-1}\right) \text{ for all } 1 \leqslant j \leqslant \ell.$$

We then set $\gamma \cdot x \coloneqq x_\ell$. Note that we have $N^k(x) = \{\gamma \cdot x : \gamma \in D\}$.

The remainder of the argument utilizes a construction similar to the one in the proof of Theorem 1.1 given in § 2. Consider the graph R with the same vertex set as Q in which two distinct vertices are adjacent if and only if they are joined by a path of length at most $2k$ in Q. Since every connected component of Q is a d-regular tree, each vertex in R has the same finite number of neighbors, so, by Theorem 2.1, the Borel chromatic number $\chi_{\mathrm{B}}(R)$ is finite. Let $N \coloneqq \chi_{\mathrm{B}}(R)$ and fix a Borel function $f \colon V(Q) \to N$ such that $f(u) \neq f(v)$ whenever u and v are adjacent in R. Then for each $x \in V(Q)$, the restriction of f to the set $N^k(x)$ is injective. Now, to each mapping $\varphi\colon N \to 2$, we associate function $\pi_\varphi\colon C_k \to 2^D$ as follows:

$$\pi_\varphi(x)(\gamma) \coloneqq (\varphi \circ f)(\gamma \cdot x) \quad \text{for all } x \in C_k \text{ and } \gamma \in D.$$

Let $I_\varphi \coloneqq \{x \in C_k : \pi_\varphi(x) \in \Phi\}$. The independence of I implies that the set I_φ is Q-independent. We will show that for some choice of $\varphi\colon N \to 2$, $\mu(I_\varphi) \geqslant (1-\varepsilon)\beta(I)$. Since ε is arbitrary, this yields the desired bound $\alpha_\mu(Q) \geqslant \beta(I)$ and completes the proof of Lemma 3.3.

Consider any $x \in C_k$ and let

$$S_x \coloneqq \{f(\gamma \cdot x) : \gamma \in D\}.$$

Since f is injective on $N^k(x)$, S_x is a subset of N of size $|D|$. Whether or not x is in I_φ is determined by the restriction of φ to S_x; furthermore, there are exactly $|\Phi|$ such restrictions that put x in I_φ. Thus, the number of mappings $\varphi\colon N \to 2$ for which $x \in I_\varphi$ is

$$|\Phi| 2^{N-|D|} = \beta(I) 2^N.$$

Since this holds for all $x \in C_k$, we conclude that

$$\sum_{\varphi\,:\,N \to 2} \mu(I_\varphi) \geqslant \mu(C_k)\,\beta(I)2^N \geqslant (1-\varepsilon)\beta(I)2^N,$$

where the second inequality uses that $\mu(C_k) \geqslant 1-\varepsilon$. In other words, the average value of $\mu(I_\varphi)$ over all $\varphi\colon N \to 2$ is at least $(1-\varepsilon)\beta(I)$. Thus, the maximum is at least $(1-\varepsilon)\beta(I)$ as well, and the proof is complete. $\square$

Acknowledgement

I am grateful to the anonymous referees for carefully reading this paper and providing helpful feedback.

BIBLIOGRAPHY

[AW13] Miklós Abért and Benjamin Weiss, *Bernoulli actions are weakly contained in any free action*, Ergodic Theory Dyn. Syst. **33** (2013), no. 2, 323–333. ↑1156, 1157

[Ber19] Anton Bernshteyn, *Measurable versions of the Lovász Local Lemma and measurable graph colorings*, Adv. Math. **353** (2019), 153–223. ↑1155

[CKTD13] Clinton T. Conley, Alexander S. Kechris, and Robin D. Tucker-Drob, *Ultraproducts of measure preserving actions and graph combinatorics*, Ergodic Theory Dyn. Syst. **33** (2013), no. 2, 334–374. ↑1156, 1157

[GG10] David Gamarnik and David A. Goldberg, *Randomized greedy algorithms for independent sets and matchings in regular graphs: Exact results and finite girth corrections*, Comb. Probab. Comput. **19** (2010), no. 1, 61–85. ↑1155

[Gre22] Jan Grebík, *Approximate Schreier decorations and approximate König's line coloring Theorem*, Ann. Henri Lebesgue **5** (2022), 303–315. ↑1157

[Kec95] Alexander S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, vol. 156, Springer, 1995. ↑1154

[KM20] Alexander S. Kechris and Andrew S. Marks, *Descriptive Graph Combinatorics*, `http://www.math.caltech.edu/~kechris/papers/combinatorics20book.pdf` (preprint), 2020. ↑1152, 1153, 1154, 1156

[KST99] Alexander S. Kechris, Sławomir Solecki, and Stevo B. Todorcevic, *Borel chromatic numbers*, Adv. Math. **141** (1999), no. 1, 1–44. ↑1154

[LW07] J. Lauer and N. Wormald, *Large independent sets in regular graphs of large girth*, J. Comb. Theory **97** (2007), 999–1009. ↑1155

[Mar16] Andrew S. Marks, *A determinacy approach to Borel combinatorics*, J. Am. Math. Soc. **29** (2016), no. 2, 579–600. ↑1154

[Mee18] C. Meehan, *Definable combinatorics of graphs and equivalence relations*, Ph.D. thesis, California Institute of Technology, Pasadena, CA, USA, 2018, `https://resolver.caltech.edu/caltechthesis:06012018-160828760`. ↑1152, 1154

[Pik21] Oleg Pikhurko, *Borel combinatorics of locally finite graphs*, Surveys in combinatorics 2021 (K.K. Dabrowski et al., eds.), London Mathematical Society Lecture Note Series, vol. 470, Cambridge University Press, 2021, pp. 267–319. ↑1152

[RV17] Mustazee Rahman and Bálint Virág, *Local algorithms for independent sets are half-optimal*, Ann. Probab. **45** (2017), no. 3, 1543–1577. ↑1155, 1156

[STD16] Brandon Seward and Robin D. Tucker-Drob, *Borel structurability on the* 2*-shift of a countable group*, Ann. Pure Appl. Logic **167** (2016), no. 1, 1–21. ↑1153

[SU97] Edward R. Scheinerman and Daniel H. Ullman, *Fractional Graph Theory. A rational approach to the theory of graphs*, John Wiley & Sons, 1997, with a foreword by Claude Berge. ↑1152

[Tho20] Riley Thornton, *Factor of i.i.d. processes and Cayley diagrams*, `https://arxiv.org/abs/2011.146044v1` (preprint), 2020. ↑1157

[Tót21] Lázló M. Tóth, *Invariant Schreier decorations of unimodular random networks*, Ann. Henri Lebesgue **4** (2021), 1705–1726. ↑1157

Manuscript received on 5th September 2021,
accepted on 22nd February 2022.

Recommended by Editors S. Gouëzel and D. Aldous.
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eISSN: 2644–9463
This journal is a member of Centre Mersenne.

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