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OPTIMAL SHAPES FOR GENERAL INTEGRAL FUNCTIONALS FORMES OPTIMALES POUR DES FONCTIONNELLES INTÉGRALES GÉNÉRALES

ABSTRACT. — We consider shape optimization problems for general integral functionals of the calculus of variations, defined on a domain Ω that varies over all subdomains of a given bounded domain D of \mathbb{R}^d . We show in a rather elementary way the existence of a solution that is in general a quasi open set. Under very mild conditions we show that the optimal domain is actually open and with finite perimeter. Some counterexamples show that in general this does not occur.

RÉSUMÉ. — On considère des problèmes d'optimisation de forme pour des fonctionnelles intégrales générales, parmi les domaines Ω parcourant tous les sous-domaines d'un domaine borné D donné de \mathbb{R}^d . Nous montrons de manière assez élémentaire l'existence d'une solution qui en général est un ensemble quasi ouvert. Sous des conditions très faibles, nous prouvons que le domaine optimal est en fait ouvert et de périmètre fini. Des contre-exemples montrent que ce n'est pas toujours le cas.

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1. Introduction

In this paper we consider a shape optimization problem for a general integral functional of the form

(1.1)
$$F(u,\Omega) = \int_{\Omega} f(x,u,\nabla u) \, dx.$$

Setting

$$\mathcal{F}(\Omega) = \min\left\{F(u,\Omega) : u \in W_0^{1,p}(\Omega)\right\}$$

the problem we are dealing with is written as

(1.2)
$$\min\left\{\mathcal{F}(\Omega) : \Omega \subset D\right\}.$$

Here p > 1 is a fixed real number, D is a given bounded domain of \mathbb{R}^d , and the function $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to] - \infty, +\infty]$ is a general integrand satisfying suitable rather mild assumptions. Note that in problem (1.2) the volume constraint can be incorporated into the cost functional \mathcal{F} by means of a Lagrange multiplier of the form $\lambda |\Omega|$ or more generally $\int_{\Omega} \lambda(x) dx$. For a detailed presentation of shape optimization problems we refer the interested reader to the books [BB05] and [HM05].

The first result is Theorem 2.1, which gives the existence of an optimal domain Ω_{opt} . This optimal domain belongs to the class of *p*-quasi open sets, defined as the sets $\{u > 0\}$ for some function $u \in W_0^{1,p}(D)$. As a consequence, if p > d these optimal sets are actually open, but if $p \leq d$ this fact does not occur any more under the very general assumptions we made, see Example 4.3.

The existence of optimal sets Ω_{opt} could have been obtained through a generalization of a result in [BDM93] to the case p > 1, making use of a γ_p -convergence on the class of p-quasi open sets. However, we have preferred to give an independent proof that, in the particular case of integral functionals of the form (1.1), is much simpler.

In order to obtain that the optimal sets Ω_{opt} are open, we need slightly stronger assumptions: this is the goal of Theorem 3.4, in which we use the Hölder continuity result of [GG84, Giu03] on the minimizers of general integral functionals.

Finally, in Theorem 5.1 we prove, under rather general assumptions on the integrand f, that Ω_{opt} has a finite perimeter. This result is obtained by adapting a previous result of [Buc12] to the general case of an integrand f with a p-growth.

2. Setting of the problem and existence result

We recall here some well-known notions from the Sobolev spaces theory; for all details we refer to [BB05] and to [Maz11]. In all the paper p > 1 will be a fixed real number.

For every set $E \subset \mathbb{R}^d$ the *p*-capacity of *E* is defined as

$$\operatorname{cap}_p(E) = \inf \left\{ \|u\|_{W^{1,\,p}(\mathbb{R}^d)}^p : u \in W^{1,\,p}(\mathbb{R}^d), \ u \ge 1 \text{ a.e. in a neighborhood of } E \right\}.$$

We say that a property $\mathcal{P}(x)$ holds p-quasi everywhere (shortly q.e.) in a set E if the set of points of $x \in E$ for which $\mathcal{P}(x)$ does not hold has p-capacity zero; the expression almost everywhere (shortly a.e.) refers, as usual, to the Lebesgue measure.

A set Ω is called p-quasi open if $\Omega = \{u > 0\}$ for a suitable function $u \in W^{1,p}(\mathbb{R}^d)$. This is equivalent to require that for every $\varepsilon > 0$ there exists an open subset A_{ε} of \mathbb{R}^d with $A_{\varepsilon} \supset A$ and such that $\operatorname{cap}_p(A_{\varepsilon} \setminus A) < \varepsilon$. It has to be noticed that, since for p > d the $W^{1,p}(\mathbb{R}^d)$ functions are Hölder continuous, we have that in this case p-quasi open sets are actually open. On the contrary, since for $p \leq d$ functions in $W^{1,p}(\mathbb{R}^d)$ are not continuous, in general p-quasi open sets are not open.

A function $f : D \to \mathbb{R}$ is said to be *p*-quasi continuous (respectively *p*-quasi lower semicontinuous) if for every $\varepsilon > 0$ there exists a continuous (respectively lower semicontinuous) function $f_{\varepsilon} : D \to \mathbb{R}$ such that

$$\operatorname{cap}_p\left(\left\{x \in D : f(x) \neq f_{\varepsilon}(x)\right\}\right) < \varepsilon.$$

It is well known (see for instance Ziemer [Zie89]) that every function u of the Sobolev space $W^{1,p}(D)$ has a p-quasi continuous representative \tilde{u} , which is uniquely defined up to a set of capacity zero. The function \tilde{u} is given by

$$\tilde{u}(x) = \lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) \, dy$$

in the sense that the limit above exists for p-quasi every $x \in D$. In the following we always identify the function u with its p-quasi continuous representative \tilde{u} , so that pointwise conditions can be imposed on u(x) for p-quasi every $x \in D$. Again, when p > d p-quasi continuous functions are continuous, because points have a positive p capacity.

If Ω is a *p*-quasi open set we may define the Sobolev space $W_0^{1,p}(\Omega)$ as

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\mathbb{R}^d) : u = 0 \text{ q.e. on } \mathbb{R}^d \setminus \Omega \right\}.$$

We notice that this definition coincides, in the case when Ω is open, with the usual one obtained as the closure of the class of smooth functions compactly supported in Ω with respect to the $W^{1,p}(\mathbb{R}^d)$ norm

$$||u||_{W^{1,p}} = \left(\int_{\mathbb{R}^d} |\nabla u|^p + |u|^p \, dx\right)^{1/p}.$$

It is also important to stress that two sets Ω_1 and Ω_2 which are equivalent in the Lebesgue sense, that is $|\Omega_1 \triangle \Omega_2| = 0$, where \triangle denotes the symmetric difference for sets, may produce very different Sobolev spaces. For instance, in \mathbb{R}^2 if Ω_1 is the unit disk in \mathbb{R}^2 and Ω_2 is the unit disk minus a radius S, the Sobolev spaces $W_0^{1,p}(\Omega_1)$ and $W_0^{1,p}(\Omega_2)$ differ a lot: the functions in the second one vanish on the radius S, which is not the case for functions in the first space. Similarly, in \mathbb{R}^d an open set Ω and Ω without a k-dimensional manifold S provide very different Sobolev spaces $W_0^{1,p}(\Omega_2)$ different Sobolev spaces $W_0^{1,p}(\Omega_2)$ different Sobolev space in the first space. Similarly, in \mathbb{R}^d and Ω without a k-dimensional manifold S provide very different Sobolev spaces $W_0^{1,p}(\Omega_2)$ different Sobolev space is the space of Ω and Ω without a k-dimensional manifold S provide very different Sobolev space space space space space space space $W_0^{1,p}(\Omega_2)$ different space $W_0^{1,p}(\Omega_2)$ different Sobolev space space space Ω and Ω without a k-dimensional manifold S provide very different Sobolev space space space $W_0^{1,p}(\Omega_2)$ different Sobolev space space space $W_0^{1,p}(\Omega_2)$ space $M_0^{1,p}(\Omega_2)$ space Ω and Ω without a k-dimensional manifold S provide very different Sobolev space space $W_0^{1,p}(\Omega_2)$ space Ω and Ω without a $M_0^{1,p}(\Omega_2)$ space Ω and Ω space Ω and Ω space Ω and Ω without a $M_0^{1,p}(\Omega_2)$ space Ω and Ω without a $M_0^{1,p}(\Omega_2)$ space Ω and Ω space Ω and

In the following we fix a bounded domain D of \mathbb{R}^d and we consider the admissible class

$$\mathcal{A} = \left\{ \Omega \subset D : \Omega \ p \text{-quasi open} \right\}.$$

For every $\Omega \in \mathcal{A}$ and $u \in W_0^{1,p}(\Omega)$ we define the integral functional

(2.1)
$$F(u,\Omega) = \int_{\Omega} f(x,u,\nabla u) \, dx$$

where the integrand f is assumed to verify the following conditions:

- (f1) f(x, s, z) is measurable in x, lower semicontinuous in (s, z), convex in z;
- (f2) there exist c > 0, $a \in L^1(D)$, and $\alpha < \lambda_{1,p}(D)$ such that

$$c(|z|^p - \alpha|s|^p - a(x)) \leq f(x, s, z)$$
 for every $x, s, z,$

being $\lambda_{1,p}(D)$ the first Dirichlet eigenvalue of the *p*–Laplacian on *D*, defined as

$$\lambda_{1,p}(D) = \min\left\{\int_D |\nabla u|^p \, dx \, : \, u \in W_0^{1,p}(D), \, \int_D |u|^p \, dx = 1\right\}.$$

(f3) $f(x, 0, 0) \ge 0$.

It is well-known (see for instance [But89]) that under conditions (f1) and (f2) for every $\Omega \in \mathcal{A}$ the functional $F(\cdot, \Omega)$ defined in (2.1) is lower semicontinuous with respect to the weak convergence in $W_0^{1,p}(\Omega)$ and that the minimum problem

(2.2)
$$\min\left\{F(u,\Omega) : u \in W_0^{1,p}(\Omega)\right\}$$

admits a solution. Let us denote by $\mathcal{F}(\Omega)$ the minimum value in (2.2). The shape optimization problem we deal with is

(2.3)
$$\min\left\{\mathcal{F}(\Omega) : \Omega \in \mathcal{A}\right\}$$

In the following theorem we prove that the shape optimization problem above admits a solution. For the proof we could use the general theory of γ -convergence and weak γ -convergence (see [BB05]), and the fact that the shape functional \mathcal{F} has some monotonicity properties with respect to the set inclusion; however, in our case a simpler proof is available and we report this one.

THEOREM 2.1. — Under assumptions (f1),(f2),(f3) the shape optimization problem (2.3) admits a solution.

Proof. — Consider the auxiliary minimum problem

(2.4)
$$\min \bigg\{ \int_D f(x, u, \nabla u) \mathbf{1}_{\{u \neq 0\}} dx : u \in W_0^{1, p}(D) \bigg\}.$$

Since

$$\int_{D} f(x, u, \nabla u) \mathbf{1}_{\{u \neq 0\}} dx = \int_{D} f(x, u, \nabla u) dx - \int_{D} f(x, 0, 0) \mathbf{1}_{\{u = 0\}} dx$$
$$= \int_{D} \left[f(x, u, \nabla u) - f(x, 0, 0) \right] dx + \int_{D} f(x, 0, 0) \mathbf{1}_{\{u \neq 0\}} dx,$$

Problem (2.4) can be rewritten as

$$\min\left\{\int_D \left[f(x, u, \nabla u) - f(x, 0, 0) + f(x, 0, 0)\mathbf{1}_{\{u \neq 0\}}\right] dx \, : \, u \in W^{1, p}_0(D)\right\}$$

and, thanks to assumptions (f1) and (f2), it verifies the lower semicontinuity and coercivity properties that guarantee it admits a solution \bar{u} . We claim that the *p*-quasi

open set $\Omega_0 = \{ \bar{u} \neq 0 \}$ solves the shape optimization Problem (2.3). Indeed, let $\Omega \in \mathcal{A}$ and let u_{Ω} be the solution of the minimum Problem (2.2); then we have

$$\mathcal{F}(\Omega) = \int_{\Omega} f(x, u_{\Omega}, \nabla u_{\Omega}) dx$$

= $\int_{D} f(x, u_{\Omega}, \nabla u_{\Omega}) dx - \int_{D \setminus \Omega} f(x, 0, 0) dx$
= $\int_{D} f(x, u_{\Omega}, \nabla u_{\Omega}) \mathbf{1}_{\{u_{\Omega} \neq 0\}} dx + \int_{D} f(x, 0, 0) \left[\mathbf{1}_{\{u_{\Omega} = 0\}} - \mathbf{1}_{D \setminus \Omega} \right] dx$
 $\geqslant \int_{D} f(x, \bar{u}, \nabla \bar{u}) \mathbf{1}_{\{\bar{u} \neq 0\}} dx + \int_{\Omega} f(x, 0, 0) \mathbf{1}_{\Omega \cap \{u_{\Omega} = 0\}} dx \geqslant \mathcal{F}(\Omega_{0})$

where the last inequality follows from the definition of Ω_0 and from assumption (f3).

Remark 2.2. — Notice that, when f(x, 0, 0) = 0, the functional $\mathcal{F}(\Omega)$ is decreasing with respect to the set inclusion. Indeed, in this case we have for every $u \in W_0^{1,p}(\Omega)$

$$\int_{\Omega} f(x, u, \nabla u) \, dx = \int_{D} f(x, u, \nabla u) \, dx$$

so that, if $\Omega_1 \subset \Omega_2$

$$\mathcal{F}(\Omega_1) \min\left\{\int_D f(x, u, \nabla u) \, dx \, : \, u \in W_0^{1, p}(\Omega_1)\right\}$$

$$\geq \min\left\{\int_D f(x, u, \nabla u) \, dx \, : \, u \in W_0^{1, p}(\Omega_2)\right\} = \mathcal{F}(\Omega_2).$$

Therefore, when f(x, 0, 0) = 0, if Ω_0 is a solution of the shape optimization problem (2.3), then every $\Omega \supset \Omega_0$ is also a solution. In particular, the whole set D is a solution of (2.3).

3. Existence of open optimal domains

In the present section we show that, under mild additional assumptions on the integrand f, the optimal domain Ω_0 of problem (2.3), obtained in Theorem 2.1 is actually an open set. To do this we show that the solution \bar{u} of the auxiliary minimum problem (2.4) is a continuous function. This follows by means of a well-known result of Giaquinta and Giusti in [GG84] (see also [Giu03]), that we summarize here below for the sake of completeness.

THEOREM 3.1. — Let \bar{u} be a solution of the minimum problem

$$\min\left\{\int_D h(x, u, \nabla u) \, dx \, : \, u \in W^{1, p}_0(D)\right\}$$

where the integrand h satisfies the condition

(3.1)
$$c(|z|^p - b(x)|s|^\gamma - g(x)) \leq h(x, s, z) \leq C(|z|^p + b(x)|s|^\gamma + g(x))$$

for all x, s, z, where p > 1, $0 < c \leq C$, $p \leq \gamma < p^*$, $b \in L^q_{loc}(D)$, $g \in L^{\sigma}_{loc}(D)$ being $p^* = dp/(d-p)$ ($p^* = +\infty$ if $p \geq d$) the Sobolev exponent relative to $p, \sigma > d/p$, $q > p^*/(p^* - \gamma)$. Then \bar{u} is locally Hölder continuous in D.

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Remark 3.2. — In the paper [GG84] the integrand h above was assumed of Carathéodory type, but in fact condition (f1) is still enough, provided condition (3.1) is satisfied. Actually, as the authors say, even the convexity of h with respect to z is not needed, if we assume that a solution \bar{u} exists.

Remark 3.3. — Of course, the result above is nontrivial only in the case $p \leq d$; indeed, if p > d the Hölder continuity of \bar{u} simply follows from the Sobolev embedding theorem.

We can now apply Theorem 3.1 to obtain that in a large number of situations the optimal set Ω_0 obtained in Theorem 2.1 is actually an open set.

THEOREM 3.4. — Assume that the integrand f satisfies conditions (f1),(f2), (f3), (3.1). Then the optimal domain Ω_0 of problem (2.3), obtained in Theorem 2.1 is an open set.

Proof. — Since $\Omega_0 = \{ \bar{u} \neq 0 \}$ where \bar{u} is a solution of the auxiliary problem (2.4), it is enough to show that the function \bar{u} is continuous on D. We have for every $u \in W_0^{1,p}(D)$

$$\int_{D} f(x, u, \nabla u) \mathbf{1}_{\{u \neq 0\}} dx = \int_{D} \left[f(x, u, \nabla u) - f(x, 0, 0) \mathbf{1}_{\{u = 0\}} \right] dx$$

and the integrand

$$h(x, s, z) = f(x, s, z) - f(x, 0, 0) \mathbf{1}_{\{s=0\}}$$

satisfies the conditions of Theorem 3.1. Then the Hölder continuity of \bar{u} follows. \Box

Remark 3.5. — In general, under the sole existence assumptions (f1), (f2), (f3), we do not expect that the optimal domain Ω_0 be open. In [BHP05] the authors consider the particular case

$$F(u,\Omega) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \langle g, u \rangle$$

under a volume constraint of the form $|\Omega| = m$, and refer to [Hay99] for a counterexample to the fact that the solution Ω_0 is open, when the function g is in $H^{-1}(D)$. In the following section we show that a counterexample can be constructed even in the case $g \in H^{-1}(D) \cap L^1(D)$.

4. Optimal domains that are not open

As we have seen in Theorem 3.4 quite mild assumptions on the integrand f imply the existence of open optimal domains Ω_{opt} . In this section we show that, when these assumptions are not satisfied, there may exist optimal domains which are not better than quasi open sets, even in very simple cases as the Dirichlet energy

$$F(u,\Omega) = \int_{\Omega} \left[\frac{1}{p} |\nabla u|^p - g(x)u \right] dx.$$

We start by a preliminary result.

PROPOSITION 4.1. — Let f be an integrand satisfying conditions (f1) and (f2), and assume that f(x,0,0) = 0. Let \bar{u} be a solution of the minimum problem

$$\min\left\{\int_D f(x, u, \nabla u) \, dx \, : \, u \in W^{1, p}_0(D)\right\}$$

and denote by Ω_0 the p-quasi open set $\{\bar{u} \neq 0\}$. Then Ω_0 is a solution of the shape optimization problem (2.3).

Proof. — Setting

$$\mathcal{F}(\Omega) = \min\left\{\int_{\Omega} f(x, u, \nabla u) \, dx \, : \, u \in W_0^{1, \, p}(\Omega)\right\}$$

we have to show that for every p-quasi open set $\Omega \subset D$ we have

(4.1)
$$\mathcal{F}(\Omega) \ge \mathcal{F}(\Omega_0).$$

Using the optimality of \bar{u} and the fact that f(x, 0, 0) = 0 we obtain

$$\min\left\{\int_{\Omega} f(x, u, \nabla u) \, dx \, : \, u \in W_0^{1, p}(\Omega)\right\} = \min\left\{\int_{D} f(x, u, \nabla u) \, dx \, : \, u \in W_0^{1, p}(\Omega)\right\}$$
$$\geqslant \min\left\{\int_{D} f(x, u, \nabla u) \, dx \, : \, u \in W_0^{1, p}(D)\right\} = \int_{D} f(x, \bar{u}, \nabla \bar{u}) \, dx$$
$$= \int_{\Omega_0} f(x, \bar{u}, \nabla \bar{u}) \, dx \geqslant \min\left\{\int_{\Omega_0} f(x, u, \nabla u) \, dx \, : \, u \in W_0^{1, p}(\Omega_0)\right\},$$
nich implies (4.1).

which implies (4.1).

We consider here two shape optimization problems for the Dirichlet energy; we set

(4.2)
$$\mathcal{F}(\Omega) = \min\left\{\int_{\Omega} \left[\frac{1}{p}|\nabla u|^p - g(x)u\right]dx : u \in W_0^{1,p}(\Omega)\right\}$$

The first problem, that we may call *penalized problem*, has the form

$$(\mathcal{P}_{\lambda}) \qquad \min\left\{\mathcal{F}(\Omega) + \lambda |\Omega| : \Omega \in \mathcal{A}\right\},$$

with $\lambda > 0$, while the second one, that we may call *constrained problem* has the form

$$(\mathcal{Q}_m)$$
 min $\Big\{ \mathcal{F}(\Omega) : \Omega \in \mathcal{A}, |\Omega| \leq m \Big\},$

with m > 0. From what seen in the previous sections both the shape optimization problems (\mathcal{P}_{λ}) and (\mathcal{Q}_m) admit a solution.

PROPOSITION 4.2. — Let $\lambda > 0$ and let Ω_0 be an optimal domain for the shape optimization problem (\mathcal{P}_{λ}) . Then there exists m > 0 such that Ω_0 solves the shape optimization problem (\mathcal{Q}_m) .

Proof. — Take $m = |\Omega_0|$ and let $\Omega \in \mathcal{A}$ with $|\Omega| \leq m$. By the optimality of Ω_0 for (\mathcal{P}_{λ}) we have

$$\mathcal{F}(\Omega) + \lambda |\Omega| \ge \mathcal{F}(\Omega_0) + \lambda |\Omega_0|,$$

hence

$$\mathcal{F}(\Omega) \ge \mathcal{F}(\Omega_0) + \lambda (m - |\Omega|) \ge \mathcal{F}(\Omega_0),$$

which proves that Ω_0 solves the shape optimization problem (\mathcal{Q}_m) too.

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While Proposition 4.2 is rather simple, the opposite implication, showing that an optimal domain for (\mathcal{Q}_m) also solves (\mathcal{P}_{λ}) for some λ , is a very delicate issue, which has been studied in [Bri04]. In particular, a rigorous proof of the equivalence of the two formulations is not available in full generality, see for instance [Bri04], [BHP05] and [Hay99] for a discussion on this matter.

Here we show that problem (\mathcal{Q}_m) may have optimal domains that are not open, when the summability of the datum g is not strong enough. We consider the unconstrained problem

(4.3)
$$\min\left\{\mathcal{F}(\Omega):\Omega\in\mathcal{A}\right\};$$

if this problem has a solution Ω_0 which is not open, taking $m = |\Omega_0|$ we have that Ω_0 also solves problem (\mathcal{Q}_m) .

The function g is always assumed in $W^{-1,p'}(D) \cap L^1(D)$. By Theorem 3.4, when in addition $g \in L^q(D)$ with q > d/p, we obtain that the optimal domains Ω_{opt} for problem (4.3) are open sets. In the following example we show that Ω_{opt} may be not open if $p \leq d$ and q = 1. More precisely, we show that for every p-quasi open set Ω we can find $g \in W^{-1,p'}(D) \cap L^1(D)$ such that Ω is the optimal domain for the shape optimization problem (4.3).

Example 4.3. — Let p > 1 and let Ω_0 be any quasi open subset of D. Let w be the torsion function associated to Ω_0 , that is the unique solution of the PDE

$$-\Delta_p w = 1 \text{ in } \Omega_0, \quad w \in W_0^{1, p}(\Omega_0),$$

intended as the minimizer on $W_0^{1,p}(\Omega_0)$ of the functional

$$\int_{\Omega} \left[\frac{1}{p} |\nabla u|^p - u \right] dx$$

or equivalently as the solution of the PDE in its weak form

$$\int_{D} |\nabla w|^{p-2} \nabla w \nabla \phi \, dx = \int_{D} w \phi \, dx \quad \text{for every} \quad \phi \in W_0^{1, p}(\Omega_0).$$

By the maximum principle (see for instance [BB05, Lemma 5.3.2]) the solution w turns out to be positive, more precisely

$$w(x) > 0$$
 for q.e. $x \in \Omega_0$,

so that we have $\Omega_0 = \{w > 0\}.$

We claim that the function $g = -\Delta_p(w^{p'})$ is in $W^{-1,p'}(D) \cap L^1(D)$. Indeed, by the maximum principle w is bounded and, since $\nabla(w^{p'}) = p'w^{p'-1}\nabla w$, we get that

$$\begin{split} w^{p'} \in W_0^{1,p}(D) \text{ and so } g \in W^{-1,p'}(D). \text{ Moreover, for every } \psi \in C_c^{\infty}(D) \text{ we have} \\ \langle g, \psi \rangle &= \int_D |\nabla(w^{p'})|^{p-2} \nabla(w^{p'}) \nabla \psi \, dx \\ &= \left(\frac{p}{p-1}\right)^{p-1} \int_D |\nabla w|^{p-2} w \nabla w \nabla \psi \, dx \\ &= \left(\frac{p}{p-1}\right)^{p-1} \int_D |\nabla w|^{p-2} \nabla w \Big(\nabla(w\psi) - \psi \nabla w\Big) \, dx \\ &= \left(\frac{p}{p-1}\right)^{p-1} \int_D \psi \Big(w - |\nabla w|^p \Big) \, dx \end{split}$$

which gives

$$g = \left(\frac{p}{p-1}\right)^{p-1} \left(w - |\nabla w|^p\right).$$

Since $w \in L^{\infty}(D)$ and $w \in W_0^{1,p}(D)$ we obtain that $g \in L^1(D)$.

Now, we apply Proposition 4.1 with g as above; since the functional in the minimization problem (4.2) is strictly convex, its minimizer is unique and so this implies that the function \bar{u} coincides with $w^{p'}$. Hence Ω_{opt} is the set $\{w^{p'} \neq 0\}$, which coincides with Ω_0 .

Remark 4.4. — We have seen that if p > d or if the function g in (4.2) is in $L^q(D)$ with q > d/p then the optimal set Ω_{opt} is open. On the contrary, if q = 1 we can construct a counterexample showing that Ω_{opt} is merely a p-quasi open set, and actually every p-quasi open set Ω_0 can be optimal for some $g \in L^1(D)$. This picture is sharp when p = d in the sense that in this case q = 1 is the borderline situation and $q \leq 1$ gives a counterexample, while q > 1 gives that Ω_{opt} is open. When p < d we do not know if similar counterexamples hold in the case $1 < q \leq d/p$. In addition, it would be interesting to provide counterexamples showing that also the optimal domains for the *penalized problem* (\mathcal{P}_{λ}) may be not open if the data are not summable enough.

5. Cases when optimal domains have finite perimeter

In this section we show that, under some assumptions slightly stronger than (f1), (f2), (f3) the optimal set Ω_0 obtained in Section 2 has a finite perimeter. We adapt the proof contained in [Buc12] to our general case. The assumptions we need are:

(f2") there exist c > 0 and $\alpha < \lambda_{1,p}(D)$ such that for every x, s, z

$$c(|z|^p - \alpha|s|^p + 1) \leqslant f(x, s, z);$$

(f3") there exist K > 0 and $a \in L^1(D)$ such that for every x, s, t, z

$$\left| f(x,s,z) - f(x,t,z) \right| \leq K |s-t| \left(a(x) + |s|^{p^*} + |t|^{p^*} + |z|^p \right),$$

where $p^* = dp/(d-p)$ (with p^* any positive number if p = d and $|\cdot|^{p^*}$ replaced by any continuous function if p > d) is the Sobolev exponent associated to p. Note that in particular condition (f2") implies a condition stronger than (f3):

$$f(x,0,0) \ge c$$
 with $c > 0$.

THEOREM 5.1. — Under assumptions (f1), (f2"),(f3"), the optimal domain Ω_0 obtained in the existence Theorem 2.1 has a finite perimeter.

Proof. — Let u be a solution of the auxiliary minimization problem (2.4) and let, for every $\varepsilon > 0$

$$u_{\varepsilon} = (u - \varepsilon)^+ - (u - \varepsilon)^-, \quad A_{\varepsilon} = \{0 < |u| \le \varepsilon\}.$$

Note that

$$u_{\varepsilon} = \begin{cases} u - \varepsilon & \text{if } u > \varepsilon \\ u + \varepsilon & \text{if } u < -\varepsilon \\ 0 & \text{if } |u| \leqslant \varepsilon \end{cases} \quad \text{and} \quad \nabla u_{\varepsilon} = \begin{cases} \nabla u & \text{a.e. on } \{|u| > \varepsilon\} \\ 0 & \text{a.e. on } \{|u| \leqslant \varepsilon\}. \end{cases}$$

Then, by the optimality of the function u, we have

$$\int_D f(x, u, \nabla u) \mathbf{1}_{\{u \neq 0\}} dx \leqslant \int_D f(x, u_\varepsilon, \nabla u_\varepsilon) \mathbf{1}_{\{u_\varepsilon \neq 0\}} dx = \int_{D \setminus A_\varepsilon} f(x, u_\varepsilon, \nabla u) dx,$$

where the last equality follows from the fact that $u_{\varepsilon} = 0$ on A_{ε} . Now, using assumption (f3"),

$$\int_{A_{\varepsilon}} f(x, u, \nabla u) \, dx \leq \int_{D \setminus A_{\varepsilon}} \left| f(x, u_{\varepsilon}, \nabla u) - f(x, u, \nabla u) \right| \, dx$$
$$\leq C \int_{D} |u_{\varepsilon} - u| \left(a(x) + |u_{\varepsilon}|^{p^{*}} + |u|^{p^{*}} + |\nabla u|^{p} \right) \, dx \leq C\varepsilon.$$

We now use assumption (f2") and we obtain

$$C\varepsilon \ge \int_{A_{\varepsilon}} f(x, u, \nabla u) \, dx \ge c \int_{A_{\varepsilon}} \left(|\nabla u|^p - \alpha |u|^p \right) dx + c |A_{\varepsilon}|,$$

which implies, thanks to Poincaré inequality and the fact that $\alpha < \lambda_{1,p}(D)$,

$$\int_{A_{\varepsilon}} |\nabla u|^p \, dx + |A_{\varepsilon}| \leqslant C\varepsilon.$$

By Hölder inequality this gives

$$\int_{A_{\varepsilon}} |\nabla u| \, dx \leqslant C \varepsilon.$$

We use now the coarea formula and we deduce

$$\int_0^{\varepsilon} \mathcal{H}^{d-1} \Big(\partial^* \{ |u| > t \} \Big) \, dt \leqslant C \varepsilon.$$

Thus there exists a sequence $\delta_n \to 0$ such that

$$\mathcal{H}^{d-1}(\partial^*\{|u| > \delta_n\}) \leqslant C \quad \text{for every } n$$

and finally this implies that

$$\mathcal{H}^{d-1}(\partial^*\Omega_0) = \mathcal{H}^{d-1}\left(\partial^*\{|u| > 0\}\right) \leqslant C,$$

as required.

ANNALES HENRI LEBESGUE

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