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JEAN BOURGAIN

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SIDON SETS AND RIESZ PRODUCTS

by Jean BOURGAIN

1. Notations.

In what follows, G will be a compact abelian group and $\Gamma = \hat{G}$ the dual group. According to the context, we will use the additive or multiplicative notation for the group operation in Γ. For $1 \leqslant p \leqslant \infty$, $L^p(G)$ denotes the usual Lebesgue space. For $\mu \in M(G)$, let $\|\mu\|_{PM} = \sup_{\gamma \in \Gamma} |\hat{\mu}(\gamma)|$.

A subset Λ of Γ is called a Sidon set provided there is a constant C such that the inequality

$$\sum_{\gamma \in \Lambda} |\alpha_\gamma| \leqslant C \| \sum_{\gamma \in \Lambda} \alpha_\gamma \gamma \|_\infty \tag{1}$$

holds for all finite scalar sequences $(\alpha_\gamma)_{\gamma \in \Lambda}$. The smallest constant $S(\Lambda)$ fulfilling (1) is called the Sidon constant of Λ. The reader is referred to [3] for elementary Sidon set theory.

$|A|$ stands for the cardinal of the set A.

Assume A a subset of Γ and $d \geqslant 0$. We will consider the set of characters

$$P_d[A] = \left\{ \sum_{\gamma \in A} z_\gamma \gamma \,|\, z_\gamma \in \mathbf{Z}(\gamma \in A) \quad \text{and} \quad \sum_{\gamma \in A} |z_\gamma| \leqslant d \right\}.$$

Then $\quad |P_d[A]| \leqslant \left(\frac{C|A|}{d}\right)^d \quad$ if $\quad d \leqslant |A|$

and $\quad |P_d[A]| \leqslant \left(\frac{C d}{|A|}\right)^{|A|} \quad$ if $\quad d > |A|$

where C is a numerical constant (cf. [7], p. 46).

Mots-clefs : Ensemble de Sidon, Ensemble quasi-indépendant, Produits de Riesz.

We say that $A \subset \Gamma$ is quasi-independent, if the relation $\Sigma_A z_\gamma \gamma = 0$, $z_\gamma = -1, 0, 1$ $(\gamma \in A)$ implies $z_\gamma = 0$ $(\gamma \in A)$.

If A is quasi-independent, the measure

$$\mu = \prod_{\gamma \in A} (1 + \operatorname{Re} a_\gamma \gamma)$$

where $a_\gamma \in \mathbf{C}$, $|a_\gamma| \leqslant 1$, is positive and $\|\mu\|_{M(G)} = 1$.

We call it a Riesz product.

Say that $A \subset \Gamma$ tends to infinity provided to each finite subset Γ_0 of Γ corresponds a finite subset A_0 of A such that

$$\gamma, \delta \in A \backslash A_0, \ \gamma \neq \delta \implies \gamma - \delta \notin \Gamma_0 .$$

A Sidon set Λ is of first type provided there is a constant $C < \infty$ and, for each nonempty open subset I of G, there is a finite subset Λ_0 of Λ so that

$$\sum_{\gamma \in \Lambda \backslash \Lambda_0} |\alpha_\gamma| \leqslant C \Big\| \sum_{\gamma \in \Lambda \backslash \Lambda_0} \alpha_\gamma \gamma \Big\|_{C(I)} \tag{2}$$

for finite scalar sequences $(\alpha_\gamma)_{\gamma \in \Lambda \backslash \Lambda_0}$, where

$$\|f\|_{C(I)} = \sup_{x \in I} |f(x)| .$$

2. Interpolation by averaging Riesz products.

In this section, we will prove the following result :

THEOREM. – *For a subset Λ of Γ, the following conditions are equivalent :*

(1) Λ *is a Sidon set*

(2) $\|\Sigma_\Lambda \alpha_\gamma \gamma\|_p \leqslant C p^{1/2} (\Sigma |\alpha_\gamma|^2)^{1/2}$ *for all finite scalar sequences* $(\alpha_\gamma)_{\gamma \in \Lambda}$ *and* $p \geqslant 1$.

(3) *There is* $\delta > 0$ *such that each finite subset* A *of* Λ *contains a quasi-independent subset* B *with* $|B| \geqslant \delta |A|$.

(4) *There is* $\delta > 0$ *such that if* $(\alpha_\gamma)_{\gamma \in \Lambda}$ *is a finite sequence of scalars, there exists a quasi-independent subset* A *of* Λ *such that*

$$\sum_{\gamma \in \Lambda} |\alpha_\gamma| \geqslant \delta \sum_{\gamma \in \Lambda} |\alpha_\gamma| .$$

Implication (1) $\Longrightarrow$ (2) is a consequence of Khintchine's inequalities and is due to W. Rudin [8]. The standard argument that quasi-independent sets are Sidon sets yields (4) $\Longrightarrow$ (1). We will not give it here since it will appear in the next section in the context of an application. Finally, the results (2) $\Longrightarrow$ (1) and (1) $\Longrightarrow$ (3) are due to G. Pisier (see [4], [5] and [6]. The characterization (4)is new. It has the following consequence (by a duality argument):

COROLLARY 1. – *If* Λ *is a Sidon set, there is* $\delta > 0$ *such that whenever* $(a_\gamma)_{\gamma \in \Lambda}$ *is a finite scalar sequence and* $|a_\gamma| \leqslant \delta$, *then we have*

$$\hat{\mu}(\gamma) = \int_G \overline{\gamma}(x)\, \mu\,(dx) = a_\gamma \quad \text{for} \quad \gamma \in \Lambda$$

where μ *is in the* σ*-convex hull of a sequence of Riesz products.*

Recall that the σ-convex hull of a bounded subset P of a complex Banach space X is the set of all elements $\sum_{i=1}^{\infty} \lambda_i x_i$ where $x_i \in \mathrm{P}$, $\sum_{i=1}^{\infty} |\lambda_i| \leqslant 1$.

The remainder of the paragraph is devoted to the proof of (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4).

Let us point out that in the case of bounded groups, i.e. which elements are of bounded order, they can be simplified using algebraic arguments.

LEMMA 1. – *Condition (2) implies (3) with* $\delta \sim \mathrm{C}^{-2}$.

Proof. – We first exhibit a subset A_1 of A, $|\mathrm{A}_1| \gtrsim \mathrm{C}^{-2} |\mathrm{A}|$, such that if $\sum_{\gamma \in \mathrm{A}_1} \epsilon_\gamma \gamma = 0$ and $\epsilon_\gamma = -1, 0, 1$, then $\Sigma |\epsilon_\gamma| < \frac{1}{2} |\mathrm{A}_1|$. If $\sum_{\gamma \in \mathrm{A}_2} \epsilon_\gamma \gamma = 0$, $\epsilon_\gamma = \pm 1$ and $\mathrm{A}_2 \subset \mathrm{A}_1$ is choosen with $|\mathrm{A}_2|$ maximum, the set $\mathrm{B} = \mathrm{A}_1 \backslash \mathrm{A}_2$ will be quasi-independent and $|\mathrm{B}| > \delta\, |\mathrm{A}|$.

The set A_2 is obtained using a probabilistic argument. Fix $\tau = C_1^{-1} C^{-2}$ and $\ell = \frac{1}{4} \tau |A|$ (C_2 is a fixed constant, choosen to fulfil a next estimation). Let $(\xi_\gamma)_{\gamma \in A}$ be independent $(0,1)$-valued random variables in ω and define

$$F_\omega(x) = \sum_{m=\ell}^{|A|} \sum_{\substack{S \subset A \\ |S|=m}} \prod_{\gamma \in S} \xi_\gamma(\omega)(\gamma(x) + \overline{\gamma}(x)).$$

Notice that the property $\int_G F_\omega(x)\, dx = 0$ is equivalent to the fact $\int_G \prod_{\gamma \in S} (\gamma + \overline{\gamma}) = 0$ whenever S is a subset of the random set $\{\gamma \in A \mid \xi_\gamma(\omega) = 1\}$ with $|S| \geqslant \ell$.

Thus the random set does not present (± 1)-relations of length at least ℓ.

Using condition (2) and the choice of τ, ℓ, we may evaluate

$$\iint_G F_\omega(x)\, dx\, d\omega \leqslant \sum_{m=\ell}^{|A|} \tau^m \frac{1}{m!} \int_G |\Sigma_A(\gamma + \overline{\gamma})|^m$$

$$\leqslant \sum_{m \geqslant \ell} \tau^m (6\,C)^m \left(\frac{|A|}{m}\right)^{m/2} < 2^{-\ell/2}.$$

Hence

$$\frac{\tau |A|}{2} + 2^{\ell/2} \iint_G F_\omega(x)\, dx\, d\omega < \int \sum_{\gamma \in A} \xi_\gamma(\omega)$$

implying the existence of ω s.t.

$$|A_1| > \frac{\tau |A|}{2}$$

where $A_1 = \{\gamma \in A \mid \xi_\gamma(\omega) = 1\}$

and

$$\int_G F_\omega(x) < 2^{-\ell/2} |A| < 1, \text{ so } \int_G F_\omega(x) = 0.$$

By definition of F_ω and the choice of ℓ, it follows that A_1 has the desired properties.

The key step is the following construction:

LEMMA 2. – *Assume* $\Lambda_1, \ldots, \Lambda_J$ *disjoint quasi-independent subsets of* Γ and

$$\frac{|\Lambda_{j+1}|}{|\Lambda_j|} > \mathrm{R} \quad \text{for} \quad j = 1, \ldots, \mathrm{J} - 1$$

where the ratio $\mathrm{R} > 10$ *is some fixed numerical constant (appearing through later computations).*

Then there are subsets Λ_j' *of* $\Lambda_j (1 \leqslant j \leqslant \mathrm{J})$ *s.t.*

(1) $|\Lambda_j'| > \frac{1}{10} |\Lambda_j|$ *and* (2) $\bigcup_{j=1}^{j} \Lambda_j'$ *is quasi-independent.*

Proof. – Fixing $j = 1, \ldots, \mathrm{J}$, we will exhibit a subset Λ_j' of Λ_j satisfying the following condition (*)

$$\eta_1 \ldots \eta_{j-1} \eta_{j+1} \ldots \eta_{\mathrm{J}} \neq 0$$

if

$$0 \neq \eta_j = \sum_{\gamma \in \Lambda'} \epsilon_\gamma \gamma \, (\epsilon_\gamma = -1, 0, 1)$$

and for each $k \neq j$

$$\eta_k \in \mathrm{P}_{d_k}(\Lambda_k) \quad \text{where} \quad d_k = \frac{|\Lambda_j|}{|\Lambda_k|} \sum_{\gamma \in \Lambda_j'} |\epsilon_\gamma| .$$

Those sets Λ_j' satisfy (2). Indeed if

$$\eta_1 \ldots \eta_{\mathrm{J}} = 0 \quad \text{and} \quad \eta_j = \sum_{\gamma \in \Lambda_j'} \epsilon_\gamma \gamma \, (\epsilon_\gamma = -1, 0, 1)$$

then, defining $d_j = \sum_{\Lambda_{j'}} |\epsilon_\gamma|$, either $d_j = 0$ or $d_k |\Lambda_k| > d_j |\Lambda_j|$ for some $k \neq j$. If the d_j are not all 0, we may consider j' s.t. $d_{j'} |\Lambda_{j'}|$ is maximum, leading to a contradiction.

The construction of Λ_j' for fixed j is done in the spirit of Lemma 1. It suffices to construct first $\overline{\Lambda}_j \subset \Lambda_j$, $|\overline{\Lambda}_j| > \frac{1}{5} |\Lambda_j|$, fulfilling (*) under the additional restriction

$$\sum_{\gamma \in \overline{\Lambda}_j} |\epsilon_\gamma| > \frac{1}{2} |\overline{\Lambda}_j| . \tag{**}$$

This set $\overline{\Lambda}_j$ is again found randomly. Consider independent $(0, 1)$-valued random variable $\{\xi_\gamma \mid \gamma \in \Lambda_j\}$ of mean $\frac{1}{4}$ and define the random function on G

$$\mathrm{F}_\omega = \sum_{m=|\Lambda_j|/10}^{|\Lambda_j|} \sum_{\substack{S\subset\Lambda_j\\ |S|=m}} \prod_{\gamma\in S} \xi_\gamma(\omega)(\gamma+\bar{\gamma}) \prod_{k\neq j} \Sigma\,\{\eta\in \mathrm{P}_{d_k(m)}(\Lambda_k)\}$$

where $d_k(m) = \dfrac{|\Lambda_j|}{|\Lambda_k|}m$. Write

$$\int\int_G \mathrm{F}_\omega(x)\,dx\,d\omega \leqslant \sum_{m=|\Lambda_j|/10}^{|\Lambda_j|} 2^{-m}$$

$$\int_G \prod_{\gamma\in\Lambda_j}(1+\operatorname{Re}\gamma)\prod_{k\neq j}\Sigma\,\{\eta\in\mathrm{P}_{d_k(m)}(\Lambda_k)\},$$

and using the estimation on $|\mathrm{P}_d(\mathrm{A})|$ mentioned in the introduction, it follows the majoration by

$$2^{-\frac{|\Lambda_j|}{10}}\prod_{k\neq j}|\mathrm{P}_{d_k}(\Lambda_k)| \qquad \left(d_k = d_k(|\Lambda_j|) = \frac{|\Lambda_j|^2}{|\Lambda_k|}\right)$$

$$\leqslant 2^{-\frac{|\Lambda_j|}{10}}\exp\left\{2\sum_{k<j}|\Lambda_k|\log C\frac{|\Lambda_j|}{|\Lambda_k|} + 2\sum_{k>j}\frac{|\Lambda_j|^2}{|\Lambda_k|}\log C\frac{|\Lambda_k|}{|\Lambda_j|}\right\}.$$

Since $\log x < 2\sqrt{x}$ for $x \geqslant 1$, we may further estimate by

$$2^{-\frac{|\Lambda_j|}{10}}\exp\left\{C_1\sum_{k<j}\left(\frac{|\Lambda_k|}{|\Lambda_j|}\right)^{1/2} + C_1\sum_{k>j}\left(\frac{|\Lambda_j|}{|\Lambda_k|}\right)^{1/2}\right\}|\Lambda_j| < 2^{-\frac{|\Lambda_j|}{11}}$$

for an appropriate choice of the ratio R.

So again, since we may assume $|\Lambda_j| > 20$

$$\frac{1}{5}|\Lambda_j| + 2^{\frac{|\Lambda_j|}{11}}\int\int_G \mathrm{F}_\omega(x)\,dx\,d\omega < \int\sum_{\Lambda_j}\xi_\gamma(\omega)\,d\omega$$

and there exists therefore some ω s.t. if $\bar{\Lambda}_j = \{\gamma\in\Lambda_j \mid \xi_\gamma(\omega)=1\}$ we have

$$|\bar{\Lambda}_j| > \frac{1}{5}|\Lambda_j| \quad \text{and} \quad \int_G \mathrm{F}_\omega(x)\,dx = 0.$$

But the latter property means that (*) holds under the restriction (**).

This proves lemma 2.

We derive now the implication (3) $\Longrightarrow$ (4).

LEMMA 3. – *If* (3) *of the theorem holds, then* (4) *is valid with* $\delta(4) \sim \delta(3)$.

Proof. – From Lemma 2, the argument is routine. Let R be the constant appearing in Lemma 2 and fix a sequence $(\alpha_\gamma)_{\gamma \in \Lambda}$ s.t. $\Sigma |\alpha_\gamma| = 1$.

Define for $k = 0, 1, 2, \ldots$

$$\Lambda_k = \{\gamma \in \Lambda \mid R_1^{-k} \geqslant |\alpha_\gamma| > R_1^{-k-1}\}$$

where R_1 is a numerical constant with $R_1 > 4R$.

By hypothesis, there exists for each k a quasi-independent subset Λ_k^1 of Λ_k s.t.

$$|\Lambda_k^1| > \delta\, |\Lambda_k|. \tag{1}$$

Defining

$$\Omega_e = \bigcup_{k \text{ even}} \Lambda_k^1 \quad \text{and} \quad \Omega_0 = \bigcup_{k \text{ odd}} \Lambda_k^1$$

we have

$$\sum_{\gamma \in \Omega_e} |\alpha_\gamma| + \sum_{\gamma \in \Omega_0} |\alpha_\gamma| \geqslant \frac{\delta}{R_1}$$

and may for instance assume

$$\sum_{\gamma \in \Omega_e} |\alpha_\gamma| \geqslant \frac{\delta}{2R_1}. \tag{2}$$

Define inductively the sequence $(k_j)_{j=1,2,\ldots}$ by

$$k_1 = 0 \quad \text{and} \quad k_{j+1} = \min\{k > k_j \mid |\Lambda_{2k}^1| > R\, |\Lambda_{2k_j}^1|\}.$$

If we take $\Lambda_j^2 = \Lambda_{2k_j}^1$, it follows by construction that $\dfrac{|\Lambda_{j+1}^2|}{|\Lambda_j^2|} > R$.

Moreover

$$\sum_j \sum_{k_j<k<k_{j+1}} \sum_{\gamma\in\Lambda^1_{2k}} |\alpha_\gamma|$$
$$\leqslant \sum_j \sum_{k>k_j} R_1^{-2k} R\, |\Lambda^1_{2k_j}|$$
$$\leqslant \frac{2R}{R_1} \sum_j R_1^{-2kj-1}\, |\Lambda^1_{2k_j}|$$
$$\leqslant \frac{2R}{R_1} \sum_{\gamma\in\Omega_e} |\alpha_\gamma|$$

and since $R_1 > 4R$, it follows thus by (2)

$$\sum_j \sum_{\gamma\in\Lambda_j^2} |\alpha_\gamma| > \frac{1}{4R_1}\,\delta\,. \tag{3}$$

Application of Lemma 2 to the sequence $(\Lambda_j^2)_{j=1,2,\ldots}$ leads to further subsets $\Lambda_j^3 \subset \Lambda_j^2$ satisfying

$$|\Lambda_j^3| \geqslant \frac{1}{10}|\Lambda_j^2| \quad \text{and} \quad A = \cup\, \Lambda_j^3 \text{ is quasi-independent.}$$

It remains to write

$$\sum_{\gamma\in A} |\alpha_\gamma| \geqslant \sum_j R_1^{-2k_j-1} |\Lambda_j^3| \geqslant \frac{1}{10R_1} \sum_j R_1^{-2k_j} |\Lambda_j^2|$$
$$\geqslant \frac{1}{10R_1} \sum_j \sum_{\gamma\in\Lambda_j^2} |\alpha_\gamma|$$

and use (3).

Remark. – Say that a subset A of the dual group Γ is d-independent $(d = 1,2,\ldots)$ provided the relation

$$\sum_{\gamma\in A}{}' \epsilon_\gamma\, \gamma = 0 \quad (\epsilon_\gamma = -d, -d+1, \ldots, d)$$

implies $\epsilon_\gamma = 0$ $(\gamma \in A)$.

With this terminology, 1-independent corresponds to quasi-independent.

Assume G a torsion-free compact, abelian group. Fixing an integer d, statements (3) and (4) of the theorem can be reformulated for d-independent sets. The proof is a straightforward modification.

3. Sidon sets of first type.

As an application of previous section, we show

COROLLARY 2. – *A sidon set tending to infinity is a Sidon set of first type.*

Notice that conversely each set of first type tends to infinity (see [2]). Also, each Sidon set is the finite union of sets tending to infinity (see [3], p. 141 and [1] for the general case).

Proof of Cor. 2. – Fix a Sidon set Λ tending to infinity and a nonempty open subset I of G. Choose $\delta > 0$ s.t. (4) of the previous theorem holds.

Let $p \in L^1(G)$ be a polynomial s.t. $p \geqslant 0$, $\hat{p} \geqslant 0$, $\int_G p = 1$ and $|p| < \epsilon$ on $G \backslash I$ (where $\epsilon > 0$ will be defined later). Denote Γ_0 the spectrum of p. By hypothesis, we may assume

$$\gamma - \delta \notin \Gamma_0 \quad \text{for} \quad \gamma \neq \delta \quad \text{in} \quad \Lambda. \tag{1}$$

We claim the existence of a finite subset Λ_0 of Λ s.t. if $(\alpha_\gamma)_{\gamma \in \Lambda \backslash \Lambda_0}$ is a finite scalar sequence, there exists a quasi-independent subset A of $\Lambda \backslash A_0$ s.t.

$$\sum_{\gamma \in A} |\alpha_\gamma| > \frac{\delta}{2} \Sigma |\alpha_\gamma| \tag{2}$$

and

$$\int p \prod_{\gamma \in A} (1 + \operatorname{Re} \gamma) < 2. \tag{3}$$

The existence of Λ_0 is shown by contradiction. Indeed, one should otherwise obtain finite disjointly supported systems

$$(\alpha_\gamma)_{\gamma \in \Lambda_1}, \ldots, (\alpha_\gamma)_{\gamma \in \Lambda_r}, \ldots (\Lambda_r \subset \Lambda)$$

with

$$\sum_{\gamma \in \Lambda_r} |\alpha_\gamma| = 1$$

and for which a quasi-independent set fulfilling (2), (3) does not exist.

Fix R large and apply (4) of the Theorem to the system

$$\left\{ \alpha_\gamma \mid \gamma \in \bigcup_{r=1}^{R} \Lambda_r \right\}.$$

This yields a quasi-independent set $B \subset \Lambda$ so that

$$\sum_{r=1}^{R} \sum_{\gamma \in \Lambda_r \cap B} |\alpha_\gamma| > \delta\, R. \tag{4}$$

Also, since $\hat{p} \geqslant 0$

$$\sum_{r=1}^{R} \int p \left\{ \prod_{\gamma \in B \cap \Lambda_r} (1 + \operatorname{Re} \gamma) - 1 \right\}$$

$$\leqslant \int p \prod_{\gamma \in B} (1 + \operatorname{Re} \gamma) \leqslant \|p\|_\infty \leqslant |\Gamma_0|. \tag{5}$$

As a consequence of (4), (5), there must be some $r = 1, \ldots, R$ for which $\sum_{\gamma \in \Lambda_r \cap B} |\alpha_\gamma| > \frac{\delta}{2}$ as well as

$$\int p \prod_{\gamma \in B \cap \Lambda_r} (1 + \operatorname{Re} \gamma) < 1 + \int p = 2,$$

provided R is chosen large enough. Since $A = B \cap \Lambda_r$ is quasi-independent, a contradiction follows. This ensures the existence of Λ_0. We assume $\Gamma_0 \subset \Lambda_0$.

Let now $(\alpha_\gamma)_{\gamma \in \Lambda \setminus \Lambda_0}$ a finite scalar sequence and A a quasi-independent set fulfilling (2), (3). Clearly, whenever $|a_\gamma| \leqslant 1$ $(\gamma \in A)$, by construction of p,

$$\left| \int \prod_{\gamma \in A} (1 + \operatorname{Re} a_\gamma\, \gamma) (\Sigma\, \alpha_\gamma\, \gamma)\, p \right| \leqslant 2\, \|\Sigma\, \alpha_\gamma\, \gamma\|_{C(I)} + \epsilon\, \Sigma\, |\alpha_\gamma|.$$

We now analyze the left side, defining $a_\gamma = \kappa\, b_\gamma$ $(|b_\gamma| = 1)$, κ to be specified later. Write

$$\prod_{\gamma \in A} (1 + \operatorname{Re} a_\gamma\, \gamma) = 1 + \kappa \sum_{\gamma \in A} \operatorname{Re} b_\gamma\, \gamma + \sum_{\ell \geqslant 2} \kappa^\ell\, Q_\ell$$

where $Q_\ell = \sum_{\substack{S \subset A \\ |S| = \ell}} \prod_{\gamma \in S} \operatorname{Re} b_\gamma\, \gamma$ and, since $\int (\Sigma\, \alpha_\gamma\, \gamma)\, p = 0$,

minorate consequently the left member as

$$\kappa \left| \int \left(\sum_{\gamma \in A} \operatorname{Re} b_\gamma \gamma \right) (\Sigma \alpha_\gamma \gamma) p \right| - \sum_{\ell \geq 2} \kappa^\ell \left| \int Q_\ell p (\Sigma \alpha_\gamma \gamma) \right| . \qquad (*)$$

Since $\hat{p} \geq 0$, we have for fixed ℓ (from (3))

$$\left| \int Q_\ell p (\Sigma \alpha_\gamma \gamma) \right| \leq \| Q_\ell p \|_{PM} \cdot \Sigma |\alpha_\gamma|$$

and

$$\| Q_\ell p \|_{PM} \leq \left\| \left(\sum_{\substack{S \subset A \\ |S| = \ell}} \prod_{\gamma \in S} \operatorname{Re} \gamma \right) p \right\|_{PM}$$

$$\leq \left\| \prod_{\gamma \in A} (1 + \operatorname{Re} \gamma) \cdot p \right\|_1 < 2 .$$

Thus $(*)$ can be minorated as

$$\kappa \left| \int \left(\sum_{\gamma \in A} \operatorname{Re} b_\gamma \gamma \right) (\Sigma \alpha_\gamma \gamma) p \right| - 3 \kappa^2 \Sigma |\alpha_\gamma| .$$

Since $\operatorname{Re} b_\gamma \gamma$ can be replaced by $\operatorname{Im} b_\gamma \gamma$, we see that

$$2 \| \Sigma \alpha_\gamma \gamma \|_{C(I)} \geq \frac{\kappa}{2} \left| \int (\Sigma_A b_\gamma \bar{\gamma}) (\Sigma_\Lambda \alpha_\gamma \gamma) p \right| - (\epsilon + 3\kappa^2) \Sigma |\alpha_\gamma| .$$

Now, for $\gamma \in A \subset \Lambda$ and $\delta \in \Lambda$, either $\gamma = \delta$ or $\int \bar{\gamma} \delta p = 0$.

This as a consequence of (1). Thus, taking $b_\gamma = \dfrac{\overline{\alpha_\gamma}}{|\alpha_\gamma|}$,

$$\int (\Sigma_A b_\gamma \bar{\gamma}) (\Sigma_\Lambda \alpha_\gamma \gamma) p = \Sigma_A |\alpha_\gamma| > \frac{\delta}{2} \Sigma |\alpha_\gamma| .$$

Choosing ϵ, κ appropriately, the proof is completed.

Remark. – Let G be a compactly generated, locally compact abelian group and B the dual group. A subset Λ of Γ is called a topological Sidon set provided there exists a compact subset K of G satisfying $\sum_{\gamma \in \Lambda} |\alpha_\gamma| \leq C \sup_{x \in K} \left| \sum_{\gamma \in \Lambda} \alpha_\gamma \gamma(x) \right|$ where C is a fixed constant.

Similarly to the case of compact groups, we define Sidon sets of first type. Then Cor. 2 remains valid. It is indeed easy using the stability property of topological Sidon sets for small perturbations (see [2] for details) to reduce the problem to the periodic case.

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Jean BOURGAIN,
Dept. of Mathematics
Vrije Universiteit Brussel
Pleinlaan 2-F7
1050 Brussels (Belgium).