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# BOUNDED DOUBLE SQUARE FUNCTIONS

by Jill PIPHER (*)

## 1. Introduction.

Suppose $f(x)$, $x \in \mathbf{R}^d$, is integrable with respect to $(1+|x|^2)^{-(d+1)/2}$, and let $f(t,y)$ be its Poisson integral in the half space $\mathbf{R}^{d+1}_+ = \{t \in \mathbf{R}^d, y>0\}$. For $0 < \gamma < \infty$, the area function of $f(t,y)$ is defined as

$$A_\gamma f(x) = \left( \int_{\Gamma_{\gamma(x)}} |\nabla f(t,y)|^2 y^{1-d}\, dt\, dy \right)^{1/2}$$

where $\Gamma_\gamma(x)$ is the cone $\{(t,y) : |t-x| < \gamma y\}$. The following is proved in [4].

THEOREM 1. — *If* $A_\gamma f \in L^\infty$, *then for all cubes* Q,

$$\int_Q \exp\left( \frac{c_1 |f-(f)_Q|^2}{\|A_\gamma f\|_\infty^2} \right) dx \leqslant C_2,$$

*where* $c_1 > 0$ *and* $c_2 < \infty$ *are constants depending only on the dimension d and the aperture* $\gamma$ *and where* $(f)_Q$ *is the average of* $f(x)$ *over* Q.

Because $f \in \mathrm{BMO}$ if $A_\gamma f \in L^\infty$, exponential integrability of $|f-(f)_Q|$ over the cube Q follows from the John-Nirenberg Theorem and it is the exponential square integrability, which is sharp, that is the assertion of Theorem 1. In this note we extend Theorem 1 to the bidisc case of two parameter kernels.

Just as averages over Q of functions of $|f-(f)_Q|$ occur in the classical definition of BMO, the extension of Theorem 1 to the two parameter case

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will involve the expressions used by Chang and Fefferman [3] in their characterization of BMO in the bidisc. Fix a smooth even function $\psi(z)$ supported on $[-1,1]$ and satisfying $\int \psi(x)\,dx = 0$ and $\int |\hat{\psi}(x)|^2/x\,dx = 1$. Write $(t,y)$ for the point $(t_1,y_1,t_2,y_2)$ of $\mathbf{R}^2_+ \times \mathbf{R}^2_+$, so that $t = (t_1,t_2) \in \mathbf{R}^2$ and $y = (y_1,y_2) \in \mathbf{R}_+ \times \mathbf{R}_+$, and define

$$\psi_y(t) = \frac{1}{y_1}\psi\left(\frac{t_1}{y_1}\right)\frac{1}{y_2}\psi\left(\frac{t_2}{y_2}\right). \tag{1.1}$$

When I is a dyadic interval, let $\mathrm{I}^+$ denote $\{(t,y) \in \mathbf{R}^2_+ : t \in \mathrm{I}$ and $|\mathrm{I}|/2 < y < |\mathrm{I}|\}$ and when $\mathrm{R} = \mathrm{I} \times \mathrm{J}$ is a dyadic rectangle, set $\mathrm{R}^+ = \mathrm{I}^+ \times \mathrm{J}^+$. Then for $0 < \alpha < \infty$ and for $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^2)$, we define

$$f_{\mathrm{R},\alpha}(x) = \iint_{\mathrm{R}^+} f * \psi_{y\alpha}(t)\psi_y(t-x)\,dt\,\frac{dy}{y}$$

where $\psi_{y\alpha}$ means replacing $y$ by $\alpha y$ in (1.1) and

$$\frac{dt\,dy}{y}$$

abbreviates

$$dt_1\,dt_2\,\frac{dy_1}{y_1}\frac{dy_2}{y_2}.$$

Also for $\Omega \subset \mathbf{R}^2$ open, we define

$$\mathrm{F}_{\Omega,\alpha}(x) = \sum_{\substack{\mathrm{R} \subseteq \Omega \\ \mathrm{R}\text{ dyadic}}} f_{\mathrm{R},\alpha}(x).$$

Then Chang and Fefferman [3] have characterized the bidisc space BMO (dual to the space $\mathrm{H}^1$ of functions whose square functions are integrable) by the condition

$$\|\mathrm{F}_{\Omega,\alpha}\|_2^2 \leqslant \mathrm{C}_\alpha|\Omega| \tag{1.2}$$

for all $\Omega$ and for any $\alpha$. Moreover, if $f \in \mathrm{BMO}$, then

$$\int_\Omega \exp\,[c_\alpha|\mathrm{F}_{\Omega,\alpha}|]^{1/2} \leqslant \mathrm{C}|\Omega|,$$

which is the appropriate analog of the John-Nirenberg Theorem.

For $0 < \gamma < \infty$, the square function of $f \in L^1_{loc}(\mathbf{R}^2)$ is defined as

$$S_\gamma f(x_1,x_2) = \left[\int_{\Gamma_\gamma(x_1)}\int_{\Gamma_\gamma(x_2)} |f * \psi_y(t)|^2 \frac{dt\,dy}{y^2}\right]^{1/2}$$

where

$$dt\,\frac{dy}{y^2}$$

abbreviates

$$dt_1\,dt_2\,\frac{dy_1}{(y_1)^2}\,\frac{dy_2}{(y_2)^2}.$$

Then $S_\gamma$ is a two parameter form of $A_\gamma$ and we have

THEOREM 2. – *Suppose $f \in L^1_{loc}(\mathbf{R}^2)$ and $S_\gamma f \in L^\infty$. Then there exist constants $c_1$ and $\alpha$, depending only on $\gamma$, and $c_2$, independent of $\gamma$, such that for all open $\Omega \subseteq \mathbf{R}^2$,*

$$\int_\Omega \exp\left(\frac{c_1|F_{\Omega,\alpha}|}{\|S_\gamma f\|_\infty}\right] dt \leqslant c_2|\Omega|.$$

We wish to make a few remarks about the dependence on $\gamma$ of the expression $F_{\Omega,\alpha}$ in Theorem 2, which is in contrast with the situation in Theorem 1. Let us set, for $x \in \mathbf{R}^d$,

$$(1.3) \qquad F_\alpha(x) = \iint_{(t,y)\in\mathbf{R}^d_+} f * \psi_{\gamma\alpha}(t)\psi_y(t-x)\,dt\,\frac{dy}{y},$$

an idea which originates with Calderon. Taking the Fourier transform of both sides and invoking the normalization $\int |\hat{\psi}(x)|^2\,dx/x = 1$, we find that $F_\alpha(x) = c_\alpha f(x)$, where $|c_\alpha| \leqslant 1$. Consequently, for any cube $Q$,

$$|F_\alpha - (F_\alpha)_Q| = c_\alpha|f-(f)_Q|,$$

and in $\mathbf{R}^d$ we may just as well use $F_\alpha$, rather than $f$.

There are two reasons we must use $F_{\Omega,\alpha}$ in the setting of product domains. First, Carleson [2] has shown that the dual of $H^1(\mathbf{R}\times\mathbf{R})$ cannot be defined in terms of mean oscillations over rectangles, and thus there is no simple analog of the clean geometric definition of $BMO(\mathbf{R}^d)$ for product domains. The second reason is technical: $\alpha$ must be so large that certain rectangles fit into certain cones (see the proof of Theorem 2 at the end of Section 3).

We also note that one usually sees $\alpha = 1$ in definition (1.2). The proof of Theorem 2 will show that one can take $\alpha = 1$ when $S_\gamma f \in L^\infty$ for a cone $\Gamma_\gamma$ which is sufficiently wide.

The crux of the argument for Theorem 2 is a corresponding result for double dyadic martingales. In section 2 we prove a vector-valued form of Theorem 1 which, by an iteration, yields the double martingale result. In section 3 we derive Theorem 2 from its martingale analog using a technique from [4].

This work appears in my doctoral thesis and I would like to thank my advisor, Professor J. B. Garnett, for his tremendous support and encouragement while I was completing this work. I would also like to thank Professor S.-Y. A. Chang for bringing this problem to my attention and for many helpful conversations.

## 2. Double dyadic martingales.

Let $\mathscr{F}_n$ be the $\sigma$-field generated by the dyadic intervals of length $2^{-n}$ in $[0,1]$. The expectation of $f$ on $\mathscr{F}_n$ is

$$E(f|\mathscr{F}_n) = \sum_{|I|=2^{-n}} (f)_I \chi_I(x).$$

A *dyadic martingale* is a sequence $\langle f_0, f_1, f_2, \ldots \rangle$ such that $f_n$ is measurable to $\mathscr{F}_n$ and $E(f_{n+1}|\mathscr{F}_n) = f_n$ for all $n$. Set $d_n = f_n - f_{n-1}$ and define the square function of $f$ by $Sf(x) = \left(\sum_n d_n^2(x)\right)^{1/2}$. We assume that $f_0 = d_0 = 0$. A *double dyadic martingale* can be written as a doubly indexed sequence $\{f_{n,m}\}$ where $\{f_{n,m}\}$ is a dyadic marginale relative to $n$ for each fixed $m$, and also a dyadic martingale in $m$ for each fixed $n$. In particular, $f_{n,0} = f_{0,m} = 0$ for each $n$ and $m$. If $d_{n,m} = f_{n,m} - f_{n-1,m} - f_{n,m-1} + f_{n-1,m-1}$, then the square function of $f$ is defined to be

$$Sf(x) = \left(\sum_n \sum_m d_{n,m}^2\right)^{1/2}.$$

In this section we determine the sharp order of integrability of a double dyadic martingale whose square function is uniformly bounded. The strategy is to find the precise dependence on $p$ of the constant $c_p$ in the inequality $\|f\|_p \leqslant c_p \|Sf\|_p$. The following lemma appears in [4].

LEMMA 2.1 (*H. Rubin*). — *Let* $f_N = \sum_{n=0}^{N} d_n$ *be a dyadic martingale and let* $\alpha$ *be any positive number. Then*

$$\int \exp\left(\alpha f_N - \frac{\alpha^2}{2}\sum_{n=0}^{N} d_n^2\right) \leqslant 1 .$$

COROLLARY 2.1(*a*). — *There exists a constant* $c$, *independent of* N, *such that*

$$\int \exp\left(\frac{c|f_N|^2}{\|Sf_N\|_\infty^2}\right) < \infty .$$

COROLLARY 2.1(*b*). — *For* $p \geqslant 2$, *and* $f^* = \sup_n |f_n|$,

$$\|f^*\|_p \leqslant C\sqrt{p}\|Sf\|_p .$$

For the proofs of these corollaries, see the arguments given later in connection with Lemma 2.2.

If $f_{n,m}$ is a double dyadic martingale, Lemma 2.1 immediately yields

$$\int \exp\left(\alpha f_{N,N} - \frac{\alpha^2}{2}\sum_{n=0}^{N}\left(\sum_{m=0}^{N} d_{n,m}\right)^2\right) dx_1 \leqslant 1 .$$

Set

$$S_1^2 f_{N,N}(x_1,x_2) = \sum_{n=0}^{N}\left(\sum_{m=0}^{N} d_{n,m}\right)^2 (x_1,x_2),$$

the square function taken with respect to the single index $n$. Then we have

$$\|f_{N,N}\|_p \leqslant C\sqrt{p}\|S_1 f_{N,N}\|_p, \qquad p \geqslant 2$$

with, of course, C independent of N and $p$. We need, then, an $L^p$ norm inequality between $S_1 f$ and $Sf$. This can be given in a more general framework.

LEMMA 2.2. — *Suppose* $X_n^j = \sum_{q=0}^{n} d_q^j$, $j = 1, 2, \dots, m$, *is a sequence of dyadic martingales and set* $SX_n^j = \left(\sum_{q=0}^{n} (d_q^j)^2\right)^{1/2}$, *the square functions of the* $X_n^j$. *Then*

$$\int \exp\left(\sqrt{1 + \sum_{j=1}^{m} (X_n^j)^2} - \sum_{j=1}^{m} S^2 X_n^j\right) dx \leqslant e .$$

*Proof.* – Set

$$A_k = \sum_{j=1}^{m} (X_k^j)^2 = \sum_{j=1}^{m} \left(\sum_{q=0}^{k} d_q^j\right)^2, \qquad D_k = \sum_{j=1}^{m} (d_k^j)^2,$$

and

$$r_k = 2 \sum_{j=1}^{m} X_{k-1}^j \, d_k^j .$$

Consider

$$E(e^{\sqrt{1+\sum_{j=1}^{m}(X_n^j)^2}} | \mathcal{F}_{n-1}) = E(e^{\sqrt{1+\sum_{j=1}^{m}(X_{n-1}^j+d_n^j)^2}} | \mathcal{F}_{n-1})$$
$$= E(e^{\sqrt{1+A_{n-1}+D_n+r_n}} | \mathcal{F}_{n-1})$$
$$= \frac{1}{2}\{e^{\sqrt{1+A_{n-1}+D_n+|r_n|}} + e^{\sqrt{1+A_{n-1}+D_n-|r_n|}}\}.$$

This last equality follows from the fact that both $A_{n-1}$ and $D_n$ are measurable with respect to $\mathcal{F}_{n-1}$ and $r_n$ merely changes sign on either half of the intervals of length $2^{-n+1}$. Using the estimates $\sqrt{1+x} \leqslant 1 + x/2$ and $\cosh x \leqslant e^{x^2/2}$, we have

$$E(e^{\sqrt{1+A_n}} | \mathcal{F}_{n-1})$$
$$\leqslant 1/2 \exp\left[\sqrt{1+A_{n-1}}\left(1+\frac{D_n+|r_n|/2}{2(1+A_{n-1})}\right)\right]$$
$$+ 1/2 \exp\left[\sqrt{1+A_{n-1}}\left(1+\frac{D_n-|r_n|}{2(1+A_{n-1})}\right)\right]$$
$$= e^{\sqrt{1+A_{n-1}}} e^{D_n/2\sqrt{1+A_{n-1}}}\left(\frac{e^{|r_n|/2\sqrt{1+A_{n-1}}} + e^{-|r_n|/2\sqrt{1+A_{n-1}}}}{2}\right)$$
$$\leqslant e^{\sqrt{1+A_{n-1}}} e^{D_n/2} \cosh\left(\frac{|r_n|}{2\sqrt{1+A_{n-1}}}\right)$$
$$\leqslant e^{\sqrt{1+A_{n-1}}} e^{D_n/2} e^{|r_n|^2/8(1+A_{n-1})}.$$

Since, by Schwarz, $r_n^2 \leqslant 4A_{n-1}D_n$, the last expression is at most

$$e^{\sqrt{1+A_{n-1}}} e^{D_n/2} e^{A_{n-1}D_n/2(1+A_{n-1})} \leqslant e^{\sqrt{1+A_{n-1}}} e^{D_n/2} e^{D_n/2} = e^{(\sqrt{1+A_{n-1}}+D_n)}.$$

Therefore,

$$E(e^{\sqrt{1+A_n}-\sum_{k=0}^{n} D_k} | \mathcal{F}_{n-1}) \leqslant \exp\left[\sqrt{1+A_{n-1}} - \sum_{k=0}^{n-1} D_k\right].$$

Consequently, the sequence $\{g_n\}_{m=0}^{n}$, where

$$g_m = \exp\left(\sqrt{1 + A_m} - \sum_{k=0}^{m} D_k\right)$$

is a supermartingale, and $E(g_n) \leqslant E(g_0) = e$. □

We adapt Lemma 2.2 to double dyadic martingales by regarding $S_1^2 f_{N,N} = \sum_{n=0}^{N}\left(\sum_{m=0}^{N} d_{n,m}\right)^2 (x_1,x_2)$ as a sum of squares of dyadic martingales with respect to $m$ in the variable $x_2$ with $x_1$ fixed. Each $X_n^j$ in Lemma 2.2 can be replaced by $\alpha X_n^j$ and we obtain the following corollary.

COROLLARY 2.2(*a*). – $\|S_1 f\|_{L^p(dx_2)} \leqslant C\sqrt{p}\|Sf\|_{L^p(dx_2)}$.

*Proof.* – We first want to estimate $|\{S_1 f > 2\lambda, Sf \leqslant \varepsilon\lambda\}|$. The proof uses some standard arguments.

Let

$$S_1^r f = \sqrt{\sum_{n=0}^{N}\left(\sum_{m=0}^{r} d_{n,m}\right)^2}$$

and set $S_1^* f = \sup_{0\leqslant r\leqslant N} S_1^r f$. For fixed $x_1$, $\{x_2 : S_1^* f > \lambda\}$ determines maximal dyadic intervals $\{J_k\}$ such that

(i) $\sum_{n=0}^{N}\left(\sum_{m=0}^{r_k} d_{n,m}\right)^2 > \lambda^2$ on $J_k$,

(ii) $\sum_{n=0}^{N}\left(\sum_{m=0}^{r_{k-1}} d_{n,m}\right)^2 \leqslant \lambda^2$ on $J_k$,

(iii) $\{S_1^* f > \lambda\} = \cup J_k$, a disjoint union.

Fix such a $J_k$. Then Lemma 2.2 yields

$$(2.3)\quad \int_{J_k} \exp\left[\alpha\left(\sum_{n=0}^{N}\left(\sum_{n=r_k+1}^{\ell} d_{n,m}\right)^2\right)^{1/2} - \alpha^2 \sum_{n=0}^{N}\sum_{m=r_k+1}^{\ell} d_{n,m}^2\right] \leqslant e|J_k|.$$

Decompose $\{S_1^* f > 2\lambda\} \cap J_k$ into intervals $\{J_k^i\}$ such that

$$t_i = \inf\left\{t : \sum_{n=0}^{N}\left(\sum_{m=0}^{t} d_{n,m}\right)^2 > (2\lambda)^2\right\}.$$

By the definitions of $t_i$ and $r_k$ we have $t_i \geqslant r_k$. If $t_i = r_k$, it follows that $J_k^i \cap \{Sf \leqslant \varepsilon\lambda\} = \varnothing$, so we consider only those $J_k^i$ for which $t_i \geqslant r_k + 1$. Since $\ell$ is arbitrary in (2.3) we have

$$\int_{J_k^i \cap \{Sf \leqslant \varepsilon\lambda\}} \exp\left[\alpha \sqrt{\sum_{n=0}^{N}\left(\sum_{m=r_k+1}^{t_i} d_{n,m}\right)^2} - \alpha^2 \sum_{n=0}^{N}\sum_{m=r_k+1}^{t_i} d_{n,m}^2\right] dx_2 \leqslant e|J_k|.$$

But

$$\sum_{n=0}^{N}\left(\sum_{m=r_k+1}^{t_i} d_{n,m}\right)^2 \geqslant 2\lambda - \lambda - \varepsilon\lambda,$$

so that

$$|J_k^i \cap \{Sf \leqslant \varepsilon\lambda\}| e^{\alpha(1-\varepsilon)\lambda} e^{-\alpha^2\varepsilon^2\lambda^2} \leqslant e|J_k|.$$

Summing over the $J_k^i$ in each $J_k$ and then summing on the $J_k$ yields

$$|\{S_1^* f > 2\lambda, Sf \leqslant \varepsilon\lambda\}| \leqslant e^{-\alpha(1-\varepsilon)\lambda} e^{\alpha^2\varepsilon^2\lambda^2} |S_1^* f > \lambda\}|.$$

This good-$\lambda$ inequality is used in the usual way to estimate $\|[S_1 f\|_{L^p(dx_2)}$. Take $\alpha = (1-\varepsilon)/2\varepsilon^2\lambda$. Then

$$\int (S_1^* f)^p \, dx_2 = 2^p p \int_0^\infty \lambda^{p-1} |\{S_1^* f > 2\lambda\}| \, d\lambda$$
$$\leqslant \frac{2^p}{\varepsilon^p} \int (Sf)^p \, dx_2 + 2^p e^{-1/4(1-\varepsilon/\varepsilon)^2}/4 \int (S_1^* f)^p \, dx_2.$$

Solving for $\varepsilon$ to insure that $2^p \; e^{-1/4(1-\varepsilon/\varepsilon)^2}/4 \approx 1/2$ gives $1/\varepsilon^2 \approx Cp$, with C an absolute constant, and so

$$\int S_1^p f \, dx_2 \leqslant (C\sqrt{p})^p \int S^p f \, dx_2. \qquad \square$$

We can now integrate in each variable separately, obtaining

$$\text{(2.4)} \quad \|f_{N,N}\|_{L^p(dx_1 dx_2)} \leqslant C\sqrt{p}\|S_1 f\|_{L^p(dx_1 dx_2)} \leqslant Cp\|Sf\|_{L^p(dx_1 dx_2)}.$$

And, if $\|Sf\|_\infty \leqslant 1$,

$$\int e^{c|f_{N,N}|} \, dx_1 \, dx_2 \leqslant \sum_{p=0}^{N} \frac{c^p}{p!} \|f_{N,N}\|_p^p < \infty,$$

when $c$ is sufficiently small. This proves the double dyadic form of Theorem 2.

*Remarks.* – There are several. First, Lemmas 2.1 and 2.2 hold for continuous local martingales, with the Ito calculus replacing the computations for conditional expectation. R. Banuelos (personal communication) has proved, by probabilistic means, results like Theorem 1, and we would expect that similar results could be obtained for the square functions generated by Brownian motion in two independent variables.

Second, although the iteration method does not give the sharp constant $c$ for which $\int e^{c|f_{N,N}|/\|Sf\|_\infty} < \infty$, the following example shows that there exists a $c$ such that $\lim_{N\to\infty} \int e^{c|f_{N,N}|} = \infty$. Let $\{r_n\}$ be the sequence of Rademacher functions on $[0,1]$ and set

$$f_{N,N}(x,y) = \frac{1}{N} \sum_{n=0}^{N} \sum_{m=0}^{N} r_n(x) r_m(y).$$

Then $S^2 f_{N,N}(x,y) \equiv 1$. However,

$$\begin{aligned}\iint e^{cf_{N,N}(x,y)}\,dx\,dy &= \int \left[\cosh\left(\frac{c}{N}\sum_{m=0}^{N} r_m(y)\right)\right]^N dy \\ &= \frac{1}{2^N}\sum_{\ell=0}^{N}\binom{N}{\ell}\int \exp\left(\frac{c}{N}(N-2\ell)\sum_{m=0}^{N} r_m(y)\right)dy \\ &= \frac{1}{4^N}\sum_{\ell=0}^{N}\binom{N}{\ell}\sum_{k=0}^{N}\binom{N}{k}\exp\left[\frac{c}{N}(N-2\ell)(N-2k)\right] \\ &\geqslant \frac{e^{cN}}{4^N}\end{aligned}$$

which tends to $\infty$ with N if $c$ is large enough.

Finally, suppose $\{\varepsilon_{n,m}\}$ is any sequence such that $\varepsilon_{n,m} = \pm 1$. Inequality (2.4), together with the fact that $\|Sf\|_{L^p(dx_1\,dx_2)} \leqslant Cp\|f\|_{L^p(dx_1\,dx_2)}$ for $p \geqslant 2$, yields

$$\left\|\sum_n\sum_m \varepsilon_{n,m} d_{n,m}\right\|_p \leqslant B_p \left\|\sum_n\sum_m d_{n,m}\right\|_p,$$

where $B_p$ is asymptotic to $(p)^2$. For single index dyadic martingales, Burkholder has given the sharp result, namely that $B_p$ is $p-1$.

## 3. Proof of Theorem 2.

We reduce to the double dyadic case, using a method from [4], where the following lemma appears.

LEMMA 3.1. — *Let* $\mathcal{G}$ *be the family of all dyadic intervals of length at most* $2^A$. *Then* $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$, *where :*

(1) *There exists an* $x_j$ *such that* $I \in \mathcal{G}_j \Rightarrow \tilde{I} \subseteq I' + x_j$ *for some dyadic interval* $I'$ *of length at most* $8|I|$. *(Here* $\tilde{I}$ *is a 3-fold enlargement of* I.)

(2) *If* $I_1, I_2 \in \mathcal{G}_j, I_1 \neq I_2$, *then* $I'_1 \neq I'_2$.

We are assuming $S_\gamma f \leqslant 1$. Fix $\Omega \subseteq [0,1] \times [0,1]$, fix an $\alpha$ to be determined later, and set

$$F_{\Omega,\alpha}(x) = \sum_{\substack{R \subseteq \Omega \\ R\,:\,\text{dyadic}}} \iint_{R^+} f * \psi_{\gamma\alpha}(t)\psi_y(t-x)\, dt \frac{dy}{y} = \sum_{R \subseteq \Omega} F_{R,\alpha}(x).$$

Each $F_R(x)$ has mean value zero in each variable separately and has support in $\tilde{R}$. Using Lemma 3.1, we split the family of dyadic rectangles into four distinct families $\mathcal{L}_j$ with the properties :

(1)′ If $R \in \mathcal{L}_j$, then there exists a dyadic $R'$ such that

$$\tilde{R} \subseteq I' + x_j \times J' + y_j \qquad \text{and} \qquad |R'| \leqslant 64|R|.$$

(2)′ If $R_1, R_2 \in \mathcal{L}_j$ and $R_1 \neq R_2$, then $R'_1 \neq R'_2$.

Then $F_\Omega(x) = \sum_j \sum_{\substack{R \subseteq \Omega \\ R \in \mathcal{L}_j}} F_R(x)$. We claim

$$\left\| \sum_{\substack{R \subseteq \Omega \\ R \in \mathcal{L}_j}} F_R \right\|_p \leqslant Cp|\Omega|^{1/p}. \tag{3.2}$$

Theorem 2 follows immediately from (3.2).

The left side of (3.2) is the $L^p$ norm of

$$g_i(x_1 x_2) = \sum_{\substack{R \subseteq \Omega \\ R \in \mathcal{L}_j}} F_R(x_1 - x_i, x_2 - y_i).$$

If R′ is the rectangle associated with R as in (1)′, set

$$g_{R'} = F_R(x_1 - x_1', x_2 - x_i)$$

and observe that $g_{R'}$ has support in R′. Moreover, each R′ in this sum is contained in $\Omega_i$, a translate of an enlargement of $\Omega$. We express $g_i$ as a double dyadic martingale by expanding in Haar series:

$$g_i = \sum_{R_0}\left(\sum_{R'}(g_{R'},h_{R_0})\right)h_{R_0}, \tag{3.3}$$

where $h_{R_0} = h_{I_0}(x_1)h_{J_0}(x_2)$ and the Haar function $h_{I_0}$ equals $|I_0|^{-1/2}$ on the left half of I, $(-1)|I_0|^{-1/2}$ on the right half of I, and zero elsewhere. If a rectangle $R_0$ appearing in the expansion (3.3) is not contained in R′, then $(g_{R'},h_{R_0}) = 0$, because either $h_{I_0}$ or $h_{J_0}$ will be constant on the support of $g_i$. (Recall that $g_i$ has mean value zero.) Therefore

$$g_i = \sum_{R_0}\left(\sum_{R'\supseteq R_0}(g_{R'},h_{R_0})\right)h_{R_0}.$$

The martingale square function of $g_i$ is then

$$S_d^2 g_i(x_0) = \sum_{R_0\ni x}\frac{1}{|R_0|}\left\{\sum_{R'\supseteq R_0}(g_{R'},h_{R_0})\right\}^2$$

and we claim

$$S_d^2 g_i(x_0) \leqslant C_\alpha,\ \text{all } x_0 \tag{3.4}$$

where $C_\alpha$ depends only on $\alpha = \alpha(\gamma)$. It is easy to see that (3.2) follows from (3.4). Indeed, by (2.4),

$$\|g_i\|_p \leqslant Cp\|S_d g_i\|_p \leqslant Cp|\Omega|^{1/p}$$

since $S_d g_i$ is bounded and has support in $\Omega_i$.

To prove (3.4), we need an estimate for $(g_{R'},h_{R_0})$.

LEMMA 3.2. — $|(g_{R'},h_{R_0})| \leqslant \dfrac{c|R_0|^{3/2}}{|R'|^2}\displaystyle\iint_{R+}|f*\psi_{y\alpha}(t)|^2\frac{dt\,dy}{y}.$

*Proof.* —
$$\left|\int \psi_{y_1}(t_1-x_1)h_{I_0}(x_1)\,dx_1\right|$$
$$= \left|\int[\psi_{y_1}(t_1-x_1) - \psi_{y_1}(t_1-\bar{x}_1)]h_{I_0}(x_1)\,dx_1\right|$$
$$\leqslant \frac{\|\psi'\|_\infty}{(y_1)^2}\int_{I_0}|h_{I_0}|\,|x_1-\bar{x}_1|\,dx_1 \leqslant \|\psi'\|_\infty\frac{|I_0|^{3/2}}{(y_1)^2}.$$

So

$$|(g_{R'}, h_{R_0})| \leqslant C \iint_{R^+} |f * \psi_{y\alpha}(t)|^2 \frac{|R_0|^{3/2}}{y^2} \frac{dt\, dy}{y}$$
$$\leqslant C \frac{|R_0|^{3/2}}{|R'|^2} \iint_{R^+} |f * \psi_{y\alpha}(t)|^2 \frac{dt\, dy}{y},$$

because $y \sim |R|$ and $|R'| \leqslant C|R_0|$.

*Proof of* (3.4). – We have

$$S_d^2 g_i = \sum_{R_0 \ni x_0} \frac{1}{|R_0|} \left\{ \sum_{R' \supseteq R_0} (g_{R'}, h_{R_0}) \right\}^2$$
$$\leqslant C \sum_{R_0 \ni x_0} \frac{1}{|R_0|} \left\{ \sum_{R' \supseteq R_0} \frac{|R_0|^{3/2}}{|R'|^2} \iint_{R^+} |f * \psi_{y\alpha}(t)|^2\, dt \frac{dy}{y} \right\}^2$$
$$\leqslant C \sum_{R_0 \ni x_0} \left\{ \sum_{R' \supseteq R_0} \frac{|R_0|}{|R'|^2} \left[ \iint_{R^+} |f * \psi_{y\alpha}(t)|^2 \frac{dt\, dy}{y^2} \right]^{1/2} |R| \right\}^2,$$

by the Cauchy-Schwarz inequality. Hence for $0 < \beta < 1$,

$$S_d^2 g_i \leqslant C \sum_{R_0 \ni x_0} \sum_{R' \supseteq R_0} \left(\frac{|R_0|}{|R|}\right)^{2\beta} \iint_{R^+} |f * \psi_{y\alpha}(t)|^2 \frac{dt\, dy}{y^2} \cdot \sum_{R' \supseteq R_0} \left(\frac{|R_0|}{|R|}\right)^{2(1-\beta)}$$

Observe that

$$\sum_{R' \supseteq R_0} \left(\frac{|R_0|}{|R|}\right)^{2(1-\beta)} = \sum_{m,k} \sum_{|I'| = 2^m |I_0|} \sum_{|J'| = 2^k |J_0|} (2^{-m} 2^{-k})^{2(1-\beta)}$$

and this is finite whenever $2(1-\beta) > 0$ since only one $I'$ and $J'$ of each possible size appears in the sum. Therefore

$$S_d^2 g_i(x_0) \leqslant C \sum_{R_0 \ni x_0} \sum_{R_0 \supseteq R'} \left(\frac{|R_0|}{|R'|}\right)^{2\beta} \iint_{R^+} |f * \psi_{y\alpha}(t)|^2 \frac{dt\, dy}{y^2},$$
$$\leqslant C \sum_{R' \ni x_0} \sum_{R_0 \subseteq R'} \left(\frac{|R_0|}{|R'|}\right)^{2} \iint_{R^+} |f * \psi_{y\alpha}(t)|^2 \frac{dt\, dy}{y^2},$$

where this last inequality comes from the estimate

$$\sum_{R_0 \subseteq R'} \left(\frac{|R_0|}{|R'|}\right)^{2\beta} = \sum_n \sum_\ell \sum_{|I_0| = 2^{-n}|I'|} \sum_{|J_0| = 2^{-\ell}|J'|} (2^{-n} 2^{-\ell})^{2\beta},$$

which is finite if $\beta > 1/2$. From the definition of $\psi_{y\alpha}(x)$,

$$\iint_{R^+} |f * \psi_{y\alpha}(t)|^2 \frac{dt\, dy}{y^2} \leqslant \alpha \iint_{\{t \in R, \alpha|R|/2 < y \leqslant \alpha|R|\}} |f * \psi_y(t)|^2 \frac{dt\, dy}{y^2}.$$

At this point, we choose $\alpha$ large enough so that $R_\alpha^+$ is contained in $\Gamma_\gamma(x_{0,1}+x_i) \times \Gamma_\gamma(x_{0,2}+y_i)$ (for $x_0 = (x_{0,1}, x_{0,2})$) for each rectangle R in this sum. (See fig. 3.5.) This is possible since $R' \ni x_0$ and $R \subseteq R' + (x_i, y_i)$, with $|R'| \leqslant C|R|$. Furthermore, to each R′ there is associated a unique R, and therefore

$$\sum_{R' \ni x_0} \iint_{R_\alpha^+ = \{t \in R, y \approx \alpha|R|\}} |f * \psi_y(t)|^2 \frac{dt\,dy}{y^2} \leqslant S_\gamma^2 f(x_{0,1}+x_i, x_{0,2}+y_i) \leqslant 1 .$$

This proves (3.4). □

Observe that by introducing the kernel $\psi_{y\alpha}$, $\alpha = \alpha(\gamma)$, we have obtained, in the estimate above, a sum over elongated tops of rectangles which corresponds to $S_\gamma f$. For a smaller $\alpha$, one obtains a sum corresponding to $S_{\gamma' f}$, a square function with larger aperture. There is, however, no guarantee that $S_{\gamma' f} \in L^\infty$. Peter Jones (personal communication) has constructed a function such that $S_{\gamma f} \in L^\infty$ but $S_{\gamma' f} \notin L^\infty$.

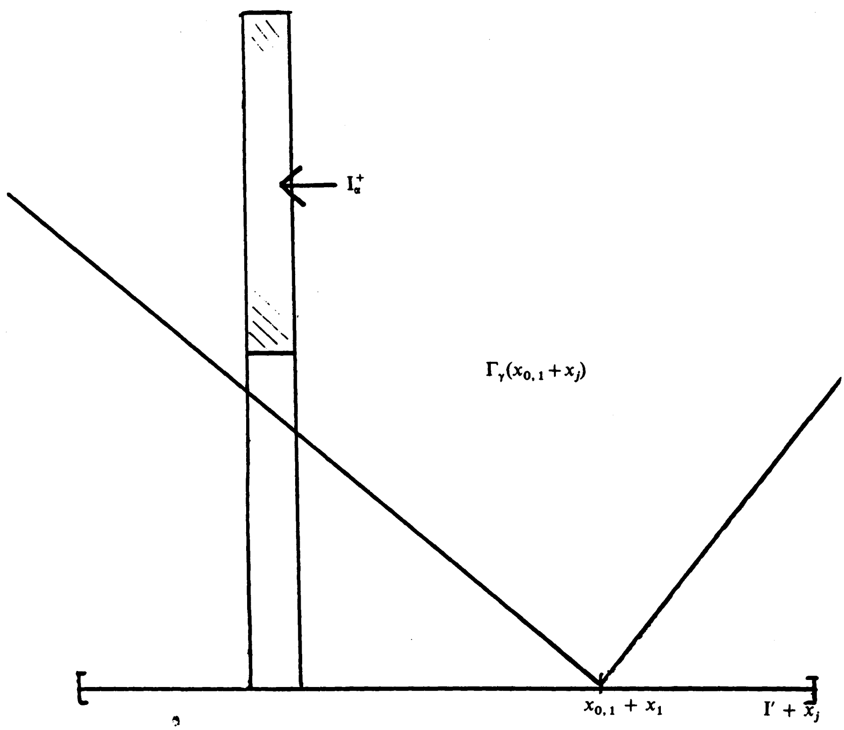


Fig. (3.5). -The one-variable representation of the situation in the proof of (3.3).

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