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# GEOMETRY OF AN ÉTALE COVERING OF THE $p$ -ADIC UPPER HALF PLANE

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## Introduction.

In this paper we describe the rigid geometry of the first layer in the tower of coverings of the  $p$ -adic upper half plane obtained from the division points of the formal group constructed in [2]. This covering is accessible because it is abelian and in some sense "tame." Using our results, we are able to describe the stable special fiber at  $p$  of Shimura curves with a very small amount of level  $p$  structure.

## Preliminaries.

Let  $\hat{\mathcal{H}}_p$  denote the formal scheme over  $\mathbf{Z}_p$  constructed by Mumford ([4]) and commonly referred to as the  $p$ -adic upper half plane. Naively,  $\hat{\mathcal{H}}_p$  is the complement of the  $\mathbf{Q}_p$ -rational points in  $\mathbf{P}^1$ . We let  $\mathcal{H}_p$  be the rigid analytic space associated to  $\hat{\mathcal{H}}_p$ .

In [2], Drinfeld shows that  $\hat{\mathcal{H}}_p$  is a parameter space for two-dimensional formal groups with a certain endomorphism structure. As a

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result, there is a universal family of formal groups  $\mathcal{G}$  over  $\hat{\mathcal{H}}_p$ . The subgroups obtained as the division points of this family of formal groups yield a tower of coverings of  $\hat{\mathcal{H}}_p$ . The rigid spaces associated to these coverings are a family of étale coverings of  $\mathcal{H}_p$ . Our goal in this work is to describe the simplest of these coverings.

For a detailed description of Drinfeld's universal formal group, we refer the reader to [8]. We recall here the basic definitions which we will require.

Let  $D$  be the quaternion division algebra over  $\mathbb{Q}_p$ , and let  $\mathcal{O}_D$  be the maximal order in  $D$ . A formal group  $G$  of dimension 2 and height 4 over a ring  $R$  on which  $p$  is nilpotent is called a special, formal  $\mathcal{O}_D$ -module (abbreviated *SFD*-module) provided that  $\mathcal{O}_D$  acts on  $G$  and, at each maximal ideal  $m$  of  $R$ , both characters of the residue field of  $\mathcal{O}_D$  occur in the tangent space to  $G$  at  $m$ . Over  $\bar{\mathbb{F}}_p$ , all *SFD*-modules are isogenous, so fix one such module  $\Phi$ . With these conventions, we can state Drinfeld's theorem.

**THEOREM (Drinfeld).** —  $\hat{\mathcal{H}}_p \otimes \mathrm{Spf} \hat{\mathbb{Z}}_p^{ur}$  (over  $\mathbb{Z}_p$ ) represents the functor which assigns to a ring  $R$  on which  $p$  is nilpotent the set of isomorphism classes of triples  $(\psi, G, \rho)$  where

1.  $\psi : \hat{\mathbb{Z}}_p^{ur} / p\hat{\mathbb{Z}}_p^{ur} \rightarrow R/p$  is a homomorphism,
2.  $G$  is an *SFD*-module over  $R$ ,
3. and  $\rho : \psi_*\Phi \rightarrow G \otimes R/p$  is a "quasi-isogeny of height zero," which means that  $\rho$  is an isogeny with a certain normalization condition which will not be important in our work.

We let  $(\Psi, \mathcal{G}, \mathbb{P})$  be the universal triple over  $\hat{\mathcal{H}}_p \otimes \mathrm{Spf} \hat{\mathbb{Z}}_p^{ur}$ , and  $\mathcal{G}_\Pi$  be the kernel of multiplication by  $\Pi$  on  $\mathcal{G}$ . This is a finite, flat group scheme of order  $p^2$  over  $\hat{\mathcal{H}}_p \otimes \mathrm{Spf} \hat{\mathbb{Z}}_p^{ur}$ . If  $\mathrm{PGL}_2^+(\mathbb{Q}_p)$  denotes the image in  $\mathrm{PGL}_2(\mathbb{Q}_p)$  of the elements of  $\mathrm{GL}_2(\mathbb{Q}_p)$  with determinants of even  $p$ -adic order, then the action of  $\mathrm{PGL}_2^+(\mathbb{Q}_p)$  on  $\hat{\mathcal{H}}_p$  extends to an action on the universal triple over  $\hat{\mathcal{H}}_p \otimes \mathrm{Spf} \hat{\mathbb{Z}}_p^{ur}$ , and therefore to  $\mathcal{G}_\Pi$ . Furthermore, the residue field  $\mathcal{O}_D/\Pi = \mathbb{F}$  acts on  $\mathcal{G}_\Pi$ , and this action commutes with the action of  $\mathrm{PGL}_2^+(\mathbb{Q}_p)$ . In fact,  $\mathrm{GL}_2(\mathbb{Q}_p)$  acts on the universal triple; see [2] p. 109 for details.

Let us fix a map  $\psi_0 : \hat{\mathbb{Z}}_p^{ur} \rightarrow \hat{\mathbb{Z}}_p^{ur}$  and consider the fiber of the induced projection map  $\hat{\mathcal{H}}_p \otimes \mathrm{Spf} \hat{\mathbb{Z}}_p^{ur} \rightarrow \mathrm{Spf} \hat{\mathbb{Z}}_p^{ur}$ . The resulting formal scheme is isomorphic to  $\hat{\mathcal{H}}_p \otimes \mathrm{Spf} \hat{\mathbb{Z}}_p^{ur}$  viewed as a formal scheme over  $\hat{\mathbb{Z}}_p^{ur}$  with structure map  $\psi_0$ . The geometry of the formal scheme obtained in this

way does not depend on the choice of  $\psi_0$ , so for the remainder of this paper we will abuse notation, suppress reference to  $\psi_0$ , and denote by  $\hat{\mathcal{H}}_p$  and  $\mathcal{H}_p$  respectively the formal and rigid  $p$ -adic upper half planes over  $\mathbb{Z}_p$  (resp.  $\mathbb{Q}_p$ ), base-changed up to  $\hat{\mathbb{Z}}_p^{ur}$  (resp.  $\hat{\mathbb{Q}}_p^{ur}$ .) Similarly, we base change the covering  $\mathcal{G}_\Pi$  to obtain a finite flat group scheme of order  $p^2$  over (our base-changed)  $\hat{\mathcal{H}}_p$ . We let  $\hat{\Sigma}$  be the complement of the zero section in this group scheme, and  $\Sigma$  be the associated rigid space. This paper is devoted to describing the rigid geometry of  $\Sigma$ .

**Classification of  $\Sigma$  as  $\mu_{p^2-1}$ -torsor.**

The action of the endomorphism ring of  $\mathcal{G}$  induces an action of  $\mathbb{F}^\times$  on the covering  $\Sigma$ , as we mentioned before. The two embeddings

$$\sigma_0, \sigma_1 : \mathbb{F} \rightarrow \hat{\mathbb{Z}}_p^{ur} / p\hat{\mathbb{Z}}_p^{ur}$$

induce two different actions of  $\mu_{p^2-1}$  on our fixed *SFD*-module  $\Phi$  over  $\bar{\mathbb{F}}_p$ , and therefore, via the universal isogeny  $P$ , two different actions on  $\Sigma$ . This allows us to view  $\Sigma$  as a  $\mu_{p^2-1}$  torsor in two different ways.

DEFINITION 1. — *Let*

$$c_i(\Sigma) \in H_{et}^1(\mathcal{H}_p \otimes \text{Spf } \mathbb{C}_p, \mu_{p^2-1})$$

*be the class representing the covering  $\Sigma \otimes \text{Spf } \mathbb{C}_p$  viewed as a  $\mu_{p^2-1}$  torsor via the embedding*

$$\tilde{\sigma}_i : W(\mathbb{F}) \rightarrow \hat{\mathbb{Z}}_p^{ur} \subset \mathbb{C}_p$$

*induced by  $\sigma_i$ .*

The following lemma relates the two classes.

LEMMA 2. —  $c_i(\Sigma) = pc_{i+1}(\Sigma)$ , *reading subscripts mod 2.*

*Proof.* — Changing the choice of  $\sigma_i$  twists the  $\mu_{p^2-1}$  action on  $\Sigma$  by  $\zeta \mapsto \zeta^p$ . □

Our goal now is to determine the classes  $c_i$  precisely. Let  $\mathcal{T}$  be the tree of  $\text{SL}_2(\mathbb{Q}_p)$ . We fix a reduction map  $r : \mathcal{H}_p \rightarrow \mathcal{T}$  which is compatible with the action of  $\text{SL}_2(\mathbb{Q}_p)$ . For one of many detailed descriptions of the relation between  $\mathcal{T}$  and  $\mathcal{H}_p$ , see [8], pp. 648–660.

The following theorem of Drinfeld ([1]) relates the cohomology of  $\mathcal{H}_p$  to the tree  $\mathcal{T}$ .

**THEOREM 3** (Drinfeld [1]). — *If  $N$  is an integer prime to  $p$ , there is an isomorphism*

$$\partial : H_{\text{ét}}^1(\mathcal{H}_p \otimes \text{Spf } \mathbb{C}_p, \mu_N) \rightarrow C_{\text{har}}^1(\mathcal{T}, \mathbb{Z}/N\mathbb{Z})$$

where  $\mathcal{T}$  is the tree of  $\text{SL}_2$  and  $C_{\text{har}}^1(\mathcal{T}, \mathbb{Z}/N\mathbb{Z})$  is the group of harmonic 1-cochains on  $\mathcal{T}$  — that is, the set of functions  $f$  on the edges of  $\mathcal{T}$  such that, for each vertex  $v$ ,  $f$  satisfies

$$\sum_{e \rightarrow v} f(e) = 0$$

where the sum is over the oriented edges of  $\mathcal{T}$  meeting  $v$ .

Let us briefly recall how the map  $\partial$  of the theorem is constructed. Suppose  $Y$  is a torsor over  $\mathcal{H}_p \otimes \text{Spf } \mathbb{C}_p$ . Let  $U$  be the admissible open set in  $\mathcal{H}_p$  corresponding to a vertex  $v$  in  $\mathcal{T}$ , together with its bounding edges. It follows from Lemma 2 of [8] that  $U$  is a  $\text{GL}_2(\mathbb{Q}_p)$  translate of the standard open set

$$V = \{P \in \mathcal{H}_p : 1/p < |z(P)| < p\} - \bigcup_{i=1}^{p-1} B_{1/p}^+(i)$$

where  $B_r^+(i)$  denotes the closed ball centered at  $i$  of radius  $r$ . Therefore  $\text{Pic}(U) = 0$  and so

$$\mathcal{O}_Y = \mathcal{O}_U[T]/(T^{p^2-1} - f)$$

where  $\mu_{p^2-1}$  acts by multiplication on  $T$ , and  $f$  is uniquely determined up to  $(p^2 - 1)^{\text{st}}$  powers.

Let  $e = \{v, v'\}$  be an (oriented) edge leaving  $v$ . To evaluate  $\partial(Y)(e)$ , choose a coordinate function  $z$  on  $U$  such that if  $P \in U$  reduces to  $v$  then  $|z(P)| = 1$  while if  $P \in U$  reduces to  $e$  then

$$|p| < |z(P)| < 1.$$

Then we let

$$\partial(Y)(e) = \text{Res}_e df/f \pmod{p^2 - 1}$$

where  $\text{Res}_e$  denotes the rigid analytic “annular residue” computed with respect to the selected coordinate  $z$  (which determines the sign of  $\text{Res}$ .)

Since  $\text{PGL}_2^+(\mathbb{Q}_p)$  acts on  $\Sigma$ , the class  $c_i(\Sigma)$  is invariant by  $\text{PGL}_2^+(\mathbb{Q}_p)$ . As we see in the next lemma, this is a very strong condition.

**LEMMA 4.** — *Suppose  $Y$  is a  $\text{PGL}_2^+(\mathbb{Q}_p)$ -invariant torsor over  $\mathcal{H}_p \otimes \text{Spf } \mathbb{C}_p$ , and  $\partial(Y)$  is the associated harmonic 1-cocycle. Then  $\partial(Y)$  satisfies the following conditions:*

1.  $\partial(Y)(e) = \partial(Y)(\gamma e)$  for all  $\gamma \in \text{PGL}_2^+(\mathbb{Q}_p)$  and all oriented edges  $e$  of  $\mathcal{T}$ .

2.  $\partial(Y)(e) \equiv 0 \pmod{p-1}$  on all edges  $e$  of  $\mathcal{T}$ .

*Proof.* — The first property is a consequence of the invariance of  $Y$  by  $\text{PGL}_2^+(\mathbb{Q}_p)$  and the fact that  $\text{PGL}_2^+(\mathbb{Q}_p)$  preserves the orientation of edges in  $\mathcal{T}$ . For the second, observe that all the edges  $e$  leaving any vertex  $v$  are permuted transitively by  $\text{PGL}_2^+(\mathbb{Q}_p)$ . Therefore  $\partial(Y)(e)$  is a constant  $x$  on all edges  $e$  leaving  $v$ . From the harmonicity condition we obtain

$$\sum_{e \rightarrow v} \partial(Y)(e) = (p+1)x \equiv 0 \pmod{p^2-1}$$

since there are  $p+1$  edges leaving  $v$ . This proves the lemma. □

In order to give a precise statement of our theorem, we must invoke the relationship between orientations on  $\mathcal{T}$  and the embeddings  $\sigma_i$ . As Drinfeld shows, and we explain in Lemma 14 of [8], the action of  $\Pi$  on the tangent space  $T$  to  $\mathcal{G}$  allows us to partition the vertices of  $\mathcal{T}$  into two classes labeled with the  $\sigma_i$ . To describe this partition, first decompose  $T$  into  $\sigma_i$ -eigenspaces for the action for the quadratic unramified extension of  $\mathbb{Z}_p$  inside  $\mathcal{O}_D$ . A vertex  $v$  is labelled with  $\sigma_i$  if  $\Pi T^i \subset pT^{i+1}$  over the affinoid reducing to  $v$  where  $T^i$  is the  $\sigma_i$ -eigenspace in  $T$ . Vertices of the two classes alternate in the tree.

Since  $\partial_i(\Sigma)$  is  $\text{PGL}_2^+(\mathbb{Q}_p)$ -invariant, and this group permutes the edges of  $\mathcal{T}$  transitively, it suffices to specify the value of  $\partial_i(\Sigma)$  on a single edge. This is the content of our theorem.

**THEOREM 5.** — *Let  $e = [v, v']$  be an edge of  $\mathcal{T}$ . Suppose that  $v$  is labeled with  $\sigma_i$ . Then*

$$\partial_i(\Sigma)(e) = p - 1 .$$

*Proof.* — Notice first of all that, by Lemmas 2 and 4,

$$\begin{aligned} \partial_i(\Sigma)([v, v']) &= -\partial_i(\Sigma)([v', v]) \\ &= p\partial_i(\Sigma)([v', v]) \\ &= \partial_{i+1}(\Sigma)([v', v]) \end{aligned}$$

and therefore we may assume that  $v$  is labeled with  $\sigma_0$ . Let  $U$  be the affine open set of  $\mathcal{H}_p$  corresponding to  $e$ . It follows from Lemma 2 of [8] that  $U$  is a  $\text{PGL}_2^+(\mathbb{Q}_p)$  translate of the standard open set

$$U(1) = \{P \in \mathcal{H}_p : 1/p \leq |z(P)| \leq 1\} - \bigcup_{i=1}^{p-1} (B_1(i) \cup B_{1/p}(pi))$$

where  $B_r(i)$  denotes the open disc of radius  $r$  centered at  $i$ . Therefore the coordinate ring of  $U$  is isomorphic to

$$\hat{R} = \varprojlim R/p^n R$$

where

$$R = \frac{\mathbf{Z}[z_0, z_1]}{(z_0 z_1 - p)} \left[ \frac{1}{1 - z_0^{p-1}}, \frac{1}{1 - z_1^{p-1}} \right].$$

As we recalled prior to stating this theorem, the tangent space  $T$  to  $\mathcal{G}$  over  $U$  is free of rank 2, and it carries a grading coming from the action of the maximal unramified extension of  $\mathbf{Z}_p$  in  $\mathcal{O}_D$ . Write  $T = \hat{R}t_0 \oplus \hat{R}t_1$ . Referring to [8] p. 656, we see that the  $\Pi$  action on  $T$  is  $\Pi t_0 = z_1 t_1$  and  $\Pi t_1 = z_0 t_0$ . With these conventions, the vertex  $v$  of  $T$  labeled with  $\sigma_0$  corresponds to the region where  $z_0$  is a unit; the vertex  $v'$  labeled with  $\sigma_1$  corresponds to the region where  $z_1$  is a unit.

Let  $\Omega$  be the cotangent space to  $\mathcal{G}$ . Then  $\Omega_\Pi = \Omega/\Pi\Omega$  is naturally the cotangent space to  $\mathcal{G}_\Pi$ . If  $\omega_0$  and  $\omega_1$  generate the graded pieces of  $\Omega$ , then we must have  $\Pi\omega_i = z_i\omega_{i+1}$ . It follows that

$$(1) \quad \Omega_\Pi = (\hat{R}/z_1\hat{R})\omega_0 \oplus (\hat{R}/z_0\hat{R})\omega_1.$$

The finite flat group scheme  $\mathcal{G}_\Pi$ , together with its action by  $\mathbf{F}$  is of the type classified by Raynaud. Applying his classification (see [5], Corollary 1.5.1) we see that the coordinate  $B$  ring of  $\mathcal{G}_\Pi$  must have the form

$$B = \hat{R}[X_0, X_1]/(X_0^p - \delta_0 X_1, X_1^p - \delta_1 X_0)$$

where the functions  $\delta_i$  and  $p/\delta_i$  belong to  $\hat{R}^\times$ . In addition, Raynaud shows that the natural identification of  $\Omega_\Pi$  with  $I/I^2$  ( $I$  being the augmentation ideal) means that

$$(2) \quad \Omega_\Pi = (\hat{R}/\delta_1\hat{R})X_0 \oplus (\hat{R}/\delta_0\hat{R})X_1.$$

Combining (1) and (2), we see that  $z_i$  and  $\delta_i$  differ by a unit of  $\hat{R}$ .

We now have enough information to compute the class of  $\Sigma$ . Indeed,  $\Sigma$  is defined over  $U$  by the equation

$$X_0^{p^2-1} - \delta_0^p \delta_1 = 0.$$

The group  $\mathbf{F}^\times$  acts on  $X_0$  through the embedding  $\sigma_0$ . Let us write  $f = \delta_0^p \delta_1$ . Then

$$\partial_0(\Sigma)(e) = \text{Res}_{z_0} df/f \pmod{p^2 - 1}.$$

Our results above tell us that  $\delta_0^p \delta_1 = pz_0^{p-1}h(z)$  where  $h$  is a unit in  $\hat{R}$ . This residue is clearly  $p - 1$ .  $\square$

Thanks to this theorem, we can construct  $\Sigma$  over  $\mathbb{C}_p$ . Let  $X$  be the non-singular projective curve over  $\mathbb{C}_p$  defined by the affine equation

$$(3) \quad Y^{p+1} = z - z^p .$$

Let  $W \subset X$  be the admissible open set of points  $P$  on  $X$  such that

$$(4) \quad |p^{1/(p+1)}| < |Y(P)| < |p^{-p/(p+1)}| .$$

**COROLLARY 6.** — *Over  $\mathbb{C}_p$ ,  $\Sigma$  consists of  $p-1$  isomorphic connected components. Each such component has a covering by admissible open sets isomorphic to  $W$ . The nerve of this covering is the tree  $\mathcal{T}$ . If  $W_1$  and  $W_2$  are two elements of the covering, and  $E = W_1 \cap W_2 \subset W_1$ , then  $E$  is one of the boundary annuli of  $W_1$ .*

*Proof.* — Let  $U$  be the subset of  $\mathcal{H}_p \otimes \text{Spf } \mathbb{C}_p$  consisting of one vertex (say, labeled with  $\sigma_0$ ), and its bounding edges. Then by Theorem 5,  $\Sigma$  over  $U$  is obtained by extracting the  $p^2 - 1$  root of a function with order congruent to  $p-1 \pmod{p^2-1}$  on each bounding annulus. If  $z$  is a coordinate on  $U$ , then the function  $f(z) = (z - z^p)^{p-1}$  clearly meets this condition. Thus  $\Sigma$  is defined over  $U$  by the equation

$$Y_0^{p^2-1} = (z_0 - z_0^p)^{p-1}$$

where  $z_0$  is an appropriate parameter on  $U$ . Notice first that this equation factors, so that  $\Sigma$  consists of  $p-1$  connected components, and is built up out of pieces of the curve in (3) as claimed. It is a simple matter to check that the subset of  $W$  satisfying the inequality (4) has genus  $(p^2 - p)/2$ . Thus the reduction of  $\Sigma$  consists of curves meeting in double points, like the reduction of  $\mathcal{H}_p$ , except that the rational curves which appear in the reduction of  $\mathcal{H}_p$  are replaced by the curves of equation (3).

With somewhat more care one can determine the equations for  $\Sigma$  over  $\hat{\mathbb{Z}}_p^{ur}$ , instead of just over  $\mathbb{C}_p$ . Examining the end of the proof of Theorem 5 we see that over  $\hat{\mathbb{Z}}_p^{ur}$  we can take  $\delta_0^p \delta_1 = p(z - z^p)^{p-1}$  and therefore  $\Sigma$  is defined over  $U$  by the equation

$$Y_0^{p^2-1} = p(z_0 - z_0^p)^{p-1} .$$

From this one can obtain a minimal regular model for  $\Sigma$  over  $\hat{\mathbb{Z}}_p^{ur}$  by blowing-up. This is a straightforward computational problem whose solution we omit, although we do point out that the minimal model has no components of multiplicity one.

Finally, notice that the action of  $\text{PGL}_2^+(\mathbb{Q}_p)$  on  $\Sigma$  extends to an action of  $\text{PGL}_2(\mathbb{Q}_p)$ . Indeed, choose  $\tau \in \text{PGL}_2(\mathbb{Q}_p) - \text{PGL}_2^+(\mathbb{Q}_p)$ . Then



$c_i(\tau^*\Sigma) = -c_i(\Sigma)$  since  $\tau$  reverses orientations. It follows that  $\tau^*\Sigma$  is isomorphic to  $\Sigma$  as a rigid space but that the action of  $\mu_{p^2-1}$  on  $\tau^*\Sigma$  is twisted by Frobenius. This could have been deduced, of course, from the general construction of Drinfeld.

### Application to Shimura curves.

Now we examine the implications of Theorem 5 for the geometry of Shimura curves. Let  $\Delta$  be an indefinite quaternion algebra over  $\mathbf{Q}$  ramified at  $p$  and let  $L$  be a maximal order in  $\Delta$ . Suppose  $n \geq 3$  is prime to  $p$ . Let  $S_n$  be the scheme representing the functor which associates to a scheme  $S$  the set of isomorphism classes of abelian surfaces over  $S$  with an  $L$  action and a level  $n$ -structure, and let  $S_n^{\text{an}}$  be the associated  $p$ -adic rigid space.

Let  $\wp \subset L$  be the unique prime ideal above  $p$ . Let  $S_{n,\wp}$  be the covering of  $S_n$  which classifies abelian  $L$  surfaces together with level  $n$  structure and level  $\wp$  structure. As before,  $S_{n,\wp}^{\text{an}}$  is the associated  $p$ -adic rigid space.

Let

$$X_n = U_n \backslash (\Delta' \otimes \mathbf{A}^f)^\times / \Delta'^\times$$

where  $\Delta'$  is the definite quaternion algebra obtained from  $\Delta$  by interchange of invariants at  $p$  and  $\infty$ . Let  $U_n$  be the principal congruence subgroup

$$U_n \subset \prod_{l \neq p} (L \otimes \mathbf{Z}_l).$$

With this notation, we can state (a slightly simplified form of) Drinfeld's theorem.

THEOREM (Drinfeld [2]). — *There are isomorphisms*

$$(5) \quad \text{GL}_2(\mathbf{Q}_p) \backslash \mathcal{H}_p \otimes \text{Spf } \hat{\mathbf{Z}}_p^{ur} \times X_n \rightarrow S_n^{\text{an}}$$

and

$$(6) \quad \text{GL}_2(\mathbf{Q}_p) \backslash \Sigma \times X_n \rightarrow S_{n,\wp}^{\text{an}}.$$

Furthermore, as Drinfeld points out, the quotient in (5) is actually the union of a finite number of components, each of the form

$$\Gamma \backslash \mathcal{H}_p \otimes \text{Spf } \hat{\mathbf{Z}}_p^{ur}$$

where  $\Gamma$  is a Schottky group.

Combining this with our geometric description of  $\Sigma$ , we obtain the following theorem.

**THEOREM 8.** — *Let  $S_n(\Gamma) = \Gamma \backslash \mathcal{H}_p \otimes \text{Spf } \hat{\mathbf{Z}}_p^{ur}$  be one of the components of  $S_n^{\text{an}}$ . Suppose  $\Phi = \mathcal{T}/\Gamma$  is the intersection graph of  $S_n(\Gamma)$ . Then the covering  $S_{n,p}^{\text{an}}$  over  $S_n(\Gamma)$  has a stable model over  $\mathbf{C}_p$  consisting of  $p - 1$  components. The reduction of each such component has intersection graph  $\Phi$ , but the vertices correspond to curves with the equation (3) rather than to rational curves.*

For the sake of concreteness, we supply an example. Suppose that  $\Delta$  has discriminant 26 and that  $p = 2$ . Then  $\Delta'$  has discriminant 13. Choose an embedding  $\Delta' \otimes \mathbf{Z}_p \hookrightarrow M_2(\mathbf{Q}_p)$ . Let  $A$  be a maximal  $\mathbf{Z}[1/2]$  order in  $\Delta'$ , and let

$$\Gamma = \{ \gamma \in A : nr(\gamma) = 2^k, k \text{ even} \} .$$

The Shimura curve  $S_1^{\text{an}}$  of level 1 (over  $\mathbf{C}_p$ ) is the quotient

$$\Gamma \backslash \mathcal{H}_p \otimes \text{Spf } \mathbf{C}_p .$$

We are allowed to consider level 1 since  $\Delta'$  has no multiplicative torsion. Since  $\Delta'$  has class number 1, it is not hard to check that the special fiber of  $S_1$  consists of two rational curves meeting in 3 points — see [3] or [6]. By the theorem, the special fiber of  $S_{1,p}$  consists of two copies of the elliptic curve  $Y^3 = z - z^2$  crossing in three points.

### Conclusions.

In conclusion, we mention two questions related to our subject matter. The first, rather naturally, is to obtain information on the higher coverings of the  $p$ -adic upper half plane; and, in particular, on the covering obtained from the  $p$ -torsion on Drinfeld's formal group. This is clearly a much harder problem than the one we have solved, since the higher coverings are not abelian and are in some sense "wildly ramified."

From a rigid analytic point of view, however, it would also be interesting to study the class of curves which admit a uniformization by  $\Sigma$ . Such curves are a type of generalized Mumford curve, and it would be worthwhile to extend the  $p$ -adic analytic theory of Mumford curves to this more general setting. In particular, the Jacobians of these curves are semi-abelian schemes, and it would be interesting to obtain some form of the Manin-Drinfeld theory of  $p$ -adic automorphic forms on  $\Sigma$ .

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