ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 40, nº 4 (1990), p. 951-963
<http://www.numdam.org/item?id=AIF_1990__40_4_951_0>

Ann. Inst. Fourier, Grenoble
40, 4 (1990), 951-963.

REMARKS ON THE LICHNEROWICZ-POISSON COHOMOLOGY

by Izu VAISMAN

The Lichnerowicz-Poisson (LP) cohomology of a Poisson manifold was defined in [L], and it provides a good framework to express deformation and quantization obstructions [L], [VK], [H], [V2]. The LP cohomology spaces are, generally, very large, and their structure is known only in some particular cases [VK], [X]. The homological algebraic place of these spaces was clarified in [H]. In the present note, we make a number of further remarks on the LP cohomology, most of them related with a certain natural spectral sequence which shows that, in the case of a regular Poisson manifold, the LP cohomology is connected with the cohomology of the sheaves of germs of foliated (i.e., projectable) forms of the symplectic foliation of the manifold (e.g., [V1]).

1. General remarks.

Let M^m be a Poisson manifold with the Poisson bivector Π, and put $\mathscr{V}^0(M) \overset{\text{def}}{=} C^\infty(M)$, $\mathscr{V}(M) = \mathscr{V}^1(M) \overset{\text{def}}{=}$ the space of C^∞ vector fields of M, $\mathscr{V}^k(M) \overset{\text{def}}{=}$ the space of k-vector fields (i.e., antisymmetric k-contravariant tensor fields of M), $\mathscr{V}^*(M) \overset{\text{def}}{=}$ the space of Pfaff forms of M, and, finally $\mathscr{L}(M) \overset{\text{def}}{=} \bigoplus_{k=0}^{m} \mathscr{V}^k(M) =$ the contravariant Grassmann algebra of M. The bivector Π has an associated morphism $\# : T^*M \to TM$, defined by $\beta(\alpha^\#) = \Pi(\alpha,\beta)$, $\forall \alpha, \beta \in T^*M$, and it yields the Poisson bracket of functions $\{f,g\} = \Pi(df,dg)$, as well as Hamiltonian vector fields X_f, $\forall f \in \mathscr{V}^0(M)$, given by $X_f g = \{f,g\}$. These fields define a generalized foliation with symplectic leaves called the *symplectic foliation* of (M,Π) (i.e., $\{X_f\}$ generate the tangent spaces of

Key-words : Poisson manifolds - LP cohomology.
A.M.S. Classification : 58A12 - 58F05.

the leaves). It is important to remember that the Poisson bracket induces a bracket of Pfaff forms which is the unique natural extension of the formula $\{df,dg\} = d\{f,g\}$, and is given by

$$(1.1) \qquad \{\alpha,\beta\} = L_{\alpha^{\#}}\beta - L_{\beta^{\#}}\alpha - d(\Pi(\alpha,\beta)).$$

The basic Poisson condition $[\Pi,\Pi] = 0$, where $[\ ,\ ]$ denotes the Schouten-Nijenhuis bracket, ensures that $(\mathscr{V}^0(M),\{\ ,\ \})$ and $(\mathscr{V}^*(M),\{\ ,\ \})$ are Lie algebras. The same condition also shows that the operator $\sigma Q = -[\Pi,Q]$ is a couboundary on $\mathscr{L}(M)$ (i.e., $\sigma^2=0$), and the cohomology of the cochain complex $(\mathscr{L},\sigma)$ is, by definition, the LP cohomology of (M,Π). Its spaces will be denoted by $H^k_{\mathrm{LP}}(M,\Pi)$. It is also important to remind that, for $Q = \mathscr{V}^k(M)$, one has [BV]

$$(1.2) \quad (\sigma Q)(\alpha_0,\ldots,\alpha_k) = \sum_{i=1}^{k} \alpha_i^{\#}(Q(\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_k)) + \sum_{i<j=0}^{k} (-1)^{i+j} Q(\{\alpha_i,\alpha_j\},\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\hat{\alpha}_j,\ldots,\alpha_k),$$

where $\alpha_i \in \mathscr{V}^*(M)$, and ^ denotes the absence of an argument.

Now, the definitions given above have some easy consequences such as

a) [X], [VK]. $H^0_{\mathrm{LP}}(M,\Pi) = \{f \in C^\infty(M)/\forall g \in C^\infty(M),\ X_g f = 0\}$. (Since $\sigma f = -X_f$.)

b) [X], [VK]. $H^1_{\mathrm{LP}}(M,\Pi) = \mathscr{V}_\pi(M)/\mathscr{V}_{\mathscr{H}}(M)$, where

$$\mathscr{V}_\pi(M) \overset{\text{def}}{=} \{X \in \mathscr{V}(M)/L_X\Pi = 0\}, \qquad \mathscr{V}_{\mathscr{H}}(M) \overset{\text{def}}{=} \{X_f/f \in \mathscr{V}^0(M)\}.$$

(Since $\sigma X = -L_X\Pi$ [L].)

c) [L], $\sigma\Pi = 0$, and Π defines a *fundamental class* $[\Pi] \in H^2_{\mathrm{LP}}(M,\Pi)$.

d) The LP cohomology satisfies the Mayer-Vietoris exact sequence property i.e., if U, V are open subsets of M, there is an exact sequence of the form

$$(1.3) \quad \ldots \to H^k_{\mathrm{LP}}(U\cup V,\Pi) \to H^k_{\mathrm{LP}}(U,\Pi) \oplus H^k_{\mathrm{LP}}(V,\Pi) \to H^k_{\mathrm{LP}}(U\cap V,\Pi) \to H^{k+1}_{\mathrm{LP}}(U\cup V,\Pi) \to \ldots$$

The definition of the arrows and the proof of the exactness are the same as for the de Rham cohomology (e.g., [BT]).

e) [L], [K]. Natural homomorphisms $\rho : H^k(M,\mathbb{R}) \to H^k_{LP}(M,\Pi)$, which are isomorphisms in the symplectic case, exist. Namely, ρ is defined by the extension of $\#$ to k-forms λ by

$$\lambda^{\#}(\alpha_1, \ldots, \alpha_k) = (-1)^k \lambda(\alpha_1^{\#}, \ldots, \alpha_k^{\#}), \tag{1.4}$$

since (1.2) shows that $\sigma(\lambda^{\#}) = (-1)^k (d\lambda)^{\#}$.

Because of *e*), it is natural to ask for a covariant interpretation of the whole LP cohomology via a Riemannian metric, and such an interpretation can be obtained by using Koszul's generating operators of the Schouten-Nijenhuis bracket. If we change signs such as to agree with [L], Koszul's formula for [A, B] where $A \in \mathcal{V}^i(M)$, $B \in \mathcal{V}^j(M)$ is [K]

$$[A,B] = D_\nabla(A \wedge B) - (D_\nabla A) \wedge B - (-1)^i A \wedge (D_\nabla B), \tag{1.5}$$

where ∇ is a torsionless linear connection on M, and D_∇ is defined by the coordinatewise formula

$$(D_\nabla A)^{h_2, \ldots, h_i} = \nabla_k A^{kh_2, \ldots, h_i}. \tag{1.6}$$

If ∇ is the Riemannian connection of a metric g, (1.6) means $D_\nabla = - \#_g \delta_g \#_g^{-1}$, where $\#_g : T^*M \to TM$ is the well known musical isomorphism, and δ_g is the codifferential of (M,g). Now, if we denote $\pi = \#_g^{-1}\Pi$, $B = \#_g\lambda$, and take $A = \Pi$ in (1.5), we obtain $\sigma(\#_g\lambda) = \#_g\delta_\pi$, where, if e (i) denotes the exterior (interior) multiplication by a form, one has

$$\delta_\pi = \delta_g e(\pi) - e(\pi)\delta_g - e(\delta_g \pi). \tag{1.7}$$

Hence, $H^k_{LP}(M,\Pi)$ are isomorphic to the cohomology spaces of the Grassmann complex ΛM endowed with the coboundary δ_π.

Of course, π must satisfy the condition $\delta_\pi \pi = 0$, which is equivalent to $[\Pi,\Pi] = 0$ i.e., we must have

$$\delta_g(\pi \wedge \pi) = 2\pi \wedge (\delta_g \pi), \tag{1.8}$$

and this is a new characterization of a Poisson structure which may have some usefulness. For instance, it shows that the parallel 2-forms of a Riemannian manifold (if any) and the harmonic 2-forms of a compact Riemannian symmetric space (where the exterior product of two harmonic forms is again a harmonic form) define Poisson structures. Formulas (1.7), (1.8) may also be used if we are looking for compatible

Poisson structures on a given symplectic manifold M with symplectic form ω i.e., Poisson bivectors Π such that $[\omega^{-1},\Pi] = 0$ (e.g., [G]). After the choice of a metric g on M, this problem amounts to solving the equations

$$\text{(1.9)} \qquad \delta_{\#_g^{-1}(\omega^{-1})}\pi = 0, \qquad \delta_g(\pi \wedge \pi) = 2\pi \wedge \delta_g \pi),$$

where also, if we ask g to be almost Hermitian ω-compatible, then $\#_g^{-1}(\omega^{-1}) = \omega$. For instance, (1.9) shows that, if M is a compact Hermitian symmetric space, and ω is its Kähler form, then any harmonic form of M defines an ω-compatible Poisson structure. On the other hand, we shall notice that, in case M is compact and oriented, δ_π has the formal adjoint

$$\text{(1.10)} \qquad d_\pi = i(\pi)d - di(\pi) - i(\delta_g \pi),$$

and we may expect to be able to apply the abstract Hodge decomposition theorem of [LT]. (From the expression of the Schouten-Nijenhuis bracket [L], it follows easily that the complex

$$\ldots \to \mathscr{V}^k(M) \xrightarrow{\sigma} \mathscr{V}^{k+1}(M) \to \ldots$$

is elliptic along the leaves of the symplectic foliation of (M,Π).)

Finally, we make a remark which will be important for the next sections of this paper. Namely, that there is a Serre-Hochschild spectral sequence associated with the LP cohomology. Let $\mathscr{V}_0^*(M) \overset{\text{def}}{=} \ker \# =$ the space of *conormal 1-forms* of the symplectic foliation of (M,Π). Since the bracket (1.1) satisfies $\{\alpha,\beta\}^\# = [\alpha^\#,\beta^\#]$ [BV], $\mathscr{V}_0^*(M)$ is an abelian ideal of $(\mathscr{V}^*(M), \{\ ,\ \})$, and we may define the *filtration degree* of $Q \in \mathscr{V}^k(M)$ to be h if $Q(\alpha_1, \ldots, \alpha_k) = 0$ as soon as $\geqslant k - h + 1$ of the arguments are conormal. This yields a differential filtration of the LP complex $\mathscr{L}(M)$, where $S_h^k(M) \overset{\text{def}}{=}$ the space of k-vector fields of filtration degree h is equal to the locally finite span of $\{f_0 X_{f_1} \wedge \ldots \wedge X_{f_h} \wedge Y_1 \wedge \ldots \wedge Y_{k-h} / f_i \in \mathscr{V}^0(M),\ Y_j \in \mathscr{V}^1(M)\}$. Now, the spectral sequence which we have in mind, and which we shall denote by $E_r^{pq}(M,\Pi)$, is the one associated with this filtration i.e., the Serre-Hochschild sequence of the pair of Lie algebras $(\mathscr{V}^*(M), \mathscr{V}_0^*(M), \{\ ,\ \})$. This sequence converges to $H_{\mathrm{LP}}^*(M,\Pi)$, and one has (e.g., [F])

$$\text{(1.11)} \qquad E_2^{pq}(M,\Pi) = H^p(V^*(M)/_{\mathscr{V}_0^*(M)}; H^q(\mathscr{V}_0^*(M); C^\infty(M)).$$

2. The regular case.

In the remaining part of this paper we assume that Π is of the constant rank $2n$, and $m = 2n + s$. This is the *regularity condition*, and then the symplectic foliation of (M,Π), hereafter to be denoted by $\mathcal{S}$, is regular. Hence, we can and shall define a transversal distribution $\mathcal{S}'$, and $TM = \mathcal{S}' \oplus T\mathcal{S}$, $T^*M = \mathcal{S}'^* \oplus T^*\mathcal{S}$ induce a bigrading of the covariant and contravariant tensors of M. A tensor whose transversal degree is p and whose leafwise degree is q is said to be of the type (p,q). We shall denote by $\mathcal{V}^{p,q}(M)$ and $\Lambda^{p,q}(M)$ the spaces of k-vector fields and k-forms $(k=p+q)$ of the type (p,q) of M, respectively. For instance, it is easy to understand that $\ker \#$ (i.e., $\mathcal{V}^*_0(M)$) is just $\mathcal{S}'^*$ = the space of the 1-forms of type $(1,0)$, and that type $\Pi = (0,2)$. (E.g., see [V1] for details on the bigrading of differential forms.)

Now, if $Q \in \mathcal{V}^k(M)$ is of type $(p,q)(p+q=k)$, and if we use bihomogeneous arguments α_i in (1.2), we see that $\sigma = \sigma' + \sigma''$ where type $\sigma' = (-1,2)$, type $\sigma'' = (0,1)$, and, for arguments α of type $(1,0)$ and β of type $(0,1)$, one has

$$(2.1)\quad (\sigma' Q)(\alpha_0, \ldots, \alpha_{p-2}, \beta_0, \ldots, \beta_{q+1}) = \sum_{i<j=0}^{q+1} (-1)^{i+j} Q(\{\beta_i, \beta_j\}, \alpha_0, \ldots, \alpha_{p-2}, \beta_0, \ldots, \hat{\beta}_i, \ldots, \hat{\beta}_j, \ldots, \beta_{q+1}),$$

$$(2.2)\quad (\sigma'' Q)(\alpha_0, \ldots, \alpha_{p-1}, \beta_0, \ldots, \beta_q) = \sum_{i=0}^{q} (-1)^{p+i} \beta_i^{\#}(Q(\alpha_0, \ldots, \alpha_{p-1}, \beta_0, \ldots, \hat{\beta}_i, \ldots, \beta_q) + \sum_{i=0}^{p-1}\sum_{j=0}^{q} (-1)^{p+i+j} Q(\{\alpha_i, \beta_j\}, \alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_{p-1}, \beta_0, \ldots, \hat{\beta}_j, \ldots, \beta_q) + \sum_{i<j=0}^{q} (-1)^{p+i+j} Q(\alpha_0, \ldots, \alpha_{p-1}, \{\beta_i, \beta_j\}'', \beta_0, \ldots, \hat{\beta}_i, \ldots, \hat{\beta}_j, \ldots, \beta_q).$$

Remember that type $\alpha = (1,0)$ means $\alpha \in \mathcal{V}^*_0(M)$, and that the latter is an ideal of $\mathcal{V}^*(M)$. On the other hand, we denoted by $\{\ ,\ \}'$, $\{\ ,\ \}''$ the type $(1,0)$ and $(0,1)$ components of $\{\ ,\ \}$. Particularly, if type $X = (1,0)$, we get easily

$$(2.3)\qquad \{\beta_1, \beta_2\}'(X) = (L_X \pi)(\beta_1, \beta_2).$$

In this section we use the type decomposition of σ in order to indicate a recurrent computational process of the LP cohomology which, in fact, is similar to the one used in [VK] for the case where $\mathcal{S}$ is a fibration. Take $Q \in \mathcal{V}^k(M)$, and decompose it as

$$Q = Q^{k,0} + Q^{k-1,1} + \cdots + Q^{0,k}, \tag{2.4}$$

where the indices denote the type of the components. Then, $\sigma Q = 0$ means

$$\sigma'' Q^{i,k-i} + \sigma' Q^{i+1,k-i-1} = 0 \quad (i=0,\ldots,k). \tag{2.5}$$

For $i = k$, (2.5) gives $\sigma'' Q^{k,0} = 0$, and, on the other hand, $(Q + \tilde{Q})^{k,0} = Q^{k,0}, \forall \tilde{Q} \in \mathcal{V}^{k-1}(M)$. Therefore, there exist homomorphisms

$$p_{k,0}\colon H^k_{\mathrm{LP}}(M,\Pi) \to \mathcal{V}^{k,0}_0(M), \tag{2.6}$$

where $\mathcal{V}^{k,0}_0(M)$ is the space of σ''-closed k-vectors of type $(k,0)$, and, furthermore, (2.5) shows that $\operatorname{im} p_{k,0}$ consists of k-vectors $Q^{k,0} \in \mathcal{V}^{k,0}_0(M)$ which satisfy the following sequence of existence conditions of k-vectors $Q^{k-1,1}, \ldots, Q^{0,k}$ such that

$$\begin{aligned}
&(c_1) \quad \sigma' Q^{k,0} = \sigma''\text{-exact} \overset{\text{def}}{=} -\sigma'' Q^{k-1,1},\\
&(c_2) \quad \sigma' Q^{k-1,1} = \sigma''\text{-exact} \overset{\text{def}}{=} -\sigma'' Q^{k-2,2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(c_k) \quad \sigma' Q^{1,k-1} = \sigma''\text{-exact} \overset{\text{def}}{=} -\sigma'' Q^{0,k}.
\end{aligned}$$

In this case we shall say that $\sigma' Q^{k,0}$ *satisfies k times the σ''-exactness condition*, and we shall denote by $\mathcal{V}^{k,0}_{0(k)}(M)$ the space of such $Q^{k,0}$. If we also denote $\ker p_{k,0} = {}^0H^k_{\mathrm{LP}}(M,\Pi) =$ the space of k-dimensional LP cohomology classes whoses cocycles are (2.4) with $Q^{k,0} = 0$, we obtain the result of the first recurrence step

$$H^k_{\mathrm{LP}}(M,\Pi) \approx {}^0H^k_{\mathrm{LP}}(M,\Pi) \oplus \mathcal{V}^{k,0}_{0(k)}(M). \tag{2.7}$$

Now, in the next step we have to compute ${}^0H^k_{\mathrm{LP}}(M,\Pi)$, and for this purpose we take the subcomplex ${}^0\mathcal{L}(M)$ of $\mathcal{L}(M)$ consisting of multivectors Q with a vanishing $(.,0)$ component, and denote by $H^k({}^0\mathcal{L}(M))$ its cohomology spaces. Then ${}^0H^k_{\mathrm{LP}}(M,\Pi)$ is the image of $H^k({}^0\mathcal{L}(M))$ with respect to the inclusion: ${}^0\mathcal{L}(M) \subseteq \mathcal{L}(M)$. It is clear that the complex $\mathcal{L}(M)/{}^0\mathcal{L}(M)$ has coboundary zero, therefore, $H^k(\mathcal{L}/{}^0\mathcal{L}) = (\mathcal{L}/{}^0\mathcal{L})^k = \mathcal{V}^{k,0}(M)$. This gives us the exact sequence

$\mathscr{V}^{k-1,0}(M) \xrightarrow{\sigma} H^k({}^0\mathscr{L}(M)) \xrightarrow{\iota_*} H^k(\mathscr{L}(M))$, and we get

$$(2.8) \qquad {}^0H^k_{\mathrm{LP}}(M,\Pi) \approx H^k({}^0\mathscr{L}(M))/\sigma(\mathscr{V}^{k-1,0}(M)).$$

Hence, the second step will have to consist of an analysis of $H^k({}^0\mathscr{L}(M))$, which can be made in the same way as in step 1, and resulting in a formula similar to (2.7), and so on.

For $k = 1$, we get easily

$$(2.9) \quad {}^0H^1_{\mathrm{LP}}(M,\Pi) = \{X \in \mathscr{V}^{0,1}(M)/\sigma''X=0\}/\sigma''(\mathscr{V}^0(M)).$$

For $k = 2$, we have first

$$(2.10) \quad H^2({}^0\mathscr{L}(M)) = \frac{\{Q^{1,1}+Q^{0,2}/\sigma''Q^{1,1}=0,\ \sigma''Q^{0,2}+\sigma'Q^{1,1}=0\}}{\{\sigma''X^{0,1}\}},$$

and the analysis which gave (2.7) now yields

$$(2.11) \qquad H^2({}^0\mathscr{L}(M)) \approx {}''H^2(\mathscr{L}^{0,*}(M)) \oplus \mathscr{V}^{1,1}_{0(1)}(M),$$

where $\mathscr{L}^{0,*}(M) = \oplus_k \mathscr{V}^{0,k}(M)$, and $''H$ is its cohomology with respect to σ'', and

$$(2.12) \quad \mathscr{V}^{1,1}_{0(1)}(M) = \{Q^{1,1}/\sigma''Q^{1,1}=0 \text{ and } \sigma'Q^{1,1}=\sigma''\text{-exact}\}.$$

(We shall see in Section 3 that, if the foliation $\mathscr{S}$ is either transversally Riemannian or transversally symplectic, then

$$''H^i(\mathscr{L}^{0,*}(M)) \approx H^i(M,\Phi^0(\mathscr{S})),$$

where $\Phi^0(\mathscr{S})$ is the sheaf of germs of functions which are constant along the leaves of $\mathscr{S}$.) Summing up the results we get

$$(2.13) \quad H^2_{\mathrm{LP}}(M,\Pi) \approx ({}''H^2(\mathscr{L}^{0,*}(M)) \oplus ((\mathscr{V}^{1,1}_{0(1)}(M))/\sigma(\mathscr{V}^{1,0}(M)))\ .\ \oplus \mathscr{V}^{2,0}_{0(2)}(M)),$$

Etc.

3. The spectral sequence.

In this section we continue to refer to a regular Poisson manifold (M,Π), and use the notation introduced in Section 2, while we are focussing on the spectral sequence $E^{pq}_r(M,\Pi)$ defined at the end of Section 1. We have :

PROPOSITION 3.1. — *The first terms of the* LP *Serre-Hochschild spectral sequence of a regular Poisson manifold* (M,Π) *are given by*

$$(3.1)\qquad \begin{aligned} E_0^{pq}(M,\Pi) &= E_1^{pq}(M,\Pi) = \mathscr{V}^{q,p}(M),\\ E_2^{pq}(M,\Pi) &= H^p(\oplus\mathscr{V}^{q*},\sigma''). \end{aligned}$$

The reader can prove this by noticing that the h-filtering subcomplex of $\mathscr{L}(M)$ as defined in Section 1 is equal to $S_h(M) = \bigoplus_{i\geq h}\bigoplus_{p} \mathscr{V}^{p,i}(M)$, and then following the usual definition of E_r^{pq}. Here, we just prefer to observe that $\{\mathscr{L}(M)=\oplus\mathscr{W}^{i,j}(M),\ \sigma=\Sigma d_{hk}\}$, where $\mathscr{W}^{i,j}(M) = \mathscr{V}^{j,i}(M)$, and the terms of σ are $d_{01} = 0$, $d_{10} = \sigma''$, $d_{2,-1} = \sigma'$, is a double semipositive cochain complex in the sense of [V1], p. 76-77, and then (3.1) follows from this reference.

Now, let G be a metric of the vector bundle $\mathscr{S}'^*$ of Section 2, and let $\tilde{\#} \overset{\text{def}}{=} \#_G \oplus \# : \mathscr{S}'^* \oplus T^*\mathscr{S} \to \mathscr{S}' \oplus T\mathscr{S}$ be the corresponding musical isomorphism also extended to $\Lambda^k(M) \to \mathscr{V}^k(M)$. Then, if λ is a differential form of type (p,q), $\lambda^{\tilde{\#}}$ is a multivector of the same type, and we have

$$(3.2)\quad \begin{aligned}&(\tilde{\#}^{-1}\sigma''\lambda^{\tilde{\#}})(X_0,\ldots,X_{p-1},Y_0,\ldots,Y_q)\\ &\qquad = (-1)^{q+1}(\sigma''\lambda^{\tilde{\#}})(\#_G^{-1}X_0,\ldots,\#_G^{-1}X_{p-1},\#^{-1}Y_0,\ldots,\#^{-1}Y_q).\end{aligned}$$

In this relation, and in the sequel, we agree that type $X = (1,0)$ and type $Y = (0,1)$. Furthermore, in order to compute $\sigma''\lambda^{\tilde{\#}}$ by (2.2) we establish first

$$\{\#^{-1}Y_i,\#^{-1}Y_j\}''^{\#} = \{\#^{-1}Y_i,\#^{-1}Y_j\}^{\#} = [Y_i,Y_j]$$

(remember that $\{\alpha,\beta\}^{\#}=[\alpha^{\#},\beta^{\#}]$ [BV]), and using (1.1))

$$\{\#_G^{-1}X_i,\#^{-1}Y_j\}(X) = -(L_{Y_j}G^*)(X_i,X) - G^*([Y_j,X_i],X),$$

where G^* is the dual metric of G on $\mathscr{S}'$. If these formulas are used, and the result is compared with the formula of the $\mathscr{S}$-leafwise exterior differential d_f [V1], p. 184, one gets

$$(3.3)\quad \begin{aligned}&(\tilde{\#}^{-1}\sigma''\lambda^{\tilde{\#}})(X_0,\ldots,X_{p-1},Y_0,\ldots,Y_q)\\ &\qquad = -(d_f\lambda)(X_0,\ldots,X_{p-1},Y_0,\ldots,Y_q)\\ &\qquad + \sum_{i=0}^{p-1}\sum_{j=0}^{q}(-1)^{p+i+j}\lambda([(L_{Y_j}G^*)(X_i,.)]^{\#_G},\\ &\qquad X_0,\ldots,\hat{X}_i,\ldots,X_{p-1},Y_0,\ldots,\hat{Y}_j,\ldots,Y_q).\end{aligned}$$

Remark. — The same result holds if G is a symplectic structure on $\mathcal{S}'^*$.

This computation leads to

PROPOSITION 3.2. — *If the symplectic foliation $\mathcal{S}$ of the regular Poisson manifold (M,Π) is either transversally Riemannian or transversally symplectic, one has*

$$E_2^{pq}(M,\Pi) = E_1^{pq}(\mathcal{S}) = H^p(M,\Phi^q(\mathcal{S})) \tag{3.4}$$

where $E_r^{pq}(\mathcal{S})$ is the spectral sequence of the foliation $\mathcal{S}$ (e.g., [KT]), *and $\Phi^q(\mathcal{S})$ is the sheaf of germs of $\mathcal{S}$-foliated q-forms of M (e.g.,* [V1]). *Particularly,* (3.4) *holds if $\mathcal{S}$ is a fibration.*

Indeed, under the hypotheses, $L_{Y_j}G = 0$ in (3.3), and in view of (3.1) we get an isomorphism $E_2^{pq}(M,\Pi) = H^p(\oplus\Lambda^{q,*}(M),d_f)$. But then (3.4) is known [V1], p. 216, 222, 77. (Remember that an $\mathcal{S}$-foliated q-form is a q-form which, locally, is the pull-back of a form of a local transversal manifold of the foliation $\mathcal{S}$.)

Now, let us define an interesting special class of Poisson manifolds. A vector field V of M is *$\mathcal{S}$-foliated* if it sends leaves to leaves or, equivalently, $\forall Y \in T\mathcal{S}$, $[V,Y] \in T\mathcal{S}$. For instance, this happens if V is an infinitesimal automorphism of Π i.e., $L_V\Pi = 0$, a condition which is easily seen to be equivalent to each of the following two conditions, where $f,\ g \in C^\infty(M)$,

$$V\{f,g\} = [V,X_f](g) - [V,X_g](f), \tag{3.5}$$

$$[V,X_f] = X_{V(f)}. \tag{3.6}$$

A regular Poisson structure Π of M will be called *transversally constant* if $\mathcal{S}$ has a transversal distribution $\mathcal{S}'$ such that every local foliate vector field $V \in \mathcal{S}'$ is a local infinitesimal automorphism of Π. For instance, if $M = S \times N$, and Π is defined by a symplectic structure of S, the distribution $\mathcal{S}' = TN$ has this property. Particularly, the existence of the local canonical coordinates of Π in the sense of [L] p. 256-257, shows that every regular Poisson manifold is locally transversally constant. Another example is the *Dirac bracket* defined as follows. Let (M,ω) be a symplectic manifold endowed with a foliation $\mathcal{F}$ such that ω induces symplectic structures of its leaves. These induced structures yield a Poisson bivector Π such that $\mathcal{S}(\Pi) = \mathcal{F}$, and $\{\ ,\ \}_\Pi$ is the Dirac bracket of $(M,\omega,\mathcal{F})$. It follows that every $\mathcal{F}$-foliate vector field V which is ω-orthogonal to $\mathcal{F}$ is an infinitesimal automorphism of Π. Indeed, for such V, (3.5) is equivalent to $(L_V\omega)(X_f, X_g) = 0$, and this is an easy consequence of $d\omega = 0$. Using this definition, we have

PROPOSITION 3.3. — *If* Π *is transversally constant,* $\sigma' = 0$, *and*

$$H^k_{LP}(M,\Pi) = \bigoplus_{k=0}^{q} E_2^{k-q,q}(M,\Pi). \tag{3.7}$$

Proof. — Of course, the proposition refers to σ' of (2.1) taken with respect to the distribution $\mathcal{S}'$ involved in the definition of a transversally constant Poisson structure. Let us use the notation of (2.1), and evaluate there $\{\beta_i, \beta_j\}'_p(X_p)(p \in M,\ X_p \in \mathcal{S}'_p)$. This may be done by extending X_p to a local foliate (1,0)-vector field X, and using (2.3). Since Π is transversally constant, $L_X\Pi = 0$ and we get $\sigma' = 0$. Then, (3.7) follows from (3.1). Q.e.d.

We shall finish by giving various corollaries of Propositions 3.1, 3.2, 3.3.

COROLLARY 3.1. — *If* (M,Π) *is a transversally constant Poisson manifold whose symplectic foliation is either transversally Riemannian or transversally symplectic, one has*

$$H^k_{LP}(M,\Pi) = \bigoplus_{q=0}^{k} E_1^{q,k-q}(\mathcal{S}) = \bigoplus_{q=0}^{k} H^q(M, \Phi^{k-q}(\mathcal{S})). \tag{3.8}$$

COROLLARY 3.2. — *Let* Π *be a Dirac bracket of a symplectic manifold* (M,ω) *endowed with a leafwise symplectic foliation* $\mathcal{S}$, *and its* ω-*orthogonal distribution* $\mathcal{S}'$. *Assume that the bihomogeneous components of* ω *with respect to the decomposition* $TM = \mathcal{S}' \oplus T\mathcal{S}$ *are closed. Then, again, formula* (3.8) *holds good.*

Proof. — Being a Dirac bracket, Π is transversally constant. On the other hand, if $\omega = \omega_{(2,0)} + \omega_{(0,2)}$; the hypothesis $d\omega_{(2,0)} = 0$ implies $(L_Y\omega_{(2,0)})(X_1, X_2) = 0$ for $(Y \in T\mathcal{S},\ X_{1,2} \in \mathcal{S}')$, and we see that $\omega_{(2,0)}$ defines a transversal symplectic structure of $\mathcal{S}$. Q.e.d.

COROLLARY 3.3 [X]. — *Let* Π *be the Poisson structure defined on* $M = S \times N$ *by a fixed symplectic structure of* S, *and assume that* S *has finite Betti numbers. Then one has*

$$H^k_{\mathrm{LP}}(M,\Pi) = \bigoplus_{q=0}^{k} [H^q(S,\mathbb{R}) \otimes \Lambda^{k-q}(N)]. \tag{3.9}$$

This result follows from (3.8) and from

PROPOSITION 3.4. — *Let $\mathscr{F}$ be the foliation of $M = F \times N$ by the leaves $F \times \{x\}$ $(x \in N)$, and assume that F has finite Betti numbers. Then*

$$H^q(M, \Phi^p(\mathscr{F})) = H^q(F, \mathbb{R}) \otimes \Lambda^p(\mathrm{N}). \tag{3.10}$$

Proof. — For $q = 0$ the result was proven in [E] by a spectral sequence argument. Generally, we have the following straightforward argument. By the foliated de Rham theorem [V1], p. 216, we have

$$H^q(M, \Phi^p(\mathscr{F})) = \frac{\ker [d_f : \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M)]}{\operatorname{im} [d_f : \Lambda^{p,q-1}(M) \to \Lambda^{p,q}(M)]}. \tag{3.11}$$

In our case, $\Lambda^{p,q}(M)$ is isomorphic to the space $\Lambda^q(F, \Lambda^p(N))$ of $\Lambda^p(N)$-valued q-forms on F by the mapping which sends $\lambda \in \Lambda^{p,q}(M)$ to $\tilde{\lambda} \in \Lambda^p(F, \Lambda^p(N))$ defined by

$$(\tilde{\lambda}_y(Y_1, \ldots, Y_q))_x(X_1, \ldots, X_p) = (-1)^p \lambda_{(x,y)}(X_1, \ldots, X_p, Y_1, \ldots, Y_q),$$

$y \in F$, $x \in N$, $Y_i \in T_yF$, $X_j \in T_xN$. Moreover, this isomorphism sends d_f to the exterior differential of $\Lambda^p N$-valued forms. Hence (3.11) becomes

$$H^q(M, \Phi^p(\mathscr{F})) = H^q(F, \Lambda^p(N)) = H^q(F, \mathbb{R}) \otimes \Lambda^p(N),$$

where the last equality follows from the hypothesis on F. Q.e.d.

Remark. — If $M = S \times N$ of Corollary 3.3 is given a Poisson structure Π which has the symplectic foliation $S \times \{x\}$ $(x \in N)$, but where each leaf has a different symplectic structure (e.g., the structure studied in [X]), Π is no more transversally constant, but we may use Propositions 3.2. and 3.4, and get

$$E_2^{pq}(M, \Pi) = H^p(S, \mathbb{R}) \otimes \Lambda^q(N). \tag{3.12}$$

COROLLARY 3.4. — *Let (M, Π) be an arbitrary regular Poisson manifold. Then every $x \in M$ has a connected open neighbourhood Y such that*

$$H^k_{\mathrm{LP}}(U, \Pi/_U) = \Gamma(\Phi^k(\mathscr{S}/_U)), \tag{3.13}$$

i.e., the space of the $\mathscr{S}$-foliated k-forms over U.

Indeed, we may take $U = S \times N$ where S is contractible, and such that the product coordinates are canonical for Π in the sense of [L], p. 257. Then Corollary 3.3 holds on U, and we get (3.13). We shall say that such a neighbourhood U is LP-simple.

COROLLARY 3.5 (*The* LP *Poincaré Lemma* [L]). – *Let* (M,Π) *be a regular Poisson manifold, and* $x \in M$. *Then, there exists an open neighbourhood* U *of* x *in* M *such that, if* $Q \in \mathcal{V}^k(U)$ *and* $\sigma Q = 0$, *one has* $Q = A + \sigma B$ *for some* $B \in \mathcal{V}^{k-1}(U)$ *and a k-vector field* A *over* U *which is projectable to a k-vector field of a local transversal submanifold of* $\mathcal{S}$ *in* U.

Proof. – Take U LP-simple, and with Π-canonical coordinates. The latter define a bigrading, and we may write $Q = \sum_{p=0}^{k} (\lambda^{p,k-p})^{\tilde{\#}}$, where λ are differential forms, and $\tilde{\#}$ is like in (3.2). The use of the canonical coordinates makes $\Pi/_U$ transversally constant and transversally Riemannian hence, by Proposition 3.3 and formula (3.3), $\sigma = \sigma''$, and $\sigma Q = 0$ is equivalent to $d_f\lambda^{p,k-p} = 0$ $(k = 0, \ldots, p)$. But d_f satisfies a local Poincaré lemma [V1], p. 215, hence, there are local forms μ such that $\lambda^{p,k-p} = d_f\mu^{p,k-p-1}$ for $k - p > 0$, while $\lambda^{k,0}$ is a foliate form. The conclusion follows by using again (3.3). Q.e.d

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Manuscrit reçu le 6 novembre 1990.

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