ANNALES DE L'INSTITUT FOURIER

NIKOLAI S. NADIRASHVILI

Metric properties of eigenfunctions of the Laplace operator on manifolds

Annales de l'institut Fourier, tome 41, nº 1 (1991), p. 259-265

<http://www.numdam.org/item?id=AIF_1991__41_1_259_0>

Ann. Inst. Fourier, Grenoble
41, 1 (1991), 259-265

METRIC PROPERTIES OF EIGENFUNCTIONS OF THE LAPLACE OPERATOR ON MANIFOLDS

by Nikolai S. NADIRASHVILI

In this note, we prove two theorems which express a quasi-symmetry relation between the positive and the negative part of the distribution function of an eigenfunction of the Laplace operator on a Riemannian manifold.

1. An estimate of the volume of a domain on which an eigenfunction of the Laplace operator on a Riemannian surface has constant sign.

Let M be a two-dimensional compact real analytic Riemannian manifold, $u_1, u_2, \ldots$-eigenfunctions of the Laplace operator on M, $\Delta u_i = \lambda_i u_i$.

THEOREM. — *There exists a positive constant C which depends on M such that, for every $i = 1, 2, \ldots$,*

$$\mathrm{vol}\{x \in M, u_i(x) > 0\} > C\ .$$

The proof is based on the two following lemmas.

Let f be a bounded function, continuous on $[0,1]$. Let us denote by $N(f)$ the number of changes in the sign of the function f on $[0,1]$.

Key-words : Beltrami-Laplace operator – Eigenfunctions – Nodal sets.
A.M.S. Classification : 58G25.

LEMMA 1. — *Let f_n be a sequence of non-zero continuous functions, defined on* $\mathbf{R}$*, with values in* $\mathbf{R}$*, with support in* $[0,1]$ *and assume that* $N(f_n)$ *is bounded by some fixed number* N.

Then there exists a subsequence n_i *of* $\mathbf{N}$ $(n_i \to \infty)$*, real numbers* α_{n_i}*, such that :* $\alpha_{n_i}.f_{n_i}$ *converges as* $i \to \infty$*, for the (weak-)topology of the space of distributions* $\mathcal{D}'$ *to a non-zero distribution of order less than* N.

Remark. — First, we recall that a distribution is said of order less than N if it is a sum of derivatives of order less than N of Radon measures. Moreover, if P is a polynomial of degree N and μ a Radon measure, the set of all $T \in \mathcal{D}'$ satisfying $PT = \mu$ is an N-dimensionnal affine subspace of the space of all distributions of order less than N.

Proof of Lemma 1 (suggested by Y. Colin de Verdière). — Let $P_n = \prod_{k=1}^{N}(x - x_k)$ be a sequence of polynomials of degree N such that $P_n.f_n$ is ≥ 0. By renormalisation and taking a subsequence, we may assume that $\int_0^1 P_n.f_n = 1$, that $P_n.f_n$ converges to a probability measure μ and that P_n converges to a polynomial P of degree exactly N.

Let $T_0 \in \mathcal{D}'$ be such that : $PT_0 = \mu$.

Let $T_n = f_n - T_0$, then we get :

$$\lim P_n.T_n = 0 \ .$$

We introduce now the following decomposition of the space of distributions :

$$\mathcal{D}' = Z_P \oplus W \ ,$$

where $Z_P = \{T \in \mathcal{D}' | PT = 0\}$, and W is a topological complement of Z_P.

W is a complement to Z_{P_n} if n is big enough. Now we can write uniquely :

$$T_n = z_n + w_n \ ,$$

where $z_n \in Z_{P_n}$ and $w_n \in W$. Now $P_n.w_n \to 0$ and we deduce that $w_n \to 0$, because the multiplication by P_n is uniformly invertible in W.

Now $z_n \in Z_{P_n}$ and Z_{P_n} converges to Z_P.

Two cases are possible :

First case : z_n is bounded and we can extract a convergent subsequence converging to T_1 in Z_P. Then $T_0 + T_1$ is not zero and we get the conclusion.

Second case : z_n is unbounded; then there exists a sequence $\beta_{n_i} \to 0$ such that $\beta_{n_i}.z_{n_i}$ converges to a non-zero distribution T_1 and then :

$$\beta_{n_i}.f_{n_i}$$

converges to T_1.

Let us denote by B the unit disk in $\mathbf{R}^2$, $S = \partial B$, if f is a continuous function on S then $N(f)$ is the number of changes of sign of the function f on S.

LEMMA 2. — *Let u be a harmonic function in B which is continuous in $\overline{B}$, $u\,|_S = f$, $u(0) = 0$. Let $N(f) = k < \infty$. Define*

$$G_u = \{x \in B\ ,\ u(x) > 0\}\ .$$

Then mes $G_u > C$, *where constant $C > 0$ is dependent on k .*

Proof. — Let us assume the contrary. This means that there exists a sequence of harmonic functions u_n in B, $u_n|_S = f_n$, $u_n(0) = 0$, $N(f_n) \leq k$, mes $G_{u_n} \to 0$, $n \to \infty$. According to lemma 1 there exists a real valued sequence α_m and a subsequence f_{n_m} such that, $\alpha_m f_{n_m} \to \tilde{f} \not\equiv 0$ in the sense of distributions. From the convergence of the distributions $\alpha_m f_{n_m}$ on S it follows that in an arbitrary compact subdomain of B the convergence of functions $\alpha_m u_{n_m}$ is uniform. Let $\alpha_m u_{n_m} \to U$ in B. From [1] it follows that $U \not\equiv 0$ in B if $\tilde{f} \not\equiv 0$ on S. We have $U(0) = 0$. From the assumption mes $G_{u_n} \to 0$, $n \to \infty$, it follows that $U \leq 0$ in B. Equality $U(0) = 0$ and inequality $U \leq 0$ in B contradicts the maximum principle for harmonic functions.

Proof of the theorem.

1. Let us denote by B_r^x, $x \in M$, r, the geodesic circle on M with centre x and radius r.

There is a constant $C_0 > 0$, such that for every $\varepsilon > 0$ there exists points $x_1 \dots x_N \in M$, $N > C_0/\varepsilon^2$, such that the circles $B_\varepsilon^{x_2} \dots B_\varepsilon^{x_N}$ mutually have no intersections.

2. There exists a constant r_0, such that for every $x \in M, 0 < r < r_0$, B_r^x is diffeomorphic to a disk.

3. There is a constant $C_1 > 0$, such that for all $x \in M$, $\lambda > 0$ in the circle $B^x_{1/C_1\sqrt{\lambda}}$ there exists a positive solution of the equation $\Delta u + \lambda u = 0$.

4. Let $x \in M$, $\lambda > 0$, $r \doteq 1/C_1\sqrt{\lambda} < r_0$, u is a solution of the equation $\Delta u + \lambda u = 0$ in B_r^x. Then there exists a diffeomorphism h of the

unit disk B on B_r^x, $h(0) = x$, and a function s in B, $0 < s < \infty$, such that $s.u(h)$ is a harmonic function in B (by a representation theorem in quasiconformal mapping theory, [2]). From the compactness of M it follows that the Jacobian of the mapping h is uniformly bounded.

5. There is a constant $C_2 > 0$, such that for all $x \in M$, $\lambda > 0$ in the circle $B^x_{1/C_2\sqrt{\lambda}}$ every solution $u \not\equiv 0$ of the equation $\Delta u + \lambda u = 0$ changes its sign [3].

6. Let u_i be an eigenfunction, $\Delta u_i = \lambda_i u_i$ on M, γ is a nodal line of the function u_i. For a two-dimensional real analytic manifold the following estimate is true, [4],

$$\text{length } \gamma \le C_3\sqrt{\lambda_i}$$

where constant $C_3 > 0$ is dependent on M.

7. Let u_i be an eigenfunction, $\Delta u_i = \lambda_i u_i$ on M. According to 1 we can choose circles $B_\varepsilon^{x_1} \dots B_\varepsilon^{x_n}$ with $\varepsilon = 2/C_2\sqrt{\lambda_i}$. We have $N > C_0 C_2^2 \lambda_i/4$.

According to 5 there exist points $y_n \in B^{x_n}_{\varepsilon/2}$, $n = 1\dots N$, such that $u_i(y_n) = 0$.

According to 6 at least $N/2$ points $y_{k_1} \dots y_{k_J}$, $J > N/2$, from the set $\{y_n\}$ have the following property : for all $j = 1\dots J$ there exist r_j,

$$\frac{1}{2C_1\sqrt{\lambda_i}} < r_j < \frac{1}{C_1\sqrt{\lambda_i}} \ ,$$

such that restriction of the function u_j on $\partial B_{r_j}^{y_{k_j}}$ has no more than

$$\frac{8C_1C_3}{C_2^2C_0}$$

zeros.

According to 4 and lemma 2 for all $j = 1\dots J$

$$\text{mes}\{x \in B_{r_j}^{y_{k_j}}\ ,\ u_i(x) > 0\} > C_4\varepsilon^2\ .$$

We have $J > C_0/2\varepsilon^2$ and so the theorem is proved.

2. An estimate of the relation between the positive and the negative extrema of an eigenfunction.

Let M be a n-dimensional compact smooth Riemannian manifold, $u_1, u_2, \dots$-eigenfunctions of the Laplace operator on M, $\Delta u_i = \lambda_i u_i$.

THEOREM 2. — *There exists a positive constant C which depends only on n and a positive constant N which depends on M such that, for every $i > N$,*

$$\frac{1}{C} < \frac{\sup_M u_i(x)}{|\inf_M u_i(x)|} < C .$$

We denote by $B_r \subset \mathbb{R}^n$ the ball centered at 0 of radius r.

In B_r we consider a uniformly elliptic second order operator L defined by

$$L = \sum_{i,j} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) , \tag{2.1}$$

where a_{ij} is a symmetric positive definite matrix in B_r. If the eigenvalues of the matrix $\|a_{ij}(x)\|$ lie on the segment $[e^{-1}, e]$, $e \geq 1$ we say that the operator L has an ellipticity constant e.

LEMMA 3. — *Let u be a solution of the equation*

$$a(x)Ly + \lambda u = 0$$

in the ball B_1, $1/A < a(x) < A$, $A > 0$, L is an elliptic operator with the ellipticity constant e, λ is a constant such that $|\lambda| < C$. Let us assume that $u(x_0) > 0$ and that there exists $x_0 \in B_{1/2}$ with $u(x_0) = 0$. Then

$$|\inf_{B_1} u| > \delta u(0) ,$$

where the constant $\delta > 0$, $\delta = \delta(n, A, e, C)$.

Proof.

1. We shall prove Lemma 2 under the assumption that $\lambda = 0$. Denote

$$\varphi_1 = \sup\{0, u\,|_{\partial B_1}\}$$
$$\varphi_2 = \inf\{0, u\,|_{\partial B_1}\} .$$

Let u_1, u_2 be the solutions of the following Dirichlet problems :

$$Lu_1 = 0 \quad \text{in } B_1 ,\ u_1\,|_{\partial B_1} = \varphi_1,$$
$$Lu_2 = 0 \quad \text{in } B_2 ,\ u_2\,|_{\partial B_1} = \varphi_2.$$

Then, $u = u_1 + u_2$, $u_1 > 0$ in B_1, $u_1(0) \geq u(0)$. From the Harnack inequality [5] it follows that there exists a constant $\delta > 0$, $\delta = \delta(n, e)$ such that

$$u_1\,|_{B1/2} > \delta u_1(0) .$$

Since $u(x_0) = 0$, $x_0 \in B_{1/2}$, then

$$\inf \varphi_2 < -\delta u_1(0) < -\delta u(0) .$$

2. Let $\lambda \notin 0$. Let us make a cylindric extension of the functions $u(x), a(x)$ and the operator L in the new coordinate x_{n+1}. After this extension we shall keep the notations u, a, L. Denote

$$v = ue^{\sqrt{\lambda}\, x_{n+1}}$$

clearly the function v is a solution of the elliptic equation

$$aLv + \frac{\partial^2 v}{\partial x_{n+1}^2} = 0 .$$

Now the statement of Lemma 3 follows from the assertion 1 to the function v in the unit ball in $\mathbb{R}^{n+1}$.

Proof of Theorem 2.

1. There are constant $C_1 = C_1(M) > 0$, $C_2 = C_2(M) > 0$ such that for all $x \in M$, $\lambda > C_2$ any solution of the equation $\Delta u + \lambda u = 0$ in the ball $B^x_{C_1/\sqrt{\lambda}}$ change its sign.

2. There exists a constant $N > C_2$, $N = N(M)$, such that for all $x \in M$ there exists a diffeomorphism

$$d : B^x_{2C_1/\sqrt{\lambda}} \subset M \to B_1 \subset \mathbb{R}^n$$

such that the equation $\Delta u + \lambda u = 0$ in $B^x_{2C_1/\sqrt{\lambda}}$ viewed in the ball B_1 has the form

$$a(x)Lu + \lambda' u = 0 \tag{2.2}$$

where L is an elliptic operator of the type (2.1), $e = 2$, $A = 2$, $|\lambda| < C = C(n) > 0$. We can obtain such a diffeomorphism d if we introduce in the ball $B^x_{2C_1/\sqrt{\lambda}}$ a normal coordinate system. Applying Lemma 3 to the solution u of the equation (2.2) we obtain the statement of Theorem 2.

BIBLIOGRAPHY

[1] J.-P. Lions, E. Magenes, Problèmes aux limites non homogènes et application, vol. 1, Dunod, Paris, 1968.

[2] L. Bers, F. John, M. Schechter, Partial differential equations, Providence, R.I, 1974.

[3] J. Brüning, Uber Knoten von Eigenfunnktionen des Laplace-Beltrami Operators, Math. Z., 158 (1978), 15–21.

[4] H. DONNELLY, C. FEFFERMAN, Nodal sets of eigenfunctions on Riemannian manifolds, Invent. Math., 93 (1988), 161–183.

[5] D. GILBARG, N.S. TRUDINGER, Elliptic partial differential equations of second order, Second Edition, Springer, 1983.

Manuscrit reçu le 15 février 1991.

N.S. NADIRASHVILI,
Universität Bielefeld
Fakultät für Mathematik
Universität Strasse
Postfach 8640
4800 BIELEFELD
(Allemagne).