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## MEROMORPHIC EXTENSION SPACES

by LE MAU HAI and NGUYEN VAN KHUE

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The extension of meromorphic maps from a spreaded domain over a Stein manifold to its envelope of holomorphy has been investigated by some authors. This problem for meromorphic functions was proved by Kajiwara and Sakai [9], for meromorphic maps with values in a compact subalgebraic space by Hirschowitz [8].

The extension of meromorphic maps with values in a compact Kahler manifold through an analytic set of codimension  $\geq 2$  has been established first by P. Griffiths [6] in a particular case and by Siu [17] in general. In the present paper we shall prove the following two theorems are based on ideas of Dloussky [2].

**THEOREM 2.2** — *Let  $\theta : X \rightarrow Y$  be a Hartogs meromorphic extension map. Assume that  $Y$  is a Hartogs meromorphic extension space. Then for every meromorphic mapping  $f$  from a domain  $D$  over a Stein manifold to  $X$ , there exists an analytic subset  $A$  of codimension at least 2 in  ${}^{\wedge}D$  such that  $f$  extends meromorphically to  ${}^{\wedge}D \setminus A$ . Moreover, if  $X$  is a compact Kahler manifold and  $Y$  is a Hartogs meromorphic extension space and  $\theta$  is a Hartogs meromorphic extension map then  $X$  is a meromorphic extension space.*

**THEOREM 3.1.** — *Let  $\theta : X \rightarrow Y$  be a finite proper surjective holomorphic map. Then  $X$  is a meromorphic extension space if and only if  $Y$  has the same property.*

Using Theorem 3.1 we prove that every compact non-singular elliptic Kahler surface is a meromorphic extension space. Moreover using Theorem 2.2 we also prove that every complex Lie group is a meromorphic extension space.

We would like to thank referees for their helpful remarks.

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*Key words* : Meromorphic map - Meromorphic extension spaces - Branched covering map - Elliptic Kahler surface.

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### 1. Meromorphic extension spaces.

We first recall that a meromorphic map  $f: X \rightarrow Y$  is an analytic set  $\Gamma(f)$  in  $X \times Y$  such that the canonical projection  $p(f): \Gamma(f) \rightarrow X$  is proper and there exists an open subset  $X(f)$  of  $X$  such that  $\Gamma(f) \cap (X(f) \times Y)$  is the graph of a holomorphic map from  $X(f)$  into  $Y$ .  $\Gamma(f)$  is called the graph of  $f$ . It is known [14] that in the case where  $X$  is normal, the indeterminacy locus of  $f$

$$I(f) = \{x \in X : f \text{ is not holomorphic at } x\} = \{x \in X : \dim p(f)^{-1}(x) > 0\}$$

is an analytic set of codimension  $\geq 2$ .

We now give the following

**DEFINITION 1.1.** — Let  $X$  be a complex space. We say that  $X$  is a meromorphic extension space if the two following conditions are satisfied :

H) every meromorphic map from a spreaded domain  $D$  over a Stein manifold into  $X$  can be extended meromorphically to  $\hat{D}$ , the envelope of holomorphy of  $D$ .

R) Every meromorphic map from  $Z \setminus S$  into  $X$ , where  $Z$  is a normal complex space and  $S$  is an analytic set of codimension  $\geq 2$  in  $Z$  can be meromorphically extended to  $Z$ .

In the case where only the condition H) (resp. R)) holds,  $X$  is called a Hartogs (resp. Riemann) meromorphic extension space.

We have the following

**PROPOSITION 1.2.** — Let  $X$  be a complex space. Then the following conditions are equivalent :

(i) every meromorphic map from a Hartogs domain to  $X$  can be meromorphically extended to its envelope of holomorphy.

(ii)  $\mu_R^X$  is Stein for every Stein manifold  $R$ , where  $\mu_R^X$  denotes the spread domain over  $R$  associated to the sheaf of germs of meromorphic maps on  $R$  with values in  $X$ .

(iii)  $X$  is a Hartogs meromorphic extension space.

*Proof.* — (i)  $\rightarrow$  (ii). By the Docquier-Grauert theorem [3] it suffices to show that  $\mu_R^X$  is  $p_7$ -convex, i.e. every holomorphic embedding  $\sigma : H_k(r) \rightarrow \mu_R^X$  can be holomorphically extended to  $\Delta^k$ , where  $\Delta$  denotes the unit disc in  $C$  and  $H_k(r)$  is given by

$$H_k(r) = \{(z_1, z_2, \dots, z_k) \in \Delta^k : |z_j| < r, j=1, 2, \dots, k-1\} \\ \cup \{(z_1, z_2, \dots, z_k) \in \Delta^k : |z_k| > 1-r\}, \quad 0 < r < 1,$$

$k = \dim R$ .

Let  $\mathcal{O}_R^X$  denote the spread domain over  $R$  associated to the sheaf of germs of holomorphic maps on  $R$  with values in  $X$ . Obviously  $\mathcal{O}_R^X$  is dense open in  $\mu_R^X$ . Consider the canonical map  $e : \mathcal{O}_R^X \rightarrow X$  given by

$$e(g_z) = g_z(z) \quad \text{for } z \in R \quad \text{and} \quad g_z \in (\mathcal{O}_R^X)_z.$$

It is easy to see that  $e$  extends meromorphically by definition  $\mu_R^X$ . Hence  $e\sigma : H_k(r) \rightarrow X$  is meromorphic. By hypothesis it is extended to a meromorphic map  $\wedge e : \Delta^k \rightarrow X$ . Let  $p : \mu_R^X \rightarrow R$  denote the locally biholomorphic canonical map and let  $\mathcal{E} : \Delta^k \rightarrow R$  be a holomorphic extension of  $p\sigma$ . Since every hypersurface in  $\Delta^k$  meets  $H_k(r)$ , it follows that  $\mathcal{E}$  is a locally biholomorphic map. Define now a holomorphic extension  $\wedge \sigma : \Delta^k \rightarrow \mu_R^X$  by

$$\wedge \sigma(z) = (\mathcal{E}(z), \wedge e(\mathcal{E}|_{U_z})^{-1}_{\mathcal{E}(z)}) \quad \text{for } z \in \Delta^k$$

where  $U_z$  is a neighbourhood of  $z$  in  $\Delta^k$  on which  $\mathcal{E}$  is biholomorphic. Therefore (ii) is proved.

(ii)  $\rightarrow$  (iii). Given a meromorphic map  $f : D \rightarrow X$ , where  $D$  is a spread domain over a Stein manifold. Consider  $D$  as a spread domain over  $\wedge D$  with the canonical map  $e : D \rightarrow \wedge D$ . By  $D(f)$  we denote the envelope of meromorphy of  $f$ . Then  $D(f)$  is a Stein manifold and  $f$  has a canonical meromorphic extension  $\tilde{f}$  to  $D(f)$ . By the Steiness of  $D(f)$  the canonical map  $\beta : D \rightarrow D(f)$  can be extended to a holomorphic map  $\wedge \beta : \wedge D \rightarrow D(f)$ . Therefore  $\tilde{f} \wedge \beta$  is a meromorphic extension of  $f$  to  $\wedge D$ .

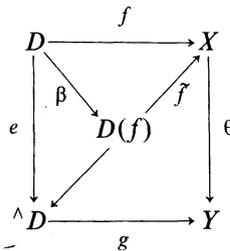
(iii)  $\rightarrow$  (i) is trivial.

**2. Meromorphic extension maps.**

DEFINITION 2.1. — Let  $\theta : X \rightarrow Y$  be a holomorphic map between complex space. We say that  $\theta$  is a Hartogs (resp. Riemann) meromorphic extension map if for each  $y \in Y$  there exists a neighbourhood  $U$  of  $y$  such that  $\theta^{-1}(U)$  is a Hartogs (resp. Riemann) meromorphic extension space. If both conditions of Hartogs and Riemann meromorphic extension are satisfied, then  $\theta$  is called a meromorphic extension map.

THEOREM 2.2. — Let  $\theta : X \rightarrow Y$  be a Hartogs meromorphic extension map. Assume that  $Y$  is a Hartogs meromorphic extension space. Then for every meromorphic mapping  $f$  from a domain  $D$  over a Stein manifold to  $X$ , there exists an analytic subset  $A$  of codimension at least 2 in  ${}^{\wedge}D$  such that  $f$  extends meromorphically to  ${}^{\wedge}D \setminus A$ . Moreover if  $X$  is a compact Kahler manifold and  $Y$  is a Hartogs meromorphic extension space and  $\theta$  is a Hartogs meromorphic extension map then  $X$  is a meromorphic extension space.

Proof. — (i) Given  $f : D \rightarrow X$  a meromorphic map, where  $D$  is a spread domain over a Stein manifold. By hypothesis we have a following commutative diagram



where  $g$  is a meromorphic extension of  $\theta \cdot f$ .

We show that  $\gamma_0 = \gamma|_{D(f)\gamma^{-1}(I(g))} : D(f)\gamma^{-1}(I(g)) \rightarrow {}^{\wedge}D \setminus I(g)$  is locally pseudoconvex. Let  $z \in {}^{\wedge}D \setminus I(g)$ . Take a neighbourhood  $V$  of  $g(z)$  in  $Y$  and a Stein neighbourhood  $U$  of  $z$  in  ${}^{\wedge}D \setminus I(g)$  such that  $g(U) \subset V$ . As in Proposition 1.2 ((i)  $\rightarrow$  (ii)), it follows that  $\gamma^{-1}(U)$  is  $p_r$ -convex. Therefore  $\gamma^{-1}(U)$  is Stein [3], and the local pseudoconvexity of  $D(f)\gamma^{-1}(I(g))$  over  ${}^{\wedge}D \setminus I(g)$  is proved.

We now write  $I(g) = \cap Z(h_\alpha)$ , where  $h_\alpha$  is holomorphic on  $\wedge D$  and vanishes on  $I(g)$  and  $Z(h_\alpha)$  denotes the zero-set of  $h_\alpha$ . Since  $\gamma_0 : D(f) \setminus \gamma^{-1}(I(g)) \rightarrow \wedge D \setminus I(g)$  is locally pseudoconvex and  $\wedge D \setminus Z(h_\alpha)$  is Stein,  $\gamma_0^{-1}(\wedge D \setminus Z(h_\alpha))$  also is Stein for every  $\alpha$ . For each  $\alpha$  consider the holomorphic map  $\beta_\alpha = \beta|_{D \setminus Z(h_\alpha)} : D \setminus Z(h_\alpha) \rightarrow \gamma_0^{-1}(\wedge D \setminus Z(h_\alpha))$ . Then  $\beta_\alpha$  can be extended to a holomorphic map  $\wedge \beta_\alpha : \wedge (D \setminus Z(h_\alpha)) = \wedge D \setminus Z(h_\alpha) \rightarrow D(f)$  [2]. By uniqueness the maps  $\wedge \beta_\alpha$  define a holomorphic map  $\wedge \beta : \wedge D \setminus I(g) \rightarrow D(f)$  such that  $\wedge \beta \cdot e = \beta$  on  $D \setminus e^{-1}(I(g))$ . The map  $\wedge f = \tilde{f} \cdot \wedge \beta : \wedge D \setminus I(g) \rightarrow X$  is meromorphic and is a meromorphic extension of  $f$ .

(ii) Now assume that  $X$  is a compact Kahler manifold and  $Y$  is a Hartogs meromorphic extension space and  $\theta$  is a Hartogs meromorphic extension map. By [17] and since (i) it implies that  $X$  is a meromorphic extension space.

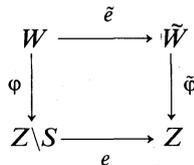
**3. Finite proper holomorphic surjections and meromorphic extension spaces.**

The aim of this section is to prove Theorem 3.1 on invariance of meromorphic extendibility under finite proper holomorphic surjections.

**THEOREM 3.1.** — *Let  $\theta : X \rightarrow Y$  be a finite proper holomorphic surjective map. Then  $X$  is a meromorphic extension space if and only if  $Y$  has the same property.*

For the proof of the theorem we need following four lemmas.

**LEMMA 3.2.** — *Let  $\varphi : W \rightarrow Z \setminus S$  be unbranched finite covering map, where  $W, Z$  are complex manifolds and  $S$  is an analytic set in  $Z$  of codimension  $\geq 1$ . Then there exists a following commutative diagram*

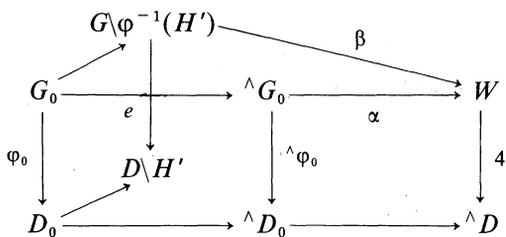


where  $(\tilde{W}, \tilde{\varphi}, Z)$  is a branched covering map and  $\tilde{e}$  is an open embedding.

*Proof.* — See [5] and [18].

LEMMA 3.3. — Let  $\varphi : G \rightarrow D$  be a branched covering map, where  $G$  is a normal complex space and  $D$  is a spread domain over a Stein manifold such that points of  $D$  are separated by holomorphic functions on  $D$ . Assume that  $H$  is the branch locus of  $\varphi$  and  $D_0 = D \setminus H$ ,  $G_0 = G \setminus \varphi^{-1}(H)$ ,  $\varphi_0 = \varphi|_{G_0}$ .

Then there exists an analytic set  $H'$  in  $D$  contained in  $H$  such that  $\wedge(D \setminus H') = \wedge D$  and a commutative diagram of normal complex spaces

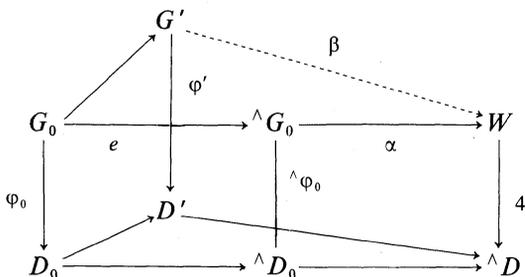


where  $\wedge \varphi_0$ ,  $4$ ,  $\beta : G \setminus \varphi^{-1}(H') \rightarrow \text{Im } \beta$  are branched covering maps,  $\alpha$  is an open embedding and  $\beta^{-1}(\alpha e(G_0)) = G_0$ .

*Proof.* — Since  $D$  and  $G$  are normal, it follows that either  $H$  is a hypersurface in  $D$  or  $H = \emptyset$ . The case where  $H = \emptyset$  is trivial. Therefore we can assume that  $H$  is a hypersurface. Then there exists an analytic set  $\wedge H$  in  $\wedge D$  such that

$$\wedge D_0 = \wedge D \setminus \wedge H \quad [2].$$

Observe that  $\wedge H \cap D \subset H$ . We write  $H = (\wedge H \cap D) \cup H'$ , where  $H'$  is an analytic set in  $D$  such that  $\wedge(D \setminus H') = \wedge D$ . By [11] the map  $\wedge \varphi_0 : \wedge G_0 \rightarrow \wedge D_0$  is an unbranched covering map and using Lemma 3.2 to  $\wedge \varphi_0$ , we can construct a commutative diagram



where  $D' = D \setminus H'$ ,  $G' = G \setminus \varphi^{-1}(H')$  and  $\varphi' = \varphi|_{G'}$ , in which  $4$  is a branched covering map of the normal complex space  $W$  onto  $\wedge D$  and

$\alpha$  is an open embedding. Put  $\hat{\alpha} = \alpha e$ . We shall prove  $\hat{\alpha}$  can be extended to a holomorphic map  $\beta$  from  $G'$  to  $W$ . Since the Steiness is invariant under finite proper holomorphic surjections [13],  $W$  is Stein. Thus by the normality of  $G'$  it suffices to show that  $\hat{\alpha}$  is locally compact on  $G'$ , i.e for every  $z \in G'$  there exists a neighbourhood  $U$  of  $z$  such that  $\hat{\alpha}(U \cap G_0)$  is relatively compact in  $W$ . Assume that  $z_0 \in \varphi'^{-1}(H')$  and  $\{z_n\} \subset G_0$  converging to  $z_0$ .

Then

$$\lim 4 \hat{\alpha}(z_n) = \lim \varphi'(z_n) = \varphi_0(z_0) \in D' \hookrightarrow \hat{D}.$$

Thus from property of 4, it follows that  $\{\hat{\alpha}(z_n)\}$  is relatively compact in  $W$ . This yields the local compactness of  $\hat{\alpha}$ .

Let  $\beta : G' \rightarrow W$  be a holomorphic extension of  $\hat{\alpha}$ . Since  $\varphi'$  and 4 are finite proper maps and  $D'$  is contained in  $\hat{D}$  as an open subset, it is easy to see that  $\beta : G' \rightarrow \beta(G')$  is finite proper. Hence by the normality of  $W$  and by the equality  $\dim G' = \dim W$ , it follows that  $\beta(G')$  is open in  $W$  and  $\beta : G' \rightarrow \beta(G')$  is a branched covering map. Finally, if  $\hat{\alpha}(z_0) = \beta(z_1)$ , where  $z_0 \in G_0$  and  $z_1 \in G'$ , then

$$\varphi'(z_1) = 4\beta(z_1) = 4 \hat{\alpha}(z_0) = \varphi_0(z_0).$$

This implies  $z_1 \in G_0$ . Hence  $\beta^{-1}(\hat{\alpha}(G_0)) = G_0$ .

The lemma is proved.

LEMMA 3.4. — Let  $X$  be a meromorphic extension space and  $Z$  a normal Stein space. Assume that  $H$  is a hypersurface of  $Z$  and  $G$  is an open subset of  $Z$  meeting every irreducible branch of  $H$ . Then every meromorphic map  $f : (D \setminus H) \cup G \rightarrow X$  can be meromorphically extended to  $Z$ .

*Proof.* — Since  $Z$  is normal,  $\text{codim } S(Z) \geq 2$  [4], where  $S(Z)$  denotes the singular locus of  $Z$ . We write by the Steiness of  $ZS(Z)$  in the form

$$S(Z) = \cap \{Z(h) : h \text{ is holomorphic on } Z, h|_{S(Z)} = 0 \text{ and } h \neq 0 \text{ on every irreducible branch of } H\}.$$

From hypothesis, it suffices to show that for every such  $h$  the map  $f_h = f|_{Z_h \setminus H}$ , where  $Z_h = Z \setminus Z(h)$ , can be meromorphically extended on  $Z_h$ . Put  $G_h = G \setminus Z(h)$  and  $H_h = H \cap Z_h$ . Then  $G_h$  also meets every

irreducible branch of  $H_h$ . Consider the meromorphic map  $f_h|_{(Z_h \setminus H_h) \cup G_h}$ . Since  $\hat{\wedge}((Z_h \setminus H_h) \cup G_h) = Z_h$  [2] it follows that  $f_h|_{(Z_h \setminus H_h) \cup G_h}$  can be extended to a meromorphic map  $\hat{\wedge}f_h$  to  $Z_h$ .

The lemma is proved.

LEMMA 3.5. — *Let  $\pi : Z \rightarrow W$  be a bunched covering map and  $f : Z \rightarrow X$  a meromorphic map which can be factorized through  $\pi|_{\pi^{-1}(V)}$  for some non-empty open subset  $V$  of  $W$ . Then  $f$  can be factorized through  $\pi$ .*

*Proof.* — Let  $H$  denote the branch locus of  $\pi$ . It is easy to check that there exists a holomorphic map  $g$  from  $W \setminus (H \cup \pi(I(f)))$  to  $X$  such that  $g\pi = f$  on  $\pi^{-1}(W \setminus (H \cup \pi(I(f))))$ . Since  $\pi \times \text{id} : Z \times X \rightarrow W \times X$  is proper,

$$\overline{\Gamma(g)} = (\pi \times \text{id})\Gamma(f)$$

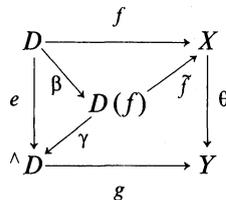
is an analytic set in  $W \times X$ . Hence from property of  $\pi$  and  $p(f)$ , it follows that  $\overline{\Gamma(g)}$  defines a meromorphic map  $g_1$  on  $W$  such that  $g_1 \cdot \pi = f$ .

The lemma is proved.

We now can prove Theorem 3.1.

a) First prove sufficiency of the theorem.

(i) Given  $f : D \rightarrow X$  a meromorphic map, where  $D$  is a spread domain over a Stein manifold. From hypothesis we have a following commutative diagram



where  $g$  is a meromorphic extension of  $\theta \cdot f$ .

As in Theorem 2.2,  $\beta|_{D \setminus e^{-1}(I(g))}$  can be extended to a holomorphic map  $\hat{\wedge}\beta : \hat{\wedge}D \setminus I(g) \rightarrow D(f)$ . Put  $A = (\text{id}^x \theta)^{-1}(p(g)^{-1}(I(g)))$ . Then  $\Gamma(\hat{\wedge}f) \subset (\hat{\wedge}D \setminus I(g)) \times X \subset (D(f) \times X) \setminus A$ , where  $\hat{\wedge}f = \tilde{f} \wedge \beta$ , and is closed in  $(\gamma(D(f)) \times X) \setminus A$ . Indeed, let  $\{(x_n, z_n)\} \subset \Gamma(\hat{\wedge}f)$  converge to

$(x_0, z_0) \in \gamma(D(f)) \times X \setminus A$ . Since  $(x_0, z_0) \notin A$ ,  $(\text{id} \times \theta)(x_0, z_0) = (x_0, \theta z_0) \in p(g)^{-1}(I(g))$ . If  $x_0 \in I(g)$ , then  $(x_0, z_0) \in (\wedge D \setminus I(g)) \times X$ . Hence  $(x_0, z_0) \in \Gamma(\wedge f)$ . In the case where  $x_0 \in I(g)$ , we have  $(x_0, \theta z_0) \in \Gamma(g)$ . This is impossible, because of the relation  $\Gamma(g) \supset \{(x_n, z_n)\} \rightarrow (x_0, \theta z_0) \in \Gamma(g)$ . Therefore  $\Gamma(\wedge f)$  is closed in  $(\gamma(\wedge D(f)) \times X) \setminus A$ . Since  $\dim \Gamma(\wedge f) = \dim \wedge D > \dim A$ , by the Remmert-Stein theorem [7]  $\overline{\Gamma(\wedge f)}$  is an analytic set in  $\wedge D \times X$ . Since  $\theta$  is proper, it follows that  $\overline{\Gamma(\wedge f)}$  defines a meromorphic extension of  $f$  to  $\wedge D$ .

(ii) Let now  $f: Z \setminus S \rightarrow X$ , where  $Z$  is a normal complex space and  $S$  is an analytic set in  $Z$  of codimension  $\geq 2$ . From the Riemann meromorphic extendibility of  $Y$  we have a following commutative diagram

$$\begin{array}{ccc} Z \setminus S & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ Z & \xrightarrow{g} & Y \end{array}$$

Similarly as in (i), where  $D$ ,  $\wedge D \setminus I(g)$  and  $\tilde{f}^\beta$  are replaced by  $Z$ ,  $Z \setminus (I(g) \cup S)$  and  $f$  respectively we obtain a meromorphic extension  $\wedge f$  of  $f$  to  $Z$ .

b) We now prove necessity of the theorem.

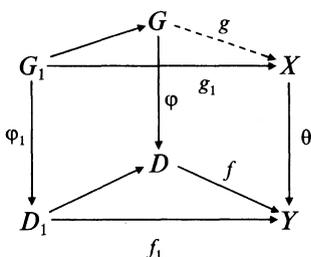
(i) Let  $f$  be a meromorphic map from a spread domain  $D$  over a Stein manifold to  $X$ . By Proposition 1.2 we can assume that  $D$  is a Hartogs domain. Consider the commutative diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{g_1} & X \\ \varphi_1 \downarrow & & \downarrow \theta \\ D_1 & \xrightarrow{f_1} & Y \end{array}$$

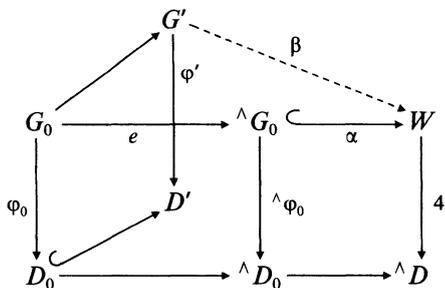
where  $D_1 = D \setminus I(g)$ ,  $G_1 = (D_1 \times_Y X)_{\text{red}}$  is the fiber product,  $f_1 = f|_{D_1}$  and  $\varphi_1, g_1$  are canonical projections.

Without loss of generality we may assume that  $G_1$  is normal. Observe that  $\varphi_1$  is a branched covering map. Let  $H_1$  denote the branched locus of  $\varphi_1$ . Since  $\dim H_1 > \dim I(f)$ , it follows that  $\overline{H_1}$  is an analytic

set in  $D$ . Using Lemma 3.2 to the unbranched covering map  $\varphi_1: G_1 \setminus \varphi_1^{-1}(H_1 \cup I(f)) \rightarrow D_1 \setminus (\bar{H}_1 \cup I(f))$  we have a following commutative diagram



in which  $\varphi$  is a branched covering map and  $G$  is normal. Since  $\dim G_1 = \dim (\varphi \times \theta)^{-1} \Gamma(f) \geq \dim (\varphi \times \theta)^{-1} p(f)^{-1}(I(f))$ , by the Remmert-Stein theorem [7],  $\Gamma(g_1)$  is an analytic set in  $G \times X$ . Hence by property of  $\theta$ , it defines a meromorphic extension of  $g$  on  $G$  such that  $\theta g = f$ . In notations of Lemma 3.3 we have a following commutative diagram of normal complexe spaces



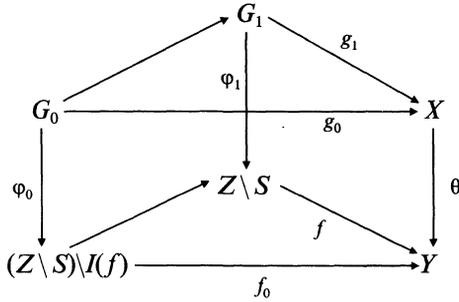
in which  $\varphi_0$  and  $\hat{\varphi}_0$  are unbranched covering maps,  $\varphi', 4, \beta: G' \rightarrow \beta(G')$  are branched covering maps. Moreover  $G_0 = \beta^{-1}(\alpha e(G_0))$ . Thus  $g|_G$  can be meromorphically factorized through  $\beta: G' \rightarrow \beta(G')$ . Let  $\hat{g}_0$  be a meromorphic extension of  $g|_{G_0}$  on  $\hat{G}_0$  and  $\tilde{g}$  a meromorphic map on  $\beta(G')$  such that  $\tilde{g}\beta = g|_{G'}$ . Define a meromorphic map  $g_2$  from  $\hat{G}_0 \cup \beta(G')$  into  $X$  by

$$g_2 = \hat{g}_0 \text{ on } \hat{G}_0 \quad \text{and} \quad g_2 = \tilde{g} \text{ on } \beta(G').$$

Since  $4$  is finite proper and every irreducible branch of  $\hat{H}$  meets  $D'$ , it follows that this holds for  $4^{-1}(\hat{H})$  and  $\beta(G')$ . Thus by Lemma 3.4 we have a meromorphic extension  $g_3$  of  $g_2$  on  $W$ . From Lemma 3.5,  $g_3$  can be meromorphically factorized through  $4$ . Hence  $f$  is extended to a meromorphic map to  $\hat{D}$ .

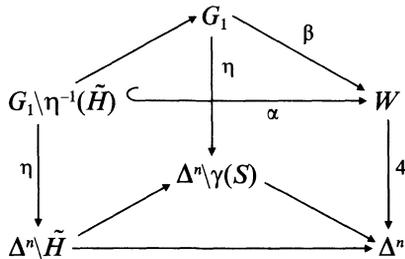
(ii) Finally we show that  $Y$  has the Riemann meromorphic extension property. Given  $f: Z \setminus S \rightarrow Y$  a meromorphic map, where  $Z$  is a normal complex space and  $S$  is an analytic set in  $Z$  of codimension  $\geq 2$  which can be assumed to contain the singular locus of  $Z$ .

As in (i) we can construct a following commutative diagram of normal complex spaces



where  $\varphi_0, \varphi_1$  are branched covering maps and  $g_0, g_1$  are meromorphic maps. The problem is local without loss of generality we may assume that there exists a branched covering map  $\gamma: Z \rightarrow \Delta^n$ ,  $n = \dim Z$ . Let  $H$  denote the branch locus of  $\varphi_1$ .

Then  $\bar{H}$  is an analytic set in  $Z$  because of the inequality  $\text{codim } I(f) \geq 2$ . Take a hypersurface  $\tilde{H}$  in  $\Delta^n$  containing the branch locus of  $\gamma$  such that  $\gamma(S \cup H) \subset \tilde{H}$ . Using Lemma 3.3 we give a following commutative diagram



where  $\eta = \gamma\varphi_1, 4, \beta: G_1 \rightarrow \beta(G_1)$  are branched covering maps.

Obviously  $\beta^{-1}(\alpha(G_1 \setminus \eta^{-1}(\tilde{H}))) = G_1 \setminus \eta^{-1}(\tilde{H})$ . Thus  $g_1$  can be meromorphically factorized through  $\beta: G_1 \rightarrow \beta(G_1)$ . Hence  $g_0$  and  $g_1$  induce a meromorphic map  $g_2$  on  $G_1 \setminus \eta^{-1}(\tilde{H}) \cup \beta(G_1)$  with values in  $X$ . Since

every irreducible branch of  $\tilde{H}$  meets  $\Delta^n \setminus \gamma(S)$ , it follows that this holds for  $4^{-1}(\tilde{H})$  and  $\beta(G_1)$ . By Lemma 3.4,  $g_2$  can be extended to a meromorphic map  $g_3$  on  $W$ . Thus from Lemma 3.5 we give a meromorphic extension of  $f$  to  $\Delta^n$ .

Theorem 3.1 is completely proved.

#### 4. Some applications.

We first recall that an elliptic surface is a compact regular surface  $V$  equipped with a holomorphic map  $\theta$  from  $V$  onto a non-singular curve  $C$  such that  $\theta^{-1}(x)$  is an elliptic curve outside a finite set in  $C$ .

Using now Theorem 3.1 we prove the following.

**THEOREM 4.1.** — *Let  $V$  be an elliptic Kahler surface. Then  $V$  is a meromorphic extension space.*

*Proof.* — From a result of Siu [17],  $V$  is a Riemann meromorphic extension space. Thus it remains to prove that  $V$  has the Hartogs meromorphic extension property.

(i) In [12] Kodaira constructed for  $V$  a branched covering map  $\alpha$  from an elliptic surface  $\tilde{V}$  on  $V$  such that for each  $x \in C$  there exists a sufficiently small disc  $U_x$  containing  $x$  for which  $(\theta\alpha)^{-1}(U_x)$  is biholomorphic to a locally pseudoconvex open subset of a projective surface  $P_x$ . Put  $\eta = \theta \cdot \alpha$ . Given  $f: D \rightarrow \eta^{-1}(U_x)$  a meromorphic map, where  $D$  is a spread domain over a Stein manifold.

Let  ${}^\wedge f: {}^\wedge D \rightarrow P_x$  be a meromorphic extension of  $f|_{D \setminus \{0\}}$ . Put

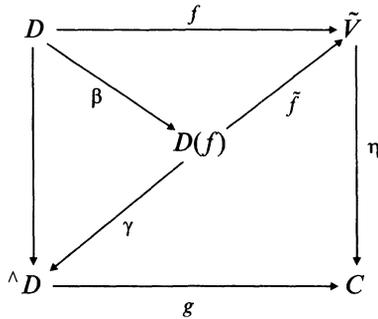
$$G = {}^\wedge f_0^{-1}(\eta^{-1}(U_x)), \quad \text{where} \quad {}^\wedge f_0 = {}^\wedge f|_{{}^\wedge D \setminus I({}^\wedge f)}.$$

We may suppose that  $D$  is a Hartogs domain. Since  $D \setminus I(f) \subset G$  we have  ${}^\wedge G = {}^\wedge D$ . Let now  $G \neq {}^\wedge D \setminus I({}^\wedge f)$ . Then we can find a point  $z_0 \in \partial G$  in  ${}^\wedge D \setminus I({}^\wedge f)$  and two Stein neighbourhoods of  $z_0$  and  ${}^\wedge f_0(z_0)$  in  ${}^\wedge D \setminus I({}^\wedge f)$  and  $P_x$  respectively such that  ${}^\wedge f_0(U) \subset W$  and  $z_0 \in {}^\wedge(U \cap G)$ . Since  $W \cap \eta^{-1}(U_x)$  is Stein and  ${}^\wedge f_0(U \cap G) \subset W \cap \eta^{-1}(U_x)$ , it follows that  ${}^\wedge f_0(z_0) \in W \cap \eta^{-1}(U_x)$ . This yields  $z_0 \in G$ . Hence  $G = {}^\wedge D \setminus I({}^\wedge f)$ . On the other hand, since  $\alpha \cdot {}^\wedge f_0$  and  $\eta \cdot {}^\wedge f_0$  are extended to meromorphic maps  $g: {}^\wedge D \rightarrow V$  and  $h: {}^\wedge D \rightarrow U_x$  respectively. We have  $\theta g = h$ .

It is easy to see that  $\Gamma({}^\wedge f_0)$  is contained and closed in  $(\text{id} \times \alpha)^{-1}\Gamma(g) \setminus (\text{id} \times \alpha)^{-1}p(g)^{-1}(I(g))$ , by the Remmert-Stein theorem,  $\Gamma({}^\wedge f_0)$  defines a meromorphic extension  $\tilde{f}$  of  $f$ .

From the relation  $\eta\tilde{f} = f$ , it follows that  $\tilde{f}$  induces a meromorphic extension of  $f$  with values in  $\eta^{-1}(U_x)$ .

(ii) Let now  $f$  be a meromorphic map from a spread domain  $D$  over a Stein manifold into  $\tilde{V}$ . Consider the following commutative diagram



By (i) as in Theorem 2.1 we can find a holomorphic extension  $\hat{\beta}$  of  $\beta|_{D \setminus I(g)}$  on  $\hat{D} \setminus I(g)$ . Let  $\hat{f}_1: \hat{D} \rightarrow V$  be a meromorphic extension of  $f_1 = \alpha\tilde{f} \wedge \beta: D \setminus I(g) \rightarrow V$ . Then as in Theorem 2.1, it follows that  $\Gamma(\hat{f} \wedge \hat{\beta})$  defines a meromorphic extension of  $f$ . Hence  $\tilde{V}$  is a Hartogs meromorphic extension space.

(iii) Given a meromorphic map  $f$  from  $Z \setminus S$  into  $\tilde{V}$ , where  $Z$  is a normal complex space and  $S$  is an analytic set in  $Z$  of codimension  $\geq 2$ . Let  $g: Z \rightarrow V$  be a meromorphic extension of  $\alpha f$ . Then as in (i) we infer that  $\Gamma(f)$  defines a meromorphic extension of  $f$ .

(iv) From (ii) and (iii),  $\tilde{V}$  is a meromorphic extension space. Hence by Theorem 3.1,  $V$  is a meromorphic extension space.

The theorem is proved.

**THEOREM 4.2.** — *Every complex Lie group is a meromorphic extension space.*

*Proof.* — Let  $G$  be a complex Lie group.

(i) Given  $f: D \rightarrow G$  a meromorphic map, where  $D$  is a spread domain over a Stein manifold. Since  $\text{codim } I(f) \geq 2$ ,  $f|_{D \setminus I(f)}$  can be holomorphically extended to  $\hat{D}$  [1]. Thus  $G$  is a Hartogs meromorphic extension space.

(ii) Given now  $f$  a meromorphic map from  $Z \setminus S$  to  $G$ , where  $Z$  is a normal complex space and  $S$  is an analytic set in  $Z$  of codimension  $\geq 2$ . Let  $\varphi$  be a plurisubharmonic exhaustion function [10] on  $G$ . Since  $\text{codim } I(f) \geq 2$  and  $\text{codim } S \geq 2$ ,  $\varphi f$  is plurisubharmonic on  $Z$  [8]. By [19] there exists a holomorphic bundle map  $\theta$  from  $G$  onto a complex torus  $T$  such that the fibers of  $\theta$  are Stein manifolds. Consider the holomorphic map  $\theta f|_{(Z \setminus S) \setminus I(f)}$ . Then, by the Kählerness of the torus  $T$ ,  $\theta f$  is meromorphic on  $Z$  [17]. Let  $\gamma: \hat{Z} \rightarrow Z$  be the Hironaka singular resolution of  $Z$ . By (i),  $h = \theta f \gamma$  is holomorphic on  $\hat{Z}$ . For each  $z_0 \in \gamma^{-1}(S)$  take the two neighbourhoods  $U$  and  $V$  of  $z_0$  and  $h(z_0)$  respectively such that  $h(U) \leq V$  and  $\theta^{-1}(V)$  is a Stein manifold. Then we have  $f \gamma(U \setminus \gamma^{-1}(S)) \leq \theta^{-1}(V)$ . By the upper semi-continuity of  $\varphi f \gamma$  on  $\hat{Z}$  and since  $\varphi$  is an exhaustion function on  $G$  it follows that  $f \gamma|_{U \setminus \gamma^{-1}(S)}$  can be extended holomorphically at  $z_0$ . Since  $z_0$  is arbitrary  $f \gamma$  is extended holomorphically to  $\hat{Z}$ . Then  $(f \gamma) \gamma^{-1}$  is a meromorphic extension of  $f$ .

The theorem is proved.

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