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STOCHASTIC CALCULUS AND DEGENERATE BOUNDARY VALUE PROBLEMS

by **Patrick CATTIAUX**

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0. INTRODUCTION

Let A be a second order differential operator written as

$$A = 1/2 \sum_{i=1}^{m} Y_i^2 + Y_0,$$

where Y_i, $(0 \leqslant i \leqslant m)$, are $C_b^\infty(\mathbb{R}^d)$ vector fields, h be a $C_b^\infty(\mathbb{R}^d)$ function and D be a regular open set of $\mathbb{R}^d$.

$h - A$ is said to be hypoelliptic in D if for $u \in \mathscr{D}'(D)$, $u \in C^\infty(D)$ as soon as $(h-A)u \in C^\infty(D)$. In 1967, L. Hörmander [21] showed that a sufficient condition for $h - A$ to be hypoelliptic is :

(0.1) (H.G) Lie $(Y_i, 0 \leqslant i \leqslant m)$ *spans* $\mathbb{R}^d$ *at every* $y \in D$.

This condition is also necessary when the $Y_i's$ and h are analytic [16].

Let T_t^h be the semi group on $C_b^0(\mathbb{R}^d)$ whose infinitesimal generator coincides with $A - h$ on $C_b^\infty(\mathbb{R}^d)$. The Ito's theory allows to give a stochastic representation of T_t^h, namely

$$T_t^h f(y) = E^y\left[f(y_t) \exp - \int_0^t h(y_s) ds \right]$$

where $y_.$ is the generic element of $C^0(\mathbb{R}^+, \mathbb{R}^d)$ and E^y is the expectation relative to P_y, where P_y denotes the Probability measure defined on $C^0(\mathbb{R}^+,\mathbb{R}^d)$ and solution of the stochastic differential system of Stratonovitch

$$(0.2) \qquad \begin{cases} dy_t = Y_0(y_t)\,dt + Y_i(y_t)\,dw_t^i \\ y_0 = y \end{cases}$$

Here the w^i's are independent Brownian motions for $1 \leqslant i \leqslant m$, and we do the usual summation convention for repeated indices.

Actually one can build on the standard Wiener space a stochastic process $\varphi_.(\omega,y)$ whose law is P_y and which is for every ω a flow of C^∞ diffeomorphisms of $\mathbb{R}^d$ ([2], [24]).

P. Malliavin ([26]) has introduced a differential calculus on the Wiener space (known as Malliavin calculus) which allows to give a probabilistic proof of the C^∞ smoothness for the heat kernel $\left(\text{associated with } \frac{\partial}{\partial t}+A\right)$, under the natural Hörmander's condition

$$(0.3) \quad (\text{H.R}) \quad \text{Lie}\left(\frac{\partial}{\partial t}+Y_0, Y_i; 1 \leqslant i \leqslant m\right) \textit{ spans } \mathbb{R}^{d+1} \textit{ at every } (t,y) \in \mathbb{R}_+^* \times \mathbb{R}^d, \text{ (cf. [30])}.$$

In [13], using the Malliavin calculus and the diffeomorphism property of $\varphi_t(\omega,\cdot)$, we proved (probabilistically) that under (H.G) (see (0.1)) the resolvant operators of the semigroup T_t^h have C^∞ kernel out of the diagonal. We then obtained a probabilistic proof of the celebrated Hörmander's result, and a decomposition theorem for the space $C^\infty(M)$ (when M is a compact C^∞ manifold), similar to the classical decomposition result for elliptic pseudo differential operators ([15]).

Consider now the boundary value problem

$$(0.4) \qquad (\text{L.P.}) \begin{cases} (h-A)u = f \text{ in } D \\ \qquad Lu = g \text{ on } \partial D \end{cases}$$

where L is an operator defined on a suitable space.

There are several methods to regard such a problem. A powerful one consists in transforming (L.P.) into a pseudo differential problem on the boundary.

This method requires having a good knowledge on the first boundary value problem (Dirichlet problem denoted by (D.P)), that is with $L = \gamma_0$, where γ_0 is the sectional trace of order 0 on ∂D.

Indeed assume that the Green and the harmonic operators G and H are well defined, that is :

Gf (resp. Hg) is the unique solution of (D.P) with data $(f,0)$ (resp. $(0,g)$),

and moreover G (resp. H), maps $C^\infty(\bar{D})$ (resp. $C^\infty(\partial D)$) into $C^\infty(\bar{D})$.

Then for $f \in C^\infty(\bar{D})$ and $g \in C^\infty(\partial D)$, (0.4) has a solution $u \in C^\infty(\bar{D})$ if and only if a solution $\varphi \in C^\infty(\partial D)$ exists to the following boundary problem :

$$\text{(0.5)} \qquad \text{(B.P)} \quad L\cdot(H\varphi) = g - L\cdot(Gf) \text{ on } \partial D,$$

and then $u = H\varphi + Gf$.

In particular if L maps $C^\infty(\bar{D})$ into $C^\infty(\partial D)$, the smoothness of any solution of (L.P) is equivalent to the hypoellipticity of L.H.

The Dirichlet problem has been studied for a long time in the p.d.e theory when A is uniformly elliptic. When A only satisfies (H.G) and $\bar{D}$ is compact, it was solved by Derridj ([16]), with two additional assumptions : the boundary is non characteristic for A, and a coercivity assumption for h. Derridj proved that if the data (f,g) are C^∞, there exists a C^∞ solution u, and uniqueness holds in an intermediary space between L^2 and $H^1(\bar{D})$. Here again the solution u admits a stochastic representation

$$\text{(0.6)} \qquad \begin{aligned} u(y) = {} & E^y\left[\int_0^T f(y_t)\exp-\left(\int_0^t h(y_s)\,ds\right)dt\right] \\ & + E^y\left[g(y_T)\exp-\left(\int_0^T h(y_t)\,dt\right)\right] \end{aligned}$$

where $T = \inf\{t \geqslant 0, y_t \notin D\}$.

Conversely a natural assumption for (0.6) to make sense is :

$$\text{(0.7)} \qquad \text{(H.C)} \quad \sup_{y\in\bar{D}} E^y\left[\int_0^T \exp-\left(\int_0^t h(y_s)\,ds\right)dt\right] < +\infty$$

offently called the (uniform) Gauge condition for the Dirichlet problem. If (H.C) holds and if ∂D is non characteristic, one can prove by probabilistic methods (the martingales methods of Stroock and Varadhan [32]), that u given by (0.6) is a solution of (D.P). In [14] we applied the Malliavin calculus (in the formalism of [3]) and developped a stochastic calculus of variations for the stopping time T (following [11] and [4]) in order to give a probabilistic proof of the smoothness of u. Actually we obtained more.

First of all uniqueness holds in $\mathscr{D}'(\bar{D})$. This is a consequence of the following analogue of Hörmander's result proved in [14] Thm 5.3 :

(0.8) Let U be an open domain of $\bar{D}$, every solution $u \in \mathscr{D}'(\bar{D})$ of (D.P) with $f \in C^\infty(U)$ and $g \in C^\infty(U \cap \partial D)$ is in $C^\infty(U)$.

Seconds, all these results extend to the case of non compact $\bar{D}$, with $\mathscr{S}'(\bar{D})$ (that is the $u \in \mathscr{D}'(\bar{D})$ which extend to $\mathscr{S}'(\mathbb{R}^d)$) in place of $\mathscr{D}'(\bar{D})$. Finally, the Gauge condition (0.7) is, at least when $\bar{D}$ is compact, weaker than the coercivity assumption of Derridj. Notice that a probabilistic study of the Poisson kernel was first done in [1].

In this paper we shall study the boundary value problem (0.4) for a Ventcel's boundary condition, that is $L = v - \Gamma$, with

$$\Gamma = 1/2 \sum_{\ell=1}^{r} V_\ell^2 + \bar{V}_0 + \alpha \frac{\partial}{\partial n} - \rho A .$$

The V^ℓ's, and $\bar{V}_0$ are C_b^∞ vector fields acting on ∂D, v, ρ and α are $C_b^\infty(\partial D)$ functions with ρ and $\alpha \geqslant 0$, and n is the unitary normal relative to ∂D pointed inward.

This condition was introduced by Ventcel [35] in order to characterize the Feller semi groups on $\bar{D}$ generated by A, via the Hille-Yosida theorem (see [8], [28], [29], [33] and the references therein).

When the V^ℓ's $(1 \leqslant \ell \leqslant r)$ and ρ are identically 0, we get the classical oblique derivative problem. If one of the V^ℓ's $(1 \leqslant \ell \leqslant r)$ does not vanish identically, the problem (L.P) (see (0.4)) can fail to be elliptic ([20]).

For such an L, (L.P) is only solved when A is uniformly elliptic, $\bar{D}$ is compact, and Γ is transversal on ∂D, that is

* $\alpha > 0$ (strong transversality) in [8],
* $\alpha + \rho > 0$ (weak transversality) in [28] and [33] ;

of course with some supplementary assumptions on h and v, essentially $h + v > 0$, h and $v \geqslant 0$.

On the other hand the construction of a semi group T_t^h whose generator is $A - h$ on $M^\infty = \{f \in C^\infty(\bar{D}),\ (v-\Gamma)f=0 \text{ on } \partial D\}$, can be done directly by probabilistic methods, without any assumption of ellipticity for A (see [31], [18], [23], [36], [19]).

Namely for $f \in M^\infty$, one has

$$T_t^h f(y) = E^y\left[f(y_t)\exp-\left(\int_0^t h(y_s)\,ds\right)\right],$$

where y is the generic element of $C^0(\mathbb{R}^+, \bar{D})$ and E^y the expectation relative to an appropriate law Q_y on the previous space.

We are interested here in the case when Q_y is the (unique) law solution of the following stochastic differential system of reflection of Stratonovitch (this is not always true, see the references given above):

$$(0.9)\quad \begin{cases} dy_t = 1_D(y_t)\,Y_0(y_t)\,dt_t + 1_D(y_t)\,Y_i(y_t)\,dw_t^i \\ \qquad\qquad + 1_{\partial D}(y_t)\,V_0(y_t)\,dL_t + 1_{\partial D}(y_t)\,V_\ell(y_t)\,dM_t^\ell \\ y_0 = y \\ \rho(y_t)\,dL_t = 1_{\partial D}(y_t)\,dt \end{cases}$$

where $V_0 = \bar{V}_0 + \alpha\dfrac{\partial}{\partial n}$.

The precise meaning of this statement and the definition of the local time L_t are given at § 1.

If the boundary is (uniformly) non characteristic and Γ (uniformly) strongly transversal (see (1.1) in § 1) this is always possible.

The Ito formula gives us the natural candidate u to be the solution of (L.P) that is:

$$(0.10)\quad u(y) = E^y\left[\int_0^\infty f(y_t)\exp-\int_0^t h(y_s)\,ds\cdot\exp-\int_0^t v(y_s)\,dL_s\,dt\right]$$
$$+ E^y\left[\int_0^{+\infty} g(y_t)\exp-\int_0^t h(y_s)\,ds\cdot\exp-\int_0^t v(y_s)\,dL_s\,dL_t\right]$$

and the Gauge condition for (L.P) is then the following

(0.11)
$$\sup_{y \in \bar{D}} E^y\left[\int_0^{+\infty} \exp\left(-\int_0^t h(y_s)\,ds - \int_0^t v(y_s)\,dL_s\right)(dt+dL_t)\right] < +\infty .$$

We show in § 4 that if (1.1), (0.11) and (H.G) are fulfilled in $\bar{D}$, then for C^∞ data f and g, the function u given by (0.10) is in $C^\infty(\bar{D})$, and is the unique solution of (L.P) in $\mathscr{D}'(\bar{D})$ if $\bar{D}$ is compact, or $\mathscr{S}'(\bar{D})$ if it is not (assuming in that case an uniform condition (H.G)).

We now describe the contents of the paper and the relationship between the probabilistic and the p.d.e formalisms.

In § 1 we define (0.9) and recall some basic facts on existence and uniqueness. We also give a construction of the solution well adapted for the sequel. From a geometric point of view the main difference between (0.2) and (0.9) consists in that we cannot represent the solution of (0.9) as the image law of a flow of C^∞ diffeomorphisms. This is because L_t, which depends on the starting point y, is Markovian in y and hence very irregular.

Fortunately if $\rho \equiv 0$ (non sticky case) we can find local charts in which the tangential coordinates form again, for each fixed time, a C^∞ diffeomorphism of $\mathbb{R}^{d-1}$. We also describe in this case the inverse flow with the help of time reversal on the probability space.

In § 2, we study the Q_y law of y_t, denoted by $P(t,y,dx)$. In [9], [10], [11], we have gotten sufficient conditions for $P(t,y,dx)$ to be of the form :

$$P(t,y,dx) = 1_D(x)p(t,y,x)\,dx + 1_{\partial D}(x)\rho(x)p(t,y,x)\,d\mu(x)$$

where $d\mu$ is the area measure on ∂D and $p(t,y,\cdot) \in C^\infty(\bar{D})$ (see (2.4) and (2.6)). We then study the smoothness of $p(t,\cdot,\cdot)$. The main tools are the local stochastic calculus of variations on the reflected process (see [11]) and the expression of the inverse flow. This part is very similar to what we have done in [14] for the stopped process. The relevant results are (2.9) and (2.25).

In § 3 we study the smoothness of u given by (0.10), under the hypothesis (H.G) (see (0.1)). This is done by introducing an auxiliary process on an extended space, satisfying the hypotheses of § 2, and which is related to the initial process by a regular time change and a projection. The time change is clearly smoothed by the integration in dt. We then obtain ((3.4)) the smoothness out of the diagonal of the kernels associated to the operators $(f,0) \to u$ and $(0,g) \to u$. The remarks (3.28) at the end of the paragraph are also of great importance for the sequel.

In § 4 we solve the problem (L.P). We start by proving in § 4.A the analogue of Hormander's result for (L.P), that is if (0.7) is satisfied and (H.G) (see (0.1)) is fulfilled in an open domain U of $\bar{D}$, any solution u ($u \in \mathscr{D}'(\bar{D})$) of (L.P) with regular data $f \in C^\infty(U)$ and $g \in C^\infty(U \cap \partial D)$ belongs to $C^\infty(U)$. ((4.2)).

The method of the proof is the following.

First we show that the harmonic operator H extends continuously to $\mathscr{E}'(\partial D)$ (4.32), then that so does $\frac{\partial}{\partial n} H$. This is done by studying the associated Fourier transform. The key point is the expression of the inverse flow.

Finally we prove that the boundary operator is hypoelliptic, by an approximation procedure and integration by parts as in [13] and [14].

One crucial point in the proof is a theorem of existence and continuity for sectional traces on the boundary ((4.6)). This result is our debt to the p.d.e theory.

Though $(v-\Gamma)\, H$ extends to $\mathscr{E}'(\partial D)$ and has a kernel which is C^∞ out of the diagonal, we haven't succeeded in proving that it is a pseudo differential operator (in the sense of [15]). This point is discussed at the end of § 5.

In § 4.B we show by a standard martingale method that the u defined in (0.10) is a solution of (L.P), and conclude to the existence, uniqueness and regularity of the solution. These results are extended to a non compact domain, and to data (f,g) in certain Sobolev spaces.

In § 4.C, we give some simple conditions for the Gauge hypothesis (0.11), to hold.

The last paragraph is devoted to some complements.

In § 5.A, we introduce the adjoint system (A^*,Γ^*) which arises from the usual Green formula. We then obtain a decomposition result (5.29) in the non sticky case

$$C^\infty(\bar{D}) = (\ker(h-A) \cap M^\infty) \oplus (h-A^*)(M^{*\infty})$$

provided $\bar{D}$ is compact. This result is the analogue of the one we proved in the case without boundary in [13]. The proof lies on the compactness of the resolvant operator in the C^k spaces ($k < +\infty$) (see (5.5)).

This is then applied in § 5.B to the study of the invariant (and reversible) measures of the process y. Since the semi group has the Feller property, the existence of an invariant measure is ensured by the compactness of $\bar{D}$. We explain the relation between the set of invariant measures and $\ker(h-A^*) \cap M^{*\infty}$, and prove that if the Lie algebra generated by the Y^i's, $1 \leqslant i \leqslant m$, is full at each $y \in \bar{D}$, there exists a unique invariant Probability measure (5.35).

Uniqueness is closely related to the strict positivity of the density p of $P(t,y,dx)$. This property is obtained by using some elementary large deviations ideas. In that case ergodicity holds.

In § 5.C the connection between the spectrum of the generator $(h-A)$ and the Gauge condition is explained.

Finally in § 5.D we recall the connection between the boundary operator and the boundary process $y_{A.}$, where $A.$ is the right continuous inverse of the local time L. Let we say that the law of the boundary process has been studied by J. M. Bismut [4] (also see [19]). The hypoellipticity of $(v-\Gamma)H$ can certainly be proved by using the results of [4].

The results of the paper are closely related to [8], [29], and [33] where the analytical arsenal is developped in the uniformly elliptic case for compact domains. At the present stage this arsenal seems to fail to give the analogous results in the degenerate case. The present paper can be viewed as the conclusion of the work started in [13] and [14].

I am very indebted to J. M. Bismut who has initiated me into the intricacies of the Malliavin calculus. I also wish to thank C. Leonard for the time he has spended on teaching me the very little I know about large deviations and an anonymous referee for helpful criticism.

Notation. – For $0 \leqslant k \leqslant +\infty$, C^k, C^k_b, C^k_0 are respectively the spaces of k-times continuously differentiable functions, C^k functions with bounded derivatives up to order k, and C^k functions with compact support.

$\mathscr{D}'(\bar{D})$, $\mathscr{E}'(\bar{D})$, $H^k(\bar{D})$, are the spaces of distributions in D which extend to elements of $\mathscr{D}'(\mathbb{R}^d)$, $\mathscr{E}'(\mathbb{R}^d)$, $H^k(\mathbb{R}^d)$ (see e.g [15] for more details).

If Y is a C^∞ vector field and f a function, Yf denotes the action of Y on f.

If F is a C^∞ diffeomorphism of $\mathbb{R}^d$, F^*Y (resp $F^{*-1}Y$) is the vector field defined by

$$(F^*Y)(y) = (dF \,.\, Y)(F^{-1}(y))$$
$$(\text{resp. } F^{*-1}Y)(y) = (dF)^{-1}(y)\, Y(F(y)))$$

and is called the push-forward (resp. the pull-back) of Y by F.

If M_t is a continuous L^2 martingale, dM_t (resp. δM_t) denotes its differential in the sense of Stratonovitch (resp. Ito).

Throughout this work all the considered filtrations are supposed to fulfill the usual hypotheses, and we use the summation convention

$$\sum_i a_i b_i = a_i b^i = a^i b_i .$$

To help the reader, we add an index of some notation which we will use throughout the paper.

Notation	Meaning	Introduced in
(H.G)	General Hörmander's hypothesis	(0.1), (2.1)
(H.R)	Restricted Hörmander's hypothesis	(0.3), (2.1)
(H.F)	Special Hörmander's hypothesis	(2.2)
(H. ?.unif)	Uniform Hörmander's hypothesis	(2.1), (2.2)
(H.S)		(2.4)
(H.C)	« Coercivity assumption »	(0.7)
(0.11)	Gauge condition	(0.11), (3.2)
(D.P)	General Dirichlet problem	Introduction
(P.P)	Dirichlet problem	(4.8)
(L.P)	Boundary value problem	(0.4)
(B.P)	Boundary problem	(0.5)
$R^{h,v}$	Resolvant operator	(3.3)

1. DIFFUSIONS WITH A BOUNDARY CONDITION

Let D be an open regular set of $\mathbb{R}^d$, defined as $D = \{\psi(x) > 0\}$, where $\psi \in C_b^\infty(\mathbb{R}^d)$. ∂D is locally a C^∞ hypersurface defined by $\partial D = \{\psi(x)=0\}$. $\bar{D}$ denotes the closure of D.

We assume that for every $x \in \partial D$, $|\nabla\psi(x)| = 1$ so that $\nabla\psi(x) = n(x)$ is the unitary inward normal at $x \in \partial D$.

Let we consider

– $Y_0, Y_1, \ldots, Y_m$ $m + 1$ C_b^∞ vector fields defined on $\mathbb{R}^d$ with values in $\mathbb{R}^d$.

– $V_0, V_1, \ldots, V_r$ $r + 1$ C_b^∞ vector fields defined on ∂D with values in $\mathbb{R}^d$.

– $\rho \in C_b^\infty\ (\partial D, \mathbb{R}^+)$.

By Seeley's theorem [15], one can always suppose that the V_ℓ's, $0 \leqslant \ell \leqslant r$, and ρ are the restrictions to ∂D of C_b^∞ functions defined on the whole space $\mathbb{R}^d$, denoted again by V_l and ρ.

We then define the following second order differential operators

$$(1.0)\qquad \begin{cases} A = 1/2 \sum_{i=1}^{m} Y_i^2 + Y_0 \\ \Gamma = 1/2 \sum_{\ell=1}^{r} V_\ell^2 + V_0 - \rho A. \end{cases}$$

Furthermore we assume that

$$(1.1):\ \text{for any } x \in \partial D \quad \begin{cases} \text{i) } (V_\ell\psi)(x) = 0 \quad \text{for} \quad \ell = 1, \ldots, r \\ \text{ii) } (V_0\psi)(x) \geqslant v_0 > 0 \\ \text{iii) } a^2(x) = \sum_{i=1}^{m} (Y_i\psi)^2(x) \geqslant a > 0 \end{cases}$$

i) means that the vector fields V_ℓ act on ∂D,

ii) is a transversality assumption on V_0 and,

iii) means that ∂D is uniformly non characteristic.

It is then known that there exists for each $y \in \bar{D}$ a unique solution to the stochastic differential system of reflexion of Stratonovitch (see [18], [23]).

$$(0.9)\quad \begin{cases} \text{i) } dy_t = 1_D(y_t)\, Y_0(y_t)\, dt + 1_D(y_t)\, Y_i(y_t)\, dw_t^i \\ \qquad\qquad + 1_{\partial D}(y_t)\, V_0(y_t)\, dL_t + 1_{\partial D}(y_t)\, V_\ell(y_t)\, dM_t^\ell \\ \text{ii) } y_0 = y \\ \text{iii) } \rho(y_t)\, dL_t = 1_{\partial D}(y_t)\, dt \end{cases}$$

that is one can find a stochastic process $(\Omega, F_t, P_y, y_t, L_t, w_t, M_t)$ satisfying

(1.2)
i) y_t is an F_t adapted continuous process with values in $\bar{D}$,

ii) L_t is an F_t adapted non decreasing and continuous process with $L_0 = 0$ and P_y a.s for all $t \geqslant 0$, $\int_0^t 1_{\partial D}(y_s)\, dL_s = L_t$,

iii) w_t(resp. M_t) is a $(F_t, P_y) L^2$ continuous martingale with values in $\mathbb{R}^m$(resp. $\mathbb{R}^r$) whose brackets are given by $\langle w^i, w^j \rangle_t = \delta_{ij} t$; $\langle M^i, M^j \rangle_t = \delta_{ij} L_t$; $\langle w^i, M^j \rangle_t = 0$,

iv) the stochastic differential in the sense of Stratonovitch of y_t is given by (0.9) i) ; $y_0 = y$, and P a.s for all $t \geqslant 0$,

$$\int_0^t \rho(y_s)\, dL_s = \int_0^t 1_{\partial D}(y_s)\, ds .$$

(1.3) *Notation.* – We denote by Q_y the family of the P_y's laws of y. So Q_y is a strong Markov family of probabilities on $C^0(\mathbb{R}^+, \bar{D})$, we shall call following [23] an (A,Γ)-diffusion.

Remark that there is one degree of freedom in the definition of the operator Γ. Indeed if we define, for $h \in C_b^\infty(\partial D)$ with $h(x) \geqslant h_0 > 0$

$$(1.4)\quad L_t' = \int_0^t h(y_s)\, dL_s \quad \text{and} \quad M_t' = \int_0^t h^{1/2}(y_s)\, \delta M_s$$

the stochastic process $(\Omega, F_t, P_y, y_t, L_t', w_t, M_t')$ induces an $(A, \Gamma/h)$-diffusion.

The different terms can be interpreted in the following way:

* A is the generator of a diffusion (with a drift) in D,

* $1/2 \sum_{\ell=1}^{r} V_\ell^2 + \bar{V}_0$, with $\bar{V}_0 = V_0 - (V_0\psi)n$, is the one of a diffusion (with a drift) on ∂D,

* $(V_0\psi)n$ is a reflexion coefficient at the boundary and ρ a viscosity coefficient.

We shall now recall a construction of y_t, on which our study is based. First if y_t induces an (A,Γ)-diffusion, define

$$(1.5)\quad \bar{\gamma}_t = \int_0^t 1_D(y_s)\,ds\,,\ \text{and}\ \bar{\eta}_t = \bar{\gamma}_t^{-1}\ \text{its right continuous inverse.}$$

Then $y_{\bar{\eta}_t}$ induces an (A,Γ^0)-diffusion, i.e with $\rho \equiv 0$. Conversely if y_t^0 induces an (A,Γ^0)-diffusion, define

$$(1.6)\qquad \gamma_t = t + \int_0^t \rho(y_s)\,dL_s\,,\qquad \text{and}\qquad \eta_t = \gamma_t^{-1}$$

then $y^0_{\eta_t}$ induces an (A,Γ)-diffusion (cf. [23] p. 212-214).

Hence we are reduced to build a solution of (0.9) with $\rho \equiv 0$. We start with the following definition (see [1]).

(1.7) DEFINITION. – *Let $x \in \partial D$ and (U,F) a local chart at x. (U,F) will be called a good chart if*

i) *F is a C^∞ diffeomorphism from $\bar{U}$ onto $\bar{V} \subset \mathbb{R}^+ \times \mathbb{R}^{d-1}$*

ii) *If $Y_i' = F^* Y_i$ and $A' = 1/2 \sum_{i=1}^{m} Y_i'^2 + Y_0'$, then A' is written in $\bar{V}$ as*

$$1/2 \frac{\partial^2}{\partial x_1^2} + b \frac{\partial}{\partial x_1} + 1/2 \sum_{i=2}^{m} X_i^2 + X_0$$

where $b \in C_b^\infty(\mathbb{R}^d)$ and $X_i = \sum_{j=2}^{d} X_i^j \frac{\partial}{\partial x_j}$, are C_b^∞ vector fields defined on $\mathbb{R}^d$ with values in $\mathbb{R}^{d-1}$ for $i = 2, \ldots, m$.

From a probabilistic point of view, (1.7) means the following. Let $y' = (u,x) = F(y)$. Then, up to the first exit time out of $\bar{V}$, y_t' solves the system

$$(1.8)\quad \begin{cases} du_t = dw_t^1 + b(y_t')\,dt + dL_t \\ dx_t = X_0(y_t')\,dt + \sum_{i=2}^{m} X_i(y_t')\,dw_t^i + S_0(x_t)\,dL_t + \sum_{\ell=1}^{r} S_\ell(x_t)\,dM_t^\ell \\ 1_{\partial D}(y_t')\,dt = 0\,,\ y_t' = (u_t,x_t) \in R^+ \times \mathbb{R}^{d-1} \end{cases}$$

where $\begin{pmatrix}0\\ S_\ell\end{pmatrix} = F^* V_\ell$, $\ell = 1,\dots,r$ (see (1.1)i)); after choosing $(F^* V_0)^1 \equiv 1$ in virtue of (1.4).

We then have

(1.9) PROPOSITION. — *There exists* $\varepsilon > 0$ *such that for all* $x \in \partial D$ *we can find* F_x *such that* $(B(x,\varepsilon) \cap \bar D, F_x)$ *is a good local chart at* x. *Moreover*

$$\sup_{x \in \partial D} \left(\|F_x\|_{C_b^k(\bar B(x,\varepsilon)\cap \bar D)} \vee \|F_x^{-1}\|_{C_b^k(Fx(\bar B(x,\varepsilon)\cap \bar D))}\right) \leqslant M(k) < +\infty\,.$$

Proof. — See [14] Thm 4.5 and lemma 5.18. □

ε being chosen as in (1.9), in order to build the (A,Γ^0)-diffusion, we are reduced to build the A diffusion in D, and the solutions of (1.8) in a countable locally finite family of good local charts and then to piece together these processes using the methods of [18]. Thanks to (1.9), to classical estimates on exit times (for instance [13] 1.6) and to the Borel-Cantelli lemma, one shows that the so obtained process has an infinite explosion time.

We go back now to (1.8). Consider $\Omega = \Omega' \times \bar\Omega \times \Omega^*$ with $\Omega' = C^0(\mathbb{R}^+,\mathbb{R})$, $\bar\Omega = C^0(\mathbb{R}^+,\mathbb{R}^{m-1})$, $\Omega^* = C^0(\mathbb{R}^+,\mathbb{R}^r)$. The path of $(\omega',\bar\omega,\omega^*) = \omega \in \Omega$ is denoted by $(B\cdot, w^2_\cdot,\dots,w^m_\cdot, w^{*1}_\cdot,\dots,w^{*r}_\cdot)$ and $P = P' \otimes \bar P \otimes P^*$ is the Wiener measure on Ω. For $u \in \mathbb{R}$ we define

$$(1.10)\quad \begin{cases} z_t(u) = u + B_t\,; \\ u_t(u) = |z_t(u)| = |u| + \int_0^t \operatorname{sign}(z_s(u))\,\delta B_s + L_t^u \\ \qquad\quad = |u| + w_t^1(u) + L_t^u \\ M_t(u) = w^*_{L_t^u}\,. \end{cases}$$

F_t is the right continuous P completed filtration generated by B_s, w_s, $0 \leqslant s \leqslant t$, and w_s^*, $0 \leqslant s \leqslant L_t^u$.

Consider the stochastic differential system of Stratonovitch

$$(1.11)\quad \begin{cases} dx_t = X_0(u_t,x_t)\,dt + X_i(u_t,x_t)\,dw_t^i + S_0(x_t)\,dL_t^u + S_\ell(x_t)\,dM_t^\ell(u) \\ x_0 = x\,. \end{cases}$$

Following [24], for all fixed $(u,\omega') \in \mathbb{R} \times \Omega'$, one can construct a version of the P essentially unique solution of (1.11) which is a flow of C^∞ diffeomorphisms of $\mathbb{R}^{d-1}$. Let $\phi_t(\omega,u,x)$ be this process. We describe the flow of the inverses. From [24], $\phi_t^{-1}(\omega,u,x)$ is solution of the backward stochastic differential system

$$(1.12)\quad \begin{cases} \hat{d}x_s = -X_0(u_{t-s},x_s)\,ds - X_i(u_{t-s},x_s)\,\hat{d}w_s^i - S_0(x_s)\hat{d}L_s - S_\ell(x_s)\,\hat{d}M_s^\ell \\ x_0 = x\,. \end{cases}$$

Consider the transformation on $(\Omega,\underline{F}_t)$ defined by

$$(1.13)\quad \omega \to \tilde{\omega}^t = (s \to B_t - B_{t-s}\,;\, s \to w_t - w_{t-s}\,;\, s \to w^*_{L_t^u} - w^*_{L_{t-s}^u}) = (\tilde{B}_.,\tilde{w}_.,\tilde{w}_.^*)\,.$$

Since L_t^u only depends on ω', and since ω' and ω^* are independent, P is invariant under $\omega \to \tilde{\omega}^t$.

Define for $u \in \mathbb{R}$

$$(1.14)\quad \begin{cases} \tilde{z}_s(u) = u - \tilde{B}_s\,; \qquad \tilde{u}_s = |\tilde{z}_s| = |u| + \tilde{w}_s^1 + \tilde{L}_s^u \\ \tilde{M}_s = \tilde{w}^*_{\tilde{L}_s^u}|. \end{cases}$$

It is known (see [38]) that

(1.15) Conditionally to $z_t(u) = u'$ we have $\tilde{L}_s^{u'} = L_t^u - L_{t-s}^u$ and consequently, $\tilde{M}_s = M_t - M_{t-s}$, P a.s.

We can now introduce the flow $\tilde{\phi}_t(\omega,u,x)$ solution, with (u,ω') fixed of

$$(1.16)\quad \begin{cases} dx_t = -X_0(u_t,x_t)\,dt - X_i(u_t,x_t)\,dw_t^i - S_0(x_t)\,dL_t^u - S_\ell(x_t)\,dM_t^\ell \\ x_0 = x\,. \end{cases}$$

One deduces from what precedes, that for fixed t, P a.s.

$$(1.17)\quad \begin{cases} \phi_t^{-1}(\omega,u,\cdot) = \tilde{\phi}_t(\tilde{\omega}^t,-z_t(u),\cdot) \\ \phi_s(\omega,u,\cdot) = \tilde{\phi}_{t-s}(\tilde{\omega}^t,-z_t(u),\cdot) \circ \phi_t(\omega,u,\cdot) \quad \text{for} \quad s \leqslant t \\ \tilde{\phi}_{t-s}(\tilde{\omega}^t,-z_t(u),\cdot) = \phi_s(\omega,u,\cdot) \circ \tilde{\phi}_t(\tilde{\omega}^t,-z_t(u),\cdot) \quad \text{for} \quad s \leqslant t\,. \end{cases}$$

This result is very similar to the one obtained in the case without boundary (cf. [2]).

Finally we introduce the Girsanov martingale

$$(1.18)\quad G_t^b(u,x) = \exp\left\{\int_0^t b(u_s, \phi_s(\omega,u,x))\, \delta w_s^1(u) - 1/2\int_0^t b^2(u_s, \phi_s(\omega,u,x))\, ds\right\}$$

and $P^b = G_t^b P$ on F_t.

Then the P^b law of $(u_t, \phi_t(\omega,u,x))$ for $u \geqslant 0$, is the law solution of (1.8).

The construction of the (A,Γ)-diffusion is achieved.

Remark. – The construction of the flow ϕ given here differs from the ones given in [4] or [9], [10], [11]. It seems that this construction is the most convenient to express the inverse flow.

Finally recall that the local time L_t is the density of occupation time at ∂D, that is

(1.19) PROPOSITION. – *Let y_t be an (A,Γ)-diffusion with local time L_t. For $\varepsilon > 0$, and $D_\varepsilon = \{0 < \psi(x) < \varepsilon\}$, we have*

$$\lim_{\varepsilon \to 0} 1/\varepsilon \int_0^t 1_{D_\varepsilon}(y_s) a^2(y_s)\, ds = \int_0^t (V_0\psi)(y_s)\, dL_s$$

both in L^2 and almost surely, the convergence being uniform in t on $[0,T]$ for all $T \geqslant 0$.

2. SMOOTHNESS OF THE LAWS

In [9], [10], [11] we studied the problem of the existence of smooth densities for the laws of the diffusions with a boundary condition. We recall here the main results we obtained and complete them by showing the regularity in the starting point. We begin with some definitions.

(2.1) DEFINITION (cf. [13] 1.2). — *For* $x \in \mathbb{R}^d$, *let* $L'_N(x)$ *be the set of the vectors* $Y_i(x)$, $1 \leqslant i \leqslant m$, *and of the Lie brackets of length* $\leqslant N$ *of the vector fields* Y_i, $0 \leqslant i \leqslant m$, *at* x *;* $L_N(x) = L'_N(x) \cup Y_0(x)$.

Recall the Hörmander's assumptions

— (H.G) *is satisfied at* x *if there exist* $N(x)$ *and* $c(x) > 0$ *such that for all* $z \in \mathbb{R}^d$ $\sum_{Y \in L_{N(x)}(x)} \langle Y,z \rangle^2 \geqslant c(x) \|z\|^2$ *i.e if* Lie $(Y_i(x);0 \leqslant i \leqslant m)$ *spans* $\mathbb{R}^d$ *at* x.

— (H.R) *is the analogue of* (H.G) *just replacing* L_N *by* L'_N.

— (H.G unif) (*resp.* (H.R unif)) *is satisfied on* $U \subset \mathbb{R}^d$ *if there exist* $N(U)$ *and* $c(U) > 0$ *with* $N(U) \geqslant N(x)$ *and* $c(U) \leqslant c(x)$ *for all* $x \in U$.

In particular if (H..) *is satisfied at each point of a compact set* K, (H... unif) *is satisfied on* K.

(2.2) DEFINITION. — *Let* $x \in \partial D$, X_0, $X_2, \ldots, X_m$ *are the vector fields defined in* (1.6) ii). *Let* $L''_N(x)$ *be the set of the vector fields* $\frac{\partial}{\partial x_1}, X_2, \ldots, X_m$ *and of the Lie brackets of length* $\leqslant N$ *of the vector fields* $\frac{\partial}{\partial x_1}, X_0, X_2, \ldots, X_m$ *where at least one of the* X_i, $2 \leqslant i \leqslant m$, *appears, at* x. *We say that*

(H.F) *is satisfied at* x *if there exist* $N(x)$ *and* $c(x) > 0$ *such that for all* $z \in \mathbb{R}^d$, $\sum_{Y \in L''_{N(x)}(x)} \langle Y,z \rangle^2 \geqslant c(x) \|z\|^2$

(H.F unif) *is defined in the same way as* (H.G unif).

(2.3) DEFINITION (cf. [1], [12] 1.5 and [14] 3.22). — *Let* $x \in \partial D$ *and* (U,F) *be a local chart at* x. (U,F) *is said to be a very good local chart if it is a good local chart (see* (1.7)) *and if* (H.F unif) *is satisfied in* U.

We shall say that ∂D *is very good if for every* $x \in \partial D$, *one can find a very good local chart at* x.

We can state

(2.4) THEOREM (cf. [12] 1.22). — *Let* $P(t,y,dx)$ *be the* P_y *law of* y_t. *We assume that* (H.R) *holds* (*at* y), *that* ∂D *is very good and that* V_0 *is normalized so that* $V_0\psi = 1/2\, a^2$ *on* ∂D. *Under one of the supplementary* *assumptions*

$$\text{(H.S)}\quad \begin{cases} \text{i) } \rho \textit{ is constant (eventually } 0), \\ \text{ii) } V_\ell \equiv 0,\ 1 \leqslant \ell \leqslant r,\ \textit{(oblique derivative condition), and} \\ \quad \rho(x) > 0 \textit{ for all } x \in \partial D, \end{cases}$$

there exists $p(t,y,x)$ *defined on* $\mathbb{R}_+^* \times \bar{D}$ *with values in* $\mathbb{R}^+$, *which satisfies*

$$(2.5)\quad \begin{cases} \text{i) } x \to p(t,y,x) \textit{ belongs to } C^\infty(\bar{D}), \\ \text{ii) } (t,x) \to \dfrac{\partial^\alpha}{\partial x^\alpha} p(t,y,x) \textit{ is continuous on } \mathbb{R}_*^+ \times \bar{D} \textit{ for all} \\ \quad \textit{multi index } \alpha, \\ \text{iii) } \textit{For all } t > 0 \\ \quad P(t,y,dx) = 1_D(x)p(t,y,x)\,dx + 1_{\partial D}(x)\rho(x)p(t,y,x)\,d\mu \\ \quad \textit{where } d\mu \textit{ is the area measure on } \partial D \textit{ associated to the} \\ \quad \textit{Leray form } \mu \textit{ defined by : } d\psi \wedge \mu = dx^1 \wedge \cdots \wedge dx^d. \end{cases}$$

The smoothness of p in the interior of the domain is shown by using the local calculus of variations (cf. [10] 1.15), the C^∞ extension up to the boundary is the aim of [11] in case (H.S) i). The existence of a smooth density on ∂D is shown in [10] § VI. Finally the C^∞ extension up to ∂D in case (H.S) ii) is proved in [12] 1.22. The identification of the boundary density is an easy consequence of (1.19) (cf. [12] 1.10).

Theorem (2.4) can be localized as follows :

(2.6) THEOREM. — *Assume that* (H.R unif) *is satisfied on* $U \subset \bar{D}$, *that* $U \cap \partial D$ *is very good, everything else being as in* (2.4). *Then for every* $y \in \bar{D}$ *and* $t > 0$, (2.5) *holds if we replace* $\bar{D}$ *and* ∂D *by* U *and* $U \cap \partial D$.

We are now interested in the regularity of p in both y and x. According to what was done in [14] 3.35 and 4.37 we shall use the stochastic calculus of variations on the inverse flow. We start by some notation.

(2.7) $\mathscr{C} = C^0(\mathbb{R}^+,\bar{D})$ equipped with the cylindrical σ algebra and $(Q_y)_{y \in \bar{D}}$, so that $P(s,y,dx)$ is the semi group associated to the canonical process $s \to y_s$.

(2.8) On $\mathscr{W} = C^0(\mathbb{R}^+, \mathbb{R}^m)$ equipped with its canonical Wiener structure, one builds the flow $\xi_t(w,y)$ solution of

$$dy_t = Y_0(y_t)\,dt + Y_i(y_t)\,dw_t^i;\; y_0 = y.$$

We can state the first result :

(2.9) THEOREM. – *Let U and V two open sets of D. We assume that* (H.R unif) (*see* (2.1)) *is satisfied both on $\bar{U}$ and $\bar{V}$, that $\bar{V} \cap \partial D$ is very good and that* (H.S) (*see* (2.5)) *holds. Then for all* $t > 0$, $(y,x) \to p(t,y,x) \in C^\infty(U \times \bar{V})$.

Assume furthermore that $\bar{U} \cap \partial D$ is very good and that $\rho \equiv 0$. Then $p(t,\cdot,\cdot) \in C^\infty(\bar{U} \times \bar{V})$.

Proof. – In order to prove the first part of the theorem, it is enough to prove that for any compact set K in U, any $f \in C_0^\infty(U)$ with support in K, any $g \in C_0^\infty(\bar{D})$ with support in $\bar{V}$ and any multiindices α and β, the following holds (see e.g. [13], [14])

$$(2.10)\qquad \left|\int \partial^\alpha f(y)\partial^\beta g(x) P(t,y,dx)dy\right| \leqslant c(\alpha,\beta,t)\,\|f\|_\infty \|g\|_\infty.$$

If $2\varepsilon = d(K,\partial D)$ and $S = \inf\{s \geqslant 0, d(y_s,K) \geqslant \varepsilon\}$, we may apply the Markov property at time $T = 1/2\,(t \wedge S)$. Therefore

$$\begin{aligned}(2.11)\qquad &\int \partial^\alpha f(y)\,\partial^\beta g(x) P(t,y,dx)\,dy\\ &= \int \partial^\alpha f(y)\,\partial^\beta g(x) P(T,y,dz) P(t-T,z,dx)\,dy\\ &= \int \partial^\alpha f(y) E^w\left[\int \partial^\beta g(x) P(t-T(y), \xi_{T(y)}(w,y), dx)\right] dy\end{aligned}$$

where $S(y) = \inf\{s \geqslant 0, d(\xi_s(w,y),K) \geqslant \varepsilon\}$ and $T(y) = 1/2\,(t \wedge S(y))$.

Indeed starting from $y \in U$ and before to leave U, the Q_y law of y and the law of $\xi_{\cdot}(w,y)$ are the same. On the other hand we can write for all $y \in K$ and almost all w

$$1 = 1_{S(y) > t/2} + \sum_{n=2}^{\infty} 1_{t/(n+1) < S(y) \leqslant t/n}.$$

For simplicity we write

$$(2.12)\qquad S_{n+1} = \inf\{s \geqslant 0, d(\xi_s(w', \xi_{t/n+1}(w,y)), K) \geqslant \varepsilon\}.$$

Therefore (2.11) becomes

$$(2.13)\quad \int \partial^\beta f(y)\, \partial^\beta g(x) P(t,y,dx)\, dy = \sum_{n=2}^{+\infty} \int \partial^\alpha f(y) E^w\Big[1_{t/n+1 < S(y)} E^{w'}\Big[1_{S_{n+1} \leqslant t/(n+n^2)} \int \partial^\beta g(x) P((nt/n+1) - S_{n+1}, \xi_{S_{n+1}}(w', \xi_{t/n+1}), dx) \Big]\Big] dy + \int \partial^\alpha f(y) E^w\Big[1_{t/2 < S(y)} \int \partial^\beta g(x) P(t/2, \xi_{t/2}(w,y), dx) \Big] dy .$$

For fixed (w,w') (resp. w) we make the change of variable $y \to \xi_{(t/n+1)}(w,y)$ (resp. $y \to \xi_{t/2}(w,y)$).

Remember that if $\tilde{\xi}\cdot(w,y)$ is the flow solution of

$$d\tilde{\xi}_t = -\, Y_0(\tilde{\xi}_t)\, dt - Y_i(\tilde{\xi}_t)\, dw_t^i ; \qquad \tilde{\xi}_0 = y$$

and if

$$\tilde{w}_s^n = w_{t/n+1} - w_{(t/n+1)-s}, \qquad 0 \leqslant s \leqslant t/n + 1$$

then

$$\tilde{\xi}_{(t/n+1)-s}(\tilde{w}^n, \cdot) \circ \xi_{t/n+1}(w, \cdot) = \xi_s(w, \cdot) \quad \text{for} \quad 0 \leqslant s \leqslant t/n + 1 \text{ a.s.}$$

Moreover the Wiener measure on $\mathscr{W}$ is still unchanged if we change w into $\tilde{w}^n$. As in [14] 3.35, we then deduce that the above (2.13) is equal to

$$(2.14)\quad = \sum_{n=2}^{\infty} \int E^w[\partial^\alpha f(\tilde{\xi}_{t/n+1}(w,y)) 1_{t/n+1 < \tilde{S}(y)} \text{ Jacobian } \tilde{\xi}_{t/n+1}(w,y)] E^{w'}\Big[1_{S(y) \leqslant t/(n+n^2)} \int \partial^\beta g(x) P((nt/n+1) - S(y)), \xi_{S(y)}(w',y), dx) \Big] dy + \int E^{w[\partial\alpha} f(\tilde{\xi}_{t/2}(w,y)) 1_{t/2 < \tilde{S}(y)} \text{ Jacobian } \tilde{\xi}_{t/2}(w,y)\, \partial^\beta g(x)\, P(t/2, y, dx)]\, dy$$

where $\tilde{S}(y) = \inf\{s \geqslant 0, d(\tilde{\xi}_s(w,y), K) \geqslant \varepsilon\}$.

Remark that since $\tilde{\xi}_0 = \text{id}$, Jacobian $\tilde{\xi}_s > 0$ for all $s \geqslant 0$.

It remains now to integrate by parts in w and in x, and to add in n. To this end we remark first that since $\tilde{S}(y) > 0$, the integration in y holds in $K_\varepsilon = \{y, d(y,K) \geqslant \varepsilon\}$ which is compact. We apply the local calculus of variations in $E^{w'}$ (cf. [13]). Since $\operatorname{supp}.f \subset K$, we obtain for $n \geqslant 1$

$$(2.15)\quad \begin{cases} \text{i)} \sup\limits_{y \in K_{\varepsilon/2}} |E^{w}[\partial^{\alpha} f(\tilde{\xi}_{t/n+1}(w,y))1_{(t/n+1) < \tilde{S}(y)} \text{ Jacobian } \tilde{\xi}_{t/n+1}(w,y)]| \\ \qquad \leqslant c(\alpha,\varepsilon,t)(n+1)^{k(|\alpha|+d/2)}\|f\|_{\infty} \\ \text{ii)} \sup\limits_{y \in K_{\varepsilon}-K_{\varepsilon/2}} |E^{w}[\partial^{\alpha} f(\tilde{\xi}_{t/n+1}(w,y))1_{(t/n+1) < \tilde{S}(y)} \text{ Jacobian } \tilde{\xi}_{t/n+1} \\ (w,y)]| \\ \qquad \leqslant c(\alpha,\varepsilon,t)\left(\dfrac{n+1}{t} \wedge \dfrac{1}{\varepsilon}\right)^{k(|\alpha|+d/2)} \|f\|_{\infty} \exp - c((n+1)\varepsilon^2/t) \end{cases}$$

where k is a positive constant only depending on the vector fields. The exponential control in ii) is a consequence of the following fact : if $d(y,K) \geqslant \varepsilon/2$, $t/n+1$ must be greater than the first time the process hits K, for $\partial^{\alpha} f$ not to vanish (cf. [13] 1.6).

We then apply the local techniques of [10], [11] to $P(\cdot,\cdot,dx)$. Since $(nt/n+1) - S(y) \geqslant t/2$ on $S(y) \leqslant t/(n+n^2)$, we obtain

$$(2.16)\quad \begin{cases} \text{i)} \sup\limits_{y \in K_{\varepsilon/2}} |E^{w'}[1_{S(y) \leqslant t/n+n^2} \displaystyle\int \partial^{\beta} g(x) P((nt/n+1) - S(y), \\ \xi_{S(y)}(w',y),dx)]| \\ \qquad \leqslant c(\beta,\varepsilon,t)\|g\|_{\infty} \sup\limits_{y \in K_{\varepsilon/2}} P(S(y) \leqslant t/n+n^2) \\ \qquad \leqslant c(\beta,\varepsilon,t)\|g\|_{\infty} \exp - (c(n^2+n)\varepsilon^2/t). \\ \text{ii)} \sup\limits_{y \in K_{\varepsilon}-K_{\varepsilon/2}} |E^{w'}[1_{S(y) \leqslant t/n+n^2} \displaystyle\int \partial^{\beta} g(x) P((nt/n+1) - S(y), \\ \qquad \xi_{S(y)}(w',y),dx)]| \leqslant c(\beta,\varepsilon,t)\|g\|_{\infty}. \end{cases}$$

Hence we can make the summation in n, which proves (2.10) and the first part of (2.9).

It remains to study the existence of a C^{∞} extension up to $(\bar{U} \cap \partial D) \times \bar{V}$. Indeed $y \to Q_y$ is continuous (cf. [18]), hence such an extension necessarily coincides with the a priori defined density $p(t,y,\cdot)$ of $P(t,y,dx)$ for $y \in \bar{U} \cap \partial D$. To show the existence of such an extension if suffices to work in a very good local chart (B,F) at $y \in \bar{U} \cap \partial D$, if such a chart exists, and to prove (2.10) with $f \in C_0^{\infty}(B)$. Define

$$B_{\varepsilon} = \{z \in B, d(z,\partial B) > \varepsilon\}.$$

For ε small enough, $B_{2\varepsilon}$ is a neighborhood of y and we can restrict us to $f \in C_0^\infty(B_{2\varepsilon})$. Let

$$S = \inf\{s \geqslant 0, d(y_s, B_{2\varepsilon}) \geqslant \varepsilon\} \quad \text{and} \quad T = 1/2\,(t \wedge S).$$

$\phi_t(\omega,u,x)$ is the flow builded in § 1, for the X_i's given by (1.7).

We define naturally

$$\text{(2.17)}\quad \begin{aligned} S(u,x) &= \inf\{s \geqslant 0,\ (u_s(u), \phi_s(\omega,u,x)) \notin F(B_\varepsilon)\}\,; \\ T(u,x) &= 1/2(t \wedge S(u,x)). \end{aligned}$$

In order to prove the second part of the Theorem, it is enough to prove (2.10) with $\operatorname{supp} f \subset (B_{2\varepsilon} \cap D)$, for a very good relatively compact local chart (B,F) at $y \in \bar{U} \cap \partial D$.

(2.18) From now we assume that $\rho \equiv 0$ and that $\bar{U} \cap \partial D$ is very good.

We may write

$$\text{(2.19)}\quad \int \partial^\alpha f(y)\, \partial^\beta g(z) P(t,y,dz)\, dy = \int_{\mathbb{R}^+ \times \mathbb{R}^{d-1}} \partial^\alpha f(F^{-1}(u,x))\, |\text{Jacobian } F^{-1}(u,x)|$$
$$E^{P^b}\left[\int \partial^\beta g(z) P(t - T(u,x), F^{-1}(u_{T(u,x)}, \phi_{T(u,x)}(\omega,u,x), dz)\right] du\, dx.$$

One extends every $h \in C_b^0(\mathbb{R}^+ \times \mathbb{R}^{d-1})$ to $\bar{h} \in C_b^0(\mathbb{R}^d)$ by

$$\bar{h}(u,x) = h(|u|,x) \quad \text{for} \quad u \in \mathbb{R}.$$

Since the P laws of

$$(u\cdot(-u), \phi\cdot(\omega,-u,x), G^{\bar{b}}\cdot(-u,x)) \quad \text{and} \quad (u\cdot(u), \phi\cdot(\omega,u,x), G^{\bar{b}}\cdot(u,x))$$

are the same, (2.19) is equal to

$$\text{(2.20)}\quad = 1/2 \int_{\mathbb{R}^+ \times \mathbb{R}^{d-1}} \partial^\alpha f((\bar{F})^{-1}(u,x))\, |\overline{\text{Jacobian } F^{-1}(u,x)}|$$
$$E^P\left[G^{\bar{b}}_{T(u,x)} \int \partial^\beta g(z) P(t - T(u,x), (\bar{F})^{-1}(u_{T(u,x)}, \phi_{T(u,x)}), dz)\right] du\, dx$$

after extending F^{-1} and Jacobian F^{-1} into elements of $C_b^0(\mathbb{R}^+ \times \mathbb{R}^{d-1})$. As in (2.12) we put

$$S_{n+1} = \inf\{s \geqslant 0, (u_s(\omega', z_{t/n+1}(u)), \phi_s(\omega', z_{t/n+1}(u), \phi_{t/n+1}(u,x))) \notin F(B_\varepsilon)\}$$

which allows to rewrite (2.20) into the form

$$(2.21) \quad = 1/2 \sum_{n=2}^{+\infty} \int \partial^\alpha((\bar{F})^{-1}(u,x)) \,|\overline{\text{Jacobian}\,(\bar{F})^{-1}(u,x)}|$$
$$E^{P^{\bar{b}}}[1_{S(u,x) > t/n+1} E^{P^{\bar{b}}}[1_{S_{n+1} > t/n+2} \int \partial^\beta g(z)$$
$$P((nt/n+1) - S_{n+1}, \bar{F}^{-1}(u_{S_{n+1}}(\omega', z_{t/n+1}(u)),$$
$$\phi_{S_{n+1}}(\omega', z_{t/n+1}, \phi_{t/n+1})), dz)]\, du\, dx$$
$$+ 1/2 \int \partial^\alpha((\bar{F})^{-1}(u,x)) \,|\overline{\text{Jacobian}\,(\bar{F})^{-1}(u,x)}|\, E^{P^{\bar{b}}}[1_{S(u,x) > t/2}$$
$$\int \partial^\beta g(z) P(t/2, (\bar{F})^{-1}(u_{t/2}, \phi_{t/2}), dz)]\, du\, dx\,.$$

We then make the two successive changes of variables

$$x \to \phi_{t/n+1}(\omega, u, x) \quad \text{next} \quad u \to -\, z_{t/n+1}(\omega, u)$$

and endly we change $\tilde{\omega}^{t/n+1}$ (which appears because of the change on x) in ω. The formula $w_t^1 = \int_0^t \operatorname{sign}(u + B_s)\,\delta B_s$ immediately shows that this transformation on the probability space induces a change of $P^{\bar{b}}$ in $P^{-\bar{b}}$.

Hence from (1.14), (1.17) and what precedes, (2.21) becomes

$$(2.22) \quad = 1/2 \sum_{n=2}^{+\infty} \int E^{P^{-\bar{b}}}[\partial^\alpha f((\bar{F})^{-1}(-z_{t/n+1}(u), \tilde{\phi}_{t/n+1}(u,x)))$$
$$1_{\tilde{S}(u,x) > t/n+1} \text{Jacobian}\, \tilde{\phi}_{t/n+1}(u,x) |\overline{\text{Jacobian}\, F^{-1}(-z_{t/n+1}, \tilde{\phi}_{t/n+1})}|$$
$$E^{P^{\bar{b}}}[1_{S(u,x) \leqslant t/n+n^2} \int \partial^\beta g(z) P((nt/n+1)$$
$$- S(u,x), (\bar{F})^{-1}(u_{S(u,x)}, \phi_{S(u,x)}), dz)]\, du\, dx$$
$$+ 1/2 \int E^{P^{-\bar{b}}}[\partial^\alpha f((\bar{F})^{-1}(-z_{t/2}, \tilde{\phi}_{t/2})) 1_{\tilde{S}(u,x) > t/2} \text{Jacobian}\, \tilde{\phi}_{t/2}$$
$$|\overline{\text{Jacobian}\, F^{-1}(-z_{t/2}, \tilde{\phi}_{t/2})}|\, \partial^\beta g(z) P(t/2, (\bar{F})^{-1}(u,x), dz)]\, dx$$

where $\tilde{S}(u,x) = \inf\{s \geqslant 0, (u_s(u), \tilde{\phi}_s(\omega, u, x)) \notin F(B_\varepsilon)\}$.

(2.23) But $(\bar{F})^{-1}(u,x) = F^{-1}(|u|,x)$ (idem for $\overline{\text{Jacobian } F^{-1}}$), and the laws of $(u\cdot(u), \tilde{\phi}\cdot(u,x), \tilde{G}_\cdot^{-\bar{b}}(u,x))$ and $(u.(-u), \tilde{\phi}.(-u,x), \tilde{G}_\cdot^{-\bar{b}}(-u,x))$ are the same.

Hence we do not change (2.22) if we replace 1/2 by 1, $(\bar{F})^{-1}$ (resp. $\overline{\text{Jacobian}}$) by F^{-1} (resp. Jacobian), $-z_{t/n+1}(u)$ by $u_{t/n+1}(u)$ and $P^{-\bar{b}}$ (resp. $P^{\bar{b}}$) by P^{-b} (resp. P^{b}).

Because supp. $f \subset D \cap B_{2\varepsilon}$ we can integrate by parts in $E^{P^{-b}}$, by using the calculus of variations near the boundary ([11]). We can also integrate by parts in dz as before. Since (u,x) belongs to $F(B_\varepsilon)$ in the integral, we have estimates like (2.15), (2.16) (involving $F(B_{3\varepsilon/2})$ and $F(B_{2\varepsilon}) - F(B_{3\varepsilon/2})$ in place of $K_{\varepsilon/2}$ and $K_\varepsilon - K_{\varepsilon/2}$). The proof is complete. □

(2.24) *Remark.* – If $\rho \not\equiv 0$ one has to deal with $(u_{\eta_t}, \phi_{\eta_t}(u,x))$ (see (1.6)). If ρ is not constant, we do not know wether ϕ_{η_t} remains a diffeomorphism of $\mathbb{R}^{d-1}$, which is required in view of the change of variable $x \to \phi_{\eta_t}(u,x)$. If ρ is a non zero constant η_t only depends on u and ϕ_{η_t} is a diffeomorphism whose inverse is given by (1.17). In return we do not know whether the function $u \to z_{\eta_t}(u)$ is a change of variables.

At last as in [13] 1.12 and [14] 3.44, one can obtain for the densities the following upper estimates.

(2.25) Proposition. – *Let U and V be two open subsets of $\bar{D}$, $d' = d/2$, $t > 0$. Let U_ε (resp. V_ε) be the set $\{x \in \bar{D}, d(x,U) < \varepsilon\}$ (resp. with V). Assume that $\bar{U} \cap \partial D$ and $\bar{V} \cap \partial D$ are very good and compact, and that* (H.S) *holds. Then*

i) *If* (H.R unif) *is satisfied in V_ε for an $\varepsilon > 0$, there exist n and c positive, only depending on the data on V_ε such that for all $x \in \bar{D}$*

$$\|p(t,x,\cdot)\|_{C_b^k(\bar{V})} \leqslant c(\varepsilon \wedge t)^{-n(k+d')}$$

ii) *If in addition* (H.R unif) *is also satisfied on U_ε, and $\rho \equiv 0$, then*

$$\|p(t,\cdot,\cdot)\|_{C_b^k(\bar{U}\times\bar{V})} \leqslant c(\varepsilon \wedge t)^{-n(k+d')}$$

n and c depending this time on the data on V_ε and U_ε.
Moreover for any $\eta > 0$ and any multi index α, there exists $c' > 0$ such that

$$\sup_{y \in V,\, d(y,U) \geqslant \eta} \left\| \frac{\partial^\alpha}{\partial y^\alpha} p(t, \cdot, y) \right\|_{C_b^k(\bar U)} \leqslant c(\varepsilon \wedge \eta \wedge t)^{-n(k+|\alpha|+d')} \exp - c'(\eta^2/t \wedge \eta))$$

$$\sup_{x \in U,\, d(x,V) \geqslant \eta} \left\| \frac{\partial^\alpha}{\partial x^\alpha} p(t, x, \cdot) \right\|_{C_b^k(\bar V)} \leqslant c(\varepsilon \wedge \eta \wedge t)^{-n(k+|\alpha|+d')} \exp - c'(\eta^2/t \wedge \eta)).$$

Recall that if Lie $(Y_1(x), \ldots, Y_m(x))$ spans $\mathbb{R}^d$ at $x \in \partial D$, we can find a very good local chart at x (cf. [11] section 2). By virtue of (1.9), Proposition (2.25) extends to domains U or V with a non compact intersection with ∂D, provided they are uniformly very good i.e. when the X_i's defined by (1.7) in $F(B(x,\varepsilon) \cap \bar D)$ are satisfying (H.F. unif). In particular if we can find N and $C > 0$ such that

$$\sum_{Y \in V_N(x)} \langle Y, z \rangle^2 \geqslant C \|z\|^2 \quad \text{for all} \quad x \in \bar U \cap \partial D \quad \text{and} \quad z \in \mathbb{R}^d,$$

where $V_N(x)$ is the set of the Lie brackets of length $\leqslant N$ of the Y_i's, $1 \leqslant i \leqslant m$, at x, $\bar U \cap \partial D$ is uniformly very good.

3. RESOLVANT OPERATORS AND KERNELS

For $h \in C_b^\infty(\bar D)$ and $v \in C_b^\infty(\partial D)$ we consider the process $H_t^{h,v}$ defined by

$$\text{(3.1)} \qquad H_t^{h,v} = \exp - \int_0^t h(y_s)\, ds \cdot \exp - \int_0^t v(y_s)\, dL_s .$$

The Gauge of $H^{h,v}_\cdot$ is the function $J(y)$ defined by

$$\text{(3.2)} \qquad J(y) = E^y\left[\int_0^{+\infty} H_t^{h,v}(dt + dL_t)\right].$$

In the whole paragraph we assume the following

$$\text{(0.11)} \qquad \sup_{y \in \bar D} J(y) < +\infty \quad \text{(Gauge condition)}$$

so that for any $f \in C_b^\infty(\bar{D})$ and $g \in C_b^\infty(\partial D)$ one can define

$$(3.3) \quad R_{f,g}^{h,v}(y) = E^y\left[\int_0^{+\infty} f(y_t)H_t^{h,v}\,dt + \int_0^{+\infty} g(y_t)H_t^{h,v}\,dL_t\right].$$

The aim of the paragraph is to show that if (0.11) and (H.G) (see (0.1) or (2.1)) are fulfilled, the operator

$$R^{h,v} : (f,g) \to R_{f,g}^{h,v},$$

maps $C_b^\infty(\bar{D}) \times C_b^\infty(\partial D)$ into $C^\infty(\bar{D})$, and admits suitably regular kernels, namely

(3.4) THEOREM. – *Let U and V be two open sets of $\bar{D}$. We assume that* (H.G) *is satisfied at any $y \in U$ and any $x \in V$, and that* (1.1) *and* (0.11) *hold. Then*

1) $R^{h,v}(C_b^\infty(\bar{D}) \times C_0^\infty(\partial D)) \subset C^\infty(U)$

2) *The operators*

$$R_D^{h,v} : C_0^\infty(V) \to \mathscr{D}'(U) \quad f \to R_{f,0}^{h,v} \qquad \text{and} \qquad \mathrm{R}_{\partial D}^{h,v} : C_0^\infty(V \cap \partial D) \to \mathscr{D}'(U) \quad g \to R_{0,g}^{h,v}$$

have their kernels, respectively denoted by K_D and $K_{\partial D}$, which are C^∞ out of the diagonal.

(*i.e* $\mathrm{supp}\cdot\mathrm{sing}\cdot K_D \subset \mathrm{diag}\,(U \times V)$ *and*

$$\mathrm{supp}\cdot\mathrm{sing}\cdot K_{\partial D} \subset \mathrm{diag}\,(U \times (V \cap \partial D)).$$

To this end we first introduce an auxiliary process easily related to $y_.$, which can be submitted to the techniques of § 2. Here the situation is a little more intricate than in [13] or [14], on the one hand because of the apparition of the local time L_t, on the other hand because of the chart by chart construction of y_t.

3.A. An auxiliary process and the reduction of the problem.

We start by settling the problem of the viscosity coefficient ρ. Indeed if y_t^0 with local time L_t^0 induces under P_y^0 an (A,Γ^0)-diffusion, setting

$$\eta_t = \left(t + \int \rho(y_s^0)\,dL_s^0\right)^{-1},$$

we saw that $y^0_{\eta_t}$ induces an (A,Γ)-diffusion, with local time $L^0_{\eta_t}$. Hence

$$(3.5)\qquad R^{h,v}_{f,g}(y) = E^y_0\left[\int_0^{+\infty} f(y^0_t)H^{0h,v+\rho h}_t\,dt + \int_0^{+\infty}(g(y^0_t)+\rho f(y^0_t))H^{0h,v+\rho h}_t\,dL^0_t\right]$$

where E^y_0 is the P^0_y expectation and $H^{0h,v}_t$ is associated to y^0_t. Then if R^0 is the resolvant operator associated to y^0_t, P^0_y, L^0_t one has

$$(3.6)\qquad R^{h,v}_{f,g} = (R^0)^{h,v+\rho h}_{f,g+\rho f}.$$

So we may and will assume that $\rho \equiv 0$.

Let we consider $m+r+3$ vector fields $\bar Y_i$, $0 \leqslant i \leqslant m$, $\bar V_\ell$, $0 \leqslant \ell \leqslant r$ and Z defined on $\mathbb{R}^{d+1}$ with values in $\mathbb{R}^{d+1}$, of C^∞_b class, given by

$$(3.7)\qquad \begin{cases} \bar Y_0(y,\xi) = \begin{pmatrix} u^2(\xi)\,Y_0(y) \\ 0 \end{pmatrix} \bar Y_i(y,\xi) = \begin{pmatrix} u(\xi)\,Y_i(y) \\ 0 \end{pmatrix}, \quad 1 \leqslant i \leqslant m, \\ \bar V_\ell(y,\xi) = \begin{pmatrix} V_\ell(y) \\ 0 \end{pmatrix}, \quad 0 \leqslant \ell \leqslant r; \quad Z(y,\xi) = \begin{pmatrix} 0 \\ u(\xi) \end{pmatrix}, \\ \text{with } u(\xi) = (2+\sin\xi)/3\sqrt{2}, \quad \xi \in \mathbb{R}. \end{cases}$$

Define

$$(3.8)\qquad \begin{cases} \bar A = 1/2\,Z^2 + 1/2\sum_{i=1}^m \bar Y_i^2 + \bar Y_0 \\ \bar\Gamma = 1/2\sum_{\ell=1}^r \bar V_\ell^2 + \bar V_0. \end{cases}$$

One can build on $\bar D \times \mathbb{R}$, an $(\bar A,\bar\Gamma)$-diffusion $(\bar P_{y,\xi}, (\bar y_t, \xi_t), \bar L_t)$.

Let π be the canonical projection from $\mathbb{R}^d \times \mathbb{R}$ onto $\mathbb{R}^d$, and

$$(3.9)\qquad \delta(t) = \int_0^t u^2(\xi_s)\,ds, \qquad \gamma(t) = (\delta(t))^{-1}.$$

Then $\pi(\bar y_{\gamma(t)}, \xi_{\gamma(t)})$ induces the (A,Γ)-diffusion starting from y, with local time $L_{\gamma(t)}$.

One immediately has

$$(3.10)\qquad R^{h,v}_{f,g}(y) = (\bar R)^{hu^2,v}_{fu^2,g}(y,\xi) \qquad \text{for all } \xi \in \mathbb{R},$$

where

$$(\bar{R})^{hu^2,v}_{fu^2,g}(y,\xi)$$
$$= \bar{E}_{y,\xi}\left[\int_0^{+\infty} f(\bar{y}_t)u^2(\xi_t)\exp - \int_0^t (h(\bar{y}_s)u^2(\xi_s)\,ds + v(\bar{y}_s)\,d\bar{L}_s)\,dt\right]$$
$$+ \bar{E}_{y,\xi}\left[\int_0^{+\infty} g(\bar{y}_t)\exp - \int_0^t (h(\bar{y}_s)u^2(\xi_s)\,ds + v(\bar{y}_s)\,d\bar{L}_s)\,d\bar{L}_t\right].$$

All the interest of the introduction of the auxiliary process is explained in the following proposition.

(3.11) PROPOSITION ([14] Thm 4.5 and lemma 5.18). — 1) *Let* $U \subset \bar{D}$. *Assume that* A *fulfills* (H.G unif) (*see* (2.1)) *on* U, *then* $\bar{A}$ *fulfills* (H.R unif) *on* $U \times \mathbb{R}$.

2) *Let* $U \subset \partial D$. *If* A *fulfills* (H.G unif) *on* $\bar{U}$, *then* $U \times \mathbb{R}$ *is very good for* $\bar{A}$. *Moreover one can find* $\varepsilon > 0$, *only depending on the data in* $\bar{U}$, *such that for every* $x \in U$ *there exists a* C^∞ *diffeomorphism* $\tilde{F}_x$ *from* $(\bar{D} \cap B(x,\varepsilon)) \times \mathbb{R}$ *onto a subset* V *of* $\mathbb{R}^+ \times \mathbb{R}^d$ *satisfying*

i) $((\bar{D} \cap B(x,\varepsilon)) \times \mathbb{R}, \tilde{F}_x)$ *is a very good local chart at* (x,ξ) *for all* $\xi \in \mathbb{R}$.

ii) *All the derivatives of* $\tilde{F}_x$ *and* $\tilde{F}_x^{-1}$ *are bounded on* $(\bar{D} \cap B(x,\varepsilon)) \times \mathbb{R}$ (*resp.* V) *uniformly in* (x,ξ).

3.B. Smoothness of the operator $R^{h,v}$.

If (0.11) holds, the operator $R^{h,v} : (f,g) \to R^{h,v}_{f,g}$, is well defined, and maps $C^\infty(\bar{D}) \times C^\infty(\partial D)$ into $L^\infty(\bar{D})$. We now give the proof of Theorem (3.4).

Proof. — As in the previous section we may and will assume that $\rho \equiv 0$. To prove (3.4.1) we shall show that for all $\varphi \in C_0^\infty(U \cap D)$ and every multiindex α the following holds:

$$\left|\int \partial^\alpha \varphi(y) R^{h,v}_{f,g}(y)\,dy\right| \leqslant c(\alpha)\|\varphi\|_\infty. \tag{3.12}$$

Indeed in this case $R^{h,v}_{f,g} \in C^\infty(U \cap D)$ and extends C^∞ up to $U \cap \partial D$. This extension is equal to the a priori defined $R^{h,v}_{f,g}(y)$, $y \in \partial D$, because of the continuity of $y \to Q_y$ and of (1.19).

From (3.10) one has

$$(3.13)\quad \int \partial^\alpha \varphi(y) R^{h,v}_{f,g}(y)\, dy = \int \partial^\alpha \varphi(y)((\bar{R})^{hu^2,v}_{fu^2,0} + (\bar{R})^{hu^2,v}_{0,g}(y))\, dy\,.$$

According to (3.11) the semigroup

$$(3.14)\quad \begin{aligned} \bar{P}(t,(y,\xi),\cdot) : C^0(\bar{D}\times\mathbb{R}) &\to \mathbb{R} \\ F &\to \bar{E}_{y,\xi}[F(\bar{y}_t,\xi_t)\bar{H}^{hu^2,v}_t]\,, \end{aligned}$$

admits for all $t > 0$, a density $m(t,\cdot,\cdot) \in C^\infty((U\times\mathbb{R}) \times (V\times\mathbb{R}))$.

Of course we are not exactly in the situation of § 2 because of the apparition of the extra term $\bar{H}^{hu^2,v}_t$. But this is not relevant, and all the results of § 2 extend to that case except the upper estimates (2.25) which have to be weighted in a natural way, using the well known fact that $E[\exp L_t] < +\infty$.

Now using again the Markov property, we can control the integration after time 1 namely:

$$(3.15)\quad \begin{aligned} &\int \partial^\alpha\varphi(y)\bar{E}_{y,\xi}\left[\int_1^{+\infty} f(\bar{y}_t)u^2(\xi_t)\bar{H}^{hu^2,v}_t\, dt\right] dy \\ &= \int \partial^\alpha\varphi(y) m(1,(y,\xi),z)(\bar{R})^{hu^2,v}_{fu^2,0}(z)\, dz\, dy \\ &= (-1)^{|\alpha|}\int \varphi(y)\,\partial^\alpha_y m(1,(y,\xi),z)(\bar{R})^{hu^2,v}_{fu^2,0}(z)\, dz\, dy \end{aligned}$$

similarly

$$(3.16)\quad \begin{aligned} &\int \partial^\alpha\varphi(y)\bar{E}_{y,\xi}\left[\int_1^{+\infty} g(\bar{y}_t)\bar{H}^{hu^2,v}_t\, d\bar{L}_t\right] dy \\ &= (-1)^{|\alpha|}\int \varphi(y)\,\partial^\alpha_y m(1,(y,\xi),z)(\bar{R})^{hu^2,v}_{0,g}(z)\, dz\, dy\,. \end{aligned}$$

We now have to integrate by parts in $\int \partial^\alpha\varphi(y)\bar{E}_{y,\xi}\left[\int_0^1 \dots\right] dy$.

This will be done in the same way as in (2.9).

Introduce $U_\varepsilon = \{y \in U, d(y,\partial D) > \varepsilon\}$ and a recovering of $U \times \mathbb{R}$ by $U_{\varepsilon/2} \times \mathbb{R}$ and a family of very good local charts like in (3.11.2). Introducing a partition of the unity if necessary, we may assume that $\operatorname{supp}\varphi$ is included in the projection of a very good local chart or in $U_{\varepsilon/2}$.

Begin with the case $\operatorname{supp}\varphi \subset U_{\varepsilon/2}$.

Set $S(y) = \inf\{t>0, y_t \notin U_{\varepsilon/4}\}$ and $T(y) = 1 \wedge S(y)$.

We are following (2.11)-(2.14). So as not to exhaust the reader's patience, we only give the details on $S(y) > 1$.

Define $\tilde{\xi}_{.}(w,z)$ as the flow solution of

$$d\tilde{\xi}_t = -\bar{Y}_0(\tilde{\xi}_t)\,dt - \bar{Y}_i(\tilde{\xi}_t)\,dw_t^i - Z(\tilde{\xi}_t)\,dw_t^{m+1}, \quad \tilde{\xi}_0(w,z) = z.$$

Then

$$(3.17)\quad \int \partial^\alpha\varphi(y)\bar{E}_{y,\xi}\left[\int_0^1 f(\bar{y}_t)u^2(\xi_t)\bar{H}_t^{hu^2,v}1_{S(y)>1}\,dt\right]dy$$
$$= E^w\left[\int_{\bar{D}\times\mathbb{R}}\int_0^1 \partial^\alpha\varphi(\pi\tilde{\xi}_1(w,z))f(\pi\tilde{\xi}_t(w,z))u^2(\pi'\tilde{\xi}_t(w,z))\right.$$
$$\left.\text{Jacobian }\tilde{\xi}_1(w,z)1_{\tilde{S}(z)>1}\tilde{H}_1(w)\tilde{H}_t^{-1}(w)\,dt\,dz\right]$$

where

$\tilde{S}(z) = \inf\{t\geqslant 0, \tilde{\xi}_t(w,z)\notin U_{\varepsilon/4}\times\mathbb{R}\}$,
π' is the projection of $\mathbb{R}^d\times\mathbb{R}$ onto $\mathbb{R}$, and
$\tilde{H}_t(w) = \exp - \int_0^t h(\pi\tilde{\xi}_s(w,z))u^2(\pi'\tilde{\xi}_s(w,z))\,ds$.

Indeed starting from a point in D and before to leave D, the local time L is remained constant equal to 0.

We then only have to integrate by parts using the calculus of variations on $\tilde{\xi}_{.}$ just taking care of the fact that the derivatives of f and h also appear.

The justification of the summation in $n\left(1=\sum_{\mathbb{N}}1_{1/n+1<T\leqslant 1/n}\right)$ follows from estimates similar to (2.15) and (2.16) replacing K by $U_{\varepsilon/2}$ and ε by $\varepsilon/2$.

Still with $\operatorname{supp}\varphi\subset U_{\varepsilon/2}$, we now have to integrate by parts in

$$\int \partial^\alpha\varphi(y)\bar{E}_{y,\xi}\left[\int_0^1 \dots d\bar{L}_t\right]dy.$$

With the same notation as before we see that on $S(y) > 1$, the local time did not increase so that this term is 0. Thus we give the details on $1/2 < S(y) \leqslant 1$.

$$(3.18)\quad \int \partial^\alpha \varphi(y) \bar{E}_{y,\xi}\left[1_{1/2<S(y)\leqslant 1} \int_0^1 g(\bar{y}_t) \bar{H}_t^{hu^2,v}\, d\bar{L}_t\right] dy$$

$$= \int \partial^\alpha \varphi(y) \bar{E}_{y,\xi}\left[1_{1/2<S(y)} \bar{E}_{\bar{y}_{1/2},\xi_{1/2}}\left[\left(\left(\int_0^{1/2} g(\bar{y}_t) \bar{H}_t^{hu^2,v}\, d\bar{L}_t\right)\right.\right.\right.$$

$$\left.\left.\left. 1_{S(\bar{y}_{1/2})\leqslant 1/2}\right]\right] dy$$

from the Markov property and because $L_t = 0$ for $t \in [0,1/2]$ on $S(y) > 1/2$. Introducing the inverse flow, this is still equal to

$$= E^w\left[\int_{\bar{D}\times\mathbb{R}} \partial^\alpha \varphi(\pi \tilde{\xi}_{1/2}(w,z)) 1_{\tilde{S}(z)>1/2} \text{ Jacobian } \tilde{\xi}_{1/2}(w,z)\right.$$

$$\left. \bar{E}_z\left[1_{S(z)\leqslant 1/2}\left(\int_0^{1/2} g(\bar{y}_t) \bar{H}_t^{hu^2,v}\, d\bar{L}_t\right)\right] dz\right]$$

and we again apply the calculus of variations on $\tilde{\xi}$.

Now assume that $\operatorname{supp} \varphi \subset B$ with $(B\times\mathbb{R}, \tilde{F})$ a very good local chart as in (3.11.2).

Set $B_\varepsilon = \{y \in B, d(y,\partial B) > \varepsilon/4\}$ and $S = \inf\{t \geqslant 0, y_t \notin B\}$.

Since the B_ε's and $U_{\varepsilon/2}$ are still an open recovering of U we may and will assume that $\operatorname{supp} \varphi \subset B_\varepsilon \cap D$ (take care that ∂B is relative to the induced topology on $\bar{D}$, i.e is the intersection of the closures in $\bar{D}$ of B and $\bar{D}\backslash B$).

Consider the vector fields

$$\tilde{F}^* \bar{Y}_0 = \begin{pmatrix} b \\ \bar{X}_0 \end{pmatrix}, \quad \tilde{F}^* \bar{Y}_i = \begin{pmatrix} 0 \\ \bar{X}_i \end{pmatrix}, \quad 1 \leqslant i \leqslant m, \quad \tilde{F}^* Z = \begin{pmatrix} 0 \\ Z \end{pmatrix},$$

$$\tilde{F}^* \bar{V}_\ell = \begin{pmatrix} 0 \\ \bar{S}_\ell \end{pmatrix}, \quad 1 \leqslant \ell \leqslant r, \quad \tilde{F}^* \bar{V}_0 = \begin{pmatrix} 1 \\ \bar{S}_0 \end{pmatrix}$$

defined on $\mathbb{R}^{d+1}$ with values in $\mathbb{R}^{d+1}$.

Let $z_t(u)$, u_t, L_t^u be as in (1.10) and $\bar{\phi}_t(\omega,u,x)$ be the solution of

$$(3.18\ bis)\quad dx_t = \bar{X}_0(u_t,x_t)\, dt + \bar{X}_i(u_t,x_t)\, dw_t^i + \bar{Z}(u_t,x_t)\, dw_t^{m+1}$$
$$+ \bar{S}_0(x_t)\, dL_t^u + \bar{S}_\ell(x_t)\, dM_t^\ell(u)\,; \qquad x_0 = x \in \mathbb{R}^d$$

$\tilde{\phi}_t(\omega,u,x)$ is the reversed flow obtained by taking the opposites of the fields, $\tilde{\omega}^t$ is the time reversed path defined in (1.13) (with the apparition here of w^{m+1}). As before we introduce $S(u,x)$ resp. $\tilde{S}(u,x)$) the first exit time of $B' = \tilde{F}(B\times\mathbb{R})$ for $\bar{\phi}.$(resp $\tilde{\phi}.$) and we replace $(\bar{y}_s,\xi_s)1_{s<S}$ by $\tilde{F}^{-1}(u_s,\bar{\phi}_s)1_{s<S(u,x)}$.

Here again we give the details of the localization procedure only on $S>1$ (i.e $S(u,x)>1$)

$$(3.19)\quad \int \partial^\alpha\varphi(y)\bar{E}_{\xi,y}\left[\int_0^1 f(\bar{y}_t)u^2(\xi_t)\bar{H}_t^{hu^2,v}1_{S>1}dt\right]dy$$
$$= \int_{B'}(\partial^\alpha\varphi\circ\pi\circ\tilde{F}^{-1})(u,x)|\text{Jacobian }\tilde{F}^{-1}(u,x)|$$
$$E^{P^b}\left[1_{S(u,x)>1}\int_0^1 (f\circ\pi\circ\tilde{F}^{-1})\times(u^2\circ\pi'\circ\tilde{F}^{-1})(u_t,\bar{\phi}_t)\right.$$
$$\left(\exp-\int_0^t (h\circ\pi\circ\tilde{F}^{-1})\times(u^2\circ\pi'\circ\tilde{F}^{-1})(u_s,\bar{\phi}_s)\,ds\right.$$
$$\left.\left.\exp-\int_0^t (v\circ\pi\circ\tilde{F}^{-1})(0,\bar{\phi}_s)\,dL_s^u\right)dt\right]du\,dx.$$

Following (2.19)-(2.21) we define

$$\begin{cases}\tilde{G}_t^b = \exp\int_0^t b(u_s,\tilde{\phi}_s)\,\delta w_s^1 - 1/2\int_0^t b^2(u_s,\tilde{\phi}_s)\,ds\\ \tilde{H}_t = \exp-\int_0^t[(h\circ\pi\circ\tilde{F}^{-1})\times(u^2\circ\pi\circ\tilde{F}^{-1}))(u_s,\tilde{\phi}_s)\,ds\\ \qquad\qquad +(v\circ\pi\circ\tilde{F}^{-1})(0,\tilde{\phi}_s)\,dL_s^u]\end{cases}$$

so that (3.19) is still equal to

$$(3.20)\quad = \int_0^1 E^P\left[\int_{B'}(\partial^\alpha\varphi\circ\pi\,\tilde{F}^{-1})(u_1,\tilde{\phi}_1(u,x))\right.$$
$$|\text{Jacobian }\tilde{F}^{-1}(u_1,\phi_1)(u,x))|$$
$$1_{\tilde{S}(u,x)>1}(f\circ\pi\circ\tilde{F}^{-1})(u_t,\tilde{\phi}_t(u,x))(u^2\circ\pi'\circ\tilde{F}^{-1})(u_t,\tilde{\phi}_t(u,x))$$
$$\text{Jacobian }\tilde{\phi}_1(u,x)\tilde{H}_1\tilde{H}_t^{-1}\tilde{G}_1^{-b}dudx]\,dt.$$

We can add to (0.9) a supplementary equation

$$\begin{cases}d\tilde{H}_t = \tilde{H}_t[((h\circ\pi\circ\tilde{F}^{-1})\times(u^2\circ\pi'\circ\tilde{F}^{-1}))(u_t,\tilde{\phi}_t)\,dt\\ \qquad +(v\circ\pi\,\tilde{F}^{-1})(0,\tilde{\phi}_t)dL_t^u]\\ \tilde{H}_0 = 1.\end{cases}$$

This shows that we can use the calculus of variations of [11] to integrate by parts in (3.20) for each fixed t.

Finally since $\tilde{L}_s^{u'} = L_1^u - L_{1-s}^u$ conditionally to $z_1(u) = u'$ (see (1.15)) we have

$$(3.21)\quad \int \partial^\alpha \varphi(y) \bar{E}_{\xi,y}\left[1_{S>1}\int_0^1 g(\bar{y}_t)\bar{H}_t\, d\bar{L}_t\right] dy$$

$$= E^P\left[\int_{B'} (\partial^\alpha\varphi \circ \pi\, \tilde{F}^{-1})(u_1, \tilde{\phi}_1(u,x))\,|\text{Jacobian}\, \tilde{F}^{-1}(u_1,\tilde{\phi}_1)|\ \text{Jacobian}\, \tilde{\phi}_1\right.$$

$$\left. 1_{\tilde{S}(u,x)>1}\tilde{G}_1^{-b}\tilde{H}_1\left(\int_0^1 (g\circ\pi\circ\tilde{F}^{-1})(0,\tilde{\phi}_t)\tilde{H}_t^{-1} dL_t^u\right) du\,dx\right].$$

One more time one easily checks that we can use the calculus of variations of [11] to integrate $\partial^\alpha\varphi$ by parts. The proof of (3.4.1) is achieved.

In order to show the regularity of the kernel K_D in (3.4.2) one has to show the following :

For any compact subsets K and K' of U and V such that $K \cap K' = \varnothing$ any f and φ respectively in $C_0^\infty(\mathring{K}')$ and $C_0^\infty(\mathring{K})$ and any multiindices α and β one has

$$(3.22)\qquad \left|\int \partial^\alpha\varphi(y) R^{h,v}_{\partial^\beta f,0}(y)\, dy\right| \leqslant c(\alpha,\beta)\|\varphi\|_\infty \|f\|_\infty .$$

Hence here we have to integrate both $\partial^\alpha\varphi$ and $\partial^\beta f$.

We write (3.15) in a slightly modified form

$$(3.23)\quad \int \partial^\alpha\varphi(y)\, \bar{E}_{y,\xi}\left[\int_1^{+\infty} \partial^\beta f(\bar{y}_t) u^2(\xi_t)\bar{H}_t^{hu^2,v}\, dt\right] dy$$

$$= \int \partial^\alpha\varphi(y) m(1/2,(y,\xi),z)\bar{E}_z\left[\int_{1/2}^{+\infty} \partial^\beta f(\bar{y}_t) u^2(\xi_t)\bar{H}_t^{hu^2,v}\, dt\right] dy\, dz .$$

We integrate by parts in $\bar{E}_z$ using the local calculus of variations between the times $t - 1/2$ and t. The integrability in dt is ensured by (0.11) and (2.25) (see [13] and [14] for analogous results). Then for fixed z we integrate by parts in y.

Set $\varepsilon = d(K,K')$. Because supp $f \subset \mathring{K}'$ one introduces

$$T(y) = \inf\{t \geqslant 0, d(\bar{y}_t, K) > \varepsilon/2\}.$$

We then have

$$(3.24)\quad \int \partial^\alpha \varphi(y)\bar{E}_{y,\xi}\left[\int_0^1 \partial^\beta f(\bar{y}_t)u^2(\xi_t)\bar{H}_t^{hu^2,v}dt\right]dy = \int \partial^\alpha \varphi(y)\bar{E}_{y,\xi}\left[\int_{T(y)}^1 \partial^\beta f(\bar{y}_t)u^2(\xi_t)\bar{H}_t^{hu^2,v}dt\right]dy.$$

One uses again the localization procedure writing

$$1 = 1_{T(y)>1} + \sum_{n\geq 1} 1_{1/n+1 < T(y) \leq 1/n}.$$

If $T(y)\in[1/n+1, 1/n]$ we reverse the time at time $1/2(n+1)$. (3.24) splits into two independant parts as in (2.14) and we use the calculus of variations on $\bar{y}_.$ to integrate $\partial^\beta f$ and on the reversed process to integrate $\partial^\alpha\varphi$ just as in the above proof of (3.4.1). This shows the regularity of K_D.

Finally to study $K_{\partial D}$, using a change of coordinates induced by a very good local chart, we are reduced to the case $\bar{D}\times\mathbb{R} = \mathbb{R}^+\times\mathbb{R}^d$.

We must show that for any compact subset K of U, any compact subset K' of $V\cap\partial D = V\cap(\{0\}\times\mathbb{R}^{d-1})$ and any multiindices $\alpha = (\alpha_1,\ldots,\alpha_d)$, $\beta = (\beta_2,\ldots,\beta_d)$, we have

$$(3.25)\quad \left|\int \partial^\alpha\varphi(y)R^{h,v}_{0,\partial^\beta g}(y)\,dy\right| \leq c(\alpha,\beta)\,\|\varphi\|_\infty\|g\|_\infty.$$

We write again (omitting the diffeomorphism $\tilde{F}$ for simplicity)

$$(3.26)\quad \int \partial^\alpha\varphi(y)\bar{E}_{y,\xi}\left[\int_1^{+\infty}\partial^\beta f(\bar{y}_t)\bar{H}_t^{hu^2,v}\,d\bar{L}_t\right]dy = \int \partial^\alpha\varphi(y)m(1/2,(y,\xi),z)\bar{E}_z\left[\int_{1/2}^{+\infty}\partial^\beta g(\bar{y}_t)\bar{H}_t^{hu^2,v}\,d\bar{L}_t\right]dy$$

$$= \int \partial^\alpha\varphi(y)m(1/2,(y,\xi),z)\left(\sum_{n=1}^{\infty}E^{P^b}\left[\int_{n/2}^{n+1/2}\partial^\beta g(\Phi_t(\omega,z))\right)\bar{H}_t^{hu^2,v}\,dL_t^u\right]dz\,dy.$$

We are obliged to cut up the integration in dL_t^u because of the possible problems due to the Girsanov transformation at infinity. With ω' fixed (see § 1 for the notation) we apply the partial calculus of variations on $w^2_.,\ldots,w^{m+1}_.$ in order to integrate $\partial^\beta g$ by parts.

Indeed each term of (3.26) is written as

$$(3.27)\quad E^{P^b}\left[\int_{n/2}^{n+1/2} \partial^\beta g(\bar{\Phi}_t(\omega,z))\bar{H}_t^{hu^2,v}\,dL_t^u\right]$$
$$= E^{\omega'}\left[\int_{n/2}^{n+1/2} E^{\bar{\omega},\omega *}[\partial^\beta g(\bar{\Phi}_t(\omega,z))G^b_{n+1/2}\bar{H}_t^{hu^2,v}]\,dL_t^u(\omega')\right].$$

Finally to integrate

$$\int \partial^\alpha \varphi(y)\bar{E}_{y,\xi}\left[\int_0^1 \partial^\beta g(\bar{y}_t)\bar{H}_t^{hu^2,v}\,d\bar{L}_t\right]dy,$$

we do as for f ((3.24)) and then as before ((3.27)). □

(3.28) *Remarks and complements.* – 1) If (0.11) is fulfilled one can choose $T > 0$ such that for all $y \in \bar{D}$

$$\left|E^y\left[\int_T^{+\infty} g(y_t)H_t^{h,v}\,dL_t\right]\right| \leqslant \varepsilon$$

(since the support of g is compact).

From (1.19) one gets

$$\lim_{\varepsilon\to 0} 1/\varepsilon \int_0^T (a^2(y_s)/2(V_0\psi)(y_s))\bar{g}(y_s)H_s^{h,v}1_{D_\varepsilon}(y_s)\,ds = \int_0^T g(y_s)H_s^{h,v}\,dL_s$$

with $\bar{g}$ any Seeley's extension of g. The above limit takes place almost surely and in L^2. We deduce that for all $x \in V \cap \partial D$ and $y \in U$ with $x \neq y$ one has

$$(3.29)\qquad K_{\partial D}(y,x) = (a^2(x)/2(V_0\psi)(x))\,K_D(y,x).$$

2) One can introduce A_t the right continuous inverse of the local time L_t. Then for $y \in \partial D$,

$$E^y\left[\int_0^{+\infty} g(y_t)H_t^{h,v}\,dL_t\right]$$
$$= E^y\left[\int_0^{+\infty} 1_{A_t<+\infty}g(y_{A_t})\exp -\left(\int_0^t v(y_{A_s})\,ds + h(y_{A_s})\,dA_s\right)dt\right].$$

The process $t \to y_{A_t}$ is called the boundary process. The preceding study supplies regularity results for the potentials associated to this particular jump process. These results also can be obtained with the help of the results of [4] and [19] on the laws of (A_t, y_{A_t}), may be with some difficulties due to the joint regularity and the necessary integration in A_t. We shall return later to the analytical signification of this result.

3) Let $\eta \in C_0^\infty(\mathbb{R}^+)$. One can mimic all what we have done in this paragraph and prove that the operator

$$R^{h,v,\eta} : (f,g) \to E^y\left[\int_0^{+\infty} f(u_t)\eta(t)H_t^{h,v}dt + \int_0^{+\infty} g(y_t)\eta(t)H_t^{h,v}dL_t\right]$$

also satisfies (3.4.1) and (3.4.2). The only thing we have to remark is that the passage to $\rho \not\equiv 0$ involves $\eta\left(t + \int_0^t \rho(y_s)\, dL_s\right)$.

In particular if $\operatorname{supp} \eta \subset [\varepsilon, +\infty[$ *for an* $\varepsilon > 0$, *the formulae* (3.15), (3.16), (3.23) *and* (3.26) *and their developments show that* $R_D^{h,v,\eta}$ *and* $R_{\partial D}^{h,v,\eta}$ *are regularizing (i.e their kernels* K_D^η *and* $K_{\partial D}^\eta$ *are of* C^∞ *class on the whole considered spaces).*

This result will be crucial in the next paragraph.

4) If we replace (H.G) by (H.G unif) (see (2.1)) on $\{d(y,U) < \varepsilon\}$ and $\{d(y,V) < \varepsilon\}$ for an $\varepsilon > 0$, then $R^{h,v}(C_b^\infty(\bar{D}) \times C_b^\infty(\partial D)) \subset C_b^\infty(U)$, and we can find uniform upper estimates for the kernels K_D and $K_{\partial D}$ on the set $d((y,x), \text{diagonal}) \geqslant c > 0$.

This is particularly relevant if (H.G unif) is satisfied on $\bar{D}$.

5) It is easily seen that, if $\bar{D}$ is compact, $t \to E^y[H_t^{h,v}]$ is uniformly continuous on $\mathbb{R}$, with modulus of continuity $\sup_{y \in \bar{D}} E^y[H_{t-s}^{h,v}](t-s)$. Hence, if (0.11) holds, $\lim_{t \to \infty} E^y[H_t^{h,v}] = 0$.

4. BOUNDARY VALUE PROBLEM

In this paragraph we shall study the boundary value problem

$$\text{(L.P)} \quad \begin{cases} (h-A)u = f \text{ in } D \\ (v-\Gamma)u = g \text{ on } \partial D \end{cases}$$

with $h \in C_b^\infty(\bar{D})$ and $v \in C_b^\infty(\partial D)$. We shall call (L.P) the Ventcel problem.

If the V_ℓ's, $1 \leqslant \ell \leqslant r$, and ρ vanish identically, this is just the oblique derivative problem. As explained in the introduction (L.P) is a classical problem in p.d.e theory ([20]), but can fail to reach the classical formalism of this theory. The Ventcel condition arises from the semigroup theory. When A is uniformly elliptic, (L.P) was solved in [8] and [33] (also see [28]). In this case it is proved that the solution u is equal to $R_{f,g}^{h,v}$ ([29] Prop. 5.2).

Here on the contrary we dispose of the Feller semigroup via the stochastic process $y_.$ of § 1. We shall use its properties proved in the previous paragraphs and the martingales methods due to Stroock and Varadhan ([18], [31]) to solve (L.P) in the hypoelliptic framework.

As in [14] § 5 we begin with a result of a priori regularity for the solutions. We then show that $R_{f,g}^{h,v}$ is a solution before concluding to uniqueness. A detailed comparison with existing results is done at the end of the paragraph.

First of all let we say what we mean by a solution of (L.P). Recall that $\mathscr{D}'(\bar{D})$ is the dual space of $C_0^\infty(\bar{D})$ and can be identified with $\mathscr{D}'_{\bar{D}}(\mathbb{R}^d)$, the space of distributions with support in $\bar{D}$, thanks to a continuous operator of extension (cf. [15] I 9.3).

Let $u \in \mathscr{D}'(\bar{D})$. We shall say that u admits sectional traces on ∂D up to order k ($k \in \mathbb{N}$) if for any local chart

$$F : B \subset \bar{D} \to V \subset \mathbb{R}^+ \times \mathbb{R}^{d-1}$$

and any $\varphi \in C_0^\infty(B)$, the image of φu by F is an application of C^k class in x^1 with values in $\mathscr{D}'(\mathbb{R}^{d-1})$, for sufficiently small $x^1 \geqslant 0$.

We define the traces as

$$\gamma_0 u = u|_{\partial D'} \qquad \gamma_j u = \gamma_0[(n^j)u], \qquad 0 \leqslant j \leqslant k$$

where n is the unitary inward normal vector field.

$\gamma_j u$ is then an element of $\mathscr{D}'(\partial D)$.

The boundary condition $(v-\Gamma)u = g$ on ∂D, reads as follows :

$$(4.1)\quad (v+\rho h)\cdot\gamma_0 u - 1/2\sum_{\ell=1}^{r} V_\ell^2(\gamma_0 u) - \bar{V}_0(\gamma_0 u) - (V_0\psi)\cdot\gamma_1 u = g + \rho(\gamma_0 f)$$

with $\bar{V}_0 = \mathrm{V}_0 - (\mathrm{V}_0(\gamma\psi))\cdot n$.

We begin by stating the main results of this paragraph. The first one is the analogue of Hörmander's result for the boundary value problem :

(4.2) THEOREM (of a priori regularity). — *Assume that* (1.1), (0.7) *and* (H.G. unif) (*see* (2.1)) *are fulfilled in* $\bar{D}$. *Let* U *be an open subset of* $\bar{D}$. *Then any* $u \in \mathscr{D}'(\bar{D})$ *solution of* (L.P) *with* $f \in C^\infty(\bar{U})$ *and* $g \in C^\infty(\overline{U\cap\partial D})$ *belongs to* $C^\infty(U)$.

In other words the boundary value problem (L.P) *is hypoelliptic.*

The next results are concerned with existence and uniqueness. First recall the following definition :

(4.3) DEFINITION. — *Let* $u \in \mathscr{D}'(\bar{D})$. *One says that* u *belongs to* $\mathscr{S}'(\bar{D})$ *if, after the identification of* $\mathscr{D}'(\bar{D})$ *and* $\mathscr{D}'_{\bar{D}}(\mathbb{R}^d)$, u *belongs to* $\mathscr{S}'(\mathbb{R}^d)$.

(4.4) THEOREM. — i) *Assume that* $\bar{D}$ *is compact, that* (1.1), (0.11) *and* (H.G) *are fulfilled in* $\bar{D}$. *Then for* $f \in C^\infty(\bar{D})$, $g \in C^\infty(\partial D)$ *the boundary value problem* (L.P) *has a solution* $u \in C^\infty(\bar{D})$, *unique in* $\mathscr{D}'(\bar{D})$, *given by* $u = R^{h,v}_{f,g}$.

ii) *In the situation of* i), *we do not assume that* $\bar{D}$ *is compact, but we assume that* $f \in C^\infty_b(\bar{D})$, $g \in C^\infty_b(\partial D)$ *and that* (H.G unif) *is satisfied in* $\bar{D}$. *Then* $u = R^{h,v}_{f,g} \in C^\infty_b(\bar{D})$, *and is the unique solution of* (L.P) *in* $\mathscr{S}'(\bar{D})$.

iii) *In the situation of* ii), *let* $f \in C^\infty(\bar{D}) \cap \mathscr{S}'(\bar{D})$ *and* $g \in C^\infty(\partial D) \cap \mathscr{S}'(\partial D)$. *Then* $u = R^{h,v}_{f,g} \in C^\infty(\bar{D}) \cap \mathscr{S}'(\bar{D})$, *and* u *is the unique solution of* (L.P) *in* $\mathscr{S}'(\bar{D})$.

We finally extend the above result to certain Sobolev spaces.

(4.5) THEOREM. – *We assume that* $\bar{D}$ *is compact, that* (1.1), (0.11) *and* (H.G) *are fulfilled in* $\bar{D}$. *Then for* $k \in \mathbb{N}$, $f \in H^{2k}(\bar{D})$ *and* $g \in H^{2k}(\partial D)$ *there exists* $u \in H^{2k}(\bar{D})$ *solution of* (L.P). *Furthermore* u *is unique in* $\mathscr{D}'(\bar{D})$.

The proofs of the above theorems will use the following

(4.6) THEOREM OF TRACES. – *Let* $u \in \mathscr{D}'(\bar{D})$ *such that* $(h-A)u = f$ *in* D. *We assume that* ∂D *is non characteristic.*

i) *If* $f \in C^\infty(\bar{D})$, u *admits sectional traces on* ∂D *of any order.*

ii) *If* $f \in H^s_{\mathrm{loc}}(\bar{D})$, u *admits sectional traces on* ∂D *up to order* k *for* $0 \leqslant k \leqslant s + 3/2$.

Furthermore if M *is a subspace of* $\mathscr{D}'(\bar{D})$ *equipped with a topology such that there exists a continuous operator of extension from* M *to* $\mathscr{D}'(\mathbb{R}^d)$, *then the traces of* u, *as elements of* $\mathscr{D}'(\partial D)$, *depend continuously of*

$$(u,f) \in M \times H^s_{\mathrm{loc}}(\bar{D}), \quad \text{for} \quad s > 1/2.$$

The proof of (4.6) can be found in [15] chapter 5, Theorem 2.9 and remarks 2.10 and 2.11.

4.A. A priori regularity.

We want to show that under (H.G) any $u \in \mathscr{D}'(\bar{D})$ solution of (L.P) with data $(f,g) \in C^\infty(\bar{D}) \times C^\infty(\partial D)$, is in $C^\infty(\bar{D})$. To this end we shall show that $\gamma_0 u \in C^\infty(\partial D)$. Indeed recall the following theorem:

(4.7) THEOREM ([14] Theorem 5.3). – *Let* U *be an open subset of* $\bar{D}$ *such that* $U \cap \partial D$ *is non characteristic. Assume that* (H.G) *is satisfied at each* $y \in \bar{U}$ *(or briefly on* $\bar{U}$). *Then if* $u \in \mathscr{D}'(\bar{D})$ *is a solution of*

$$\begin{cases} (h-A)u = f \text{ in } D \\ \gamma_0 u = g \text{ on } \partial D \end{cases}$$

with $f \in C^\infty(\bar{U})$ *and* $g \in C^\infty(\bar{U} \cap \partial D)$, *then* $u \in C^\infty(U)$.

This holds of course for $U = \bar{D}$.

As in the proof of the Hörmander's theorem in [13] Thm 2.2, or in [14] Thm 5.3, we intend to approach $\gamma_0 u$ by functions $\varphi_n \in C^\infty(\partial D)$ and to use the kernels studied in § 3. The difficulty here is essentially the presence of $\gamma_1 u$ in Γ. The idea of the proof is to solve the Dirichlet problem

$$\begin{cases} (h-A)u_n = f \text{ in } D \\ \gamma_0 u_n = \varphi_n \text{ on } \partial D \end{cases}$$

and to show that u_n depends continuously of φ_n in a suitable space. We then deduce from (4.6) that $\gamma_1 u_n$ also depends continuously of φ_n.

The reader must be aware that the machinery we shall develop is really necessary. This is because we cannot simply approach u by $u_n \in C^\infty(\bar{D})$, and be sure that the sectional traces $\gamma_j u_n$ converge to $\gamma_j u$, as we can see immediately from the density of $C_0^\infty(D)$ in $\mathscr{D}'(\bar{D})$. In other words the sectional traces are not continuous operators in general.

To carry out our program, we are lead to extend some of the results on the Dirichlet problem we obtained in [14].

First we need to define the coercivity assumption

$$\text{(0.7)} \quad \sup_{y \in \bar{D}} E^y\left[\int_0^T H_t^{h,0}\,dt\right] < +\infty, \text{ where } T = \inf\{t \geqslant 0, y_t \notin D\}.$$

(4.8) THEOREM ([14] Thms 4.37 and 5.14). – *Let $\bar{D}$ be compact, we assume that* (1.1) iii), (0.7) *and* (H.G) *are fulfilled on $\bar{D}$. Then for all $g \in C^\infty(\partial D)$, the Dirichlet problem*

$$\text{(P.P)} \quad \begin{cases} (h-A)u = 0 \text{ in } D \\ \gamma_0 u = g \text{ on } \partial D \end{cases}$$

has a unique solution u_g in $\mathscr{D}'(\bar{D})$. $u_g \in C^\infty(\bar{D})$ and is given by

$$\begin{aligned} u_g(y) &= E^y\left[1_{T<+\infty} g(y_T) \exp - \int_0^T h(y_t)\,dt\right] \\ &= \int_{\partial D} K(y,x) g(x)\,d\mu(x) \quad (d\mu \text{ the area measure}) \end{aligned}$$

where $K \in C^\infty(\bar{D} \times \partial D - \operatorname{diag}(\bar{D} \times \partial D))$ is the Poisson kernel ($K(y,x) = \delta_x$ if $y \in \partial D$).

The smoothness of K in the final variable was obtained in [1]. We shall extend (4.8) and prove

(4.9) THEOREM. – *Under the hypotheses of* (4.8), *for every* $g \in \mathscr{D}'(\partial D)$, *the problem* (P.P) *has a unique solution* $u_g \in \mathscr{D}'(\bar{D})$. *Furthermore the harmonic operator*

$$\begin{gathered} H: \quad g \to u_g \\ \mathscr{D}'(\partial D) \to \mathscr{D}'(\bar{D}) \end{gathered}$$

(*or Poisson operator*) *is continuous.*

Theorems (4.8) and (4.9) say that the Poisson kernel is a very regular kernel on $\bar{D} \times \partial D$, in the sense of [34] p. 536. An interesting problem would be to study the same property for the Green kernel of [14] and for the new kernels K_D and $K_{\partial D}$. Since K is (at least formally) a two sided fundamental kernel for (P.P), this could be seen as a partial analogue to Corollary 1 p. 540 of [34], which says that an operator A and its adjoint A^* are hypoelliptic in an open domain U if and only if every point of U has a neighborhood where A has a two sided very regular fundamental kernel.

Proof of (4.9). – The operator $H : \underset{g \to u_g}{C^\infty(\partial D) \to C^\infty(D)}$ having a C^∞ kernel, is regularizing and then extends in a continuous operator from $\mathscr{E}'(\partial D) = \mathscr{D}'(\partial D)$ into $C^\infty(D)$, defining

$$u_g(y) = \langle K(y,\cdot), g \rangle .$$

Moreover $(h-A)u_g = 0$ in D.

We have to prove that $u_g \in \mathscr{D}'(\bar{D})$ and that H is continuous for the topology of $\mathscr{D}'(\bar{D})$. The properties of K are unsufficient to conclude directly. We shall give another construction of u_g using the Fourier transform. We then first have to localize in order to work in $\mathbb{R}^d$.

Since $\bar{D}$ is compact one can find $\varepsilon > 0$ and $x_1, \ldots, x_n$, n points of ∂D such that the balls $B(x_j, \varepsilon)$ form a recovering of ∂D, and are via a C^∞ diffeomorphism F_j, good local charts. One can find $\eta > 0$ such that for all $x \in \partial D$ the ball $B(x,\eta)$ is included in one of the $B(x_j, \varepsilon) = U_j$. From the recovering $\bigcup_{x \in \partial D} B(x,\eta)$, we extract a finite subrecovering $\bigcup_{1}^{n'} B(x'_j, \eta)$. One can find $\eta' > 0$ such that $B_0 = \{\psi(x) > \eta'\}$ and the $B(x'_j, \eta/2)$ form a recovering of $\bar{D}$. Set $\eta_0, \eta_1, \ldots, \eta_n$, a subordinate partition of unity.

We have to prove that for $1 \leqslant j \leqslant n'$, $y \to \langle K(y,\cdot), \eta_j g\rangle$ belongs to $\mathscr{D}'(\bar{D})$, and the preceding construction ensures that supp $\eta_j \subset U_{\sigma(j)}$, $(U_{\sigma(j)}, F_{\sigma(j)})$ being a good local chart.

So as not to introduce unnecessary Jacobians, we replace the area measure (related to the charts induced by ψ) by the area measure $d\mu'$ induced by the (U_j, F_j)'s, but conserve the notation K for the Poisson kernel after changing the variables. Consider for $\xi \in \mathbb{R}^{d-1}$

$$(4.10)\quad \begin{cases} \text{i)}\ H_\ell(\xi,y) = \displaystyle\int_{\partial D} \eta_\ell(x) K(y,x) e^{i\langle F_{\sigma(\ell)}(x),\xi\rangle} d\mu'(x) \\ \text{ii)}\ = \displaystyle\int_{\mathbb{R}^{d-1}} (\eta_\ell \circ F_{\sigma(\ell)}^{-1})(0,x) K(y, F_{\sigma(\ell)}^{-1}(0,x)) e^{i\langle x,\xi\rangle}\, dx \\ \text{iii)}\ = E^y[\eta_\ell(y_T) 1_{T<+\infty} e^{i\langle F_{\sigma(\ell)}(y_T),\xi\rangle} H_T^{h,0}]. \end{cases}$$

From (4.8) H_ℓ is the solution of

$$\begin{cases} (h-A)H_\ell(\xi,\cdot) = 0 \text{ in } D \\ \gamma_0 H_\ell(\xi,\cdot) = \eta_\ell(\cdot) e^{i\langle F_{\sigma(\ell)}(\cdot),\xi\rangle} \in C^\infty(\partial D). \end{cases}$$

Then H_ℓ satisfies

$$(4.11)\quad \begin{cases} \text{i)}\ (\xi,y) \to H_\ell(\xi,y) \in C^\infty(\mathbb{R}^{d-1} \times \bar{D}) \\ \text{ii)}\ \text{For all } y \in D,\ \xi \to H_\ell(\xi,y) \in \mathscr{S}(\mathbb{R}^{d-1}) \\ \text{iii)}\ y \to (\xi \to H_\ell(\xi,y)) \text{ is continuous from } \bar{D} \text{ into } C^\infty(\mathbb{R}^{d-1}) \\ \quad\ \text{and from } D \text{ into } \mathscr{S}(\mathbb{R}^{d-1}) \end{cases}$$

(4.11) ii) is an immediate consequence of (4.10) ii) since H_ℓ is the inverse Fourier transform of a function of $C_0^\infty(\mathbb{R}^{d-1})$. To prove (4.11) i) it is enough to see that we can differentiate in ξ under the expectation in (4.10) iii) so that

$$(4.12)\quad \frac{\partial^\alpha}{\partial\xi^\alpha} H_\ell(\xi,y) = E^y[\eta_\ell(y_T) 1_{T<+\infty} P_\alpha(F_{\sigma(\ell)}(y_T)) e^{i\langle F_{\sigma(\ell)}(y_T),\xi\rangle} H_T^{h,0}]$$

where P_α is a complex polynomial, and then $\frac{\partial^\alpha}{\partial\xi^\alpha} H_\ell(\xi,\cdot)$ which is the solution of (P.P) with $g_\xi = \eta_\ell e^{i\langle F_{\sigma(\ell)},\xi\rangle}(P_\alpha \circ F_{\sigma(\ell)})$, belongs to $C^\infty(\bar{D})$ thanks to (4.8). $H_\ell(\cdot,\cdot)$ belongs to $C^\infty(D \times \mathbb{R}^{d-1})$ (differentiation under the integral sign in (4.10) i)), and all its derivatives are bounded from what precedes, then it belongs to $C^\infty(\bar{D} \times \mathbb{R}^{d-1})$. (4.11) iii) follows from (4.10) ii) and (4.12).

On the other hand H_ℓ and u_g are related by

$$(4.13)\quad \langle \eta_\ell(\cdot)K(y,\cdot),\eta_j(\cdot)g\rangle = (2\pi)^{1-d}\int_{\mathbb{R}^{d-1}} \hat{g}_j(\xi)H_\ell(\xi,y)\,d\xi$$

where $\hat{g}_j$ is the Fourier transform of $F_{\sigma(j)*}(\eta_j g)$, $F_{\sigma(j)*}$ being the image operator from $\mathscr{D}'(\partial D)$ to $\mathscr{D}'(\mathbb{R}^{d-1})$ induced by the diffeomorphism $F_{\sigma(j)}$. Hence since $\eta_j g \in \mathscr{E}'(\partial D)$, $\hat{g}_j \in C^\infty(\mathbb{R}^{d-1})$ and is slowly increasing.

Of course for $y \in \partial D$, $H_\ell(\cdot,y) \notin \mathscr{S}(\mathbb{R}^{d-1})$ and we cannot conclude that (4.13) is continuous at $y \in \partial D$ (otherwise $u_g \in C^0(\bar{D})$! !).

To obtain the existence of a sectional trace of order 0, we first assume that y belongs to a good local chart $B(x'_k,\eta/2)$, so that we shall work in the usual half space. If $y \in \mathbb{R}^d$ we put $y = (y^1,\bar{y}) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Let $\varphi \in C_0^\infty(R^{d-1})$, we study the following quantity

$$(4.14)\quad \langle (\eta_k \circ F_{\sigma(k)}^{-1})(y^1,\cdot)\langle \eta_\ell(\cdot)K(F_{\sigma(k)}^{-1}(y^1,\cdot),\cdot),\eta_j g\rangle,\eta\rangle, \quad 1 \leqslant k \leqslant n'.$$

From (4.11) iii), for $y^1 > 0$, one can apply Fubini's theorem so that (4.14) is still equal to

$$\begin{aligned}(4.15)\quad &(2\pi)^{1-d}\int \hat{g}_j(\xi)\int \varphi(\bar{y})(\eta_k \circ F_{\sigma(k)}^{-1})(y^1,\bar{y})H_\ell(\xi,F_{\sigma(k)}^{-1})(y^1,\bar{y})\,d\bar{y}\,d\xi \\ &= (2\pi)^{1-d}\int \hat{g}_j(\xi)\iint \varphi(\bar{y})(\eta_k \circ F_{\sigma(k)}^{-1})(y^1,\bar{y})(\eta_\ell \circ F_{\sigma(\ell)}^{-1})(0,x)e^{i\langle x,\xi\rangle} \\ &\qquad K(F_{\sigma(k)}^{-1}(y^1,\bar{y}),F_{\sigma(\ell)}^{-1}(0,x))\,dx\,d\bar{y}\,d\xi.\end{aligned}$$

If $\sigma(\ell) \neq \sigma(k)$, then $d(F_{\sigma(k)}^{-1}(y^1,\bar{y}),\text{supp. }\eta_\ell) > \eta$ by construction, and

$$(y^1,\bar{y}) \to (\eta_k \circ F_{\sigma(k)}^{-1})(y^1,\bar{y})(\eta_\ell \circ F_{\sigma(\ell)}^{-1})(0,x)K(F_{\sigma(k)}^{-1}(y^1,\bar{y}),F_{\sigma(\ell)}^{-1}(0,x))$$

is C^∞ up to the boundary $y^1 = 0$ so that

(4.16) $y^1 \to$ the quantity (4.14) belongs to $C^\infty(\mathbb{R}^+)$, and vanishes on $y^1 = 0$.

The only interesting case is then $\sigma(\ell) = \sigma(k)$. In that case we suppress this subscript and simply note F^{-1}. In the same way we write $\bar{\eta}_\ell$ (resp. $\bar{H}_\ell, \bar{K}$) instead of $\eta_\ell \circ F^{-1}$(resp. $H_\ell(\xi,F^{-1}(y^1,\bar{y})$, $K(F^{-1}(\cdot),F^{-1}(\cdot)))$. We shall show

(4.17) LEMMA. – *For all* $n \in \mathbb{N}$

$$\sup_{y^1>0}\ \sup_{\xi\in\mathbb{R}^{d-1}} (1+|\xi|^2)^n \left|\int \varphi(\bar y)\bar\eta_k(y^1,\bar y)\bar H_\ell(\xi,y^1,\bar y)\,d\bar y\right| \leqslant c(n).$$

Assume that (4.17) holds. Then

$$(4.18)\quad \begin{cases} \text{i)} & \left|\hat g_j(\xi)\int \varphi(\bar y)\bar\eta_k(y^1,\bar y)\bar H_\ell(\xi,y^1,\bar y)d\bar y\right| \\ & \qquad\qquad \leqslant c((d+q+1)/2)/(1+|\xi|^2)^{(1+d/2)} \\ & \text{if } |\hat g_j(\xi)| \leqslant c(1+|\xi|^2)^q \\ \text{ii)} & \lim_{y^1\to 0} \bar H_\ell(\xi,y^1,\bar y) = \bar\eta_\ell(0,\bar y)e^{i\langle\bar y,\xi\rangle}. \end{cases}$$

Since $H_\ell(\xi,\cdot)\in C^\infty(\bar D)$ we may apply a first time the bounded convergence theorem to obtain

$$(4.19)\quad \lim_{y^1\to 0}\int \varphi(\bar y)\bar\eta_k(y^1,\bar y)\bar H_\ell(\xi,y^1,\bar y)\,d\bar y = \int \varphi(\bar y)\bar\eta_k(0,\bar y)\bar\eta_\ell(0,\bar y,\cdot)e^{i\langle\bar y,\xi\rangle}\,d\bar y$$

and then a second time (thanks to (4.18) i)) and obtain

$$(4.20)\quad \lim_{y^1\to 0}\int \hat g_j(\xi)\int \varphi(\bar y)\bar\eta_k(y^1,\bar y)\bar H_\ell(\xi,y^1,\bar y)\,d\bar y\,d\xi$$
$$= \int \hat g_j(\xi)\int \varphi(\bar y)\bar\eta_k(0,\bar y)\bar\eta_\ell(0,\bar y)e^{i\langle\bar y,\xi\rangle}\,d\bar y\,d\xi = \langle \eta_k\eta_\ell\varphi, F_*(\eta_j g)\rangle.$$

Adding in k, ℓ one gets

$$(4.21)\quad \lim_{y^1\to 0}\langle\varphi, F_*(\eta_j u_g(y^1,\cdot))\rangle = \langle\varphi, F_*(\eta_j g)\rangle.$$

Hence $\gamma_0 u_g$ exists and is equal to g, in particular $u_g\in\mathscr{D}'(\bar D)$.

It remains to prove (4.17). By definition one has

$$(4.22)\quad \int \bar\eta_\ell(0,x)\bar K(y^1,\bar y,x)e^{i\langle x,\xi\rangle}\,dx = E^{F^{-1}(y^1,\bar y)}[1_{T<+\infty}\eta_\ell(y_T)e^{i\langle F(y_T),\xi\rangle}H_T^{h,0}].$$

We shall use the fact that (U,F) is a good local chart, by using the same localization procedure as in § 2 and § 3. As the reader can see the notation is very heavy

$$
(4.23)\quad \begin{cases}
1)\ E_0 = \inf\{t \geqslant 0, y_t \in \bar{B}(x'_\ell, \eta)\} \\
\quad S_n = \inf\{t \geqslant E_n, y_t \notin U\} \\
\quad E_{n+1} = \inf\{t \geqslant S_n, y_t \in \bar{B}(x'_\ell, \eta)\} \\
2)\ \phi_t(\omega, z) \text{ is the flow solution of} \\
\quad dz_t = (F^* Y_0)(z_t)\, dt + (F^* Y_i)(z_t)\, dw_t^i,\ z_0 = z = (z^1, \bar{z}) \in \mathbb{R}^d \\
3)\ S(z) = \inf\{s \geqslant 0, \phi_s(\omega, z) \notin F(U)\} \\
\quad T(z) = \inf\{s \geqslant 0, \phi_s(\omega, z) \in \{0\} \times \mathbb{R}^{d-1}\} \\
4)\ z_n = F(y_{E_n})\,;\ z'_n = \phi_{S(z_n)}(\omega, z_n) \\
5)\ H_t = H_t^{h,0}\,;\ \bar{H}_t(z) = \exp - \int_0^t h(\phi_s(\omega, z))\, ds .
\end{cases}
$$

Since $\operatorname{supp} \eta_\ell \subset B(x'_\ell, \eta/2)$, for $y^1 > 0$ we have

$$
\begin{aligned}
(4.24)\quad & E^{F^{-1}(y^1, \bar{y})}[1_{T < +\infty} \eta_\ell(y_T) e^{i\langle F(y_T), \xi\rangle} H_T] \\
&= \sum_{n \in \mathbb{N}} E^{F^{-1}(y^1, \bar{y})}[1_{T > E_n} H_{E_n} E^{y_{E_n}}[1_{T < S_0} \eta_\ell(y_T) e^{i\langle F(y_T), \xi\rangle} H_T]] \\
&= \sum_{n \in \mathbb{N}} E^{F^{-1}(y^1, \bar{y})}[1_{T > E_n} H_{E_n} E[(1 - 1_{T(z_n) > S(z_n)}) \\
&\qquad\qquad \bar{\eta}_\ell(\phi_{T(z_n)}(\omega, z_n)) e^{i\langle \phi_{T(z_n)}(\omega, z_n), \xi\rangle} \bar{H}_{T(z_n)}(z_n)]] \\
&= \sum_{n \in \mathbb{N}} E^{F^{-1}(y^1, \bar{y})}[1_{T > E_n} H_{E_n}\{A_n - B_n\}]
\end{aligned}
$$

with

$$
A_n = E[\bar{\eta}_\ell(\phi_{T(z_n)}(z_n) \exp i \langle \phi_{T(z_n)}(z_n), \xi \rangle \bar{H}_{T(z_n)}(z_n)]
$$

and

$$
\begin{aligned}
B_n = E[1_{T(z_n) > S(z_n)} \bar{H}_{S(z_n)}(z_n) E[\bar{\eta}_\ell(\phi_{T(z'_n)}(\omega', z'_n)) \bar{H}_{T(z'_n)}(z'_n) \\
\exp i \langle \phi_{T(z'_n)}(z'_n), \xi \rangle]] .
\end{aligned}
$$

Because of the equality

$$
|\xi|^{2j} e^{i\langle z, \xi\rangle} = (-1)^j \left(\sum_{k=1}^{d} \frac{\partial^{2j}}{\partial z_i^{2j}} \right) e^{i\langle z, \xi\rangle},
$$

to control $|\xi|^{2j} A_n$ and $|\xi|^{2j} B_n$ we are tempted to integrate by parts under the expectations.

Well for $n \geqslant 1$, $z_n = F(y_{E_n})$ and $d(y_{E_n}, \operatorname{supp}\cdot\eta_\ell) = \eta/2$ and for $n \geqslant 0$, $d(F^{-1}(z'_n), \operatorname{supp}\cdot\eta_\ell) > \eta/2$ that is the starting point and the final point are far enough, and we can actually use the calculus of variations near the boundary ([14]) to integrate by parts. The summation in n can the treated by standard arguments, similar to (2.15) and (2.16).

The only remaining term to control is the one we get for $n = 0$ in A_0, that is

$$(4.25)\quad |\xi|^{2j}\int \varphi(\bar{y})\eta_k(y^1,\bar{y})E[\eta_\ell(\phi_{T(y^1,\bar{y})}(\omega,y^1,\bar{y})) \\ \exp i\langle \phi_{T(y^1,\bar{y})}(y^1,\bar{y}),\xi\rangle \bar{H}_{T(y^1,\bar{y})}(y^1,\bar{y})1_{T(y^1,\bar{y})<+\infty}]\,d\bar{y}.$$

Because (U,F) is a good local chart, ϕ_s is of the form $\phi_s(z) = (z_s^1,\bar{z}_s)$, with

$$\begin{cases} dz_s^1 = dw_s^1 \\ d\bar{z}_s = \bar{Z}_0(z_s^1,\bar{z}_s)\,ds + \sum_{i=2}^{m} \bar{Z}_i(z_s^1,\bar{z}_s)\,dw_s^i \end{cases}$$

up to a Girsanov transformation. In particular one can build $\bar{z}\cdot$ with fixed z^1 *and* $w^1_\cdot$, furthermore

$$\begin{cases} T(\omega,z^1,\bar{z}) = T(w^1_\cdot,z^1) \\ \bar{z} \xrightarrow{\bar{\phi}} \phi_{T(z^1)}(\omega,z^1,\bar{z}) \text{ is a } C^\infty \text{ diffeomorphism of } \mathbb{R}^{d-1} \text{ for all } \omega \text{ out of} \\ \text{a null set of } \{T(z^1)<+\infty\}. \end{cases}$$

Hence (4.25) splits into two parts, just writting

$$(4.27)\qquad 1_{T<+\infty} = 1_{T\leqslant 1} + 1_{1<T<+\infty}.$$

On $T > 1$ we can again carry out the calculus of variations as before.

On $T \leqslant 1$, for fixed y^1 and ω, do the change of variable

$$(4.28)\qquad \bar{y} = \bar{\phi}^{-1}(\bar{z})$$

so that

$$(4.29)\quad \left|\int |\xi|^{2j}\varphi(\bar{y})\eta_k(y^1,\bar{y})E[\eta_\ell(\phi_{T(y^1)}(\omega,y^1,\bar{y}))\bar{H}_{T(y^1)}(y^1,\bar{y}) \\ 1_{T(y^1)\leqslant 1}\exp i\langle \phi_{T(y^1)}(y^1,\bar{y}),\xi\rangle]\,d\bar{y}\right| \\ = \left|\int |\xi|^{2j}\,\eta_\ell(0,\bar{z})e^{i\langle\bar{z},\xi\rangle}E[\eta_k(y^1,\bar{\phi}^{-1}(\bar{z}))\varphi(\bar{\phi}^{-1}(\bar{z})) \\ \text{Jacobian } \bar{\phi}^{-1}(\bar{z})1_{T(y^1)\leqslant 1}\bar{H}_{T(y^1)}(y^1,\bar{\phi}^{-1}(\bar{z}))]\,d\bar{z}\right|.$$

But from the properties of $\bar{\Phi}$, the above expectation is a C^∞ function of $\bar{z}$, which is bounded, and has all its derivatives (obtained by derivation under the expectation) bounded uniformly in y^1 for $y^1 \geqslant 0$ belonging to the (compact) support of $\bar{\eta}_k$. From classical properties of the Fourier transform, (4.29) is then uniformly bounded in y^1 and ξ. The proof of the lemma is finished.

Finally, to achieve the proof of theorem (4.9) it remains to prove the continuity of H. But the proof of (4.17) is unchanged if we replace $e^{i\langle z,\xi\rangle}$ by $P_\alpha(z)e^{i\langle z,\xi\rangle} = \dfrac{\partial^\alpha}{\partial\xi^\alpha} e^{i\langle z,\xi\rangle}$.

Hence $y^1 \to \left(\xi \to \int \varphi(\bar{y})\bar{\eta}_k(y^1,\bar{y})\bar{H}_\ell(\xi,y^1,\bar{y})\,d\bar{y}\right) \in \mathscr{S}(\mathbb{R}^d)$, is continuous and is equal to $\left(\xi \to \int \varphi(\bar{y})\bar{\eta}_k(0,\bar{y})\bar{\eta}_\ell(0,\bar{y})e^{i\langle\bar{y},\xi\rangle}\,d\bar{y}\right)$ on $y^1 = 0$.

The continuity of H follows from the continuity of the Fourier transform. □

(4.30) *Remark.* — It is possible to go further in the preceding proof by identifying $\bar{\Phi}^{-1}$ with the help of time reversal on the path ω as in § 1. It is not difficult then to show that $y^1 \to (F_*(\eta_j u_g))(y^1, \cdot)$ is a C^∞ function with values in $\mathscr{D}'(\mathbb{R}^{d-1})$. This result also proceeds from (4.6).

We stress the fact that the test function φ plays a crucial role in (4.29). It is because of the integration in $\bar{y}$ that we can get out $e^{i\langle z,\xi\rangle}$ of the expectation and then get a control in $\mathscr{S}(\mathbb{R}^{d-1})$ of the interesting quantities.

Without any difficulty we can extend (4.9) to non compact domains.

(4.31) Definition. — *$\mathscr{S}(\partial D)$ is the space of functions of $C^\infty(\partial D)$ which can be extended as functions of $\mathscr{S}(\mathbb{R}^d)$. $\mathscr{S}'(\partial D)$ is then the dual space of $\mathscr{S}(\partial D)$.*

(4.32) Theorem. — i) *Assume that ∂D is compact,* (H.G unif) *is fulfilled in $\bar{D}$, everything else being as in* (4.8). *Then the conclusions of* (4.8) *and* (4.9) *are still true with $\mathscr{S}'(\bar{D})$ in place of $\mathscr{D}'(\bar{D})$.*

ii) *Assume that* (1.1) iii), (0.7) *and* (H.G unif) *are fulfilled in $\bar{D}$. Then for any $g \in \mathscr{S}'(\partial D)$,* (P.P) *has a unique solution $u_g \in \mathscr{S}'(\bar{D})$. Furthermore the harmonic operator $H : g \to u_g$, is continuous when $\mathscr{S}'(\partial D)$ and $\mathscr{S}'(\bar{D})$ are equipped with the topologies induced by $\mathscr{D}'(\partial D)$ and $\mathscr{D}'(\bar{D})$.*

Proof. — i) One part is proved in [14] Thm. 5.16, and in that case $K \in \mathscr{S}(\bar{D} \times \partial D -$ diag. $(\bar{D} \times \partial D))$, so that the proof of (4.9) is still relevant.

ii) If ∂D is no more compact, we proved in [14] lemma 5.18 that provided (1.1) iii) is fulfilled, we can find $\varepsilon > 0$ such that for every $x \in \partial D$, $B(x,\varepsilon) \cap \bar{D}$ induces a very good local chart for the auxiliary process of § 3. Since $K \in \mathscr{S}(\bar{D} \times \partial D -$ diag $(\bar{D} \times \partial D))$ one can again apply the proof of (4.9). □

We are now in position to prove Theorem (4.2).

Proof of (4.2). — From Hörmander's theorem we know that $u \in C^\infty(D \cap U)$, thus we only have to prove that u extends C^∞ up to the boundary.

Let $x \in U \cap \partial D$ and $\varepsilon > 0$ such that $B(x,3\varepsilon) \cap \bar{D} \subset U$. We shall prove that $u \in C^\infty(B(x,\varepsilon) \cap \bar{D})$. We can extend the restriction to $B(x,2\varepsilon)$ of f into an application $\tilde{f} \in C_0^\infty(\bar{D})$. From [14] 5.14 and 5.16, the Dirichlet problem

$$\begin{cases} (h-A)u = \tilde{f} \text{ in } D \\ \gamma_0 u = 0 \text{ on } \partial D \end{cases}$$

has a solution $u_0 \in \mathscr{S}'(\bar{D}) \cap C^\infty(\bar{D})$. Then $u - u_0$ checks

$$\begin{cases} (h-A)(u-u_0) = f - \tilde{f} \text{ in } D, \text{ with } f - \tilde{f} \equiv 0 \text{ on } B(x,2\varepsilon) \\ (\rho h + v - \Gamma^0)(u-u_0) = g + \rho f + (V_0 \psi) \dfrac{\partial u_0}{\partial n} \in C^\infty(U \cap \partial D). \end{cases}$$

So we may and will assume that $f \equiv 0$ on $B(x,2\varepsilon)$, and that $\rho \equiv 0$.

From (4.7) we only have to prove that $\gamma_0 u \in C^\infty(B(x,\varepsilon) \cap \partial D)$. Let $\chi \in C_0^\infty(\partial D)$ with $\chi \equiv 1$ on $B(x,2\varepsilon)$. We have $\chi . \gamma_0 u \in \mathscr{E}'(\partial D)$, hence from (4.32) ii) one can build $\bar{u} \in \mathscr{S}'(\bar{D})$ satisfying

$$\begin{cases} (h-A)\bar{u} = 0 \text{ in } D \\ \gamma_0 \bar{u} = \chi \cdot \gamma_0 u \text{ on } \partial D. \end{cases}$$

Furthermore

$$\begin{cases} (h-A)(u-\bar{u}) = 0 \text{ in } D \cap B(x,2\varepsilon) \\ \gamma_0(u-\bar{u}) = 0 \text{ on } \partial D \cap B(x,2\varepsilon) \end{cases}$$

so that because of (4.7) again, $u - \bar{u} \in C^\infty(B(x,2\varepsilon) \cap \bar{D})$.

From (4.6), $\gamma_1 \bar{u}$ exists, and so $(v - \Gamma)\bar{u}$ is well defined on ∂D.

Since $u-\bar{u}\in C^\infty(B(x,\varepsilon)\cap\bar{D})$ on has

$$(v-\Gamma)\bar{u} = \bar{g} = g + (v-\Gamma)(\bar{u}-u)\in C^\infty(B(x,2\varepsilon)\cap\partial D).$$

Of course because $\gamma_0 u = \gamma_0\bar{u}$ on $\partial D\cap B(x,2\varepsilon)$ it is enough to show that $\gamma_0\bar{u}\in C^\infty(B(x,\varepsilon)\cap\partial D)$.

Now consider $g_n\in C_0^\infty(\partial D)$ which converges in $\mathscr{E}'(\partial D)$ to $\gamma_0\bar{u}$. Then from (4.32) ii) and (4.7),

$$(4.33)\quad u_n = H.g_n\in C^\infty(\bar{D})\cap\mathscr{S}'(\bar{D}) \text{ and } u_n \underset{n\to\infty}{\to} \bar{u} \text{ in } \mathscr{D}'(\bar{D}).$$

From (4.6) we also have

$$(4.34)\qquad \gamma_1 u_n \underset{n\to\infty}{\to} \gamma_1\bar{u} \quad\text{in}\quad \mathscr{D}'(\partial D).$$

Therefore

$$(4.35)\qquad (v-\Gamma)u_n = \bar{g} + \psi_n$$

with $\psi_n\in C^\infty(\partial D\cap B(x,2\varepsilon))$ and $\psi_n \underset{n\to\infty}{\to} 0$ in $\mathscr{D}'(\partial D)$.

Now we adapt the proofs of [13] Thm 2.3 or [14] Thm 5.3. Let $\chi\in C_0^\infty(B(x,2\varepsilon))$ such that $0\leqslant\chi\leqslant 1$, $\chi\equiv 1$ on $B(x,3\varepsilon/2)$ and $\eta\in C_0^\infty(\mathbb{R}^+)$ such that $\eta\equiv 1$ on $[0,1]$ and $\operatorname{supp}\eta\subset[0,2[$.

We apply the Ito formula to the function $(t,y)\to\eta(t)\chi u_n(y)$ and to the process $y\cdot$, which gives

$$(4.36)\quad \chi u_n(y) = R^{h,v,\eta}((h-A)\chi u_n,(v-\Gamma)\chi u_n)(y) - R^{h,v,\eta'}(\chi u_n,\chi g_n)(y)$$

where $\eta' = \dfrac{\partial\eta}{ds}$, and $R^{h,v,\eta}$ was defined at (3.28.3).

For $\varphi\in C_0^\infty(B(x,5\varepsilon/4)\cap\partial D)$ and ∂^α a C^∞ differential operator of length $|\alpha|$ on ∂D whose adjoint is called ∂^α_*, we apply (4.36) and the fact that η' is identically 0 out of $[1,2[$ in order to get

$$\begin{aligned}(4.37)\quad \langle\partial^\alpha\varphi,g_n\rangle_{\partial D} = &- \langle u_n,\chi\langle\varphi,\partial^\alpha_{*_1}K^\eta_D(\cdot 1,\cdot 2)\rangle_{\partial D}\rangle_D\\ &- \langle g_n,\chi\langle\varphi,\partial^\alpha_{*_1}K^{\eta'}_{\partial D}(\cdot 1,\cdot 2)\rangle_{\partial D}\rangle_D\\ &+ \langle\varphi,\partial^\alpha_* R^{h,v,\eta}(0,\chi\bar{g})\rangle_{\partial D}\\ &+ \langle\partial^\alpha\varphi,R^{h,v,\eta}(0,\chi\psi_n)\rangle_{\partial D}\\ &- \langle\partial^\alpha\varphi,R^{h,v,\eta}(u_n\,.\,A\chi + Y_i u_n Y^i\chi, g_n\,.\,\Gamma\chi + V_\ell g_n\,.\,V^\ell\chi)\rangle_{\partial D}\end{aligned}$$

where $\partial^\alpha_{*_1}$ acts on the first variable of $K^{\eta'}$ represented by $\cdot 1$, this variable only describing ∂D.

Since $Y_i\chi \equiv 0$ on $B(x,3\varepsilon/2) \cap \bar{D}$ (similarly $V_\ell\chi \equiv 0$ on $B(x,3\varepsilon/2) \cap \partial D$) and $\text{supp.}\,\varphi \subset B(x,5\varepsilon/4)$, the integration in the last term of (4.35) holds out of the diagonal. So we can introduce the kernels K^η_D and $K^\eta_{\partial D}$. We also integrate by parts $Y_i u_n \cdot Y^i\chi$ (resp. $V_\ell g_n \cdot V^\ell\chi$) using Stokes theorem (resp. the integration by parts formula on ∂D). The last term of (4.35) becomes:

$$
\begin{aligned}
(4.38)\quad & + \langle u_n, \{\bar{A}\chi + Y_i\chi\,.\,\mathrm{div}\,Y^i\}\langle\varphi, \partial^\alpha_{*_1}K^\eta_D(\cdot 1,\cdot 2)\rangle_{\partial D}\rangle_D \\
& + \langle u_n, Y_i\chi\,.\,Y^i_{\cdot 2}\langle\varphi, \partial^\alpha_{*_1}K^\eta_D(\cdot 1,\cdot 2)\rangle_{\partial D}\rangle_D \\
& + \langle g_n, Y_i\chi\,.\,Y^i\psi\langle\varphi, \partial^\alpha_{*_1}K^\eta_D(\cdot 1,\cdot 2)\rangle_{\partial D}\rangle_{\partial D} \\
& + \langle g_n, \{\bar{\Gamma}\chi + V_\ell\chi\,.\,\mathrm{div}\,V^\ell\}\langle\varphi, \partial^\alpha_{*_1}K^\eta_{\partial D}(\cdot 1,\cdot 2)\rangle_{\partial D}\rangle_{\partial D} \\
& + \langle g_n, V_\ell\chi\,.\,V^\ell_{\cdot 2}\langle\varphi, \partial^\alpha_{*_1}K^\eta_{\partial D}(\cdot 1,\cdot 2)\rangle_{\partial D}\rangle_{\partial D}
\end{aligned}
$$

where $\bar{A} = 1/2 \sum_{i=1}^{m} Y_i^2 - Y_0$ and $\bar{\Gamma} = 1/2 \sum_{\ell=1}^{r} V_\ell^2 - V_0$.

We desire now control the fourth term of (4.37), which can be written

$$
(4.39)\quad \langle\partial^\alpha\varphi, R^{h,v,\eta}(0,\chi\psi_n)\rangle_{\partial D} = \int_{\partial D} \partial^\alpha\varphi(y) E^y\left[\int_0^2 \eta(t)\chi\psi_n(y_t)H^{h,v}_t\,dL_t\right] d\mu(y).
$$

Taking a smaller ε if necessary, we may assume that $B(x,4\varepsilon)$ is included in a good local chart. In order to simplify the notation we do not make appear the corresponding diffeomorphism, so that $y_t = (u_t, \phi_t(\omega,0,\bar{y}))$ up to the stopping time S defined by $S = \inf\{t \geqslant 0, y_t \notin B(x,4\varepsilon)\}$. (4.39) becomes

$$
\begin{aligned}
(4.40)\quad & \int_{\mathbb{R}^{d-1}} \partial^\alpha\varphi(\bar{y})\, E\left[\int_0^2 \eta(t)1_{t<S}\,\chi\psi_n(\phi_t(\omega,0,\bar{y}))H^{h,v}_t\,dL^0_t\right] d\bar{y} \\
& + \int_{\partial D} \partial^\alpha\varphi(y)\, E^y\left[\int_0^2 \eta(t)1_{t\geqslant S}\,\chi\psi_n(y_t)H^{h,v}_t\,dL_t\right] d\mu(y).
\end{aligned}
$$

The techniques of § 3 allow to show that the second term may be written as

$$\int_{\partial D} \partial^\alpha \varphi(y) \int_{\partial D} \chi(x)\psi_n(x)q(y,x)\, d\mu(x)\, d\mu(y)$$

with $q \in C^\infty(\partial D \times \partial D)$. Indeed the time S is big enough so that we have time to make the calculus of variations in both time directions.

In the first term of (4.40), for (ω,t) fixed we do the change of variable $z = \phi_t(\omega,0,\bar{y})$, and define

$$\bar{S} = \inf\{s \geqslant 0, (u_s, \phi_s \circ \phi_t^{-1}) \notin B(x,4\varepsilon)\}$$

so that

$$(4.41) \quad \int_{\mathbb{R}^{d-1}} \partial^\alpha \varphi(\bar{y})\, E\left[\int_0^2 \eta(t) 1_{t<S} \chi\psi_n(\phi_t(\omega,0,\bar{y}) H_t^{h,v} dL_t^0\right] d\bar{y}$$

$$= \int_{\mathbb{R}^{d-1}} \chi(z)\psi_n(z) E\left[\int_0^2 \eta(t) 1_{t<\bar{S}}\, \partial^\alpha \varphi(\phi_t^{-1}(\omega,0,z)) \right.$$

$$\left. \text{Jacobian } \phi_t^{-1}(\omega,0,z) H_t^{h,v} dL_t^0 \right] dz$$

where $H_t^{h,v} = \exp - \int_0^t [h(u_s, \phi_s \circ \phi_t^{-1})\, ds + v(\phi_s \circ \phi_t^{-1})\, dL_s^0]$.

Thanks to the properties of ϕ_t^{-1}, $z \to E[\cdots] \in C^\infty(\mathbb{R}^{d-1})$, hence

$$\langle \psi_n, \chi E[\cdots]\rangle \underset{n\to\infty}{\to} 0.$$

Similarly, $\left\langle \psi_n, \chi \int_{\partial D} \partial^\alpha \varphi(y) q(y,\cdot)\, d\mu(y)\right\rangle \underset{n\to\infty}{\to} 0$.

So we can go in the limit when $n \to \infty$ in (4.37) and obtain an integration by parts formula

$$(4.42) \qquad \langle \partial^\alpha \varphi, \chi \cdot \gamma_0 \bar{u}\rangle_{\partial D} = \langle \bar{u}, \theta(\varphi)\rangle_D + \langle \gamma_0 \bar{u}, \theta'(\varphi)\rangle_{\partial D}$$

where θ (resp. θ') is a continuous linear operator from $C_0^\infty(\partial D)$ into $C_0^\infty(\bar{D})$ (resp. $C_0^\infty(\partial D)$) satisfying

$$\|\theta(\varphi)\|_{C_M^k(\bar{D})} \leqslant c(k)\|\varphi\|_\infty (\text{resp. } \|\theta'(\varphi)\|_{C_{M'}^k(\partial D)} \leqslant c'(k)\|\varphi\|_\infty)$$

for any compact subset M of $\bar{D}$ (resp. M' of ∂D) and any $k \in \mathbb{N}$. So $\chi \cdot \gamma_0 \bar{u} \in C^\infty(B(x,\varepsilon))$. □

4.B. Solution of the boundary value problem (L.P).

The first part of this section is very much inspired by [22]. As in [32], [22], we introduce a stochastic version of (L.P), that is a martingale problem, which solution is shown to be $R^{h,v}_{f,g}$. Then using the regularity of $R^{h,v}_{f,g}$, it is shown that this function is a classical solution of (L.P). Uniqueness follows from (4.2). Extensions to some Sobolev spaces ((4.5)) are given at the end of the section.

(4.43) Definition. — *A measurable function u is called a stochastic solution of* (L.P) *if for any $y \in \bar{D}$, the process $t \to M_t(u)$ defined by*

$$M_t(u) = u(y_t) - u(y_0) + \int_0^t (f - hu)(y_s)\, ds + \int_0^t (g - vu)(y_s)\, dL_s$$

is a P_y, L^2 local martingale.

(4.44) Proposition (see [22] Prop. 3.2). — *Suppose that $f \in C^\infty(\bar{D})$, $g \in C^\infty(\partial D)$ and that* (1.1) *holds. Then every $u \in C^2(\bar{D})$ which is a stochastic solution of* (L.P), *is a classical solution of* (L.P). *The converse is also true.*

Proof. — That any smooth classical solution is a stochastic solution follows from Ito formula.

Conversely, if $u \in C^2(\bar{D})$ is a stochastic solution of (L.P), straightforward computations show that

$$- Au(y) = (f - hu)(y) \quad \text{for} \quad y \in D .$$

It easily follows that one can find sequence of stopping times S_n going to infinity almost surely and such that

$$\text{(4.45)} \quad \text{for all } t \geqslant 0, \ - \int_0^{t \wedge S_n} (\Gamma u(y_s) + (g - vu)(y_s))\, dL_s = 0 .$$

But it is well known that if ∂D is non characteristic, for $y \in \partial D$, P_y a.s $\inf\{t \geqslant 0, L_t > 0\} = 0$, so that from (4.45) proves that the boundary condition is satisfied. □

We can now prove Theorem (4.4).

Proof of (4.4). — i) We already know that $R_{f,g}^{h,v} \in C^\infty(\bar{D})$. To simplify we shall call it u in the rest of the proof.

We shall show that u is a stochastic solution of (L.P), and then a classical one, thanks to (4.44).

Using (0.11) and the Markov property we get immediately

$$(4.46)\quad u(y_t) = (H_t^{h,v})^{-1}\left\{E^y\left[\int_0^{+\infty} H_s^{h,v}(f(y_s)\,ds+g(y_s)\,dL_s)/F_t\right] - \int_0^t H_s^{h,v}(f(y_s)\,ds+g(y_s)\,dL_s\right\}.$$

Put $M_t = E^y\left[\int_0^{+\infty} H_s^{h,v}\,(f(y_s)\,ds+g(y_s)\,dL_s)/F_t\right]$ which is a continuous matingale thanks to (4.46). Integrating by parts we obtain:

$$(4.47)\quad \int_0^t (H_s^{h,v})^{-1}\,\delta M_s$$
$$= (H_t^{h,v})^{-1}M_t - M_0 - \int_0^t (H_s^{h,v})^{-1}M_s(h(y_s)\,ds+v(y_s)\,dL_s)$$
$$= u(y_t) - u(y) - (H_t^{h,v})^{-1}\int_0^t H_s^{h,v}(f(y_s)\,ds+g(y_s)\,dL_s)$$
$$- \int_0^t \left(u(y_s)+\left[(H_s^{h,v})^{-1}\int_0^s H_a^{h,v}(f(y_a)\,da+g(y_a)\,dL_a)\right]\right)(h(y_s)\,ds+v(y_s)\,dL_s)$$

and

$$\int_0^t (H_s^{h,v})^{-1}\left[\int_0^s H_a^{h,v}(f(y_a)\,da+g(y_a)\,dL_a)\right](h(y_s)\,ds+v(y_s)\,dL_s)$$
$$= \int_0^t\int_0^s H_a^{h,v}(f(y_a)\,da+g(y_a)\,dL_a)\,d[(H_a^{h,v})^{-1}]$$
$$= (H_t^{h,v})^{-1}\int_0^t H_s^{h,v}(f(y_s)\,ds+g(y_s)\,dL_s) - \int_0^t (f(y_s)\,ds+g(y_s)\,dL_s).$$

Report this last equality in (4.47). We see that

$$(4.48)\qquad \int_0^t (H_s^{h,v})^{-1}\,\delta M_s = M_t(u).$$

Hence $M_t(u)$ is a local martingale.

It remains to show uniqueness. If u and u' are two solutions (in $\mathcal{D}'(\bar{D})$) of (L.P), then $\bar{u} = u - u'$ solves (L.P) with $f \equiv 0$ and $g \equiv 0$. From (4.2), $\bar{u} \in C^\infty(\bar{D})$. Indeed (0.7) is automatically fulfilled if (0.11) holds. We may apply the Ito formula and get

$$E^y[\bar{u}(y_t)H_t^{h,v}] - \bar{u}(y) = 0. \tag{4.49}$$

From (0.11), $\lim_{T\to\infty} E^y\left[\int_0^T H_t^{h,v}\bar{u}(y_t)\,dt\right]$ exists. Hence $\lim_{T\to\infty} \bar{u}(y) \times T$ exists. So, $\bar{u}(y) = 0$.

ii) is immediate in view of the proof of i).

iii) $u \in \mathcal{S}'(\bar{D})$ can be shown exactly as in [14] Thm 5.16, because the kernels are rapidly decreasing ((2.25) ii). If $\chi_n \in C_0^\infty(\bar{D})$ is equal to 1 on $\bar{D}_n = \bar{D} \cap \{|x| \leqslant n\}$, one immediately sees that $R^{h,v}_{\chi_n f, \chi_n g}$ is a solution of (L.P) with data $(\chi_n f, \chi_n g)$, hence from (4.6) and what precedes $R^{h,v}_{f,g}$ is a solution of (L.P) with data (f,g). According to (4.2) (recall that (0.11) is stronger than (0.7)), if $u \in \mathcal{S}'(\bar{D})$ is a solution of (L.P) with $f \equiv 0$ and $g \equiv 0$, then $u \in C^\infty(\bar{D})$. Hence $R^{h,v}_{u,0}$ exists, so $\lim_{t\to\infty}\left(\int_0^t E^y[u(y_s)H_s^{h,v}]\,ds\right)$ exists, but this is still equal to $\lim_{t\to\infty} t \times u(y)$, it follows that $u(y) = 0$. □

To finish the section we extend, in the compact case, our study to suitable Sobolev spaces by proving Theorem (4.5).

Proof of (4.5). — The proof is very similar to the one of theorem 5.19 in [14], so we only give an outline of it.

First we may assume that $\rho \equiv 0$ thanks to section 3.A.

Then it can be shown as in [13] 2.22 that $R^{h,v}$ extends as a continuous operator from $L^2(\bar{D}) \times L^2(\partial D)$ into $L^2(\bar{D})$ (the difference with [13] here is the necessity of the localization to apply the diffeomorphism property of § 3).

Let $(f_n, g_n) \in C^\infty(\bar{D}) \times C^\infty(\partial D)$ which converges to (f,g) in $H^{2k}(\bar{D}) \times H^{2k}(\partial D)$.

Then $R^{h,v}_{f_n,g_n}$ converges to $R^{h,v}_{f,g}$ in L^2.

This proves: $(h-A)R^{h,v}_{f,g} = f$ in D.

From the continuity of the sectional traces (4.6), we conclude that $R^{h,v}_{f,g}$ is a solution of (L.P). Uniqueness was proved in the preceding theorem.

Now we have to prove that $R^{h,v}_{f,g}$ belongs to $H^{2k}(\bar{D})$. The integration by parts formula of § 2 allows to show that $R^{h,v}$ is a continuous operator from $H^{2k}(\bar{D}) \times H^{2k}(\partial D)$ into $H^{k}(\bar{D})$ (for each differentiation in the normal coordinate, we have to differentiate twice in the tangential coordinates, see [11] 1.17, and the formula at the beginning of § 1.C). From the hypoelliptic regularity theorem proved in [13] we also know that $R^{h,v}_{f,g} \in H^{2k}_{\text{loc}}(D)$. So it is enough to study the behaviour of $R^{h,v}_{f,g}$ near the boundary. By localization, we just have to look at the case

$$\bar{D} = \mathbb{R}^d_+, \qquad A = 1/2 \frac{\partial^2}{\partial y_1^2} + b(y) \frac{\partial}{\partial y_1} + \bar{A}$$

where $\bar{A}$ is a second order differential operator acting on the d-1 last variables.

Put $y = (y^1, \bar{y})$, $F = R^{h,v}_{f,g}$. For $y^1 \geqslant 0$ one has

$$\text{(4.50)} \qquad 1/2 \frac{\partial^2 F}{\partial y_1^2} + b(y) \frac{\partial F}{\partial y_1} = (h - \bar{A})F - f = d(F).$$

For f and g of C^∞ class, the derivatives in $y_2, \ldots, y_d$ of F are computed by direct differentiation under the expectation, so that it is easily shown as in [12] 2.22 that if $|\alpha| \leqslant 2k, \frac{\partial^\alpha F}{\partial \bar{y}^\alpha} \in L^2(\bar{D})$, if $f \in H^{2k}(\bar{D})$ and $g \in H^{2k}(\partial D)$.

For $k = 1$, one knows that $\frac{\partial F}{\partial y_1} \in L^2$, hence from (4.50) $\frac{\partial^2 F}{\partial y_1^2} \in L^2$. Furthermore

$$\text{(4.51)} \quad \frac{\partial F}{\partial y_1}(y_1, \bar{y}) = \left(\exp - \int_1^{y_1} 2b(a, \bar{y})\, da\right) \left\{\frac{\partial F}{\partial y_1}(1, \bar{y}) + \int_1^{y_1} 2d(F) \exp \int_1^s 2b(a, \bar{y})\, da\, ds\right\}.$$

For $k = 2$, we know that $F \in H^2$. We may differentiate in y_1 in (4.50). $\bar{A}\left(\frac{\partial F}{\partial y}\right)$ can be computed by differentiation under the integral in (4.51) and then belongs to L^2, so that $\frac{\partial^3 F}{\partial y_1^3} \in L^2$. We may differentiate another time in (4.50) to explain $\frac{\partial^4 F}{\partial y_1^4}$, and again compute $\bar{A}\left(\frac{\partial^2 F}{\partial y_1^2}\right)$ directly from (4.51).

So $\dfrac{\partial^4 F}{\partial y_1^4} \in L^2$, and $F \in H^4(\bar{D})$. This procedure can be iterated for all k. □

(4.52) *Remark.* – In the previous proof, L^2 can be replaced by $C^0(\bar{D})$ and then H^{2k} by $C^{2k}(\bar{D})$, thanks to the Feller property of the semi group. In particular $R^{h,v}$ is a continuous operator from $C^{2k}(\bar{D}) \times C^{2k}(\partial D)$ (resp $H^{2k}(\bar{D}) \times H^{2k}(\partial D)$) into $C^{2k}(\bar{D})$ (resp $H^{2k}(\bar{D})$).

The only difference is that the hypoelliptic regularity theorem of [13], says that if $f \in C^{2k}(\bar{D})$, any solution u of $(h-A)u = f$ belongs to $C^{2k-1}(D)$. Hence a priori $R^{h,v}_{f,g} \in C^{2k-1}(D)$.

But recall the proof of Theorem (3.4). In order to prove the regularity in D of $R^{h,v}_{f,g}$, it is enough to work with a $\varphi \in C_0^\infty(D)$ in (3.12). Hence the integrations by parts it remains to do are (3.17) and (3.18), with $U \supset \text{supp} \cdot \varphi$.

In these two cases we have to use the usual calculus of variations on an ordinary flow $\tilde{\xi}$, before the exit time out of U_ε for the function φ with support in U. It is easily seen that this is the usual formulation of the local calculus of variations, such that the integration by parts (3.17) up to order $2k$ involves the derivatives of f and h up to order $2k$, while the integration by parts (3.18) does not make appear any derivative of g.

So $R^{h,v}_{f,g}$ belongs to $C^{2k}(D)$ as soon as f belongs to $C^{2k}(\bar{D})$.

4.C. On the Gauge condition.

Conditions (0.11) and (0.7) are not really readable on the generator A. Then it seems useful to give some explicit conditions ensuring that (0.7) and (0.11) are satisfied. In [14] § 4.C we showed

(4.53) PROPOSITION.

If one of the three following conditions is satisfied

i) $\inf_{x \in \bar{D}} h(x) > 0$,

ii) *$\bar{D}$ is compact, $h \geqslant 0$ and A is non totally degenerate* (*that is the Y_i's, $1 \leqslant i \leqslant m$, do not all vanish at a same point*),

iii) *$\bar{D}$ is compact and for all $y \in \bar{D}$*

$$h(y) > \left(1/4 \sum_{i=1}^{m} [Y_i(\operatorname{div} Y_i) - (\operatorname{div} Y_i)^2] - 1/2 \operatorname{div} Y_0\right)(y)$$

then (0.7) *holds.*

The first analogue of (4.53) i) is

$$\inf_{x \in \bar{D}} h(x) > 0 \quad \text{and} \quad v \geqslant 0. \tag{4.54}$$

If v is not necessarily positive we may again choose a sufficiently big h for (0.11) to hold.

Actually $E[e^{vL_t}] \leqslant ce^{v^2 t/2}$ for a constant v, so that

(4.55) if $\inf_{x \in \bar{D}} h(x) - 1/2 \sup_{x \in \partial D} v^2(x) > 0$, then (0.11) holds.

It is more difficult to give sufficient conditions for (0.11) to hold when h may vanish. But we similarly have

(4.56) if $\inf_{x \in \partial D} v(x) > 0$ and $h \geqslant 0$ then

$$\sup_{y \in \bar{D}} E^y\left[\int_0^{+\infty} H_t^{h,v}\, dL_t\right] = J_{\partial D}(y) < +\infty.$$

It is easily checked that if (4.56) holds, and if for all $y \in \bar{D}$

$$\lim_{t \to +\infty} E^y[H_t^{h,v}] = 0,$$

then for $f \equiv 0$, the conclusions of (4.4) and (4.52) remain true, without assuming (0.11).

A sufficient condition for $\lim_{t \to \infty} E^y[H_t^{h,v}]$ to be equal to 0, when $v > 0$ and $h \geqslant 0$, is that for all $y \in \bar{D}$, $\lim_{t \to \infty} L_t = +\infty$, P_y a.s (indeed if it holds we can apply the dominated convergence theorem).

Assume that (4.53) ii) holds. Then from [14] 5.21, $\sup_{y \in \bar{D}} E^y[T] < +\infty$ (recall that T is the first time the process reaches the boundary).

We define by induction, the sequence of stopping times

(4.57) $T_0 = T$; $S_n = T_{n-1} + 1$; $T_n = \inf\{t \geqslant S_n, y_t \notin D\}$.

From what precedes P_y a.s for all n, $T_n < +\infty$ and because $T_{n+1} - T_n \geqslant 1$, we have $T_n \underset{n\to\infty}{\to} +\infty$ P_y a.s.

On the other hand $L_{S_{n+1}} - L_{S_n} = L_{T_{n+1}} - L_{T_n}$ in law, hence for all $y \in \bar{D}$

(4.58) $P_y(L_{S_{n+1}} - L_{S_n} > \varepsilon) \geqslant E^y[P_{y_{T_n}}(L_1 > \varepsilon)] > \eta > 0$.

We can conclude from the Markov property of the local time and the Borel Cantelli lemma that $L_t \underset{t\to\infty}{\to} +\infty$, P_y a.s.

So we can state the following theorem.

(4.59) THEOREM. — *Assume that $\bar{D}$ is compact, that* (1.1) *and* (H.G) *are fulfilled on $\bar{D}$, that A is non totally degenerate, and that* $\inf_{x\in\partial D} v(x) > 0$, $h(x) \geqslant 0$ *for all $x \in \bar{D}$. Then if $f \in C^\infty(\bar{D})$* (resp. $H^{2k+2}(\bar{D})$) *and $g \in C^\infty(\partial D)$* (resp. $H^{2k}(\partial D)$), (L.P) *has a solution $u \in C^\infty(\bar{D})$* (resp. $H^{2k}(\bar{D})$), *unique in $\mathscr{D}'(\bar{D})$.*

Proof. — According to Thm 5.12, 5.19 of [14] we may find $w \in C^\infty(\bar{D})$ (resp. $H^{2k+2}(\bar{D})$) solution of

$$\begin{cases} (h-A)w = f & \text{in } D \\ \gamma_0 w = 0 & \text{on } \partial D. \end{cases}$$

From what precedes, there exists an unique $u \in \mathscr{D}'(\bar{D})$ solution of

$$(4.60) \qquad \begin{cases} (h-A)u = 0 & \text{in } D \\ (v-\Gamma)u = g + (V_0\psi)\dfrac{\partial w}{\partial n} + \rho f & \text{on } \partial D \end{cases}$$

and $u \in C^\infty(\bar{D})$ (resp. $H^{2k}(\bar{D})$ because $\dfrac{\partial w}{\partial n} \in H^{2k+1}(\bar{D})$ admits a trace in $H^{2k+1/2}(\partial D)$). Of course $u + w$ is a solution of (L.P). Uniqueness is clear. □

In the same way, theorem (4.2) may be extended as in [13] 2.23, in the following form.

(4.61) PROPOSITION. — *We assume that* $\bar{D}$ *is compact, and that* (1.1), (0.7), (H.G) *are satisfied on* $\bar{D}$. *Any* $u \in \mathscr{D}'(\bar{D})$ *solution of* (L.P) *with* $f \in H^{2k+2}(\bar{D})$ *and* $g \in H^{2k}(\partial D)$, *belongs to* $H^{2k}(\bar{D})$.

4.D. Remarks.

Theorem (4.2), and theorem (5.3) of [14] are the analogues of Hörmander's hypoellipticity theorem in [21]. But in the Sobolev spaces context, the results we have just proved are less pleasant than the one's one has in the uniformly elliptic case, but also than the one's we proved for the Dirichlet problem in [14].

Even when the solutions are more regular than the data in the uniformly elliptic case, as regular for the Dirichlet problem in the hypoelliptic case, they become less regular in (4.59) and (4.61).

As we said at the beginning of the paragraph and in the introduction, (L.P) was only solved in the uniformly elliptic case for a compact domain $\bar{D}$. In that case our results are contained in [29], [28] or [33], except the uniqueness in $\mathscr{D}'(\bar{D})$ (that is except the *a priori* regularity). Uniqueness is proved in $C^0(\bar{D})$ in the three papers. On the other hand the boundary condition is more general in [29] and in [33] as ours. Actually [33] only assume weak transversality ($V_0\psi + \rho > 0$) and the condition can be of Ventcel-Levy type in [29]. The stochastic representation for the solution of (L.P) is given in [29] Prop. 5.2. In some recent papers (see e.g [22], [27]) the Neumann problem for the Schrödinger operator $1/2\,\Delta + q$ was discussed, for irregular potentials q or irregular domains D. In particular the authors produce the relations between uniqueness, the Gauge condition and the positivity of the first eigenvalue of $-(1/2\,\Delta + q)$. In [27], the case of an infinite Gauge is also discussed. The next paragraph contains a similar discussion, in the hypoelliptic context.

5. COMPLEMENTS

In the whole paragraph, besides explicit mention of the contrary, we assume that $\bar{D}$ is compact and that (1.1) holds.

5.A. A decomposition result.

As in [13] § 2.C we start by studying some functional spaces related to $(h-A, v-\Gamma)$.

(5.1) DEFINITION. – *For* $\infty \geqslant k \geqslant 2$, *we define*

$$M^k (\text{resp. } N^k) = \{f \in C^k(\bar{D}) \, (\text{resp. } H^k(\bar{D})), (v-\Gamma)f = 0\},$$

M^k_∞ (resp. N^k_∞) *is the closure of* M^∞ *in* M^k (resp. N^k).

The previous definition is meaningful. If $f \in C^k(\bar{D})$, $(v-\Gamma)f$ is defined in the usual sense, and if $f \in H^2(\bar{D})$, $Af \in H^0(\bar{D})$ admits sectional traces on ∂D up to order 1, when f admits sectional traces of order 0 and 1. If $\rho \equiv 0$, one can also define N^0 and N^1 since Af does not appear in Γf, in particular since $C^0(\bar{D}) \subset L^2(\bar{D})$ one can define M^0 and M^1 just taking the traces and differentiation in the sense of distributions.

We collect the results we know on the previous spaces in the following proposition.

(5.2) PROPOSITION.

1) *For* $k \geqslant 2$, M^k *is a closed subspace of* $C^k(\bar{D})$.

2) *For* $k \geqslant 3$, N^k *is a closed subspace of* $H^k(\bar{D})$. *If furthermore* $\rho \equiv 0$, N^2 *is closed in* $H^2(\bar{D})$.

3) $C^\infty_0(D)$ *is (sequentially) dense in* M^0 *and* N^0.

4) *If* $\rho \equiv 0$, *and* $V_\ell \equiv 0$, $1 \leqslant \ell \leqslant r$, *for* $k \geqslant 2$, $M^k_\infty = M^k$ *and* $N^k_\infty = N^k$.

Proof. – 1) is clear. To show 2) and since the sectional traces are not continuous, we have to consider the usual trace operators on

$H^k(\bar{D})$. If $k \geqslant 3$, $f \to (\gamma_0 f, \gamma_1 f, \gamma_0(Af))$ is continuous

$$\text{from } H^k(\bar{D}) \text{ into } H^{k-1/2}(\partial D) \times H^{k-3/2}(\partial D) \times H^{k-5/2}(\partial D).$$

If $\rho \equiv 0$, $\gamma_0(Af)$ disappears in Γ and the result is true for $k = 2$. 3) is clear since $C_0^\infty(D)$ is dense in L^2. Then we prove 4). Let $f \in M^k$ (resp. N^k). Since $V_0\psi > 0$ on ∂D, one has

$$(5.3) \quad \gamma_1 f = [v\gamma_0 f - (V_0 - (V_0\psi)n)\gamma_0 f]/V_0\psi = F(\gamma_0 f).$$

Let $g_n \in C^\infty(\partial D)$ converging to $\gamma_0 f$ in $C^k(\partial D)$ (resp. $H^{k-1/2}(\partial D)$), $h_n = F(g_n)$.

Then $h_n \in C^\infty(\partial D)$ and converges to $\gamma_1 f$ in $C^{k-1}(\partial D)$ (resp. $H^{k-3/2}(\partial D)$).

The restriction operators

$$\pi : C^k(\bar{D}) \to \prod_{0 \leqslant j \leqslant k} C^j(\partial D), \qquad 0 \leqslant k \leqslant +\infty$$

$$f \to (\gamma_j f)$$

$$\pi'_k : H^k(\bar{D}) \to \prod_{0 \leqslant j \leqslant k-1/2} H^j(\partial D)$$

are continuous surjections.

So there exists $f_n \in C^\infty(\bar{D})$ with $\gamma_0 f_n = g_n$ and $\gamma_1 f_n = h_n$. Furthermore according to the open mapping theorem f_n converges to f in $C^k(\bar{D})$ (resp. $H^k(\bar{D})$) modulo ker π_k (resp. ker $\pi'_k = H_0^k$). Since $C_0^\infty(D)$ is dense in ker π_k(resp. H_0^k), one can find $\varphi_n \in C_0^\infty(D)$ such that $f_n \to \varphi_n$ converges to f in $C^k(\bar{D})$ (resp. $H^k(\bar{D})$). By construction $(v-\Gamma)(f_n+\varphi_n) = 0$. □

Remark. – If $\rho \not\equiv 0$, or if one of the V_ℓ's does not vanish, the convergence of h_n to $\gamma_1 f$ only holds in $C^{k-2}(\partial D)$ (resp. $H^{k-5/2}(\partial D)$) and we do not know if (5.2.4) is still true.

(5.4) Definition. – *Set* $\lambda(h,v) = \inf\{\lambda_0 \in \mathbb{R}$, *for all* $\lambda > \lambda_0$, (0.11) *is satisfied for* $(h+\lambda, v)\}$. *According to* (4.55) $\lambda(h,v) < +\infty$. *For* $\lambda > \lambda(h,v)$ *we put* $G^\lambda f = R_{f,0}^{h+\lambda,v}$.

If (H.G) *is satisfied on* $\bar{D}$, G^λ *is then a continuous operator of* $C^\infty(\bar{D})$ (resp. $H^{2k}(\bar{D})$, resp. $C^{2k}(\bar{D})$) (*see* (4.52)).

(5.5) Proposition. – *If* (H.G) *is satisfied on* $\bar{D}$, *for all* $0 \leqslant k < +\infty$, G^λ *is a compact operator of* $C^{2k}(\bar{D})$.

Proof. – As in [13] 2.27 we shall show that for $|\alpha| \leqslant 2k$, $\partial^\alpha G^\lambda f$ is uniformly Hölder continuous on $\|f\|_{C^{2k}(\bar{D})} \leqslant 1$. Without loss of generality and taking

$$\lambda > \max(\lambda(h,v), \lambda(h, v+\rho h))$$

if necessary, we may assume that $\rho \equiv 0$.

Let $\eta \in C_0^\infty(\mathbb{R}^+)$ such that supp. $\eta \subset [0,\delta[$ for a $\delta < 1$ which will be chosen at the end of the proof, and $\bar{\eta} = 1-\eta$. Then for $|\alpha| \leqslant 2k$,

$$(5.6) \quad |\partial^\alpha G^\lambda f(y) - \partial^\alpha G^\lambda f(y')| \leqslant |\partial^\alpha R_{f,0}^{\lambda+h,v,\eta}(y) - \partial^\alpha R_{f,0}^{\lambda+h,v,\eta}(y')| + |\langle \partial^\alpha K_D^{\bar{\eta}}(y,\cdot) - \partial^\alpha K_D^{\bar{\eta}}(y',\cdot), f\rangle|$$

where $K_D^{\bar{\eta}}$ is the C^∞ kernel of the regularizing operator $f \to R_{f,0}^{\lambda+h,v,\bar{\eta}}$ (see (3.28.3)).

From the estimates (2.25) one shows easily that there exists $N > 0$ such that for all α

$$(5.7) \qquad \|\partial^\alpha K_D^{\bar{\eta}}(\cdot,\cdot)\|_\infty \leqslant c(\alpha)\,\delta^{-N(|\alpha|+d')}\ (d' = d/2).$$

So that

$$(5.8) \quad |\langle \partial^\alpha (K_D^{\bar{\eta}}(y,\cdot) - K_D^{\bar{\eta}}(y',\cdot)), f\rangle| \leqslant c(\alpha)\,\delta^{-N(|\alpha|+d'+1)}|y-y'|.$$

In order to estimate the first term of the right hand side of (5.6), one can localize as in (4.4) iii). If supp $f \subset D_\varepsilon = \{\psi(x) > \varepsilon\}$ we obtain by introducing the non reflected flow

$$(5.9) \qquad |\partial^\alpha R_{f,0}^{\lambda h,v,\eta}(y) - \partial^\alpha R_{f,0}^{\lambda+h,v,\eta}(y')| \leqslant c(\alpha,\varepsilon)\delta.$$

Then if supp$f \subset$ good local chart, we transport the problem in $\mathbb{R}_+^d$ with $f \in C_0^{2k}(\mathbb{R}_+^d)$. The derivatives in the tangential coordinates $y_2, \ldots, y_d$ of $R_{f,0}^{\lambda+h,v,\eta}$ are computed by differentiation under the expectation, so we have

$$(5.9\ bis) \quad \left|\frac{\partial^\alpha}{\partial \bar{y}^\alpha} R_{f,0}^{h+\lambda,v,\eta}(y) - \frac{\partial^\alpha}{\partial \bar{y}^\alpha} R_{f,0}^{h+\lambda,v,\eta}(y')\right| \leqslant c(\alpha')\delta.$$

But since $G^\lambda f \in C^{2k}(\bar{D})$ and $R_{f,0}^{h+\lambda,v,\bar{\eta}} \in C^\infty(\bar{D})$, $R_{f,0}^{h+\lambda,v,\eta} \in C^{2k}(\bar{D})$ and if we put $F = R_{f,0}^{h+\lambda,v,\eta}$(resp. $\bar{F}$ with $\bar{\eta}$), F checks the following equality

$$(5.10) \quad 1/2\,\frac{\partial^2 F}{\partial y_1^2} + b(y_1,\bar{y})\frac{\partial F}{\partial y_1} = (h+\lambda-\bar{A})F + (h+\lambda-A)\bar{F} = d(F).$$

For $k = 0$, according to (5.8) and (5.9 *bis*), and taking $\delta = |y-y'|^s$ with $s = 1/2N(1+d')$ one shows that $G^\lambda f$ is $(s \wedge 1/2)$ Hölder continuous.

For $k = 1$, one solves (5.10)

$$(5.11)\quad \frac{\partial F}{\partial y_1}(y^1,\bar{y}) = \exp\left(-\int_0^{y_1} 2b(a,\bar{y})\,da\right)\left\{\frac{\partial F}{\partial y_1}(0,\bar{y}) - \int_0^{y_1} 2d(F)\exp\int_0^s 2b(a,\bar{y})\,da\,ds\right\}.$$

All the terms in (5.11) are controlled with the help of (5.7) and (5.9), unless $\frac{\partial F}{\partial y_1}(0,\bar{y})$. But $G^\lambda f$ satisfies (L.P) and so

$$(5.12)\quad \frac{\partial F}{\partial y_1}(0,\bar{y}) = -\frac{\partial \bar{F}}{\partial y_1}(0,\bar{y}) + (1/V_0^1(\bar{y}))[(v-\bar{\Gamma})(F+\bar{F})](0,\bar{y})$$

where $\bar{\Gamma}$ is a second order differential operator acting on $\partial D(=\mathbb{R}^{d-1})$ (4.1). Therefore if U is a relatively compact open subset of $\mathbb{R}^d$ and $(y,y') \in U \times U$, one obtains

$$(5.13)\quad \left|\frac{\partial F}{\partial y_1}(0,\bar{y}) - \frac{\partial F}{\partial y_1}(0,\bar{y})\right| \leqslant c(0)\,\delta^{-N(1+d')}|y-y'|$$
$$+ \sup_{y\in\bar{U}}(1/V_0^1(\bar{y}))\{\|v\|_\infty + c(\bar{\Gamma}))\,\delta + (\|v\|_\infty + c'(\bar{\Gamma}))\,\delta^{-N(3+d')})|y-y'|\}.$$

On the other hand

$$(5.14)\quad \left|\int_0^{y_1} 2d(F)(s,\bar{y})\exp\int_0^s 2b(a,\bar{y})\,da\,ds - \int_0^{y'_1} 2d(F)(s,\bar{y}')\exp\int_0^s 2b(a,\bar{y}')\,da\,ds\right|$$
$$\leqslant c(\bar{U},b)|y-y'|\{1+\|h+\lambda\|_\infty c'(0)\,\delta + c(\bar{A})\,\delta + c(h,\lambda,A)\,\delta^{-N(2+d')})\}$$
$$+ c'(\bar{U},b)\{|y-y'| + c(h,\lambda,\bar{A})\,\delta + c(h,\lambda,A)\,\delta^{-N(3+d')})|y-y'|\}.$$

By choosing $\delta = |y-y'|^s$ with $s = 1/2\,N(3+d')$, we obtain that $\frac{\partial F}{\partial y_1}$ is s' Hölder continuous whith $s' = \min\left(s, 1/2\,\frac{1+d'}{3+d'}\right)$.

Returning to (5.10) it is immediately seen that $\frac{\partial^2 F}{\partial y_1^2}$ also is s'' Hölder continuous for a suitable s''. In the same way one shows that $\frac{\partial^2 F}{\partial y_1 \partial y_j}$, $j \neq 1$, is uniformly Hölder continuous by differentiation in y_j in (5.11) and (5.12).

For $k \geqslant 2$, one iterates the method as in (4.5), by a step by step differentiation in (5.10), (5.11) and (5.12). □

The compactness of the resolvant operators is the key point in the proof of Thm 2.30 in [13], which gives a decomposition of the spaces L^2 and C^∞ of a compact manifold (without boundary) into the sum of $\operatorname{Ker}(h-A)$ and $\operatorname{Im}(h-A^*)$. The first step of the proof consists in the (classical) identification of $\ker(h-A)$ and $\operatorname{Im}(h-A)$ with $\operatorname{Ker}(\mathrm{Id}-\lambda G^\lambda)$ and $\operatorname{Im}(\mathrm{Id}-\lambda G^\lambda)$.

The present situation is more intricate because of the boundary.

We start by the identification result.

(5.15) Proposition. — *Assume that* (H.G) *is satisfied in* $\bar{D}$. Then for $\lambda > (\lambda(h,v) \vee \lambda(h,v+\rho h))$ *and* $2 \leqslant k \leqslant +\infty$,

1) *If* $f \in C^k(\bar{D})$, $G^\lambda f \in M^k$ *and* $(h-A)G^\lambda f = (\mathrm{Id}-\lambda G^\lambda)f$.
2) *If* $f \in M^k$, $G^\lambda(h-A)f = (\mathrm{Id}-\lambda G^\lambda)f$.
3) $\ker(h-A) \cap M^k = \ker(\mathrm{Id}-\lambda G^\lambda)$, *and* $(h-A)(M^k) \cap M^k = (\mathrm{Id}-\lambda G^\lambda)(M^k)$, *where* Id *and* G^λ *are considered as operators on* $C^k(\bar{D})$.
4) *If furthermore* (H.C) *is satisfied for* h,

$$\ker(h-A) \cap M^\infty = \ker(h-A) \cap M^k .$$

Proof: 1) is a consequence of (4.4) i) and (4.52), 2) is an immediate application of the Ito formula, 4) is a consequence of (4.2). Now let $f \in M^k$ such that $(h-A)f = 0$. Then 2) implies that $f \in \ker(\mathrm{Id}-\lambda G^\lambda)$. Conversely if $f - \lambda G^\lambda f = 0$, then according to 1), $f \in M^k$ and $(h-A)f = (h-A)\lambda G^\lambda f = 0$.

Finally if $f \in C^k(\bar{D})$ satisfies $f = (h-A)g$ for a $g \in M^k$, then $(\mathrm{Id}-\lambda G^\lambda)(f+\lambda g) = f$ according to 2). Conversely if $f = (\mathrm{Id}-\lambda G^\lambda)g$ with $g \in C^k(\bar{D})$ then $(h-A)G^\lambda g = f$ and $G^\lambda g \in M^k$. So we have proved a stronger result than 3) since we have

$$(5.16)\qquad \begin{cases}(\mathrm{Id}-\lambda G^\lambda)(C^k(\bar D)) = (h-A)(M^k)\\ (\mathrm{Id}-\lambda G^\lambda)(C^k(\bar D)) \cap M^k = (\mathrm{Id}-\lambda G^\lambda)(M^k).\end{cases}$$ □

Now we are going to describe the adjoint system of (A,Γ). This requires to make the good normalization for V_0.

(5.17) DEFINITION. – *We shall call $\bar n$ the inner conormal relative to A, that is the vector field defined by : $\bar n f = n \,.\, 1/2 \sum_{i=1}^{m} (Y_i\psi)(Y_i f)$.*

Multiplying v by $2(V_0\psi)/a^2$ if necessary, we may assume that

$$(5.18)\qquad V_0\psi = 1/2\, a^2 = 1/2 \sum_{i=1}^{m} (Y_i\psi)^2$$

so that $V_0 = \bar n + \bar V_0$, where $\bar V_0$ is tangential to ∂D. Let A^* be the formal adjoint of A given by

$$(5.19)\qquad A^* = 1/2 \sum_{i=1}^{m} Y_i^2 - Y_0 + (\operatorname{div} Y_i)\,.\,Y^i - \operatorname{div}(Y_0 - 1/2(\operatorname{div} Y_i)\,Y^i).$$

Recall the classical Green formula for f and $g \in C^\infty(\bar D)$.

$$(5.20)\qquad \int_D (f\,.\,Ag - g\,.\,A^*f)(x)\,dx = -\int_{\partial D}\left\{ f\frac{dg}{d\bar n} - g\frac{df}{d\bar n} + (Y_0 - 1/2(\operatorname{div} Y_i)\,.\,Y^i)\psi\right\}(z)\,d\mu(z).$$

If we define Γ_0^* by

$$(5.21)\qquad \Gamma_0^* = 1/2 \sum_{\ell=1}^{r} V_\ell^2 - \bar V_0 + (\operatorname{div} V_\ell)\,.\,V^\ell + \frac{d}{d\bar n} - \operatorname{div}(\bar V_0 - 1/2\,(\operatorname{div} V_\ell)\,.\,V^\ell) + [1/2\,(\operatorname{div} Y_i)\,.\,Y^i - Y_0]\psi.$$

$$\Gamma^* = \Gamma_0^* - \rho A^*$$

then for $\rho \equiv 0$, f and $g \in C^\infty(\bar D)$, one gets the duality formula

$$(5.22)\qquad \int_D (f\,.\,Ag - g\,.\,A^*f)(x)\,dx = -\int_{\partial D} (f\,.\,\Gamma g - g\,.\,\Gamma^* f)(z)\,d\mu(z).$$

As in [29] we shall call (A^*,Γ^*) the adjoint system of (A,Γ).

One checks without any difficulty that (A^*,Γ^*) satisfies (1.1) and also (H.G) if (A,Γ) satisfies (H.G).

Therefore all the results of the paper are still true if we consider $h - A^*$, $v - \Gamma^*$, $G^{*\lambda}$, M^{*k} etc... for λ sufficiently large.

In the rest of the section we assume that $\rho \equiv 0$ and that (H.G) is satisfied in $\bar{D}$. From (5.22) we obtain for $2 \leqslant k \leqslant +\infty$,

(5.23) If $g \in M^k, f \in M^{*k}$, $\int_D f \,.\, Ag \, dx = \int_D g \,.\, A^* f \, dx .$

Then (5.23) and (5.16.1) give

(5.24) For λ large enough, f and $g \in C^k(\bar{D})$,

$$\int_D f \,.\, G^\lambda g \, dx = \int_D g \,.\, G^{*\lambda} f \, dx .$$

For λ large enough, we write G_k^λ for the restriction of G^λ to $C^k(\bar{D})$.

Hence G_k^λ is a continuous operator of $C^k(\bar{D})$, and G_{2k}^λ is compact. Let $\mathscr{D}'_k$ be the dual space of the Banach space $C^k(\bar{D})$, DG_k^λ be the adjoint operator of G_k^λ, which is a continuous operator of $\mathscr{D}'_k$. It is clear that DG_k^λ coïncides with $G_k^{*\lambda}$ on $C^k(\bar{D})$. Since G_{2k}^λ and $G_{2k}^{*\lambda}$ are compact operators on Banach spaces for $k < +\infty$ the following holds:

$$(5.25) \quad \left\{ \begin{aligned} &\dim \ker (\mathrm{Id} - \lambda G_{2k}^\lambda) = \dim \ker (\mathrm{Id} - \lambda DG_{2k}^\lambda) \\ &\qquad\qquad = \operatorname{codim} \operatorname{Im} (\mathrm{Id} - \lambda G_{2k}^\lambda) < +\infty \\ &\dim \ker (\mathrm{Id} - \lambda G_{2k}^{*\lambda}) = \dim \ker (\mathrm{Id} - \lambda DG_{2k}^{*\lambda}) \\ &\qquad\qquad = \operatorname{codim} \operatorname{Im} (\mathrm{Id} - \lambda G_{2k}^{*\lambda}) < +\infty \\ &\operatorname{Im} (\mathrm{Id} - \lambda G_{2k}^\lambda) (\text{resp. } \operatorname{Im} (\mathrm{Id} - \lambda G_{2k}^{*\lambda})) \\ &\qquad\qquad \text{is a closed subspace of } C^{2k}. \end{aligned} \right.$$

But,

$$\begin{aligned} \dim \operatorname{Ker} (\mathrm{Id} - \lambda DG_{2k}^\lambda) &\geqslant (\dim \ker (\mathrm{Id} - \lambda G_{2k}^{*\lambda}) = \dim \ker (\mathrm{Id} - \lambda DG_{2k}^{*\lambda}) \\ &\geqslant (\dim \ker (\mathrm{Id} - \lambda G_{2k}^\lambda) . \end{aligned}$$

Hence all the dimensions in (5.25) are equal. On the other hand it arises from (5.24) that

(5.26) $\operatorname{Im} (\mathrm{Id} - \lambda G_k^{*\lambda}) \subset (\ker (\mathrm{Id} - \lambda G_k^\lambda))^\perp$ ($\perp$ in the L^2 sense) and then for $k < +\infty$

$$(5.27)\quad C^{2k}(\bar{D}) = \ker(\mathrm{Id} - \lambda G_{2k}^{\lambda}) \oplus \mathrm{Im}(\mathrm{Id} - \lambda G_{2k}^{*\lambda})$$
$$= \ker(\mathrm{Id} - \lambda G_{2k}^{*\lambda}) \oplus \mathrm{Im}(\mathrm{Id} - \lambda G_{2k}^{\lambda})$$
$$\text{(only algebraic sum)}$$
$$= (\ker(h-A) \cap M^{2k}) \oplus (h-A^*)(M^{*2k})$$
$$= (\ker(h-A^*) \cap M^{*2k}) \oplus (h-A)(M^{2k}).$$

Let we trace the situation on $C^{\infty}(\bar{D})$.

Let $d_k = \dim\ker(\mathrm{Id} - \lambda G_{2k}^{\lambda})$. d_k is a decreasing sequence of integers. But according to (5.25), $d_k = \dim\ker(\mathrm{Id} - \lambda D G_{2k}^{\lambda})$, and is also an increasing sequence. So for all $k \geqslant 1$, d_k is a constant. Hence

$$(5.28)\quad \mathrm{Ker}(\mathrm{Id} - \lambda G_{2k}^{\lambda}) = \ker(h-A) \cap M^{2k}$$
$$= \ker(h-A) \cap M^{\infty} \quad \text{for all } f \geqslant 1.$$

Let $f \in C^{\infty}(\bar{D})$. From (5.27) and (5.28) we get for all $k \leqslant 1$,

(5.29) $f = f_{\ker} + (h-a^*)g_k$, with $f_{\ker} \in \ker(h-A) \cap M^{\infty}$ and $g_k \in M^{*2k}$.

Furthermore the decomposition (5.29) is unique, so that for $k \geqslant 1$,

$$(h-A^*)(g_k - g_1) = 0, \qquad \text{and} \qquad g_k - g_1 \in M^{*2}.$$

Hence

$$g_k - g_1 \in \ker(h-A^*) \cap M^{*2} = \ker(h-A^*) \cap M^{*\infty},$$

i.e $g_k - g_1 \in M^{*\infty}$.

So $g_1 \in M^{*2k}$ for all $k \geqslant 1$, i.e $g_1 \in M^{*\infty}$.

We have proved the following theorèm.

(5.30) Theorem of decomposition. — *Assume that* (H.G) *is fulfilled on* $\bar{D}$ *and that* $\rho \equiv 0$. *Then*

$$C^{\infty}(\bar{D}) = (\ker(h-A) \cap M^{\infty}) \oplus (h-A^*)(M^{*\infty})$$
$$= (\ker(h-A^*) \cap \mathrm{M}^{*\infty}) \oplus (h-A)(M^{\infty}).$$

Furthermore $(h-A)(M^{\infty})$(resp. $(h-A^*)(M^{*\infty})$) *is closed in* $C^{\infty}(\bar{D})$ *and*

$$\dim(\ker(h-A) \cap M^{\infty}) = \dim(\ker(h-A^*) \cap M^{\infty}) < +\infty.$$

That $(h-A)(M^\infty)$ is closed in $C^\infty(\bar{D})$ is proved in the following way: let $g_n = (h-A)f_n$ with $f_n \in M^\infty$, and g_n converges to g with $g \in C^\infty(\bar{D})$. Then since $(h-A)(M^{2k})$ is closed in $C^{2k}(\bar{D})$ (see (5.25)), $g \in (h-A)(M^{2k})$ for all $k \geqslant 1$ and we conclude as above.

Remark that in the previous proof, if $f \in C^2(\bar{D})$ is a solution of $(h-A)f \equiv 0$ and $(v-\Gamma)f = 0$, then $f \in C^\infty(\bar{D})$ even if (0.7) is not satisfied.

Finally replacing v by $v + \rho h$, one has a biunivoque correspondance between the kernels of $(h-A)$ for $\rho \not\equiv 0$ and $\rho \equiv 0$. Then

(5.31) COROLLARY. – *If* (H.G) *is fulfilled in* $\bar{D}$, *then for all* $k \geqslant 1$

$$\ker(h-A) \cap M^k = \ker(h-A) \cap M^\infty$$

and $\dim(\ker(h-A) \cap M^\infty) = \dim(\ker(h-A^*) \cap M^{*\infty})$.

5.B. Invariant and reversible measures.

In this section we furthermore assume that D is connected, $h \equiv 0$ and $v \equiv 0$. Let $y\cdot$ be the generic element of the canonical space $C^0(\mathbb{R}^+, \bar{D})$ equipped with the cylindrical σ-field and the family $(Q_y)_{y \in \bar{D}}$ of the laws solutions of the stochastic system (0.9).

(5.32) DEFINITION. – *Let* ν *be an element of the dual space* $\mathscr{D}'_0$ *of* $C^0(\bar{D})$, *that is a measure on* $\bar{D}$. *We define* $Q_\nu = \int_{\bar{D}} Q_y \nu(dy)$. *We shall say that* ν *is an invariant measure (for* $(y\cdot, Q_y)$) *if for all* $t \geqslant 0$,

$$\int_{\bar{D}} T_t f(y) \nu(dy) = \int_{\bar{D}} f(y) \nu(dy)$$

with $T_t f(y) = E^y[f(y_t)]$, *or equivalently that* $T_t^\nu f = \langle f, \nu \rangle$, *where*

$$T_t^\nu f = E^\nu[f(y_t)],$$

and E^ν *is the expectation relative to* Q_ν.

Since $\bar{D}$ is compact, and T_t has the Feller property, for any probability ν, any weak limit of $\left(\frac{1}{t}\int_0^t T_s^\nu \, ds\right)_{t \in \mathbb{R}_x^+}$ when $t \to \infty$ is an invariant probability, so that the set of invariant measures is non empty.

On the other hand since for $f \in C^0(\bar{D})$, $\lambda G^\lambda T_t f$ converges uniformly on $\bar{D}$ to $T_t f \in C^0(\bar{D})$ when λ goes to ∞, one has, if ν is a measure

$$(5.33)\quad \nu \text{ invariant} \Leftrightarrow \text{For all } f \in C^0(\bar{D}) \text{ and } \lambda > 0, \int_{\bar{D}} (\mathrm{Id} - \lambda G^\lambda) f \, d\nu = 0$$
$$\Leftrightarrow \text{For all } f \in C^\infty(\bar{D}) \text{ and } \lambda > 0, \int_{\bar{D}} (\mathrm{Id} - \lambda G^\lambda) f \, d\nu = 0$$
$$\Leftrightarrow (\mathrm{Id} - \lambda D G^\lambda)\nu = 0 .$$

From the preceding section we can state

(5.34) Proposition. — *If* (H.G) *is fulfilled in* $\bar{D}$, *the space of invariant measures is finite dimensional, with dimension* d_ν and

$$1 \leqslant d_\nu \leqslant \dim(\ker A \cap M^\infty) = \dim \ker (\mathrm{Id} - \lambda G^\lambda)$$

(*remember that the results in* 5.*A are obtained for* $k \geqslant 1$).

A generic example will prove that even if A is non totally degenerate, d_ν may be any integer.

(5.35) *Example.*

$$\bar{D} = [-1,1] \times [-1,1],$$

$$Y_1 = \frac{\partial}{\partial x}, \qquad Y_2 = \sin(ky)\frac{\partial}{\partial y}, \qquad Y_0 = \cos(ky)\frac{\partial}{\partial y}, \qquad V_0 = \frac{\partial}{\partial n}.$$

Of course ∂D is not smooth, but one can remove the corners without essential modification in what follows. It is clear that A is non totally degenerate (Y_1 is constant), that Y_1, Y_2 and $[Y_0, Y_2]$ span $\mathbb{R}^2$ at any point of $\mathbb{R}^2$ so that (H.R) is fulfilled. Furthermore for $k \in \mathbb{N}^*$, ∂D is non characteristic. Some classical results on one dimensional diffusions (cf. [23] p. 361-367) show that if $y \in [(2p-1)\pi/k, 2p\pi/k] \subset]-1,1[$, then for all $x \in [-1,1]$, the process $y\cdot$, starting from (x,y), lives in $[-1,1] \times [(2p-1)\pi/k, 2p\pi/k]$, so that there exists an invariant probability ν_p with support included in this set. Of course if $p \neq p'$ and k big enough, ν_p and $\nu_{p'}$ are linearly independant.

Nonetheless if we slightly strength (H.G), we obtain

(5.36) Theorem. — *We assume that* $\mathscr{L}\mathrm{ie}(Y_1, \ldots, Y_n)(y)$ *spans* $\mathbb{R}^d$ *at each* $y \in \bar{D}$. *Then there exists a unique invariant probability* ν_0, *and any* *invariant measure* ν *belongs to* $\mathbb{R}\nu_0$. *Under* (5.18), ν_0 *has the following decomposition*

$$d\nu_0(y) = 1_D(y)p_0(y)\,dy + 1_{\partial D}(y)(\rho p_0)(y)\,d\mu(y)$$

with $p_0 \in (\ker A^*) \cap M^{*\infty}$, *and* p_0 *is strictly positive on* $\bar{D}$.

Finally any $p \in \mathscr{D}'(\bar{D})$ *solution of* $A^*p = 0$, $\Gamma^*p = 0$, *belongs to* $\mathbb{R}p_0$.

Proof. – We first prove that $\ker(\mathrm{Id} - \lambda G^\lambda) = \mathbb{R}$, which shows the first part of the theorem. The proof is based on the following lemma.

(5.37) LEMMA. – *Under the hypotheses of* (5.36), *if* $f \in C^0(\bar{D})$, $f \not\equiv 0$ *and* $f \geqslant 0$, *for all* $\lambda > 0$, $G^\lambda f > 0$.

Indeed if (5.37) holds, let $f = \lambda G^\lambda f$ and $b = \inf_{\bar{D}} f$. Since $(\mathrm{Id} - \lambda G^\lambda)b = 0$, one has $f - b \geqslant 0$ and $f - b = \lambda G^\lambda(f-b)$. If $f \not\equiv b$, (5.37) yields $f - b > 0$ which is impossible. So $\ker(\mathrm{Id} - \lambda G^\lambda) = \mathbb{R}$.

Remark. – Of course for (5.37) to hold a sufficient condition would be for the support of the law of the process $y\cdot$ to be the full $\bar{D}$. But we do not know any readable condition for this to hold, except the one in the statement of the theorem.

In order to prove (5.37) we shall actually prove that for all $t > 0$, the density $p(t,y,x) > 0$ for all $(y,x) \in \bar{D} \times \bar{D}$. This will be done with the help of elementary large deviations results. First we recall the following lemma.

(5.38) LEMMA (see [7] Prop. 2.1 or [6] chap. 1). – *Let* $Z \in \mathscr{L}\mathrm{ie}(Y_1, \ldots, Y_m)$ *be a* C^∞ *vector field. Then every integral curve of* Z *can be uniformly approximated by piecewise differentiable curves, each differentiable path of these curves being an integral curve of one of the* Y_i*'s.*

Let y and $x \in \bar{D}$. Following [6] we define

$$(5.39)\quad \begin{cases} K_y^x = \{f \in C^0([0,1], \bar{D}), df(t) = Y_i(f(t))h^i(t)\,dt \\ \qquad \text{with } h \in L^2 \text{ and } f(0)=y, f(1)=x, f(t) \in D \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \in]0,1[\} \\ d(y,x) = \inf_{f \in K_y^x} \left(\int_0^1 |h(t)|^2\,dt \right)^{1/2}, \text{ if } K_y^x \neq \varnothing, \ +\infty \text{ otherwise.} \end{cases}$$

In the terminology of [6] and [25] an f of K_y^x is called an horizontal curve. Notice that we impose to an horizontal curve to stay in D except for the starting and the terminal points. A simple deterministic time reversal argument shows that there is a biunivoque correspondance between K_y^x and K_x^y and that if for a $z \in D$, K_y^z and K_x^z are non empty, K_y^x is also non empty.

Let $y_0 \in D$ being fixed. Since D is C^∞ path connected, for each $x \in D$, there exists $f \in C^\infty([0,1], D)$ with $f(0) = x$, $f(1) = y_0$. According to (5.38), f is uniformly approximate by an horizontal curve (because $f(t) \in D$ for all t) joining x to an y neighboring on y_0. According to the proof of theorem 1.14 of [6] one can find a neighborhood $\mathscr{V}$ of y_0 included in D such that any $y \in \mathscr{V}$ can be joined to y_0 by an horizontal curve. Therefore we have

$$\text{(5.40)} \qquad \text{For all } x \text{ and } y \in D,\ K_y^x \neq \varnothing .$$

Let $x \in \partial D$. Since ∂D is non characteristic at x, one of the $Y_i(x)$'s $(1 \leqslant i \leqslant m)$ is non tangential to ∂D, say $Y_1(x)$. So for sufficiently small $\varepsilon > 0$, the path

$$\text{(5.41)} \quad df(t) = Y_1(f(t)) \operatorname{sign} \langle Y_1(x), n(x) \rangle \, dt, \quad f(0) = x, \quad t \in [0,\varepsilon]$$

satisfies $f(t) \in D$ for $t \in]0,\varepsilon]$. Let $x_\varepsilon = f(\varepsilon)$, one can join x to $x_\varepsilon (\in D)$ by an horizontal curve and, from (5.40), x_ε to y_0 by an horizontal curve, hence

$$\text{(5.42)} \qquad \text{For all } x \text{ and } y \in \bar{D}, \qquad K_y^x \neq \varnothing .$$

Moreover it can be shown as in [6] that $(y,x) \to d(y,x)$ is continuous, so that $M = \sup_{\bar{D} \times \bar{D}} d(x,y) < +\infty$.

Let $y \in D$ and U an open subset of D.

We define $C_y(U) = \{f \in C^0([0,1], D),\ f(0) = y,\ f(1) \in U\}$ which is an open subset of $\{f \in C^0([0,1], \mathbb{R}^d),\ f(0) = y\}$.

One deduces from what precedes that the Cramer functional $\Gamma(C_y(U))$ (see [39]) is finite. Let P_y be the law of the ordinary diffusion with generator A. It follows from [39] that

$$\text{(5.43)} \quad -\Gamma(C_y(U)) \leqslant \varliminf_{\varepsilon \to 0} \varepsilon \log P_y(z^\varepsilon_\cdot \in C_y(U)) = \varliminf_{\varepsilon \to 0} \varepsilon \log Q_y(z^\varepsilon_\cdot \in C_y(U))$$

where $z_t^\varepsilon = y_{\varepsilon t}$ for $0 \leqslant t \leqslant 1$, because P_y and Q_y coïncide on $C_y(U)$.

Therefore we obtain by classical arguments

$$(5.44)\qquad -M \leqslant -d(y,U) = -\inf_{z\in U} d(y,z) \leqslant \underline{\lim}_{t\to 0}\, t\log Q_y(y_t\in U) \leqslant \overline{\lim}_{t\to 0}\, t\log Q_y(y_t\in \bar{U})$$

because $\{y_t \in U\}$ contains $\{z^\varepsilon_\cdot \in C_y(U)\}$ up to a Q_y null set.

From the Markov property one then obtains

(5.45) For all U open set of $\bar{D}$, all $y \in D$ and all $t > 0$,
$$Q_y(y_t \in U) > 0.$$

In particular if $\rho \equiv 0$, the Q_y law of y_t admits for $t > 0$, a C^∞ density $p(t,y,\cdot)$ (see § 2) and $p(t,y,\cdot)$ is strictly positive on a dense open subset of $\bar{D}$.

Let $y \in \bar{D}$, V an open neighborhood of y (in $\bar{D}$) and $T = \inf\{t \geqslant 0, y_t \notin V\}$.

Then

$$(5.46)\qquad Q_y(y_t\in V) = Q_y(y_t\in V, T\leqslant t) + Q_y(y_t\in V, T>t).$$

Since $t \to Q_y(y_t \in V)$ is lower semi continuous, there exists $t_0 > 0$ such that for $t \leqslant t_0$, $Q_y(y_t \in V) \geqslant 1/2$.

Since $Q_y(y_t \in V, T \leqslant t)$ goes to 0 when $t \searrow 0$, for t small enough one has $Q_y(t_t \in V, T > t) > 0$.

By choosing a suitable V and since $\rho \equiv 0$, one can introduce the time reversed flow if $y \in D$ or the partial time reversed process of § 1 if $y \in \partial D$. The generator of this process satisfies the same conditions than A, so that if $\tilde{Q}_y$ denotes its law, for t small enough one has $\tilde{Q}_y(y_t \in V, T > t) > 0$. Hence

$$(5.47)\qquad \int_V p(t,z,y)\,dz = \tilde{Q}_y(y_t\in V, T>t) + o(t)$$

so that for $t > 0$ small enough, there exists $z \in V \cap D$ (depending on t) with $p(t,z,y) > 0$. Therefore one can find $V(t)$ a non empty open subset of $V \cap D$ with $p(t,z,y) \geqslant c(t) > 0$ for all $z \in V(t)$.

Then if y and $x \in \bar{D}$ and $t > 0$, one can find ε with $0 < \varepsilon < t/3$, and non empty open subsets $V_1(\varepsilon)$ and $V_2(\varepsilon)$ of D with

$$(5.48)\qquad \begin{aligned} p(\varepsilon,y,z) &\geqslant c_1(\varepsilon) > 0 \quad \text{for } z\in V_1(\varepsilon),\\ p(\varepsilon,z',z) &\geqslant c_2(\varepsilon) > 0 \quad \text{for } z'\in V_2(\varepsilon). \end{aligned}$$

We then obtain (recall that $\rho \equiv 0$)

$$(5.49)\quad p(t,y,x) = \int_{D\times D} p(\varepsilon,y,z)p(t-2\varepsilon,z,z')p(\varepsilon,z',x)\,dz\,dz'$$
$$\geqslant c_1(\varepsilon)c_2(\varepsilon)\int_{V_1(\varepsilon)\times V_2(\varepsilon)} p(t-2\varepsilon,z,z')\,dz\,dz'$$
$$\geqslant c_1(\varepsilon)c_2(\varepsilon)\operatorname{vol}(V_1(\varepsilon))\inf_{z\in V_1(\varepsilon)} Q_z(y_{t-2\varepsilon}\in V_2(\varepsilon)) > 0\,,$$

because from the continuity in z of $p(t,z,z')$, the application $z \to Q_z(y_t \in U)$ is continuous, and strictly positive according to (5.45).

Since $(t,y,z) \to p(t,y,x)$ is continuous, we finally have

$$(5.50)\qquad \text{For all } t > 0\,, \qquad \inf_{\bar D\times\bar D} p(t,y,x) = c(t) \geqslant 0\,.$$

(5.37) is an immediate consequence of (5.50), if $\rho \equiv 0$. If $\rho \not\equiv 0$, taking the notation of (3.6) one has $R^{\lambda,0}_{f,0} = G^\lambda f = R^{0\lambda,\rho\lambda}_{f,0} > 0$ from (5.50).

Endly, according to (5.31) $\dim(\ker A^* \cap M^{*\infty}) = 1$. From (5.15), (5.24) and (5.33) if $\rho \equiv 0$ and $p \in \ker A^* \cap M^{*\infty}$, then $d\nu(y) = p(y)\,dy$ is an invariant measure.

Accordingly $\nu = \lambda\nu_0$, and $d\nu_0 = p_0\,dy$, with $p_0 \in \ker A^* \cap M^{*\infty}$ and $p_0 \geqslant 0$.

Remember that $A^* = 1/2\sum_{i=1}^{m} Y_i^2 + Y_0^* + h^*$, so that we can associate to (A^*,Γ^*) a reflected diffusion process $y^*_\cdot$, and $G^{*\lambda} = (R^*)^{h^*+\lambda,\upsilon^*}$. Since A and A^* have the same principal symbol, $G^{*\lambda}$ satisfies (5.37). But $p_0 \geqslant 0$ is such that $p_0 = \lambda G^{*\lambda} p_0$, so that $p_0 > 0$.

If $\rho \not\equiv 0$, proposition (1.19) shows immediately that

$$d\nu_0 = p_0(y)1_D(y)\,dy + \rho p_0(y)1_{\partial D}(y)\,d\mu(y)$$

is an invariant measure.

To finish the proof of (5.36) if $p \in \mathscr{D}'(\bar D)$ checks $A^*p = 0$, $\Gamma^*p = 0$, since $p \in \mathscr{D}'_{2k}$ for some $k \in \mathbb{N}$, according to (5.25) and what precedes, $p = \lambda p_0$. □

Remark. – One can compare this result with the one obtained in [29]. The key point of the previous proof is the large deviations estimate (5.43). One can of course, from the estimates of [25], deduce a minoration for the density. This result is certainly far from the optimal one, one can conjecture. The natural metric for the problem certainly involves the local time of the deterministic reflexion problem, in the same way that the natural covariance matrix of Malliavin involves the local time L_t (see [19]). Some results of large deviations for reflected diffusions were obtained in [17].

Among the invariant measures, the reversible measures are of particular interest

(5.51) DEFINITION. – *ν is said to be reversible if $\int_{\bar{D}} g \,.\, T_t f \, d\nu = \int_{\bar{D}} f \cdot T_t g \, d\nu$ for all $t \geqslant 0$; $f, g \in C^0(\bar{D})$.*

In particular ν invariant and for all $t \geqslant 0$, the Q_ν laws of $(y_s)_{0 \leqslant s \leqslant t}$ and $(y_{t-s})_{0 \leqslant s \leqslant t}$ are the same.

(5.52) THEOREM. – *Assume that $\mathscr{L}\mathrm{ie}\,(Y_1, \ldots, Y_m)(y)$ spans $\mathbb{R}^d$ at each $y \in D$ and that* (5.18) *holds. Then the invariant probability ν_0 of* (5.36) *is reversible if and only if*

$$(5.53) \qquad \begin{cases} Y_0 = 1/2\,(\mathrm{div}\, Y_i + Y_i \log p_0)\, Y^i \\ V_0 = 1/2\,(\mathrm{div}\, V_\ell + V_\ell \log p_0)\, V^\ell + 1/2\,\langle Y_i, n\rangle\, Y^i . \end{cases}$$

Conversely if $q_0 \in C^\infty(\bar{D})$, $p_0 = \exp q_0$ and Y_0, V_0 are as in (5.53), *then $d\nu_0(y) = 1_D(y) p_0(y)\, dy + 1_{\partial D}(y)(\rho p_0)(y)\, d\mu(y)$ is reversible.*

Proof. – See the theorem 2.5 of [12], and use the strict positivity of p_0. □

Remark. – If $\mathscr{L}\mathrm{ie}(Y_1, \ldots, Y_n)(y)$ is full at each $y \in \bar{D}$, the semigroup T_t satisfies the Doeblin's condition and then is ergodic, that is

$$\lim_{t \to +\infty} T_t f(y) = \int f \, d\nu_0, \text{ uniformly in } y \in \bar{D} \text{ and } f \in C^0(\bar{D}),$$

the convergence being exponentially fast.

5.C. Gauge, uniqueness and spectrum.

In this section we link the problem of uniqueness for (L.P), with the finiteness of the Gauge, and the spectrum of $(h-A)$ considered as a generator. This section is strongly related to [22] section 5 and some results of [27]. For simplicity we assume that $\bar{D}$ is connected and $\rho \equiv 0$. Remember that $h \in C^{\infty}(\bar{D})$ and $v \in C^{\infty}(\partial D)$, and that $H_t^{h,v}$ is defined by

$$(5.54) \qquad H_t^{h,v} = \exp - \int_0^t h(y_s)\, ds - \int_0^t v(y_s)\, dL_s .$$

Actually we can associate two Gauge functions to (L.P) given by

$$(5.55) \qquad J_d(y) = E^y\left[\int_0^{+\infty} H_t^{h,v}\, dt\right] \quad \text{and} \quad J_{\partial D}(y) = E^y\left[\int_0^{+\infty} H_t^{h,v}\, dL_t\right].$$

In § 5.C, we gave simple sufficient conditions under which J_D and $J_{\partial D}$ are bounded on $\bar{D}$. When $A = 1/2\,\Delta$, $\Gamma = \dfrac{\partial}{\partial n}$ and $v \equiv 0$, it is shown in [22] that if $J_{\partial D}$ is finite at one point $y \in \bar{D}$, it is bounded on $\bar{D}$. This result is strongly linked to the strict positivity of $p(t,y,x)$. Then it is not difficult to build a counter example from (5.35).

One can for instance take $v \equiv 0$, $h(x,y) = h(y)$ with:

* $h > 0$ on $](2p-3/2)\pi/k,\ (2p+1/4)\pi/k[$,

and

* $h < 0$ on $](2p-3/4)\pi/k,\ (2p+5/2)\pi/k[$.

One sees that J_D and $J_{\partial D}$ are

* bounded on $[-1,1] \times [(2p-1)\pi/k, 2p\pi/k]$
* infinite on $[-1,1] \times [(2p-1)\pi/k, (2p+2)\pi/k]$.

In return from the proof of (5.36) one get

(5.56) THEOREM. – *Assume that $\mathscr{L}ie(Y_1, \dots, Y_m)(y)$ spans $\mathbb{R}^d$, at each $y \in \bar{D}$. Then if $J_D(y)$ or $J_{\partial D}(y)$ is finite for an $y \in \bar{D}$, J_D and $J_{\partial D}$ are bounded on $\bar{D}$. Furthermore there exist positive constants c and β such that* $\sup_{y \in \bar{D}} E^y[H_t^{h,v}] \leqslant ce^{-\beta t}$.

Proof. – We shall follow [22] (Thm 2.2 and 2.3). First of all since the semigroup $f \to E^y[H_t^{h,v} f(y_t)]$ admits a C^∞ density, then for $f \geqslant 0$,

$$\sup_{y \in \bar{D}} E^y[H_1^{h,v} f(y_1)] \leqslant c_1 \int f(x)\, dx \leqslant +\infty . \tag{5.57}$$

Indeed (5.57) is clearly true for $f \in C^0(\bar{D})$ and extends by increasing limit. On the other hand using Schwarz inequality,

$$(E^y[f(y_1)])^2 \leqslant E^y[H_1^{h,v} f(y_1)] E^y[H_1^{-h,-v} f(y_1)] .$$

Thus

$$\left(\int_D p(1,y,x) f(x)\, dx\right)^2 \leqslant c_1' \left[\int f(x)\, dx\right] E^y[H_1^{h,v} f(y_1)] .$$

$$E^y[H_1^{h,v} f(y_1)] \geqslant \min_{\bar{D} \times \bar{D}} p(1,x,y)/c_1') \int f(x)\, dx = c_2 \int f(x)\, dx \tag{5.58}$$

with $c_2 > 0$, since $\min\limits_{\bar{D} \times \bar{D}} p(1,x,y) > 0$.

Let choose $c > c_1$ such that $1/c < c_2$. From the Markov property one get

$$\begin{cases} J_D(y) = \sum\limits_{n=0}^{+\infty} E^y\left[H_n^{h,v} E^{y_n}\left[\int_0^1 H_s^{h,v}\, ds\right]\right] \\ J_{\partial D}(y) = \sum\limits_{n=0}^{+\infty} E^y\left[H_n^{h,v} E^{y_n}\left[\int_0^1 H_s^{h,v}\, dL_s\right]\right] . \end{cases} \tag{5.59}$$

Furthermore

(5.60) LEMMA. – *Under the hypotheses of* (5.56), *there exists* $M_1(t)$ *and* $M_2(t)$ *satisfying for all* $t > 0$,

$$* \ 0 < M_1(t) \leqslant \inf_{y \in \bar{D}} \left(E^y\left[\int_0^t H_s^{h,v}\, ds\right] \wedge E^y\left[\int_0^t H_s^{h,v}\, dL_s\right]\right)$$

$$* \ M_2(t) \geqslant \sup_{y \in \bar{D}} \left(E^y\left[\int_0^t H_s^{h,v}\, ds\right] \vee E^y\left[\int_0^t H_s^{h,v}\, dL_s\right]\right)$$

and $M_2(t) \to 0$ *when* $t \to 0$.

Accept (5.60) for the moment. We deduce

$$J_D(y) \wedge J_{\partial D}(y) \geqslant M_1(1) \sum_{n=0}^{+\infty} E^y[H_n^{h,v}] \quad \text{for all} \quad y \in \bar{D}.$$

Hence if one of J_D or $J_{\partial D}$ is finite at a point $\bar{y} \in \bar{D}$, for n_0 large enough we have

$$E^{\bar{y}}[H_{n_0}^{h,v}] \leqslant 1/2c^2.$$

But $E^{\bar{y}}[H_{n_0}^{h,v}] = E^{\bar{y}}[H_1^{h,v} E^{y_1}[H_{n_0-1}^{h,v}]] \geqslant 1/c \int E^z[H_{n_0-1}^{h,v}]\, dz$.

So that $\int E^z[H_{n_0-1}^{h,v}]\, dz \leqslant 1/2c$.

But $E^y[H_{n_0}^{h,v}] \leqslant c \int E^z[H_{n_0-1}^{h,v}]\, dz \leqslant 1/2$, for all $y \in \bar{D}$.

In order to conclude, for $t > 0$, choose an integer n so that $n - 1 \leqslant t/n_0 < n$. Then

$$E^y[H_t^{h,v}] \leqslant (\sup_{y \in \bar{D}} \sup_{0 \leqslant t \leqslant n_0} E^y[H_t^{h,v}])(\sup_{y \in \bar{D}} E^y[H_n^{h,v}])^{n-1} \leqslant Ke^{-\beta t}$$

with $K = 2 \sup_{y \in \bar{D}} \sup_{0 \leqslant t \leqslant n_0} E^y[H_t^{h,v}]$, and $\beta = \log 2/n_0$.

It is easily seen now that

$$J_D(y) \wedge J_{\partial D}(y) \geqslant M_2(1) \sum_{n=0}^{+\infty} E^y[H_n^{h,v}] < +\infty, \quad \text{for all} \quad y \in \bar{D}.$$

It remains to show (5.60). The majoration is classical and does not require any assumption on the generator.

$$\text{(5.61)} \quad \exp\left[-\int_0^{t'} h(y_s)\, ds - \int_0^{t'} v(y_s)\, dL_s\right] \leqslant \exp(\|h\|_\infty t + \|v\|_\infty L_t) \quad \text{for} \quad t' \leqslant t.$$

Thus

$$\text{(5.62)} \quad \begin{cases} E^y\left[\int_0^t H_s^{h,v}\, ds\right] \leqslant t \exp(\|h\|_\infty t) E^y[\exp(\|v\|_\infty L_t)] \\ \qquad \leqslant ct \exp\{(\|h\|_\infty + c\|v\|_\infty^2)t\} \\ E^y\left[\int_0^t H_s^{h,v}\, dL_s\right] \leqslant \exp(\|h\|_\infty t) E^y[L_t \exp(\|v\|_\infty L_t)] \\ \qquad \leqslant c\sqrt{t}(\exp(\|h\|_\infty t))(\|v\|_\infty + \exp(c\|v\|_\infty^2 t)). \end{cases}$$

For the minoration we note that

$$(5.63)\qquad H_{t'}^{h,v} \geqslant \exp(-\|h\|_\infty t - \|v\|_\infty L_t), \quad \text{for} \quad t' \geqslant t,$$

so that

$$(5.64)\qquad \begin{cases} E^y\left[\int_0^t H_s^{h,v}\,ds\right] \geqslant t\exp(\|h\|_\infty t)E^y[\exp(\|v\|_\infty L_t)] \\ \qquad\qquad \geqslant t\exp(-\|h\|_\infty t - \|v\|_\infty)P_y[L_t \leqslant 1] > 0, \\ E^y\left[\int_0^t H_s^{h,v}\,dL_s\right] \geqslant t\exp-(\|h\|_\infty t)E^y[L_t \exp\|v\|_\infty L_t]. \end{cases}$$

According to (1.19), $L_t = \lim_{\varepsilon\to 0} 1/2\varepsilon \int_0^t 1_{D_\varepsilon}(y_s)(a^2/V_0\psi)(y_s)\,ds$, a.s and in L^2. So

$$E^y[L_t] = \int_0^t \int_{\partial D} (a^2/V_0\psi)(x)p(s,y,x)\,d\mu(x)\,ds$$
$$\geqslant t/2 \inf_{\partial D}(a^2/V_0\psi)(x) \inf_{\substack{s\in[t/2,t] \\ x,y\in\bar D}} p(s,y,x)\mu(\partial D) > c(t)t,$$

with $c(t) > 0$, because ∂D is non characteristic.

Now

$$E^y[L_t \exp-(\|v\|_\infty L_t)] \geqslant \sum_{n=1}^{+\infty} \exp-n(\|v\|_\infty)E^y[L_t 1_{n-1\leqslant L_t<n}] > 0.$$

□

Then we consider the spectrum of the generator $(h-A)$.

(5.65) DEFINITION. — *$\lambda \in \mathbb{R}$ is an eigenvalue of the generator $(h-A)$ if there exists $f \not\equiv 0$ with $f \in \ker(h-\lambda-A) \cap M^0$. In particular f belongs to $\mathscr{D}\mathrm{om}(h-A)$, considered as a non bounded operator of $C^0(\bar D)$.*

It is well known ([37]) that if λ is an eigenvalue of the generator $(h-A)$ for all $t > 0$, $e^{-\lambda t}$ is an eigenvalue of the operator T_t defined by

$$T_t f(y) = E^y[f(y_t)H_t^{h,v}],$$

and conversely the eigenvalues of T_t are of the form $e^{-\lambda t}$ with λ

eigenvalue of the generator $(h-A)$. As in [22] Theorem 5.4, we can state the following result :

(5.66) THEOREM. — *The three following assertions are equivalent.*

1) $\lim_{t\to+\infty} T_t 1(y) = \lim_{t\to+\infty} E^y[H_t^{h,v}] = 0$, *for all* $y \in \bar{D}$.

2) *The smallest eigenvalue* λ_1 *of the generator* $(h-A)$ *is strictly positive.*

3) J_D *and* $J_{\partial D}$ *are bounded on* $\bar{D}$.

Proof : 1) $\Rightarrow$ 2). Let λ be an eigenvalue and f an associated eigenfunction. Then

$$T_t f(y) = e^{-\lambda t} f(y) \leqslant \|f\|_\infty (T_1 1)(y) \underset{t\to+\infty}{\rightarrow} 0 \quad \text{for all} \quad y \in \bar{D}.$$

Since $f \not\equiv 0$, this implies $e^{-\lambda t} \underset{t\to+\infty}{\rightarrow} 0$, hence $\lambda > 0$.

2) $\Rightarrow$ 3) From the spectral radius formula ([37]) one has $e^{-\lambda_1} = \lim_{n\to\infty} \|T_n\|^{1/n}$ so that for n big enough, $\|T_n 1\| \geqslant 2e^{-n\lambda_1}$.

But from the majoration of lemma (5.60) (which is always true)

$$J_D(y) \vee J_{\partial D}(y) \leqslant M_2 \sum_{n=0}^{+\infty} (T_n 1)(y) < +\infty .$$

3) $\Rightarrow$ 1) since if J_D is bounded and $\bar{D}$ compact $\lim_{t\to\infty} T_t 1(y) = 0$. □

As a first consequence one sees (at least if $\rho \equiv 0$) that when A is *n.t.d*, $v > 0$ and $h \geqslant 0$, (0.11) is fulfilled (because $\lim_{t\to\infty} T_t 1(y) = 0$ according to § 4.C), so that (4.59) can be included in tbe general frame of § 4. More generally

(5.67) COROLLARY. — *If* (H.G) *is satisfied on* $\bar{D}$, *and* $\lambda_1 > 0$, *then for* f *and* g *respectively in* $C^\infty(\bar{D})$ *and* $C^\infty(\partial D)$, (L.P) *has a solution* $u \in C^\infty(\bar{D})$ *unique in* $\mathscr{D}'(\bar{D})$.

(5.68) PROPOSITION. — *If* (H.G) *is fulfilled in* $\bar{D}$ *and if* 0 *is not an eigenvalue of the generator* $(h-A)$, *then for all* $f \in C^\infty(\bar{D})$ *there exists* $u \in C^\infty(\bar{D})$, *unique in* $C^0(\bar{D})$ *so that* $(h-A)u = f$ *in* D *and* $(v-\Gamma)u = 0$ *on* ∂D.

Proof. – If 0 is not an eigenvalue of $(h-A)$, dim-ker $(h-A)\cap M^{\infty}=0$, so that according to (5.30), dim ker $(h-A^*)\cap M^{*\infty}=0$ and $C^{\infty}(\bar{D})=(h-A)(M^{\infty})$. Uniqueness follows from the vacuity of the kernel. □

On can give a probabilistic construction of the solution (cf. [27]).

5.D. Boundary operator and boundary process.

(5.69) PROPOSITION. – *We assume that* (H.G) *is fulfilled in* $\bar{D}$ *and that* (0.7) *is satisfied. Then for* $f\in C^{\infty}(\bar{D})$ *and* $g\in C^{\infty}(\partial D)$ *there exists a unique solution* $u\in C^{\infty}(\bar{D})$ *of* (L.P) *with data* (f,g) *if and only if there exists a unique solution* $\varphi\in C^{\infty}(\partial D)$ *for the equation*

$$\text{(B.P)}\quad (v-\Gamma)H\varphi = g-(v-H)Gf$$

where H *and* G *are respectively the harmonic and the Green operators. Furthermore one has* $u=H\varphi+Gf$.

Proof. – See the introduction. □

Let we consider the boundary operator $(v-\Gamma).H$, which is written as

$$(5.70)\quad (v-\Gamma).H\varphi=(v+\rho h)\varphi-1/2\sum_{\ell=1}^{r}V_{\ell}^{2}\varphi-\bar{V}_{0}\varphi-(V_{0}\psi)\frac{\partial(H\varphi)}{\partial n}\Big|_{\partial D}.$$

Actually we have proved that $(v-\Gamma)H$ is an hypoelliptic operator. But what kind of operator is it?

Since $\varphi\to\frac{\partial}{\partial n}(H\varphi)|_{\partial D}$ extends continuously to $\mathscr{E}'(\partial D)$ and has a C^{∞} kernel out of the diagonal, we guess that it is a pseudo-differential operator. When A is uniformly elliptic it is not hard to see that $\frac{\partial}{\partial n}(H\varphi)|_{\partial D}$ (and hence $(v-\Gamma).H$) is effectively a pseudo differential operator, which principal symbol is $|\xi|$, where $|\ |$ is linked to the riemanian metric induced by the inverse of the principal part of A (see [33]). If A only fulfills the hypotheses of (5.69) we did not succeed in proving that (B.P) is still pseudo differential (in the usual sense of [15]).

We shall only deal with a very degenerate (but rather homogeneous) example namely

$$(5.71)\qquad D = \mathbb{R}^2_+\,,\quad a = 1/2\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y}\,,\quad h \equiv 1/2\,,\quad (\partial D = \{x=0\})\,.$$

For a $g \in C_0^\infty(\mathbb{R})$ and $x > 0$ one has

$$(5.72)\qquad \begin{aligned} Hg(x,y) &= \int_0^{+\infty} g(y+t)\frac{1}{\sqrt{2\pi}}(x/t^{3/2})e^{-t/2}e^{-x^2/2t}\,dt \\ &= \int_0^{+\infty} g\left(y+\frac{x^2}{u}\right)\frac{1}{\sqrt{2\pi u}}e^{-u/2}e^{-x^2/2u}\,du\,. \end{aligned}$$

Then

$$(5.73)\qquad \begin{aligned} \frac{\partial}{\partial x}Hg(x,y) &= \int_0^{+\infty}\frac{1}{\sqrt{2\pi u}}e^{-t/2}e^{-x^2/2t}[2g'(y+t)-g(y+t)]\,dt \\ \frac{\partial}{\partial x}Hg(0,y) &= \int_0^{+\infty}\frac{1}{\sqrt{2\pi t}}e^{-t/2}[2g'(y+t)-g(y+t)]\,dt \\ &= \int e^{i\xi y}\hat{g}(\xi)a(\xi)\,d\xi \end{aligned}$$

where $a \in C^\infty(\mathbb{R})$ is given by $a(\xi) = \left(\int_0^{+\infty} e^{it\xi}e^{-t/2}\frac{dt}{\sqrt{2\pi t}}\right)(2i\xi - 1)\,.$

An easy computation gives

$$(5.74)\qquad a(\xi) = \frac{2i\xi - 1}{(1+4\xi^2)^{1/4}}e^{(i\operatorname{Arctg} 4\xi)/2}$$

so that a is a symbol of order $1/2$, whose principal symbol is of the form $\xi|\xi|^{-1/2}$ (here $|\ |$ is the usual metric on $\mathbb{R}$).

Finally from what precedes, we see that the closure of the boundary operator $(v-\Gamma)H$ is the infinitesimal generator of the boundary semigroup associated to the boundary process, i.e.

$$B_t f(y) = E^y\left[1_{A_t<+\infty}f(y_{A_t})\exp - \int_0^t (v(y_{A_s})\,ds + h(y_{A_s})\,dA_s)\right]$$

where A_t is the right continuous inverse of the local time L_t.

We hope that this can be another interesting approach in the study of (L.P).

As we said before, the existence of a smooth density for the joint law of (A_t, y_{A_t}) was proved in [4] and [19]. We hope that a direct study of the law of y_{A_t} is possible, including the case when only weak transversality holds $(\alpha+\rho > 0)$.

BIBLIOGRAPHY

[1] G. Ben Arous, S. Kusuoka, D. Stroock, The Poisson kernel for certain degenerate elliptic operators, J. Func. Anal., 56 (1984), 171-209.

[2] J. M. Bismut, Mécanique aléatoire, Lect. Notes in Math., 866, Springer, Berlin, 1981.

[3] J. M. Bismut, Martingales, the Malliavin's calculus and hypoellipticity under general Hörmander's conditions, Z. Wahrsch., 56 (1981), 469-506.

[4] J. M. Bismut, The calculus of boundary processes, Ann. Scient. École Normale Sup., 17 (1984), 507-622.

[5] J. M. Bismut, Last exist decompositions and regularity at the boundary of transition probabilities, Z. Wahrsch., 69 (1985), 65-98.

[6] J. M. Bismut, Large deviations and the Malliavin calculus, Progress in Math., 45, Birkhaüser, Boston, 1984.

[7] J. M. Bony, Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés, Ann. Inst. Fourier, 19-1 (1969), 277-304.

[8] J. M. Bony, P. Courrège, P. Priouret, Semi-groupes de Feller sur une variété à bord compacte et problèmes aux limites... Ann. Inst. Fourier, 18-2 (1968), 369-521.

[9] P. Cattiaux, Thèse de 3e cycle, Univ. Paris XI, 1984.

[10] P. Cattiaux, Hypoellipticité et hypoellipticité partielle pour les diffusions avec une condition frontière, Ann. Inst., H. Poincaré, 22 (1986), 67-112.

[11] P. Cattiaux, Régularité au bord pour les densités et les densités conditionnelles d'une diffusion réfléchie hypoelliptique, Stochastics, 20 (1987), 309-340.

[12] P. Cattiaux, Time reversal of diffusion processes with a boundary condition, Stochastic processes and their Applications, 28 (1988), 275-292.

[13] P. Cattiaux, Calcul stochastique et opérateurs dégénérés du second ordre. I Résolvantes, théorème de Hörmander et application, Bull. Sc. Math., 114 (1990), 421-462.

[14] P. Cattiaux, Calcul stochastique et opérateurs dégénérés du second ordre. II Problème de Dirichlet, Bull. Sc. Math., 115 (1991), 81-122.

[15] J. Chazarain, A. Piriou, Introduction à la théorie des équations aux dérivées partielles linéaires, Gauthier Villars, Paris, 1981.

[16] M. Derridj, Un problème aux limites pour une classe d'opérateurs du second ordre hypoelliptiques, Ann. Inst. Fourier, 21-4 (1971), 99-148.

[17] H. Doss, P. Priouret, Petites perturbations de systèmes dynamiques avec réflexion, Séminaire de Probas. 17, Lect. Notes in Math., 986 (1983), 353-370.

[18] N. El Karoui, Processus de diffusion associé à un opérateur elliptique dégénéré et à une condition frontière, Thèse, Université Paris VI, 1971.

[19] C. Graham, Thèse 3e cycle, Université Paris VI, 1985.

[20] L. Hörmander, Pseudo differential operators and non elliptic boundary problems, Ann. of Math., 83 (1986), 129-209.

[21] L. Hörmander, Hypoelliptic second order differential operators, Acta. Math., 119 (1967), 147-171.

[22] Hsu-Pei, Probabilistic approach to the Neumann problem, Comm. on Pure and Applied Math., 38 (1985), 445-472.

[23] N. Ikeda, S. Watanabe, Stochastic differential equations and diffusion processes, North Holland, Amsterdam, 1981.

[24] H. Kunita, Stochastic differential equations and stochastic flows of diffeomorphisms, Lect. Notes in Math., 1097 (1984), 144-305.

[25] R. Leandre, Minoration en temps petit de la densité d'une diffusion dégénérée, preprint, 1987.

[26] P. Malliavin, * Stochastic calculus of variations and hypoelliptic operators, Proc. Intern. Symp. S.D.E. of Kyoto, K-Ito ed., (1976), 195-263.
* C^k hypoellipticity with degeneracy. In « Stochastic Analysis », A. Friedman, M. Pinsky ed., (1978), 199-214 & 327-340.

[27] Ma Zhiming, On the probabilistic approach to boundary value problems, preprint, 1986.

[28] B. P. Paneyakh, Some boundary value problems for elliptic equations and the Lie algebras associated with them, Math. USSR Sbornik, 54 (1986), 207-237.

[29] P. Priouret, Processus de Markov sur une variété à bord compacte, Ann. Inst. H. Poincaré, 4 (1968), 193-253.

[30] D. Stroock, Some applications of stochastic calculus to partial differential equations, Lect. Notes in Math., 976 (1983), 267-382.

[31] D. Stroock, S. R. S. Varadhan, Diffusion processes with boundary conditions, Comm. on Pure an Applied Math., 24 (1971), 147-225.

[32] D. Stroock, S. R. S. Varadhan, On degenerate elliptic-parabolic operators of second order and their associated diffusions, Comm. on Pure and Applied Math., 25 (1972), 651-713.

[33] K. Taira, Semi-groups and boundary value problems, Duke Math. Journal, 49 (1982), 287-320.

[34] F. Trèves, Topological vector spaces, Distributions and Kernels, Academic Press, New York-London, 1967.

[35] A. D. VENTCEL, On boundary conditions for multidimensional processes, Theory Prob. Appl., 4 (1959).

[36] S. WATANABE, Construction of diffusion processes with Wentzell's boundary conditions... Prob. Theory, Banach Center Pub. 5. Polish Sci. Publ., Warsaw, (1979), 255-271.

[37] K. YOSIDA, Functional Analysis, Springer, Berlin, 1965.

[38] Temps locaux. J. AZEMA, M. YOR, éd. Astérisque SMF, 52-53 (1978).

[39] Géodésiques et diffusions en temps petit, Astérisque SMF, 84-85 (1981).

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