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## NOTES ON INTERPOLATION OF HARDY SPACES

by Quanhua XU

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### 1. Introduction and the main result.

Let  $D$  denote the unit disc of the complex plane and  $\mathbf{T}$  the unit circle, the boundary of  $D$ .  $\mathbf{T}$  is equipped with its normalized Lebesgue measure  $m$ . Let  $H_p$  ( $0 < p \leq \infty$ ) denote the usual classical Hardy space of analytic functions in  $D$ . Functions in  $H_p$  are identified with their radial limits on  $\mathbf{T}$  so that  $H_p$  is a closed subspace of  $L_p = L_p(\mathbf{T})$ . Thus for  $1 \leq p \leq \infty$ ,  $H_p = \{f \in L_p : \hat{f}(n) = 0, n < 0\}$ , where  $\{\hat{f}(n)\}_{n \in \mathbf{Z}}$  is the sequence of Fourier coefficients of  $f$  (see [6] for more information).

Let  $f$  be a measurable function on  $\mathbf{T}$ . We denote by  $f^*$  the non-increasing rearrangement of  $|f|$ . The main result of this note is the following

**THEOREM.** — *Let  $1 \leq p < \infty$ . Let  $f$  and  $g$  be functions in  $H_p (= H_p + H_\infty)$  such that*

$$\int_0^t (g^*(s))^p ds \leq \int_0^t (f^*(s))^p ds, \quad \forall 0 < t \leq 1.$$

*Then there exists a linear operator  $T$  defined on  $L_p (= L_p + L_\infty)$  such that  $T(f) = g$ , which maps  $L_p$  into  $H_p$  and  $L_\infty$  into  $H_\infty$  and whose norms on these spaces are dominated by a constant depending only on  $p$ .*

*In particular, for any  $f \in H_p$  there exists such an operator  $T$  such that  $T(f) = f$ .*

This result is closely related to the interpolation theory of Hardy spaces. Before giving its consequences, let us first recall the following similar theorem due to P.W. Jones [9].

THEOREM J. — *Let  $f$  and  $g$  be two functions in  $H_1 (= H_1 + H_\infty)$  such that*

$$\int_0^t g^*(s)ds \leq \int_0^t f^*(s)ds, \quad \forall 0 < t \leq 1.$$

*Then there exists a linear operator  $T$  defined on  $H_1$  such that  $T(f) = g$ , which maps boundedly  $H_1$  into  $H_1$  and  $H_\infty$  into  $H_\infty$ .*

The essential new point in our theorem, as compared with Jones' theorem, is that the required operator  $T$  such that  $T(f) = g$  is defined on  $L_p$  with values in  $H_p$ , instead of being defined on  $H_p$  as in Jones' theorem. This advantage will allow us to deduce very easily at the same time all interpolation results for  $H_p$ -spaces from the corresponding known  $L_p$ -space results. For instance, this theorem gives immediately the real and complex interpolation results on Hardy spaces of Jones [8]. It includes, of course, Theorem J above, which says that  $(H_1, H_\infty)$  is a Calderón-Mitjagin couple (see the discussions below).

The particular case of the preceding theorem where two functions coincide is of special interest. In this case the operator  $T$  given by the theorem for  $f$  leaves  $f$  invariant. For simplicity, we shall say that a linear operator  $T$  is a "one-point projection" at  $f$  if it maps measurable functions to analytic functions such that  $T(f) = f$ . Thus our theorem says that for any  $f \in H_p$  there exists a one-point projection at  $f$  which is bounded simultaneously from  $L_p$  to  $H_p$  and from  $L_\infty$  to  $H_\infty$ . In fact, together with known results about  $L_p$ -spaces, the existence of simultaneously bounded one-point projections already ensures the theorem in its full generality (see Section 4 below).

For stating consequences of our theorem, we shall need some basic notions from the interpolation theory. We now recall them very briefly. The reader is referred to [2] for more information.

We shall always denote by  $F$  an interpolation functor for Banach spaces, that is,  $F$  is a functor on the category of interpolation couples of Banach spaces into the category of Banach spaces such that for any two interpolation couples  $(A_0, A_1)$  and  $(B_0, B_1)$ ,  $F(A_0, A_1)$  and  $F(B_0, B_1)$  are interpolation spaces with respect to  $(A_0, A_1)$ ,  $(B_0, B_1)$  and such that for any bounded operator  $T$  from  $A_j$  into  $B_j$  ( $j = 0, 1$ ) we have  $F(T) = T$ .

The most useful interpolation functors are those constructed from the real and complex interpolation methods. Let  $0 < \theta < 1, 1 \leq q \leq \infty$ . We shall denote by  $(\cdot, \cdot)_{\theta q}, (\cdot, \cdot)_{\theta}$  real and complex interpolation spaces constructed respectively by the real and complex interpolation methods. By interpolation theorem, they are interpolation functors.

Let  $(A_0, A_1)$  be an interpolation couple of Banach spaces. Let  $t > 0$  and  $x \in A_0 + A_1$ , we define

$$K(t, x; A_0, A_1) = \inf\{\|x_0\|_{A_0} + t\|x_1\|_{A_1} : x = x_0 + x_1, x_j \in A_j, j = 0, 1\}.$$

This is the so-called Peetre's  $K$ -functional.  $(A_0, A_1)$  is called a Calderón-Mitjagin couple if for any  $x, y \in A_0 + A_1$  such that

$$K(t, y; A_0, A_1) \leq K(t, x; A_0, A_1), \forall t > 0,$$

there exists a linear operator  $T$  from  $A_0 + A_1$  to itself which maps boundedly  $A_j$  into  $A_j$  ( $j = 0, 1$ ) such that  $Tx = y$ .

Let us also recall the following well-known results (cf. [2], [5], [14], [17]). Let  $0 < \theta < 1, 1 \leq p_0, p_1 \leq \infty$ . Then

$$(L_{p_0}, L_{p_1})_{\theta p} = L_p \quad \text{and} \quad (L_{p_0}, L_{p_1})_{\theta} = L_p,$$

where  $\frac{1-\theta}{p} = \frac{\theta}{p_0} + \frac{1}{p_1}$ . Furthermore,  $(L_{p_0}, L_{p_1})$  is a Calderón-Mitjagin couple. Finally, we have for  $1 \leq p < \infty$

$$(1) \quad K(t, f; L_p, L_{\infty}) \approx \left( \int_0^{t^p} (f^*(s))^p ds \right)^{1/p},$$

where the equivalence constants depend on  $p$  only.

We can now state the first consequence of the main theorem. For a Banach space  $X$  of integrable functions on  $\mathbf{T}$  we denote by  $H(X)$  the closed subspace of  $X$  defined by

$$H(X) = \{f \in X : \hat{f}(n) = 0, \forall n < 0\}.$$

**COROLLARY 1.** — *Let  $F$  be an interpolation functor and  $1 \leq p < \infty$ . Then  $F(H_p, H_{\infty}) = H(F(L_p, L_{\infty}))$ .*

*Proof.* — It is evident that  $F(H_p, H_{\infty}) \subset H(F(L_p, L_{\infty}))$ . For the reverse inclusion, we take a function  $f$  in  $H(F(L_p, L_{\infty}))$ . Then  $f \in H_1$  and thus by our theorem there is a one-point projection  $T$  at  $f$  which is bounded from  $L_p$  to  $H_p$  and from  $L_{\infty}$  to  $H_{\infty}$ . Therefore  $T$  is also bounded from  $F(L_p, L_{\infty})$  to  $F(H_p, H_{\infty})$ ; so that  $f = T(f) \in F(H_p, H_{\infty})$  and

$$\|f\|_{F(H_p, H_{\infty})} = \|T(f)\|_{F(H_p, H_{\infty})} \leq C \|f\|_{F(L_p, L_{\infty})},$$

where  $C$  is a constant depending only on  $F$  and  $p$ . This concludes the proof.

Corollary 1 simply says that  $F$  commutes with  $H$ . Applying it to the real and complex functors  $(\cdot, \cdot)_{\theta, q}$  and  $(\cdot, \cdot)_{\theta}$ , we recover the two interpolation theorems of Jones on Hardy spaces, as we have already noted :

$$(H_p, H_{\infty})_{\theta, q} = H_q \quad \text{and} \quad (H_p, H_{\infty})_{\theta} = H_q,$$

where  $\frac{1}{q} = \frac{1-\theta}{p}$  (cf. [8]).

The existence of bounded one-point projections also yields immediately the following more precise result on  $K$ -functional, which is also due to Jones [8], and reformulated by Sharpley (cf. [1]) as follows :

$$(2) \quad K(t, f; H_p, H_{\infty}) \leq CK(t, f; L_p, L_{\infty}), \quad \forall f \in H_p, \forall t > 0,$$

where  $C$  is a constant depending on  $p$  only.

Combining (1) and (2), we get for any  $f \in H_p$  ( $1 \leq p < \infty$ ) and any  $t > 0$

$$K(t, f; H_p, H_{\infty}) \approx \left( \int_0^{t^p} (f^*(s))^p ds \right)^{1/p}.$$

Therefore, Theorem J says that  $(H_1, H_{\infty})$  is a Calderón-Mitjagin couple. Our main theorem shows that the same is true for  $(H_p, H_{\infty})$  ( $1 < p < \infty$ ). We shall see more general results of this nature.

The paper is organized as follows. In Section 2, using techniques recently developed by S.V. Kisliakov, we prove a technical lemma, which is of interest in itself and will be crucial in the proof of our theorem. The proof of the theorem is presented in Section 3. It is very elementary and only involves the boundedness of the Hilbert transform. Section 4 contains some further consequences of the theorem. Here, we prove that if  $(E_0, E_1)$  is an interpolation couple of rearrangement invariant Banach spaces on  $\mathbf{T}$ , then  $F(H(E_0), H(E_1)) = H(F(E_0, E_1))$  for any interpolation functor  $F$ . Moreover, if  $(E_0, E_1)$  is a Calderón-Mitjagin couple, so is  $(H(E_0), H(E_1))$ .

We shall denote by  $C$  a constant which is independent of functions in consideration and may vary from lines to lines.

## 2. Lemma.

The following lemma is a strengthened form of a lemma of Kisliakov [10]. A variant of it may be obtained by adapting a construction of

Bourgain[3], as was pointed out to us by Kisliakov. The only improvement of the following lemma on those of Bourgain and Kisliakov is the property (ii) below, which will be crucial later. The constructions of this type also have applications in the other contexts (cf. [3], [10]-[12]).

LEMMA. — *Let  $a$  be a function on  $\mathbf{T}$  such that*

$$(3) \quad 2^{k_0} < a < 2^{k_1} \text{ a.e. on } \mathbf{T}$$

for two integers  $k_0, k_1$  in  $\mathbf{Z}$ . Then there exist a function  $b$  on  $\mathbf{T}$  and a sequence  $\{\varphi_k\}_{k=k_0+1}^{k_1}$  in  $H_\infty$  satisfying the following properties :

(i)  $b \geq a$  a.e. on  $\mathbf{T}$ ;

(ii) for any  $0 < t \leq 1$

$$\int_0^t b^*(s)ds \leq C \int_0^t a^*(s)ds ;$$

(iii)  $\|\varphi_k\|_\infty \leq C \quad (k_0 + 1 \leq k \leq k_1)$ ;

(iv)  $|\varphi_k|^{1/8} b \leq C 2^k$  a.e.  $(k_0 + 1 \leq k \leq k_1)$ ;

(v)  $\sum_{k=k_0+1}^{k_1} |\varphi_k|^{1/8} 2^k \leq C b$  a.e.;

(vi)  $\sum_{k=k_0+1}^{k_1} \varphi_k = 1$  a.e.,

where  $C$  is an absolute constant.

*Proof.* — We define by induction three sequences  $\{E_k\}_{k=k_0+1}^{k_1}$  of measurable subsets of  $\mathbf{T}$ ,  $\{G_k\}_{k_0+1}^{k_1} \subset H_\infty$  and  $\{b_k\}_{k_0+1}^{k_1} \subset L_\infty$  as follows. Let first

$$E_{k_1} = \emptyset, \quad G_{k_1} \equiv 1, \quad b_{k_1} = a .$$

Then for  $k_0 + 1 \leq k \leq k_1 - 1$ , we define inductively

$$E_k = \{b_{k+1} > 2^k\}$$

$$\alpha_k = \max\left\{1, \left(\frac{b_{k+1}}{2^k}\right)^{1/4}\right\}$$

$$F_k = \frac{1}{\alpha_k + i\mathcal{H}(\alpha_k)}, \quad G_k = 1 - (1 - F_k^{32})^{16}$$

$$b_k = b_{k+1} + \delta |G_{k+1} - G_k|^{1/8} 2^{k+1},$$

where  $\mathcal{H}$  stands for the Hilbert transform and where  $\delta$  ( $0 < \delta \leq 1$ ) is a constant to be determined later.

It is easy to check inductively that for  $k_0 + 1 \leq k \leq k_1$ ,  $b_k \in L_\infty$ ,  $F_k$ ,  $G_k \in H_\infty$  and

$$(4) \quad |F_k| \leq \frac{1}{\alpha_k} \leq 1 \text{ a.e.};$$

$$(5) \quad |G_k| \leq C|F_k|^{32} \leq C \text{ a.e. .}$$

Now let

$$\begin{aligned} \varphi_{k_0+1} &= G_{k_0+1}, \varphi_k = G_k - G_{k-1} \text{ for } k_0 + 2 \leq k \leq k_1; \\ b &= b_{k_0+1} + \delta |\varphi_{k_0+1}|^{1/8} 2^{k_0+1}. \end{aligned}$$

Then for these functions  $b, \varphi_k$ , we clearly have (i), (iii) and (vi). It is also clear that  $b \geq b_k \geq b_{k+1} \geq a$  for  $k_0 + 1 \leq k \leq k_1 - 1$  so that  $E_k \supset E_{k+1}$  ( $k_0 + 1 \leq k \leq k_1 - 1$ ). Let additionally

$$E_{k_0} = \{b_{k_0+1} > 2^{k_0}\}.$$

From (3),  $E_{k_0} = \mathbf{T}$ . Let

$$e_k = E_k \setminus E_{k+1}, \quad k_0 \leq k \leq k_1 - 1.$$

Then  $\{e_k\}_{k=k_0}^{k_1-1}$  is a sequence of disjoint measurable subsets of  $\mathbf{T}$  such that

$$\mathbf{T} = \bigcup_{k=k_0}^{k_1-1} e_k.$$

By the definitions of  $b_k$  and  $b$  we get for  $k_0 + 1 \leq k \leq k_1$

$$(6) \quad b = b_k + \delta \sum_{j=k_0+1}^k |\varphi_j|^{1/8} 2^j = a + \delta \sum_{j=k_0+1}^{k_1} |\varphi_j|^{1/8} 2^j,$$

which, together with (5), gives for  $k_0 + 1 \leq k \leq k_1$

$$\begin{aligned} |\varphi_k|^{1/8} b &\leq |G_k|^{1/8} b_{k+1} + |G_{k-1}|^{1/8} b_k + C2^k \\ &\leq C2^k. \end{aligned}$$

This is (iv). (6) will also yield (v) (with  $C = \delta^{-1}$ ) after appropriately choosing  $\delta$  in order that (ii) is verified. Therefore it remains to check (ii) for an appropriate  $\delta$ . For this we first claim that

$$(7) \quad 2^k \leq b \leq C2^k \text{ on } e_k \quad (k_0 \leq k \leq k_1 - 1).$$

Indeed, for a fixed  $k_0 \leq k \leq k_1 - 1$ , the first inequality in (7) is evident since by (6)

$$b \geq b_{k+1} > 2^k \text{ on } E_k.$$

For the second inequality, by (6) and the definition of  $E_k$

$$b \leq b_{k+1} + C\delta 2^k \leq C2^k \text{ on } \mathbf{T} \setminus E_k;$$

whence

$$b \leq C2^k \text{ on } e_{k-1} \text{ for } k_0 + 1 \leq k \leq k_1 - 1,$$

which proves the second inequality of (7).

Now define

$$d = \sum_{k=k_0}^{k_1-1} 2^k \chi_{e_k},$$

where  $\chi_e$  denotes the characteristic function of a subset  $e$ . Then by (7)

$$(8) \quad d \leq b \leq Cd, \text{ a.e. on } \mathbf{T}.$$

Therefore for proving (ii), we may consider  $d$  instead of  $b$ . Put  $t_{k_1} = 0$ ,  $t_k = \sum_{j=k}^{k_1-1} m(e_j)$  for  $k_0 \leq k \leq k_1 - 1$ . Then  $t_{k_1} = 0 \leq t_{k_1-1} \leq \dots \leq t_{k_0} = 1$  and

$$(9) \quad d^* = \sum_{k=k_0}^{k_1-1} 2^k \chi_{((t_{k+1}, t_k])}.$$

Now fix  $k_0 \leq k \leq k_1 - 1$ . We are going to estimate  $\int_0^{t_k} d^*(s)ds$ . We have

$$\begin{aligned} \int_0^{t_k} d^*(s)ds &= \sum_{j=k}^{k_1-1} 2^j m(e_j) \\ &= \int_{E_k} d \leq \int_{E_k} b. \end{aligned}$$

It follows by (6) that

$$\begin{aligned} \int_0^{t_k} d^*(s)ds &\leq \int_{E_k} a + C\delta 2^k m(E_k) + \delta \sum_{j=k+1}^{k_1} 2^j \int_{E_k} |\varphi_j|^{1/8} \\ &\leq \int_0^{t_k} a^*(s) + C\delta \int_0^{t_k} d^*(s)ds + \delta \sum_{j=k+1}^{k_1} 2^j \int_{\mathbf{T}} |\varphi_j|^{1/8}. \end{aligned}$$

Next we majorize

$$\int_{\mathbf{T}} |\varphi_j|^{1/8} \quad \text{for } k + 1 \leq j \leq k_1.$$



By the definitions of  $\varphi_j, G_j$  and (4) and the  $L_2$ -boundedness of  $\mathcal{H}$ , we have

$$\begin{aligned} \int_{\mathbf{T}} |\varphi_j|^{1/8} &\leq \int_{\mathbf{T}} |1 - G_j|^{1/8} + \int_{\mathbf{T}} |1 - G_{j-1}|^{1/8} \\ &\leq C \left( \int_{\mathbf{T}} |1 - F_j|^2 + \int_{\mathbf{T}} |1 - F_{j-1}|^2 \right) \\ &\leq C \left( \int_{\mathbf{T}} |\alpha_j - 1|^2 + \int_{\mathbf{T}} |\alpha_{j-1} - 1|^2 \right) \\ &\leq C \left( 2^{-j/2} \int_{E_j} b_{j+1}^{1/2} + 2^{-(j-1)/2} \int_{E_{j-1}} b_j^{1/2} \right) \\ &\leq C \left( 2^{-j/2} \int_{E_j} b^{1/2} + 2^{-(j-1)/2} \int_{E_{j-1}} b^{1/2} \right). \end{aligned}$$

Combining the preceding inequalities with (7), we obtain :

$$\begin{aligned} \sum_{j=k+1}^{k_1} 2^j \int_{\mathbf{T}} |\varphi_j|^{1/8} &\leq C \sum_{j=k}^{k_1} 2^{j/2} \sum_{l=j}^{k_1-1} 2^{l/2} m(e_l) \\ &\leq C \sum_{l=k}^{k_1-1} 2^l m(e_l) = C \int_0^{t_k} d^*(s) ds. \end{aligned}$$

Therefore

$$\int_0^{t_k} d^*(s) ds \leq \int_0^{t_k} a^*(s) ds + C\delta \int_0^{t_k} d^*(s) ds.$$

Choosing  $\delta = \frac{1}{2C}$ , we then deduce that

$$\int_0^{t_k} d^*(s) ds \leq 2 \int_0^{t_k} a^*(s) ds, \quad k_0 \leq k \leq k_1 - 1.$$

Now the function  $\int_0^t d^*(s) ds$  is linear for  $t \in [t_{k+1}, t_k]$  and the function  $\int_0^t a^*(s) ds$  is concave for  $t \in [0, 1]$ . Hence, by the last inequalities we get

$$(10) \quad \int_0^t d^*(s) ds \leq 2 \int_0^t a^*(s) ds, \quad \forall 0 < t \leq 1.$$

This proves (ii) in virtue of (8) and completes the proof of the lemma.

### 3. Proof of the theorem.

We shall prove the theorem in this section. Let  $1 \leq p < \infty$ . Let  $f, g$  be two functions as in the theorem. We shall show the existence of the

required operator  $T$  only in the case where  $g$  is in  $H_\infty$  and  $|g|$  is bounded away from zero so that the lemma in the previous section can be applied to  $a = |g|$ . The general case will easily follow from this special one by a standard limit argument.

Let  $a = |g|$  and  $k_0, k_1 \in \mathbf{Z}$  such that (3) is satisfied. Applying the lemma in Section 2 to  $a$  we get  $b$  and  $\{\varphi_k\}_{k=k_0+1}^{k_1} \subset H_\infty$  which satisfy the properties (i) - (vi). We retain all the notations in that lemma and its proof. Recall the function  $d$  is defined by

$$d = \sum_{k=k_0}^{k_1-1} 2^k \chi_{e_k} .$$

By (10) and a well-known theorem of Hardy-Littlewood-Pólya (cf. [7]), we have

$$\int_0^{t^p} (d^*(s))^p ds \leq 2^p \int_0^{t^p} (a^*(s))^p ds, \quad \forall 0 < t \leq 1 .$$

Then by the assumption on  $f$  and  $g$  in the theorem

$$\int_0^{t^p} (d^*(s))^p ds \leq 2^p \int_0^{t^p} (f^*(s))^p ds, \quad \forall 0 < t \leq 1 .$$

Replacing  $f$  by  $2f$ , we may assume

$$\int_0^{t^p} (d^*(s))^p ds \leq \int_0^{t^p} (f^*(s))^p ds, \quad \forall 0 < t \leq 1 .$$

Then it follows from Lemma 1 of [4] that there exist a finite sequence  $\{\lambda_n\}_{n=0}^N$  of positive numbers and a sequence  $\{\sigma_n\}_{n=0}^N$  of measure preserving transformations of  $[0, 1]$  onto itself such that

$$(11) \quad \begin{aligned} \sum_{n=0}^N \lambda_n &= 1 \\ \sum_{n=0}^N \lambda_n (f^*(\sigma_n(s)))^p &\geq (d^*(s))^p, \quad 0 \leq s \leq 1 . \end{aligned}$$

Therefore

$$(12) \quad (d^*(s))^p = \sum_{n=0}^N \lambda_n (f^*(\sigma_n(s)))^p w(s), \quad 0 \leq s \leq 1,$$

where  $0 \leq w(s) \leq 1$  ( $0 \leq s \leq 1$ ). Set

$$\begin{aligned} \tilde{e}_{k,n} &= \sigma_n((t_{k+1}, t_k]), \quad k_0 \leq k \leq k_1 - 1, \quad 0 \leq n \leq N, \\ w_k(s) &= \sum_{n=0}^N \lambda_n w(\sigma_n^{-1}(s)) \chi_{\tilde{e}_{k,n}}(s), \quad 0 \leq s \leq 1 . \end{aligned}$$

Then by (11) and (12), we easily check for  $k_0 \leq k \leq k_1 - 1$  that

$$(13) \quad \int_0^1 (f^*(s))^p w_k(s) ds = 2^{pk} m(e_k)$$

$$(14) \quad \int_0^1 w_k(s) ds \leq m(e_k)$$

$$(15) \quad \sum_{k=k_0}^{k_1-1} w_k(s) \leq 1, \quad s \in [0, 1].$$

Now for each  $k_0 + 1 \leq k \leq k_1$ , we let

$$v_k = \left( \sum_{j=k-1}^{k_1-1} 2^{pj} m(e_j) \right)^{1/p},$$

$$u_k(s) = \sum_{j=k-1}^{k_1-1} 2^{p(j-k+1)} w_j(s), \quad s \in [0, 1].$$

By (13) we find a positive function  $g_k$  on  $[0, 1]$  such that

$$(16) \quad \int_0^1 f^*(s) g_k(s) u_k(s) ds = \left( \sum_{j=k-1}^{k_1-1} 2^{p(j-k+1)} \cdot 2^{pj} m(e_j) \right)^{1/p},$$

$$(17) \quad \int_0^1 (g_k(s))^q u_k(s) ds \leq 1, \quad (1/q + 1/p = 1).$$

From [4] Lemma 2 we deduce that there exists a linear operator  $S$  from  $L_p(\mathbf{T})$  to  $L_p(0, 1)$  and from  $L_\infty(\mathbf{T})$  to  $L_\infty(0, 1)$  whose norms on these spaces are bounded by 1, such that  $Sf = f^*$ .

Then we define for every  $h \in L_p$

$$T_k(h) = \frac{1}{v_k} \int_0^1 Sh(s) g_k(s) u_k(s) ds.$$

By (16)

$$T_k(f) = \frac{1}{v_k} \left( \sum_{j=k-1}^{k_1-1} 2^{p(j-k+1)} \cdot 2^{pj} m(e_j) \right)^{1/p} \geq 1.$$

Let  $a_k = \frac{1}{T_k(f)}$ . Then  $0 < a_k \leq 1$  for  $k_0 + 1 \leq k \leq k_1$ . The required operator  $T$  is then defined by

$$T(h) = \sum_{k=k_0+1}^{k_1} a_k T_k(h) \varphi_k g, \quad h \in L_p.$$

Clearly,  $T(f) = g$  and  $T$  takes values in  $H_p$ . Given  $h \in L_\infty$ , we have by the Hölder inequality and (17) and (14)

$$\begin{aligned} |T_k(h)| &\leq \frac{\|Sh\|_\infty}{v_k} \left( \sum_{j=k-1}^{k_1-1} 2^{p(j-k+1)} m(e_j) \right)^{1/p} \\ &\leq 2^{-k+1} \|h\|_\infty . \end{aligned}$$

Then by (i), (iii)-(v)

$$\begin{aligned} \|T(h)\|_\infty &\leq \left\| \sum_{k=k_0+1}^{k_1} a_k 2^{-k+1} |\varphi_k|^{1/2} b \cdot |\varphi_k|^{1/2} \right\|_\infty \|h\|_\infty \\ &\leq C \|h\|_\infty \left\| \sum_{k=k_0+1}^{k_1} |\varphi_k|^{1/4} b^{-1} \cdot |\varphi_k|^{1/4} b \right\|_\infty \\ &\leq C \|h\|_\infty . \end{aligned}$$

Therefore  $T$  maps  $L_\infty$  into  $H_\infty$  boundedly. To prove that  $T$  maps  $L_p$  into  $H_p$  boundedly, we first claim that for  $k_0 + 1 \leq k \leq k_1$

$$(18) \quad \int_{\mathbf{T}} |\varphi_k|^{p/2} \leq C 2^{-pk} v_k^p .$$

Indeed, by the definition of  $\varphi_k$ , we have

$$\begin{aligned} \int_{\mathbf{T}} |\varphi_k|^{p/2} &\leq C \left( \int_{\mathbf{T}} |1 - G_k|^{p/2} + \int_{\mathbf{T}} |1 - G_{k-1}|^{p/2} \right) \\ &\leq C \left( \int_{\mathbf{T}} |1 - F_k|^{4p} + \int_{\mathbf{T}} |1 - F_{k-1}|^{4p} \right) . \end{aligned}$$

Then by the  $L_{4p}$ -boundedness of  $\mathcal{H}$  and by (7)

$$\begin{aligned} \int_{\mathbf{T}} |\varphi_k|^{p/2} &\leq C \left( \int_{\mathbf{T}} |\alpha_k - 1|^{4p} + \int_{\mathbf{T}} |\alpha_{k-1} - 1|^{4p} \right) \\ &\leq C 2^{-pk} \sum_{j=k-1}^{k_1-1} \int_{e_j} b^p \\ &\leq C 2^{-pk} \sum_{j=k-1}^{k_1-1} 2^{pj} m(e_j) = C 2^{-pk} v_k^p , \end{aligned}$$

thus proving (18). From the Hölder inequality, (i) and (iii)-(v), we deduce that

$$|T(h)|^p \leq \sum_{k=k_0+1}^{k_1} |T_k(h)|^p 2^{pk} |\varphi_k|^{p/2} , \text{ a.e. on } \mathbf{T} .$$

Combining the preceding inequalities with the Hölder inequality and (17), we obtain for  $h \in L_p$

$$\begin{aligned}
 \|T(h)\|_p^p &\leq C \sum_{k=k_0+1}^{k_1} \left( \int_0^1 |Sh(s)| g_k(s) u_k(s) ds \right)^p \\
 &\leq C \sum_{k=k_0+1}^{k_1} \int_0^1 |Sh(s)|^p u_k(s) ds \\
 &= C \sum_{k=k_0+1}^{k_1} \sum_{j=k-1}^{k_1-1} 2^{p(j-k+1)} \int_0^1 |Sh(s)|^p w_j(s) ds \\
 &= C \sum_{k=k_0+1}^{k_1} \sum_{j=k-1}^{k_1-1} 2^{p(j-k+1)} \int_0^1 |Sh(s)|^p \sum_{n=0}^N \lambda_n w(\sigma_n^{-1}(s)) \chi_{\tilde{e}_{j,n}}(s) ds \\
 &= C \sum_{n=0}^N \lambda_n \sum_{j=k_0}^{k_1-1} 2^{p(j+1)} \int_0^1 |Sh(s)|^p w(\sigma_n^{-1}(s)) \chi_{\tilde{e}_{j,n}}(s) ds \sum_{k=k_0+1}^{j+1} 2^{-pk} \\
 &\leq C \sum_{n=0}^N \lambda_n \sum_{j=k_0}^{k_1-1} \int_0^1 |Sh(s)|^p w(\sigma_n^{-1}(s)) \chi_{\tilde{e}_{j,n}}(s) ds \\
 &\leq C \sum_{n=0}^N \lambda_n \|Sh\|_p^p \leq C \|h\|_p^p \quad \text{by (11)}.
 \end{aligned}$$

This proves the boundedness of  $T$  from  $L_p$  to  $H_p$  and thus completes the proof of the theorem.

#### 4. Further results.

Let  $E$  be a rearrangement invariant (r.i. for short) space on  $\mathbf{T}$  (cf. [13] for the definition of r.i. spaces and their basic properties). Then  $E$  is an interpolation space between  $L_1$  and  $L_\infty$ . We recall that  $H(E)$  is the closed subspace of  $E$  consisting of all those functions whose Fourier coefficients vanish on negative integers.

COROLLARY 2. — *Let  $(E_0, E_1)$  be a couple of r.i. spaces on  $\mathbf{T}$ . Then*

(i) *for any functor  $F$*

$$F(H(E_0), H(E_1)) = H(F(E_0, E_1)).$$

*In particular, for  $0 < \theta < 1$ ,  $1 \leq q \leq \infty$*

$$(H(E_0), H(E_1))_{\theta q} = H((E_0, E_1)_{\theta q}) \text{ and } (H(E_0), H(E_1))_\theta = H((E_0, E_1)_\theta).$$

(ii) there exists an absolute constant  $C$  such that for any  $f \in H(E_0) + H(E_1)$  and any  $t > 0$

$$K(t, f; H(E_0), H(E_1)) \leq CK(t, f; E_0, E_1).$$

*Proof.* — It is similar to that of Corollary 1. Let  $f \in H(E_0) + H(E_1)$ . By the theorem in Section 1, there exists a one-point projection  $T$  at  $f$  which is bounded simultaneously from  $L_1$  to  $H_1$  and from  $L_\infty$  to  $H_\infty$ . Then  $T$  is bounded on  $E_j$  (cf. [4], [13]). Hence  $T$  is also bounded from  $E_j$  to  $H(E_j)$  ( $j = 0, 1$ ), so the corollary follows.

*Remark.* — Let  $f \in H_1$ . Then the one-point projection  $T$  at  $f$  given by the theorem is bounded on any r.i. space  $E$  (into  $H(E)$ ). Note that  $T$  depends only on  $f$  and its norm from  $E$  to  $H(E)$  is uniformly bounded.

**COROLLARY 3.** — Let  $(E_0, E_1)$  be a Calderón-Mitjagin couple. Then for any  $f, g \in H(E_0) + H(E_1)$  such that

$$K(t, g; H(E_0), H(E_1)) \leq K(t, f; H(E_0), H(E_1)), \quad \forall t > 0,$$

there exists a linear operator  $T$  defined on  $E_0 + E_1$  which maps boundedly  $E_j$  into  $H(E_j)$  ( $j = 0, 1$ ), such that  $T(f) = g$ .

Consequently,  $(H(E_0), H(E_1))$  is a Calderón-Mitjagin couple.

*Proof.* — Let  $f, g \in H(E_0) + H(E_1)$  be as above. Then by Corollary 2(ii)

$$K(t, g; E_0, E_1) \leq CK(t, f; E_0, E_1), \quad \forall t > 0.$$

$(E_0, E_1)$  being a Calderón-Mitjagin couple, there exists an operator  $T_1$  bounded from  $E_j$  to  $E_i$  ( $j = 0, 1$ ), such that  $T_1(f) = g$ . Now applying our theorem to  $g$  we find a one-point projection  $T_2$  at  $g$  bounded from  $E_j$  to  $H(E_j)$  ( $j = 0, 1$ ). Hence the operator  $T = T_2T_1$  satisfies the requirement of Corollary 3.

*Remark.* — Corollary 3 in particular shows that  $(H_{p_0}, H_{p_1})$  ( $1 \leq p_0, p_1 \leq \infty$ ) is a Calderón-Mitjagin couple. This is the  $H_p$ -space version of the well-known  $L_p$ -space result cited in Section 1.

*Remark.* — We have considered Hardy spaces of analytic functions only in the unit disc. Our results, however, also hold for Hardy spaces of analytic functions in the upper half-plane. With some minor modifications, all the above proofs go through to the upper half-plane case.

*Remark.* — The origin of this note is a preprint from June, 1990 (see Pub. IRMA, LILLE - 1990, Vol. 22, n° III). There we gave elementary

proofs for the three interpolation theorems of P.W. Jones on Hardy spaces, which we have already cited previously. These proofs are essentially the same as that of the main theorem of the present note. Let us also mention that there exist two other new proofs (independent of ours) for the two first theorems of Jones concerning the real and complex interpolations, found respectively by P.F.X. Müller [15] and G. Pisier [16].

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