ANNALES DE L'INSTITUT FOURIER

MARCOS CRAIZER

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Annales de l'institut Fourier, tome 44, nº 5 (1994), p. 1435-1448

<http://www.numdam.org/item?id=AIF_1994__44_5_1435_0>

Ann. Inst. Fourier, Grenoble
44, 5 (1994), 1435-1448

HOMOGENIZATION OF CODIMENSION 1 ACTIONS OF $\mathbb{R}^n$ NEAR A COMPACT ORBIT

by Marcos CRAIZER

1. Introduction.

A C^∞ locally free $\mathbb{R}^n$ action on a $(n+1)$- manifold M is a C^∞ map $\Phi : \mathbb{R}^n \times M \to M$ satisfying :

(1) $\Phi(0,p) = p, \quad \forall p \in M,$

(2) $\Phi(u+v,p) = \Phi(u,\Phi(v,p)), \quad \forall p \in M, u,v \in \mathbb{R}^n$ and

(3) $D_1\Phi(0,p) : \mathbb{R}^n \to T_pM$ is an injection, $\forall p \in M$.

It is easy to see that each Φ-orbit is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$, for some $0 \leq k \leq n$, and a compact orbit T of Φ is diffeomorphic to $\mathbb{T}^n$. Here $\mathbb{T}^n$ is the standard torus $\mathbb{R}^n/\mathbb{Z}^n$. Taking M orientable, T divides its tubular neighborhood in M into 2 sides. We shall call each side a *one-sided neighborhood* of T. It is well-known ([2]) that if V is a one-sided neighborhood of an isolated compact orbit T, then there exists $0 \leq k \leq n-1$ such that the orbits of Φ intersecting V are all diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$.

A locally free action of $\mathbb{R}^n$ determines a collection of vector fields in M, given by

$$X_u(p) = D_1\Phi(0,p).u,$$

$u \in \mathbb{R}^n$. Our interest is to describe the dynamics of these vector fields in a one-sided neighborhood V of a compact orbit T of Φ.

We begin with a simple example. A locally free action of $\mathbb{R}^n$ on $\mathbb{T}^n \times [0,\infty)$ is *homogeneous* if it is invariant by transformations of the

Partially Supported by CNPq.
Key words : Homogenization – Codimension 1 actions of $\mathbb{R}^n$ – Compact orbit – Return times.
A.M.S. Classification : 34C20.

type $(x,z) \to (x+a,z)$, with $a \in \mathbb{R}^n$, where $x \in \mathbb{T}^n$, $z \in [0,\infty)$. This is equivalent to say that the components of the vector fields X_{e_i}, $1 \le i \le n$, depend only on z, and not on x, where e_i, $1 \le i \le n$, are the generators of the isotropy group of $\mathbb{T}^n \times \{0\}$. It is also easy to show, and we shall do it in section 3, that the generators of any homogeneous locally free action of R^n can be written in the form

$$X_{e_i} = \sum_{j=1}^{n}(\alpha_i a_j(z) + \delta_{ij})\frac{\partial}{\partial x_j} + \alpha_i c(z)\frac{\partial}{\partial z}, \tag{1.1}$$

where $a_i : [0,\epsilon] \to \mathbb{R}$, $1 \le i \le n$ and $c : [0,\epsilon] \to \mathbb{R}$ are functions satisfying $a_i(0) = 0$, $c(0) = 0$, $c(z) > 0$, if $z > 0$; and α_i, $1 \le i \le n$, are real numbers. We shall denote this action by $\Psi(\alpha_i, a_i, c)_{1\le i\le n}$.

The dynamics of an homogeneous action is very well understood. Let E be the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to $\alpha = (\alpha_1, \cdots, \alpha_n)$. If $u = (u_1, \cdots, u_n) \in E$, then

$$X_u = \sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j}.$$

Thus, for $u \in E$, X_u is a linear vector field tangent to the tori $z = \textit{constant}$. Also, the subspace E divides $\mathbb{R}^n$ into 2 half-spaces E_- and E_+ defined by

$$E_{-(+)} = \left\{ u \in \mathbb{R}^n \mid \sum_{i=1}^{n} u_i\alpha_i > (<)\ 0 \right\}$$

such that if $u \in E_{-(+)}$, the $\alpha(\omega)$-limit of X_u is contained in $\mathbb{T}^n \times \{0\}$ and the $\omega(\alpha)$-limit is the empty set.

Let us see what remains valid for a general action, not necessarily homogeneous. It can be shown that there still exists a $(n-1)$-dimensional subspace $E \subset \mathbb{R}^n$ dividing $\mathbb{R}^n$ into 2 half-spaces E_- and E_+ such that if $u \in E_-(E_+)$, the $\alpha(\omega)$-limit of any point $p \in V$ is contained in T, while its $\omega(\alpha)$-limit is contained in V^c. This is not difficult to show, and in fact a proof of it is given in [1] for the case $n = 2$.

A more difficult question is : What happens if $u \in E$? If the Φ-orbits intersecting V are diffeomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}$, then $\Phi|_E$ determines a locally free action on $\mathbb{T}^{n-1}$. Therefore the flow of X_u is conjugated to a linear flow on $\mathbb{T}^{n-1}$. On the other hand, if the orbits of Φ intersecting V are diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$, for some $0 \le k \le n-2$, the answer cannot be given immediately.

Let W be the smallest Φ-invariant set containing V. We prove here in this paper that, for $u \in E$, the flow of X_u is conjugated to a linear flow of $\mathbb{T}^n$, if we admit conjugations in the class $\mathcal{C}_0^\infty$ of homeomorphisms between W and $\mathbb{T}^n \times [0,\infty)$, that are C^∞ diffeomorphisms when restricted to int(W). For this result, the smoothness of the action at the compact orbit T is crucial, as the counter-example of section 5 shows us.

The result of the last paragraph will be obtained as a consequence of a theorem that asserts that any action is conjugated to a homogeneous one. For $i = 1, 2$, consider locally free $\mathbb{R}^n$ actions Φ_i, with compact orbits T_i and invariant one-sided neighborhoods W_i as above. For simplicity, consider coordinates on W_1 and W_2 such that $T_1 = T_2 = T$. We say that (Φ_1, W_1) and (Φ_2, W_2) are $\mathcal{C}_0^\infty$ conjugated if there exists a homeomorphism $H : W_1 \to W_2$, homotopic to the identity at T, whose restriction to int(W_1) is a C^∞-diffeomorphism such that $H_*\Phi_1 = \Phi_2$. The main result of the paper is the following :

1.1. THEOREM. — *Let Φ be a C^∞ locally free action of $\mathbb{R}^n$ on M with an isolated compact orbit T. Fix a one-sided neighborhood V of T, and let W be the smallest Φ-invariant set containing V. There exists a homogeneous action $\Psi(\alpha_i, a_i, c)$, $1 \le i \le n$, of class $\mathcal{C}_0^\infty$ such that (Φ, W) and $(\Psi, \mathbb{T}^n \times [0,\infty))$ are $\mathcal{C}_0^\infty$ conjugated.*

This result shows that the homogeneous actions are the models for any locally free action of $\mathbb{R}^n$ near a compact orbit. The method of proof of this theorem that we give here is very simple. The basic idea is to find a homogeneous action whose holonomies and return times are the same as those of the given action. The details of this proof are given in sections 2,3 and 4.

The behaviour of the flows of X_u, $u \in E$, is a corollary of Theorem 1.1.

1.2. COROLLARY. — *For $u \in E$, the flow of $X_u|_{\mathrm{int}(W)}$ is C^∞-conjugated to a linear flow on $\mathbb{T}^n \times (0,\infty)$.*

By Theorem 1.1, every action is conjugated to a homogeneous one. A natural question that arises is when two given homogeneous actions are conjugated near the compact orbit. We can answer it when the open orbits of them are not diffeomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}$.

Let $\Psi(\alpha_i, a_i, c)_{1 \le i \le n}$ be a homogeneous action whose open orbits are not diffeomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}$. In this case, we can assume that $\alpha_1 = 1$ and $\alpha_i \notin \mathbb{Q}$, $2 \le i \le n$.

1.4. Theorem. — *Let $\Omega(\beta_i, b_i, d)_{1\leq i\leq n}$ be a homogeneous action. Assume, by making a linear reparametrization if necessary, that the generators of the isotropy group of Ω at $\mathbb{T}^n \times \{0\}$ are also e_i, $1 \leq i \leq n$. Then Ω is C_0^∞- conjugated to Ψ if and only if $\beta_i = \alpha_i$, $1 \leq i \leq n$, and the following limits exist :*

$$K_i = \lim_{z\to 0} \int_z^1 \frac{b_i(s) - a_i(s)}{c(s)} ds$$

and

$$K_0 = \lim_{z\to 0} \int_z^1 \frac{d(s)}{c(s)} ds.$$

The proof of this theorem is simple and will be given in section 6.

Acknowledgements : I want to thank J.L. Arraut for helpful conversations and suggestions during the preparation of this paper.

2. Holonomy and return times.

Let Φ be a locally free $\mathbb{R}^n$ action on M with an isolated compact orbit T. We consider coordinates in a one-sided neighborhood V of T such that $V = \mathbb{T}^n \times [0, \epsilon)$ and $T = \mathbb{T}^n \times \{0\}$. Assume, w.l.o.g., that $\{x\} \times [0, \epsilon)$ is transversal to the foliation $\mathcal{F}$ subjacent to Φ, for any $x \in \mathbb{T}^n$. Fix $S = \{0\} \times [0, \epsilon]$ and, for each $g \in \pi_1(\mathbb{T}^n)$, denote by $H_g : S_g \subset S \to S$ the holonomy map associated to g.

For $g \in \pi_1(\mathbb{T}^n)$ and $z \in S_g$, let $X_{g,z}$ be a curve beginning in $(0, z)$ and tangent to $\mathcal{F}$ which is a lifting of a curve representing g in $\mathbb{T}^n$. Clearly, $X_{g,z}$ ends at the point $(0, H_g(z))$. The curve $X_{g,t}$ can be lifted to $\mathbb{R}^n$ by the covering map $\Phi(-, (0, z))$, and the vector difference between the final point and the initial point of this lifting does not depend on the choice of the initial point. We denote this difference by $U_g(z)$. The function $U_g : S_g \to \mathbb{R}^n$ will be called the *return time* associated with g. Clearly, H_g and U_g are related by the formula

$$\Phi(U_g(z), (0, z)) = H_g(z).$$

2.1. Proposition. — *Take $g_1, g_2 \in \pi_1(\mathbb{T}^n)$ and $z \in S_{g_1} \cap S_{g_2} \cap H_{g_1}^{-1}(S_{g_2}) \cap H_{g_2}^{-1}(S_{g_1})$. Then*

$$U_{g_1}(z) + U_{g_2}(H_{g_1}(z)) = U_{g_2}(z) + U_{g_1}(H_{g_2}(z)). \tag{2.1}$$

Proof. — The curve $X_{g_1,z}$ followed by $X_{g_2,H_{g_1}(z)}$ is a lifting of a curve representing $g_1.g_2 \in \pi_1(\mathbb{T}^n)$. Hence

$$U_{g_1.g_2}(z) = U_{g_1}(z) + U_{g_2}(H_{g_1}(z)).$$

Since $g_1.g_2 = g_2.g_1$, formula (2.1) is proved.

Take now 2 locally free actions Φ_1 and Φ_2 as above, choosing coordinates such that $T_1 = T_2 = T$ and $S_1 = S_2 = S$. Denote by $H_{i,g} : S_{i,g} \subset S \to S$ and $U_{i,g} : S_{i,g} \subset S \to \mathbb{R}^n$ the holonomy maps and the return times of Φ_i, $i \in \{1,2\}$, respectively. Also, let W_i be the smallest Φ_i-invariant set containing V_i.

2.2. Proposition. — *Suppose that $H_{1,g} = H_{2,g}$ and $U_{1,g} = U_{2,g}$, for any $g \in \pi_1(T)$. Then (Φ_1, W_1) and (Φ_2, W_2) are conjugated and this conjugation is homotopic to the identity at T.*

Proof. — For $p \in S$, define $H(p) = p$. If $p = \Phi_1(u,q)$, for some $q \in S$, $u \in \mathbb{R}^n$, define $H(p) = \Phi_2(u,q)$. It is not difficult to verify that $H : W_1 \to W_2$ is well defined and conjugates Φ_1 with Φ_2. Also, $U_{1,g}(0) = U_{2,g}(0)$ implies that the isotropy group at the compact orbit are the same for both actions. Therefore H is homotopic to the identity at T.

3. Homogeneous actions.

Let Ψ be a locally free $\mathbb{R}^n$ action on $\mathbb{T}^n \times [0,\infty)$, whose only compact orbit is $\mathbb{T}^n \times \{0\}$. Assume that Ψ is generated by vector fields of the form

$$X_{e_i} = \sum_{j=1}^{n}(a_{ij}(z) + \delta_{ij})\frac{\partial}{\partial x_j} + c_i(z)\frac{\partial}{\partial z},$$

$(x_1, x_2, \cdots, x_n) \in \mathbb{T}^n$, $z \in [0,\infty)$, where a_{ij} and c_i, $1 \le i,j \le n$, are functions on $[0,\infty)$. Since $\mathbb{T}^n \times \{0\}$ is the unique compact orbit, $c_i(z) = 0$, for any $1 \le i \le n$, if and only if $z = 0$. We shall assume, w.l.o.g., that $a_{ij}(0) = 0$, for any $1 \le i,j \le n$.

Write $a_{1j} = a_j$ and $c_1 = c$. The relation $[X_{e_1}, X_{e_i}] = 0$ gives then

(3.1) $$cc_i' = c_i c'$$

and

(3.2) $$ca_{ij}' = c_i a_j'.$$

Let $[r,s] \subset (0,\infty)$ be such that $c(z) \neq 0$, for any $z \in (r,s)$, but $c(r) = c(s) = 0$. Then Equation (3.1) implies that $\frac{c_i(z)}{c(z)} = \textit{constant}$, for any $z \in (r,s)$. Therefore $c_i(r) = c_i(s) = 0$, $1 \leq i \leq n$, which is not possible. We conclude that either $c(z) = 0$, for any $z \in [0,\infty)$, or $c(z) \neq 0$, for any $z \in (0,\infty)$ and $c_i(z) = \alpha_i c(z)$, for some $\alpha_i \in \mathbb{R}$, $1 \leq i \leq n$. We shall assume, w.l.o.g., that the second hypothesis holds. Equation (3.2) implies now that $a_{ij} = \alpha_i a_j$, $1 \leq i,j \leq n$. Therefore $\Psi = \Psi(\alpha_i, a_i, c)_{1\leq i\leq n}$, the homogeneous action defined by (1.1).

We shall now show how one can compute the holonomy map and the return times of $\Psi = \Psi(\alpha_i, a_i, c)_{1\leq i\leq n}$. We need first some preliminaires.

Given $p = (x_1, \cdots, x_n, z) \in \mathbb{T}^n \times [0,\infty)$ and $u = (u_1, \cdots, u_n) \in \mathbb{R}^n$, we shall compute $\Psi(u,p)$. For this, let $u_i' = (0, \cdots, u_i, \cdots, 0)$ and consider the points p^i, $0 \leq i \leq n$, defined by $p^0 = p$ and $p^i = \Psi(u_i', p^{i-1})$, $1 \leq i \leq n$. Observe that

$$X_{u_i'} = \sum_{j=1}^{n} (u_i \alpha_i a_j(z) + \delta_{ij}) \frac{\partial}{\partial x_j} \; + \; u_i \alpha_i c(z) \frac{\partial}{\partial z}.$$

Writing $p^i = (x_1^i, \cdots, x_n^i, z^i)$, we have

$$\alpha_i u_i = \int_{z^{i-1}}^{z^i} \frac{1}{c(s)} ds,$$

and also

$$x_j^i - x_j^{i-1} = \int_0^1 u_i \alpha_i a_j(z(t)) dt \; + \; \delta_{ij} u_i = \int_{z^{i-1}}^{z^i} \frac{a_j(s)}{c(s)} ds + \delta_{ij} u_i,$$

$1 \leq i,j \leq n$. Summing these equations from $i = 1$ to $i = n$ we obtain

$$\sum_{i=1}^{n} \alpha_i u_i = \int_{z^0}^{z^n} \frac{1}{c(s)} ds$$

and

$$x_j^n - x_j^0 = \int_{z^0}^{z^n} \frac{a_j(s)}{c(s)} ds + u_j,$$

$1 \leq j \leq n$. Fix now $g = (m_1, \cdots, m_n) \in \mathbb{Z}^n = \pi_1(\mathbb{T}^n)$. The above equations imply that

$$\int_z^{H_g(z)} \frac{1}{c(s)} ds = \sum_1^n \alpha_i U_{g,i}(z) \tag{3.3}$$

and

$$\int_z^{H_g(z)} \frac{a_j(s)}{c(s)} ds = m_j - U_{g,j}(z), \tag{3.4}$$

$z \in (0, \infty)$, $1 \leq j \leq n$.

3.1. Proposition. — *The functions H_g and U_g can be computed as follows :*

1. *$H_g(0) = 0$ and $U_{g,j}(0) = m_j$, $1 \leq j \leq n$.*

2. *For $z \in (0, \infty)$, $H_g(z)$ and $U_g(z)$ are the unique solutions of the equations (3.3) and (3.4).*

Proof. — We have to show only that the equations (3.3) and (3.4) admit a unique common solution, for any $z \in (0, \infty)$. But (3.3) and (3.4) imply that

$$\int_z^{H_g(z)} \frac{1 + \sum_1^n \alpha_i a_i(s)}{c(s)} ds = \sum_1^n \alpha_i m_i.$$

Clearly, $H_g(z)$ is the unique solution of this equation. And then $U_g(z)$ is the unique solution of (3.4).

4. Another view of Equations (3.3) and (3.4) - Proof of Theorem 1.1.

In this section, we shall look at Equations (3.3) and (3.4) with another point of view. Assume that we are given H_g and U_g, for each $g = (m_1, \cdots, m_n) \in \pi_1(\mathbb{T}^n)$. We want now to find a_i, $1 \leq i \leq n$ and c satisfying (3.3) and (3.4).

We shall assume, w.l.o.g., that $\alpha_1 = 1$ and that $H_{e_1} : S \to S$ is a contraction. Hence, by [6], there exists a C^1-vector field $\lambda(z)\frac{\partial}{\partial z}$ on S, C^∞ on $S \setminus \{0\}$ whose time 1 is equal to H_{e_1}. This is equivalent to say that

$$\int_z^{H_{e_1}(z)} \frac{1}{\lambda(s)} ds = 1, \tag{4.1}$$

for any $z \in S$.

We shall follow here the notation of [5], with $f = H_{e_1}$. Let $\Delta(z) = z - f(z)$, $z_0 = z$ and $z_i = f(z_{i-1})$, $i > 0$. It follows from Equation (4.3) that $\frac{dz_1}{dz} = \frac{\lambda(z_1)}{\lambda(z)}$ and therefore

$$\frac{dz_i}{dz} = \frac{\lambda(z_i)}{\lambda(z)}. \tag{4.2}$$

In order to simplify the notations, we shall write Equations (3.3) and (3.4) in the form

$$\int_z^{H_g(z)} \frac{\gamma_j(s)}{\lambda(s)} ds = Q_{g,j}(z), \tag{4.3}$$

$0 \leq j \leq n$, where $Q_{g,j} = m_j - U_{g,j}$ and $Q_{g,0} = \sum_1^n \alpha_i U_{g,i}$. Our task is to find $\gamma_j : S \to \mathbb{R}$, $0 \leq j \leq n$, satisfying (4.3).

4.1. LEMMA. — *For any continuous function* $\gamma : S \to \mathbb{R}$,

$$\lim_{z \to 0} \int_z^{H_g(z)} \frac{\gamma(s)}{\lambda(s)} ds = \left(\sum_1^n \alpha_i m_i \right) \gamma(0).$$

Proof. — By the lemma of N. Koppel ([3]), H_g is equal to the time $\sum_1^n \alpha_i m_i$ of $\lambda(z)\frac{\partial}{\partial z}$, what is equivalent to say that

$$\int_z^{H_g(z)} \frac{1}{\lambda(s)} ds = \sum_1^n \alpha_i m_i. \tag{4.4}$$

Given $\rho > 0$, choose $\delta > 0$ such that $|\gamma(s) - \gamma(0)| \leq \rho$, if $0 \leq s \leq \delta$. Then

$$\left| \int_z^{H_g(z)} \frac{\gamma(s)}{\lambda(s)} ds - \left(\sum_1^n \alpha_i m_i \right) \gamma(0) \right| \leq \int_z^{H_g(z)} \left| \frac{\gamma(s) - \gamma(0)}{\lambda(s)} \right| ds \leq \rho \left(\sum_1^n \alpha_i m_i \right),$$

if $0 < z \leq \delta$. This proves the lemma.

This lemma and Equation (4.3) shows that necessarily

$$\left(\sum_1^n \alpha_i m_i \right) \gamma_j(0) = Q_{g,j}(0). \tag{4.5}$$

Therefore, by Proposition 3.1, $\gamma_0(0) = 1$ and $\gamma_j(0) = 0$, if $1 \leq j \leq n$.

4.2. LEMMA. — *The series*

$$\sum_0^\infty Q'_{e_1,j}(z_i)\lambda(x_i) \tag{4.6}$$

defines a continuous function $\sigma_j : S \to \mathbb{R}$, C^∞ *on* $S \setminus \{0\}$, *for any* $0 \leq j \leq n$.

Proof. — By Lemma 2.9 of [5], there exist a constant $C_1 > 0$ such that

$$|\lambda(z_i)| \leq C_1 \Delta(z_i), \tag{4.7}$$

for any $z \in S$, $i \geq 0$. Hence

$$|Q'_{e_1,j}(z_i)\lambda(z_i)| \leq C_1 C_2 \Delta(z_i),$$

where $C_2 = \sup_{z\in S} |Q'_{e_1,j}(z)|$. Since $\sum_0^\infty \Delta(x_i) = x$, this shows that the series (4.6) is uniformly convergent in S. Therefore its sum σ_j is a continuous function.

We prove now that σ_j is C^∞ in $S \setminus \{0\}$, by induction. Assume that σ_j is of class C^k in $S \setminus \{0\}$ and that we can write $\lambda^k(z)\sigma^{(k)}(z)$ in the form

$$\sum_0^\infty g(z_i)\lambda(z_i), \tag{4.8}$$

where $g : S \setminus \{0\} \to \mathbb{R}$ is C^∞. The term by term derivation of series (4.8) gives the series

$$\sum_0^\infty (g'(z_i)\lambda(z_i) + g(z_i)\lambda'(z_i))\frac{\lambda(z_i)}{\lambda(z)},$$

where we have used (4.2). Using again estimate (4.7), one concludes that this last series is uniformly convergent in any compact subset of $S \setminus \{0\}$. This shows that σ_j is of class C^{k+1} and that $\sigma^{(k+1)}$ can also be written in the form (4.8). This completes the induction step and therefore proves the lemma.

Define $\gamma_j(z) = \gamma_j(0) - \sigma_j(z)$, $0 \leq j \leq n$. It is not hard to show that γ_j satisfies Equation (4.3) with $g = e_1$. We must show that (4.3) holds, for any $g \in \pi_1(\mathbb{T}^n)$.

Observe first that, using (2.1), one can easily prove that

$$Q_{e_1,j}(z) + Q_{g,j}(H_{e_1}(z)) = Q_{g,j}(z) + Q_{e_1,j}(H_g(z)), \tag{4.9}$$

for any $g \in \pi_1(\mathbb{T}^n)$, $z \in S_{e_1} \cap S_g \cap H_{e_1}^{-1}(S_g) \cap H_g^{-1}(S_{e_1})$, $0 \leq j \leq n$.

Take $z_0 \in S$ and let $q_j \in \mathbb{R}$ be such that

$$\int_{z_0}^{H_g(z_0)} \frac{\gamma_j(s)}{\lambda(s)} ds = Q_{g,j}(z_0) + q_j.$$

Using now Equation (4.3), with $g = e_1$, for $z = z_0$ and also for $z = H_g(z_0)$, one concludes that

$$\int_{H_{e_1}(z_0)}^{H_g(H_{e_1}(z_0))} \frac{\gamma_j(s)}{\lambda(s)} ds = Q_{e_1,j}(H_g(z_0)) - Q_{e_1,j}(z_0) + Q_{g,j}(z_0) + q_j,$$

since $H_{e_1} \circ H_g = H_g \circ H_{e_1}$. Now, by (4.9),

$$\int_{H_{e_1}(z_0)}^{H_g(H_{e_1}(z_0))} \frac{\gamma_j(s)}{\lambda(s)} ds = Q_{g,j}(H_{e_1}(z_0)) + q_j.$$

By the same argument, we can prove that

$$\int_{H_{e_1}^k(z_0)}^{H_g(H_{e_1}^k(z_0))} \frac{\gamma_j(s)}{\lambda(s)} ds = Q_{g,j}(H_{e_1}^k(z_0)) + q_j,$$

for any $k \geq 1$. Making now $k \to \infty$, we conclude from Lemma 4.1 that

$$\gamma_j(0)\left(\sum_1^n \alpha_i m_i\right) = Q_{g,j}(0) + q_j.$$

Therefore, from (4.5), $q_j = 0$. This shows that (4.3) holds for g.

We can now complete the proof of Theorem 1.1. Given the action Φ, and a one-sided neighborhood of the isolated compact orbit T, consider the holonomy map H_g and return times U_g, $g \in \pi_1(\mathbb{T}^n)$. The holonomy map H_g determine unique real numbers α_i, $1 \leq i \leq n$, satisfying Equation (4.4). And H_g together with U_g determine solutions $\gamma_j : S \to \mathbb{R}$ to the equation (4.3). Define then $c = \dfrac{\lambda}{\gamma_0}$ and $a_i = \dfrac{c\gamma_i}{\lambda}$. The functions a_i, $1 \leq i \leq n$, and c are continuous, C^∞ outside 0. Therefore $\Psi = \Psi(\alpha_i, a_i, c)_{1 \leq i \leq n}$ is of class C_0^∞.

By Proposition 3.1, the holonomy map of Ψ is H_g, and its return times are given by U_g. Therefore, by Proposition 2.2, (Φ, W) and $(\Psi, \mathbb{T}^n \times [0, \infty)$ are C_0^∞ conjugated. This proves Theorem 1.1.

5. A Counter example.

Let $h : \mathbb{T}^2 \to \mathbb{R}$ be a C^∞ function with $\int_{\mathbb{T}^2} h(x_1, x_2)dx_1dx_2 = 0$, and consider the locally free C^∞ action of $\mathbb{R}^2$ on $\mathbb{T}^2 \times R$ generated by

$$\begin{cases} X_1 = \frac{\partial}{\partial x_1} + \alpha\frac{\partial}{\partial x_2} + h(x_1, x_2)\frac{\partial}{\partial z} \\ X_2 = \frac{\partial}{\partial z}. \end{cases} \tag{5.1}$$

Using Birkhoff's theorem, one can show that

$$\lim_{t\to\infty} \frac{X_1^t(p) - L^t(p)}{t} = 0$$

for any $p \in \mathbb{T}^2 \times \mathbb{R}$, where L is the linear vector field $\dfrac{\partial}{\partial x_1} + \alpha\dfrac{\partial}{\partial x_2}$. This implies that if the above action could be homogeneizable by a homotopic to the identity conjugation, X_1 would be conjugated with L.

In what follows, we shall find a real number α and a function h such that this latter fact cannot occur. Assume then that we compactify the example by adding a compact orbit at $\mathbb{T}^2 \times \{-\infty\}$. If the action so obtained were smooth, this would contradict Theorem 1.1. We conclude that the smoothness at T is essential for this result.

5.1. Lemma. — *There exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and a sequence of integers $(k_l)_{l\in\mathbb{N}}$ such that*

$$|1 - e^{2\pi i k_l \alpha}| \leq \frac{1}{2^{k_l}}, \tag{5.2}$$

$l \in \mathbb{N}$.

For a proof of this lemma, see [4,p.178]. For each $l \in \mathbb{N}$, let $j_l \in \mathbb{Z}$ be such that $-\frac{1}{2} \leq j_l + k_l\alpha \leq \frac{1}{2}$. Then clearly

$$\left|\frac{j_l + k_l\alpha}{1 - e^{2\pi i k_l\alpha}}\right| \leq \frac{\pi^2}{2}, \tag{5.3}$$

$l \in \mathbb{N}$.

Define now

$$H(x, y) = \sum_{(j,k)\in\mathbb{Z}^2} b_{(j,k)} e^{2\pi i(jx+ky)},$$

where

$$b_{(j,k)} = \begin{cases} \frac{-2\pi i(j_l+k_l\alpha)k_l}{|1-e^{2\pi i k_l\alpha}|2^{k_l}}\ , & if\ (j,k) = (j_l, k_l) \\ 0\ , & \text{otherwise.} \end{cases}$$

By (5.3),

$$|b_{(j,k)}| \leq \frac{\pi^2}{2}\frac{k_l}{2^{k_l}},$$

which implies that H is of class C^∞.

Take $h = \text{Re } H$. Then the z-component z_n of $X^n(0,0,0)$ is given by

$$z_n = \text{Re} \int_0^n H(s, s\alpha)ds$$

$$= \text{Re} \sum_{(j,k)\in Z^2} b_{(j,k)} \frac{e^{2\pi i(j+k\alpha)n} - 1}{2\pi i(j+k\alpha)}.$$

Therefore

$$z_n = \sum_{l\in N} \frac{k_l}{2^{k_l}} \frac{\text{Re } (1 - e^{2\pi i k_l \alpha n})}{|1 - e^{2\pi i k_l \alpha}|}.$$

Fix l and take n such that $\text{Re}(1 - e^{2\pi i k_l \alpha n}) \geq 1$. Then

$$z_n \geq \frac{k_l}{2^{k_l}} \frac{1}{|1 - e^{2\pi i k_l \alpha}|} \geq k_l,$$

by (5.2). This shows that $\limsup_{n\to\infty} z_n = \infty$, which implies that the flows of X_1 and L are not conjugated.

6. Conjugation between homogeneous actions.

Let $\Psi(\alpha_j, a_j, c)_{1\leq j\leq n}$ be a homogeneous action and e_j, $1 \leq j \leq n$, the generators of the isotropy group of $\mathbb{T}^n \times \{0\}$. For simplicity, write $X_j = X_{e_j}$, $1 \leq j \leq n$. Assume that the orbits of Ψ on $\mathbb{T}^n \times (0,\infty)$ are not diffeomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}$. In this case, we can assume that $\alpha_1 = 1$ and $\alpha_j \notin \mathbb{Q}$, $2 \leq j \leq n$. We now prove Theorem 1.4.

Let $\Omega(\beta_j, b_j, d)_{1\leq j\leq n}$ be another homogeneous action and assume, w.l.o.g., that e_j are also the generators of the isotropy group of $\mathbb{T}^n \times \{0\}$. We denote its generators vector fields by Y_j, $1 \leq j \leq n$. It is clear that

$$X_j - \alpha_j X_1 = \frac{\partial}{\partial x_j} - \alpha_j \frac{\partial}{\partial x_1} \tag{6.1}$$

and

$$Y_j - \beta_j Y_1 = \frac{\partial}{\partial x_j} - \beta_j \frac{\partial}{\partial x_1}. \tag{6.2}$$

Let $H : \mathbb{T}^n \times [0,\infty) \to \mathbb{T}^n \times [0,\infty)$ be a conjugation of class C_0^∞ between Ψ and Ω. It is easy to see from (6.1), (6.2) and the fact that H is homotopic to the identity that $\beta_j = \alpha_j$ and

$$H_*\left(\frac{\partial}{\partial x_j} - \alpha_j \frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_j} - \alpha_j \frac{\partial}{\partial x_1}, \tag{6.3}$$

for each $2 \le j \le n$.

We write $x = (x_1, x_2, \cdots, x_n)$ and

$$H(x,z) = (x,0) + (H_1(x,z), H_2(x,z), \cdots, H_{n+1}(x,z)), \tag{6.4}$$

with $H_i : \mathbb{T}^n \times [0,\infty) \to \mathbb{R}$ functions, $1 \le i \le n+1$. The formula (6.3) can be written as

$$\frac{\partial H_i}{\partial x_j} - \alpha_j \frac{\partial H_i}{\partial x_1} = 0, \tag{6.5}$$

$1 \le i \le n+1$, $2 \le j \le n$.

6.2. Lemma. — *For each $1 \le i \le n+1$, $2 \le j \le n$,*

$$\frac{\partial H_i}{\partial x_j} = 0.$$

Proof. — By taking the Fourier series of H_i we write

$$H_i(x,z) = \sum_{v \in \mathbb{Z}^n} h_{i,v}(z) e^{2\pi i <v,x>},$$

where $< .,. >$ denotes the usual inner product in $\mathbb{R}^n$. Equation (6.5) implies that

$$\sum_{v \in Z^n} h_{i,v}(z)(v_j - \alpha_j v_1) e^{2\pi i <v,x>} = 0,$$

$1 \le i \le n+1$, $2 \le j \le n$. Now $\alpha_j \notin \mathbb{Q}$ implies that $h_{i,v}(z) = 0$, unless $v_1 = v_j = 0$. Since this holds for any $2 \le j \le n$, we conclude that $h_{i,v}(z) = 0$, unless $v = 0$. Therefore $H_i(x,z) = h_{i,0}(z)$, thus proving the lemma.

If we apply dH to X_1 we obtain Y_1. This fact leads to the equations

$$a_i(z) + H_i'(z)c(z) \;=\; b_i(z),$$

for any $1 \le i \le n$, and

$$H_{n+1}'(z)c(z) \;=\; d(z).$$

Therefore H must be given by

$$H_i(z) = -\int_z^1 \frac{b_i(s) - a_i(s)}{c(s)} ds + C_i, \tag{6.6}$$

where C_i is some constant, $1 \leq i \leq n$; and

$$H_{n+1}(z) = -\int_z^1 \frac{d(s)}{c(s)} ds + C_0, \tag{6.7}$$

where C_0 is another constant.

If we want H to be continuous in $\mathbb{T}^n \times (0)$, then clearly the limits K_i, $1 \leq i \leq n$, and K_0 must exist. Reciprocally, suppose that these limits exist. Take then $C_i = K_i$, $1 \leq i \leq n$, and $C_0 = K_0$, so that the equations (6.6) and (6.7) define continuous functions $H_i : [0, \infty) \to \mathbb{R}$ satisfying $H_i(0) = 0$, $1 \leq i \leq n+1$. The formula (6.4) define then $H \in \mathcal{C}_0^\infty$ satisfying $H_*(\Psi) = \Omega$ on $\mathbb{T}^n \times (0, \infty)$. But since $H = \text{id}$ on $\mathbb{T}^n \times \{0\}$, this relation remains true in $\mathbb{T}^n \times \{0\}$. This completes the proof of Theorem 1.4.

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Manuscrit reçu le 22 mars 1994.

Marcos CRAIZER,
Departamento I
Instituto de Matemática e Estatística
Universidade do Estado do Rio de Janeiro
Rua S. Francisco Xavier, 524, sala 6016, bloco B
Maracaña - CEP 20550
Rio de Janeiro, RJ (Brésil).