CORNELIU CONSTANTINESCU  
A. CORNEA

On the axiomatic of harmonic functions II


<http://www.numdam.org/item?id=AIF_1963__13_2_389_0>
ON THE AXIOMATIC OF HARMONIC FUNCTIONS II
by C. CONSTANTINESCU and A. CORNEA (Bucarest).

In this paper we shall use constantly the notations and definitions from [2].

1. The fine topology of $X$ is the least fine topology on $X$, which is finer than the given topology and with respect to which the superharmonic functions (non necessarily defined on the whole space $X$) are continuous. A set $V \ni x$ is a fine neighbourhood of $x$ if and only if either $x$ is an interior point of $V$ for the initial topology or there exists a superharmonic function $s$, defined on a neighbourhood of $x$, such that

$$s(x) < \liminf_{x \in V \ni y \to x} s(y).$$

**Theorem 1.** — Any point of $X$ possesses a fundamental system of fine neighbourhoods which are compact and connected in the initial topology of $X$.

Since the theorem has a local character we may suppose that there exists a harmonic positive function and a positive potential on $X$. Dividing the sheaf of harmonic functions by a positive harmonic function the fine topology does not change; we may suppose therefore that the constants are harmonic functions.

Let $x \in X$ and $V$ be a fine neighbourhood of $x$. It is sufficient to suppose that $x$ is not an interior point of $V$. There exists then a superharmonic function $s$ on $X$ such that

$$s(x) < \liminf_{x \in V \ni y \to x} s(y).$$

Let $\alpha$ be a real number

$$s(x) < \alpha < \liminf_{x \in V \ni y \to x} s(y)$$
and $U$ be a regular domain containing $x$ such that $s$ is greater than $\alpha$ on $\overline{U} - V$. We denote by $F$ the set

$$F = \{ y \in \overline{U} | s(y) \leq \alpha \}$$

and by $K$ the component of $F$ containing $x$. It is sufficient to prove that $K$ is a fine neighbourhood of $x$.

Let $C$ be a component of $F$ contained in $U$. There exists then an open set $G$, $C \subset G \subset U$, $F \cap \partial G = \emptyset$. Since $s > \alpha$ on $\partial G$ it follows $s > \alpha$ on $G$ which contradicts the inequality $s \leq \alpha$ on $C$. Consequently any component of $F$ has a non-empty intersection with $\partial U$.

Let $\beta$ and $\varepsilon$ be positive numbers such that $s + \beta > \alpha$ on $\partial U$ and $s(x) + \varepsilon \beta < \alpha$. Let $K'$ be a compact set in $\partial U - K$ such that

$$\omega_x^{\partial U} (\partial U - (K \cup K')) < \varepsilon.$$

We denote by $u$ the function on $\overline{U}$ equal to

$$y \rightarrow \omega_y^{\partial U} (\partial U - (K \cup K'))$$

on $U$ equal to 1 on $\partial U - (K \cup K')$ and equal to 0 on $K \cup K'$. $u$ is lower semicontinuous. We denote by $F_\beta$ the set

$$F_\beta = \{ y \in \overline{U} | s(y) + \beta u(y) \leq \alpha \}.$$

Obviously $F_\beta \subset F$ and $x \in F_\beta$. There exists an open set $G$, $K \subset G$, $K' \cap F \cap G = \emptyset$, $F \cap \partial G = \emptyset$. Let $y \in F_\beta \cap G$ and $C_y$ be the component of $F_\beta$ containing $y$. $C_y$ is contained in $G$ since $F_\beta \cap \partial G = \emptyset$. $C_y \cap \partial U$ is not empty, as it was shown above; let $z \in C_y \cap \partial U$. If $z \in K$ then

$$s(y) + \beta u(y) = s(y) + \beta > \alpha$$

which contradicts the relation $z \in F_\beta$. Hence $z \in K$ and $C_y \subset K$, $y \in K$, $F_\beta \cap G \subset K$. Since

$$\liminf_{u \rightarrow K, y \rightarrow x} (s(y) + \beta u(y)) = \liminf_{G \rightarrow F, y \rightarrow x} (s(y) + \beta u(y))$$

$$\geq \liminf_{G \rightarrow F, y \rightarrow x} (s(y) + \beta u(y)) \geq \alpha > s(x) + \beta u(x).$$

$K$ is a fine neighbourhood of $x$.

2. We shall suppose in this paragraph that there exists a positive potential on $X$. 
Theorem 2. — For any non-negative superharmonic function \( s \) on \( X \) and any set \( E \subseteq X \), \( \hat{R}_x^E \) is equal to \( s \) on the fine interior of \( E \).

Let \( x \) be a fine interior point of \( E \). Then \([3]\) (pag. 435)

\[
\lim_{[U,U]} \int_{(x-E)\cap U} d\omega_x^U = 0,
\]

where \( U \) denotes the filter of sections of regular neighbourhoods of \( x \). If \( s \) is bounded in a neighbourhood of \( x \) then

\[
s(x) \geq \hat{R}_x^E(x) = \lim_{U,U} \int \hat{R}_x^E d\omega_x^U
\]
\[
\geq \limsup_{U,U} \int_{E\cap U} R_x^E d\omega_x^U = \limsup_{U,U} \int_{E\cap U} \frac{1}{s} d\omega_x^U
\]
\[
\geq \lim_{U,U} \int s d\omega_x^U - \limsup_{U,U} \int_{(x-E)\cap U} s d\omega_x^U = s(x).
\]

In the general case let \( \mathcal{F} \) be the set of continuous finite positive superharmonic functions dominated by \( s \). We have

\[
s(x) \geq \hat{R}_x^E(x) \geq \sup_{s' \in \mathcal{F}} \hat{R}_x^{E'}(x) = \sup_{s' \in \mathcal{F}} s'(x) = s(x).
\]

Corollary 1. — A polar set has no fine interior points.

Theorem 3. — Let \( G \) be a fine open set and \( s \) be a non-negative superharmonic function. Then

a) \( \hat{R}_x^G = R_x^G \);

b) \( R_x^E = R_x^G \) for any \( E \subseteq G \);

c) \( R_x^G = \sup_{s' \in \mathcal{F}} R_x^{G'} \), where \( \mathcal{F} \) is an increasingly directed set of superharmonic functions with \( s = \sup_{s' \in \mathcal{F}} s' \) on \( G \);

d) \( \hat{R}_x^G = \sup_{K \subseteq G} \hat{R}_x^K \) where \( K \) is compact.

Corollary 2. — For any fine open set \( G \) and any measure \( \mu \) with compact carrier we have

\[
\int s d\mu^G = \int \hat{R}_x^G d\mu \quad (1)
\]

where \( s \) is an arbitrary non-negative superharmonic function.

(1) \( \mu^G \) is the balayaged measure of \( \mu \) on \( G \) \([3]\) (p. 447).
This relation follows from Theorem 3 c) taking \( \mathcal{G} \) as the set of all continuous finite positive superharmonic functions smaller than \( s \).

**Lemma 1.** — Let \( s \) be a positive superharmonic function on \( X \) and \( F \subseteq X \) be a closed non empty set, non polar if \( X \notin \mathcal{B} \). \( s \) is resolutive for the normed Dirichlet problem on \( X - F \) and we have

\[
R^F_\alpha = H^X_{\alpha - F} \tag{2}
\]
on \( X - F \).

Since \( F \) is non-polar if \( X \notin \mathcal{B} \), there exists a locally bounded positive potential on any component of \( X - F \). Let \( s_0 \) be a positive continuous superharmonic function on \( X \). We want to prove that \( X - F \) is an \( \text{MP}_0 \)-set [1]. Let \( s' \in \mathcal{G}^X_{\alpha - F, X} \). Then \( s' + \varepsilon s_0 \) is non-negative outside a compact set contained in \( U \) for any \( \varepsilon > 0 \). From [2] (Theorem 2) it follows \( s' + \varepsilon s_0 \geq 0 \). \( \varepsilon \) being arbitrary we get \( s' \geq 0 \) and \( X - F \) is an \( \text{MP}_0 \)-set. By [1] (Corollary 3) the restrictions of the functions \( \min (s, ns_0) \) on \( \partial U \) are resolutive. Since \( \min (s, ns_0) \uparrow s \) for \( n \uparrow \infty \) it follows that the restriction of \( s \) on \( \partial U \) is resolutive.

Let \( \tilde{s} \in \mathcal{G}^X_{\alpha - F, X} \) and \( s' \) be the function on \( X \) equal to \( s \) on \( F \) and equal to \( \min (s, \tilde{s}) \) on \( X - F \). \( s' \) is superharmonic and dominates \( s \) on \( F \). Hence \( \tilde{s} \geq R^F_\alpha, H^X_{\alpha - F} \geq R^F_\alpha \) on \( X - F \). The converse inequality is trivial.

**Theorem 4.** — Let \( s \) be a non-negative superharmonic function. For any \( E \subseteq X \) such that \( s \) is finite on \( E \)

\[
R^E_\alpha = \inf_{G \supseteq E} R^G_\alpha,
\]
where \( G \) is fine open.

Obviously

\[
R^E_\alpha \leq \inf_{G \supseteq E} R^G_\alpha.
\]

Let \( s' \) be a non-negative superharmonic function on \( X \), \( s' \geq s \) on \( E \) and \( \theta \) be a real number, \( 0 < \theta < 1 \). The set

\[
G = \{ x \in X | s'(x) > \theta s(x) \}
\]

[\( \ast \) The normed Dirichlet problem and the associated notions were introduced in [1].]
is fine open and contains E. We have $s' \geq s$ on G and therefore
\[ \frac{s'}{s} \geq R^G_s \geq \inf_{G \supset E} R^G_s. \]
s', 0 being arbitrary we get
\[ R^E_s \geq \inf_{G \supset E} R^G_s. \]

**Theorem 5 (\textsuperscript{a}).** — Let $s_1$, $s_2$ be non-negative superharmonic functions and E be an arbitrary set. Then
\[ R^E_{s_1 + s_2} = R^E_{s_1} + R^E_{s_2}, \quad \hat{R}^E_{s_1 + s_2} = \hat{R}^E_{s_1} + \hat{R}^E_{s_2}. \]

If E is compact the relation follows from lemma 1. By theorem 3 d) it can be extended to fine open sets and by theorem 4 to arbitrary E subjected to the condition that $s_1 + s_2$ is finite on E. In the general case we have
\[ R^E_{s_1 + s_2} \leq R^E_{s_1} + R^E_{s_2} \]
and
\[ R^E_{s_1 + s_2} = R^E_{s_1} + R^E_{s_2} \]
on E. We denote by $E'$ the set
\[ E' = \{ y \in E | s_1(y) + s_2(y) < \infty \}. \]

Let $x \in X - E$. If $\inf R^E_{s_1 + s_2}(x) = \infty$ the required equality holds at x. On the contrary case there exists a non-negative superharmonic function $s_0$ on X finite at x and $s_0 \geq s_1 + s_2$ on E. For any non-negative superharmonic function $s$ on X and any $\varepsilon > 0$ we have
\[ R^E_{s_1 + s_2}(x) = R^E_{s_1}(x) + R^E_{s_2}(x). \]
Hence
\[ R^E_{s_1 + s_2}(x) = \inf_{E'} R^E_{s_1 + s_2} \leq R^E_{s_1} + \varepsilon s_0. \]
We have therefore
\[ R^E_{s_1 + s_2}(x) = R^E_{s_1 + s_2}(x) = R^E_{s_1}(x) + R^E_{s_2}(x) = R^E_{s_1}(x) + R^E_{s_2}(x). \]
The second equality follows immediately from the first one.

\textsuperscript{a}) This theorem was proved by R.-M. Hervé under the supplementary hypothesis that X has a countable basis, and either $s_1$, $s_2$ are continuous or E is closed or E is open or the axiom D is fulfilled [3].
BIBLIOGRAPHY

