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ON THE COMPLEX AND CONVEX GEOMETRY OF OL'SHANSKIĬ SEMIGROUPS

by Karl-Hermann NEEB

Introduction.

Let K be a connected real Lie group with compact Lie algebra and $K_{\mathbb{C}}$ its complexification which is a complex reductive Lie group. Let $\mathfrak{t} \subseteq \mathfrak{k}$ be a Cartan subalgebra. Each K-biinvariant domain $D \subseteq K_{\mathbb{C}}$ can be written as $D = K \exp \mathcal{D} K$, where $\mathcal{D} \subseteq i\mathfrak{t}$ is a domain which is invariant under the Weyl group of K, we call it the *base of D*. In [AL92] Azad and Loeb give a characterization of the K-biinvariant plurisubharmonic functions on D under the assumption that $\mathcal{D}$ is convex. Moreover, they obtain a description of the holomorphy hulls of biinvariant domains which previously has been derived by Lasalle (cf. [Las78]). Of course these results have even simpler interpretations in the case G is abelian. If G is a vector space, then the G-invariant domains of holomorphy are the convex tube domains and if $G = (\mathrm{S}^1)^n$ is a torus, then $G_{\mathbb{C}} \cong (\mathbb{C}^*)^n$ and the G-invariant domains of holomorphy are the logarithmically convex Reinhardt domains.

In this paper we will generalize the "biinvariant" results to the following setting. Let $\mathfrak{g}$ be a finite dimensional real Lie algebra with a compactly embedded Cartan subalgebra $\mathfrak{t}$. Under some mild additional assumptions on the corresponding root system, there exists a generating invariant closed convex cone $W_{\max}$ in $\mathfrak{g}$ which is maximal with respect to the property that all elements in its interior are *elliptic*, i.e., conjugate to elements of $\mathfrak{t}$. In the special case of a compact Lie algebra we have

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$W_{\max} = \mathfrak{g}$, but our setting also incorporates hermitian simple Lie algebras, a certain class of solvable Lie algebras and also a lot of mixed Lie algebras which are neither reductive nor solvable.

Let G be a connected group with Lie algebra $\mathfrak{g}$ and $W_{\max}^0 = \operatorname{int} W_{\max}$ denote the interior of $W_{\max}$. To the cone $W_{\max}$ we associate a certain connected complex manifold $S_{\max} = G\operatorname{Exp}(iW_{\max}^0)$ which carries a holomorphic semigroup multiplication and which looks infinitesimally like the tube $\mathfrak{g} + iW_{\max}^0$. The semigroup $S_{\max}$ is a connected complex group $G_{\mathbb{C}}$ in the case where $\mathfrak{g}$ is compact. Our first objective, which will be achieved in Section 3, is the characterization of the G-biinvariant plurisubharmonic funtions on biinvariant domains $D \subseteq S_{\max}$. To explain this characterization, we write D as $D = G\operatorname{Exp}(D_h)$, where $D_h \subseteq iW_{\max}^0$ is a G-invariant domain. Then a biinvariant function φ on D is plurisubharmonic if and only if the corresponding function on D_h is locally convex. We also prove a strict version of this result and show that if D_h intersects a Weyl chamber in a convex set, then each biinvariant plurisubharmonic function can be extended to the domain $G\operatorname{Exp}(\operatorname{conv} D_h) \subseteq S_{\max}$.

In this paper we will significantly use the results from [Ne96a] concerning the G-invariant subsets of the open cone $W_{\max}^0$ and invariant functions on them. Let $\widetilde{C} \subseteq W_{\max}^0$ be an invariant subset and $C := \widetilde{C} \cap \mathfrak{t}$. The main result in [Ne96a] is a characterization of the locally convex functions on domains $\widetilde{C}$ by convexity and monotonicity properties of their restrictions to C. Moreover, it contains the result that the restriction mapping from smooth and continuous invariant functions on $W_{\max}^0$ to Weyl group invariant functions on $W_{\max}^0 \cap \mathfrak{t}$ is a bijection. For easier reference we collect in Section 1 the main results of [Ne96a] that will be used in this paper.

Section 2 contains a compendium of some results from the theory of holomorphic representations that will be crucial in Section 4 to prove the existence of sufficiently many biinvariant plurisubharmonic functions and also of a strictly plurisubharmonic biinvariant function (which only exists under some additional assumptions on the Lie algebra $\mathfrak{g}$). In Section 3 we collect the explicit formulas for the low dimensional cases, mainly for $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{su}(2)$ and the oscillator algebra, that will be the bridge between convexity and plurisubharmonicity on the one hand side and convexity and geodesic convexity on the other hand.

As already mentioned above, the main result of Section 4 is that plurisubharmonic biinvariant functions on $S_{\max}$ correspond to convex functions on $W_{\max}$. The method to obtain this result is to adopt the strategy from [AL92] in the sense that instead of using finite dimensional holomorphic representations of complex reductive groups, we use infinite dimensional holomorphic representations of semigroups on Hilbert spaces (cf. [Ne94a], [Ne95a,b]). We conclude this section by showing that a G-invariant function φ on $\operatorname{Exp}(D_h) \subseteq G_{\mathbb{C}}/G$ is geodesically convex if and only if the function $\varphi \circ \operatorname{Exp}$ on D_h is locally convex which in turn is equivalent to the plurisubharmonicity of the corresponding biinvariant function on $D = G \operatorname{Exp}(D_h)$.

In Section 5 we prove among other results that the semigroup $S_{\max}$ is a Stein manifold. The setup of this section needs some additional explanation. Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $W \subseteq \mathfrak{g}$ a generating invariant cone. To this data we associate several semigroups as follows. First let $G_{\mathbb{C}}$ denote the simply connected complex Lie group whose Lie algebra is the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ and $S_W := \langle \exp(\mathfrak{g} + iW^0) \rangle$ the open subsemigroup generated by the exponential image of the open cone $\mathfrak{g} + iW^0 \subseteq \mathfrak{g}_{\mathbb{C}}$. Then S_W is a complex manifold with holomorphic semigroup multiplication. We write $\Gamma(\mathfrak{g}, W^0)$ for the universal covering manifold of S_W which sits as an open subset in the universal covering space $\Gamma(\mathfrak{g}, W)$ of $\overline{S_W}$ (cf. [HiNe93, Ch. 3]). The semigroup multiplication lifts to a multiplication on $\Gamma(\mathfrak{g}, W)$ which is holomorphic on $\Gamma(\mathfrak{g}, W^0)$ and so does the exponential function $\exp\colon \mathfrak{g} + iW \to S_W$ lift to an exponential function $\operatorname{Exp}\colon \mathfrak{g} + iW \to \Gamma(\mathfrak{g}, W)$ with $\operatorname{Exp}(\mathfrak{g} + iW^0) \subseteq \Gamma(\mathfrak{g}, W^0)$. Moreover, the antiholomorphic involution $G_{\mathbb{C}} \to G_{\mathbb{C}}, g \mapsto g^* := \overline{g}^{-1}$, where $g \mapsto \overline{g}$ denotes complex conjugation on $G_{\mathbb{C}}$, lifts to a continuous involutive antiautomorphism $\Gamma(\mathfrak{g}, W) \to \Gamma(\mathfrak{g}, W), s \mapsto s^*$ which is antiholomorphic on $\Gamma(\mathfrak{g}, W^0)$.

If $D \subseteq \Gamma(\mathfrak{g}, W)$ is a central subgroup, then $\Gamma(\mathfrak{g}, W, D) := \Gamma(\mathfrak{g}, W)/D$ is called a *(closed) complex Ol'shanskiĭ semigroup* and $\Gamma(\mathfrak{g}, W^0, D) := \Gamma(\mathfrak{g}, W^0)/D$ the corresponding *open Ol'shanskiĭ semigroup*. Both inherit a canonical semigroup structure with similar properties as in the simply connected case. Note that we regain S_W as $\Gamma(\mathfrak{g}, W, \pi_1(S))$ if we identify the fundamental group $\pi_1(S)$ of S with its canonical image in $\Gamma(\mathfrak{g}, W)$ (cf. [HiNe93, Cor. 3.18]). If $W = \mathfrak{g}$, then the corresponding Ol'shanskiĭ semigroups are precisely the connected complex Lie groups with Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

The main result of Section 5 (Theorem 5.18) is that every Ol'shanskiĭ semigroup of the form $S = G \operatorname{Exp}(iW^0)$, where W^0 is an open elliptic cone, is Stein. We also include a result on holomorphic separability which can be derived with representation theoretic methods.

Since we know from Section 5 that in particular the semigroup $S_{\max}$ considered in Section 4 is Stein, it is natural to ask for the G-biinvariant domains of holomorphy in $S_{\max}$. In Section 6 we show that the biinvariant domains of holomorphy $D = G\operatorname{Exp}(D_h)$ are precisely those for which the domain $D_h \subseteq iW_{\max}^0$ is convex. Moreover, we show in Section 7 that for each connected biinvariant domain $D = G\operatorname{Exp}(D_h)$ the envelope of holomorphy is schlicht over $S_{\max}$ and that it coincides with the domain $\widehat{D} = G\operatorname{Exp}(\operatorname{conv} D_h)$. This means in particular that every holomorphic function on D extends to the domain $\widehat{D} \subseteq S_{\max}$. We note that since we do not assume that $\mathcal{D}$ is a convex subset of $i\mathfrak{t}$, even in the case where $\mathfrak{g}$ is a compact Lie algebra our results are a generalization of those in [AL92].

1. Invariant convex sets and functions in Lie algebras.

In this section we collect the basic notions and the main results of [Ne96a] which will be crucial throughout this paper.

In the remainder of this paper the notion "invariant" always refers to the group $\operatorname{Inn}(\mathfrak{g})$ of inner automorphisms of the Lie algebra $\mathfrak{g}$. In this sense the invariant subspaces are the ideals of $\mathfrak{g}$. We note in particular that if $W \subseteq \mathfrak{g}$ is a closed convex invariant cone, then its *edge* $H(W) := W \cap (-W)$ is an ideal of $\mathfrak{g}$.

DEFINITION 1.1. — (a) *Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. For a subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ we write* $\operatorname{Inn}(\mathfrak{a}) := \langle e^{\operatorname{ad}\mathfrak{a}} \rangle \subseteq \operatorname{Aut}(\mathfrak{g})$ *for the corresponding group of inner automorphisms. A subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ is said to be* compactly embedded *if* $\operatorname{Inn}(\mathfrak{a})$ *has compact closure.*

Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compactly embedded Cartan subalgebra and recall that there exists a unique maximal compactly embedded subalgebra $\mathfrak{k}$ containing $\mathfrak{t}$ (cf. [HHL89, A.2.40]).

(b) *Associated to the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ in the complexification $\mathfrak{g}_{\mathbb{C}}$ is a root decomposition as follows ([HiNe93, Ch. 7]). For a linear functional $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$ we set*

$$\mathfrak{g}_{\mathbb{C}}^{\alpha} := \{X \in \mathfrak{g}_{\mathbb{C}} : (\forall Y \in \mathfrak{t}_{\mathbb{C}})[Y, X] = \alpha(Y)X\}$$

and write $\Delta := \{\alpha \in \mathfrak{t}_{\mathbb{C}}^* \setminus \{0\} : \mathfrak{g}_{\mathbb{C}}^{\alpha} \neq \{0\}\}$ for the set of roots. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\mathbb{C}}^{\alpha}$, $\alpha(\mathfrak{t}) \subseteq i\mathbb{R}$ for all $\alpha \in \Delta$ and $\overline{\mathfrak{g}_{\mathbb{C}}^{\alpha}} = \mathfrak{g}_{\mathbb{C}}^{-\alpha}$, where $X \mapsto \overline{X}$ denotes complex conjugation on $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$.

(c) *A root* α *is said to be* compact *if* $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{k}_{\mathbb{C}}$ *and* non-compact *otherwise. We write* Δ_k *for the set of compact and* Δ_p *for the set of non-compact roots. If* $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ *is a* $\mathfrak{k}$*-invariant Levi decomposition, then we set*

$$\Delta_r := \{\alpha \in \Delta : \mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{r}_{\mathbb{C}}\} \quad \text{and} \quad \Delta_s := \{\alpha \in \Delta : \mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{s}_{\mathbb{C}}\}$$

and recall that $\Delta = \Delta_r \dot{\cup} \Delta_s$ *is a disjoint decomposition (cf. [Ne98, Def. 5.2.4]). Note also that if* $\mathfrak{u}$ *is the largest nilpotent ideal, then* $\mathfrak{u} = [\mathfrak{t}, \mathfrak{u}] + \mathfrak{z}(\mathfrak{g})$ *([HiNe93, Prop. 7.3]) and if* $\mathfrak{t} \cap \mathfrak{r} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{t}_1$*, then* $\mathfrak{l} := \mathfrak{t}_1 \oplus \mathfrak{s}$ *is a complementary subalgebra satisfying* $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{l}$*. Moreover* $\mathfrak{t} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{t}_{\mathfrak{l}}$*, where* $\mathfrak{t}_{\mathfrak{l}} = \mathfrak{t}_1 \oplus (\mathfrak{t} \cap \mathfrak{s})$ *is a compactly embedded Cartan subalgebra of* $\mathfrak{l}$*.*

If $\alpha \in \Delta_s$*, then we write* $\check{\alpha}$ *for the uniquely determined element in the one-dimensional space* $[\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{-\alpha}] \subseteq \mathfrak{t}_{\mathbb{C}}$ *satisfying* $\alpha(\check{\alpha}) = 2$*.*

(d) *A* positive system Δ^+ *of roots is a subset of* Δ *for which there exists a regular element* $X_0 \in i\mathfrak{t}$ *with* $\Delta^+ = \{\alpha \in \Delta : \alpha(X_0) > 0\}$*. We say that a positive system* Δ^+ *is* $\mathfrak{k}$-adapted *if the set* $\Delta_p^+ := \Delta_p \cap \Delta^+$ *of positive non-compact roots is invariant under the* Weyl group $\mathcal{W}_{\mathfrak{k}} := N_{\mathrm{Inn}(\mathfrak{k})}(\mathfrak{t})/Z_{\mathrm{Inn}(\mathfrak{k})}(\mathfrak{t})$ *acting on* $\mathfrak{t}$*. We recall from [Ne94a, Prop. 2.7] that there exists a* $\mathfrak{k}$*-adapted positive system if and only if* $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{z}(\mathfrak{k})) = \mathfrak{k}$*. In this case we say that* $\mathfrak{g}$ *is* quasihermitian*. Note that a simple real Lie algebra is quasihermitian if and only if it is either compact or hermitian. We say that* $\mathfrak{g}$ *has* cone potential *if for each* $Z \in \mathfrak{g}_{\mathbb{C}}^{\alpha} \setminus \{0\}$ *the bracket* $[Z, \overline{Z}]$ *is non-zero.*

If Δ^+ *is a* $\mathfrak{k}$*-adapted positive system of roots, then we conclude from* $\mathfrak{k} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{k}_{\mathfrak{l}}$ *with* $\mathfrak{k}_{\mathfrak{l}} := \mathfrak{k} \cap \mathfrak{l}$ *as in (c) that*

$$\mathfrak{z}_{\mathfrak{l}}(\mathfrak{z}(\mathfrak{k}_{\mathfrak{l}})) = \mathfrak{z}_{\mathfrak{l}}(\mathfrak{z}(\mathfrak{k})) = \mathfrak{k}_{\mathfrak{l}},$$

i.e., that $\mathfrak{l}$ *is also quasihermitian. Thus all its simple ideals are either compact or hermitian.*

(e) *We associate to a positive system* Δ^+ *the convex cones*

$$C_{\min} := \mathrm{cone}\{i[\overline{X_\alpha}, X_\alpha] : X_\alpha \in \mathfrak{g}_{\mathbb{C}}^{\alpha}, \alpha \in \Delta_p^+\}$$

and $C_{\max} := (i\Delta_p^+)^\star = \{X \in \mathfrak{t} : (\forall \alpha \in \Delta_p^+) i\alpha(X) \geqslant 0\}$*.*

(f) *An element* $X \in \mathfrak{g}$ *is called* elliptic *if the operator* $\mathrm{ad}\, X$ *on* $\mathfrak{g}$ *is semisimple with* $\mathrm{Spec}(\mathrm{ad}\, X) \subseteq i\mathbb{R}$*, i.e., if the subalgebra* $\mathbb{R}X$ *of* $\mathfrak{g}$ *is compactly embedded.*

(g) *An invariant convex cone* $W \subseteq \mathfrak{g}$ *is called* elliptic *if* W^0 *is non-empty and consists of elliptic elements of* $\mathfrak{g}$*.* □

The main result in Section 2 of [Ne96a] ([Ne96a, Th. 2.11]) is:

THEOREM 1.2. — *A Lie algebra $\mathfrak{g}$ contains an invariant elliptic cone if and only if it contains a compactly embedded Cartan subalgebra $\mathfrak{t}$ and there exists a $\mathfrak{k}$-adapted positive system Δ^+ with $C_{\min} \subseteq C_{\max}$. If this condition is satisfied, then the unqiuely determined invariant cone $W_{\max}$ with $W_{\max} \cap \mathfrak{t} = C_{\max}$ is elliptic. If $W \subseteq \mathfrak{g}$ is an invariant elliptic cone, then*

$$W^0 = \mathrm{Inn}(\mathfrak{g}).(W^0 \cap \mathfrak{t})$$

and there exists a unique $\mathfrak{k}$-adapted positive system Δ^+ such that $W \subseteq W_{\max}$. It follows in particular that the cones $W_{\max}$ are maximal elliptic. □

From now on Δ^+ denotes a $\mathfrak{k}$-adapted positive system with $C_{\min} \subseteq C_{\max}$ and $W_{\max}$ the corresponding maximal invariant elliptic cone. Note that Theorem 1.2 states in particular that $W_{\max} = \mathrm{Inn}(\mathfrak{g}).C^0_{\max}$, i.e., that each adjoint orbit $\mathcal{O}_X := \mathrm{Inn}(\mathfrak{g}).X$ in $W^0_{\max}$ intersects $\mathfrak{t}$.

We set $C_{\max,\mathrm{k}} := \mathrm{Inn}(\mathfrak{k}).C_{\max} \subseteq \mathfrak{k}$. We recall the set Δ_r^+ of positive roots of solvable type and the set $\Delta^+_{p,s}$ of non-compact positive roots of semisimple type. For $\alpha \in \Delta$ we set $[\alpha] := \{\alpha, -\alpha\}$ and $\mathfrak{g}^{[\alpha]} := \mathfrak{g} \cap (\mathfrak{g}_{\mathbb{C}}^{\alpha} + \mathfrak{g}_{\mathbb{C}}^{-\alpha})$.

We set $\mathfrak{p}_r := \bigoplus_{\alpha \in \Delta_r^+} \mathfrak{g}^{[\alpha]}$ and $\mathfrak{p}_s := \bigoplus_{\alpha \in \Delta_{p,s}^+} \mathfrak{g}^{[\alpha]}$. Note that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_r \oplus \mathfrak{p}_s$ is a $\mathfrak{k}$-invariant decomposition and that $\mathfrak{l} = (\mathfrak{k} \cap \mathfrak{l}) \oplus \mathfrak{p}_s$ is a Cartan decomposition of $\mathfrak{l}$ such that $\mathfrak{k} \cap \mathfrak{l}$ contains the center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$. The following decomposition result (cf. [Ne96a, Lemma 3.3]) is essential to obtain the reduction from $W_{\max}$ to $C_{\max}$.

LEMMA 1.3. — *The mapping*

$$\varphi \colon \mathfrak{p}_r \times \mathfrak{p}_s \times C^0_{\max,\mathrm{k}} \to W^0_{\max}, \quad (X, Y, Z) \mapsto e^{\operatorname{ad} X} e^{\operatorname{ad} Y}.Z$$

is a diffeomorphism. □

An important consequence of Lemma 1.3 is the following result.

THEOREM 1.4. — *If $\Omega \subseteq W^0_{\max}$ is an open invariant subset and $\mathfrak{t}^+ \subseteq \mathfrak{t}$ a fundamental domain for the action of $\mathcal{W}_{\mathfrak{k}}$, then the restriction maps*

$$C^\infty(\Omega)^{\mathrm{Inn}(\mathfrak{g})} \to C^\infty(\Omega \cap \mathfrak{t})^{\mathcal{W}_{\mathfrak{k}}} \quad \text{and} \quad C(\Omega)^{\mathrm{Inn}(\mathfrak{g})} \to C(\Omega \cap \mathfrak{t})^{\mathcal{W}_{\mathfrak{k}}} \to C(\Omega \cap \mathfrak{t}^+)$$

are bijections.

Proof. — [Ne96a, Cor. 3.4, Prop. 3.6]. □

DEFINITION 1.5. — *Let V be a finite dimensional real vector space and $\Omega \subseteq V$ be a subset.*

(a) *A function* $\varphi\colon \Omega \to \mathbb{R}$ *is said to be* locally convex *if each* $x \in \Omega$ *has a convex neighborhood on which* φ *is a convex function.*

(b) *If* Ω *is open, then a function* $\varphi \in C^\infty(\Omega)$ *is called* stably locally convex *if all the bilinear forms* $d^2\varphi(x)\colon V \times V \to \mathbb{R}$, $x \in \Omega$, *are positive definite. If, in addition,* Ω *is convex, then a stably locally convex function* φ *on* Ω *is called* stably convex. *Note that stable local convexity implies local convexity because a two times differentiable function with positive semidefinite second derivative is convex.*

(c) *Let* $C \subseteq V$ *be a closed convex cone. We define a quasiorder* $\leqslant_C$ *on* V *by*

$$x \leqslant_C y \quad \text{if} \quad y - x \in C.$$

A function φ *on a subset* $\Omega \subseteq V$ *is said to be* C-decreasing *if* $x \leqslant_C y$ *implies* $\varphi(y) \leqslant \varphi(x)$.

(d) *Let* $C \subseteq V$ *a convex subset. We define the recession cone of* C *by*

$$\lim(C) := \{v \in V\colon C + v \subseteq C\}. \qquad \square$$

To each $X \in \operatorname{int} C_{\max}$ we associate the cone

$$C_X := \operatorname{cone}\{i\alpha(X)i[\overline{X_\alpha}, X_\alpha]\colon \alpha \in \Delta, X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha\} = C_{\min} + C_{X,k}$$

with $C_{X,k} := \operatorname{cone}\{i\alpha(X)i[\overline{X_\alpha}, X_\alpha]\colon \alpha \in \Delta_k, X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha\}$. We also recall that

$$\mathfrak{t}^+ = \{X \in \mathfrak{t}\colon (\forall \alpha \in \Delta_k^+) i\alpha(X) \geqslant 0\}$$

is a fundamental domain for the action of the Weyl group $\mathcal{W}_{\mathfrak{k}}$ on $\mathfrak{t}$.

For the following theorem we recall that the Lie algebra $\mathfrak{g}$ is called *admissible* if $\mathfrak{g} \oplus \mathbb{R}$ contains pointed generating invariant cones (cf. [Ne98, Ch. V] for other characterizations).

THEOREM 1.6. — *Let* $\Omega \subseteq W_{\max}^0$ *be an open invariant subset,* $\widetilde{\varphi}\colon \Omega \to \mathbb{R}$ *be an invariant (smooth) function, and* $\varphi := \widetilde{\varphi}\,|_{\Omega \cap \mathfrak{t}}$. *Then the following are equivalent:*

(1) $\widetilde{\varphi}$ *is (stably) locally convex.*

(2) φ *is (stably) locally convex and* $d\varphi(X)(C_X) \subseteq \mathbb{R}^-$ ($\mathfrak{g}$ *is admissible and* $d\varphi(X) \in -\operatorname{int} C_X^\star$) *for all* $X \in \Omega \cap \mathfrak{t}$.

For the implication (2) $\Rightarrow$ (1) *it suffices that* $\mathfrak{g}$ *has cone potential. If* $\Omega \cap \mathfrak{t}^+$ *is convex, then each locally convex function on* Ω *has a convex extension to* $\operatorname{conv}(\Omega)$.

If $\Omega \cap \mathfrak{t}$ is convex, then in (2) *the cones C_X can be replaced by the smaller cone $C_{\min}$ which does not depend on X.*

Proof. — [Ne96a, Th. 3.19, Rem. 3.20, Th. 3.22, Rem. 3.23(b)]. □

The following proposition is the basic tool for showing that invariant subsets of $W_{\max}$ are convex.

PROPOSITION 1.7. — *Let $\widetilde{C} \subseteq W_{\max}^0$ be an* $\mathrm{Inn}(\mathfrak{g})$*-invariant subset and $C := \widetilde{C} \cap \mathfrak{t}$. Then* (2) $\Rightarrow$ (1) *holds for the following statements:*

(1) *$\widetilde{C}$ is convex.*

(2) *C is convex and $C_{\min} + C \subseteq C$.*

If, in addition, $\widetilde{C}$ is closed, open, or $C_{\min}$ is pointed, then also (1) $\Rightarrow$ (2).

Proof. — [Ne96a, Prop. 3.14]. □

2. Holomorphic representations.

In this short section we collect the representation theoretic results that we will need in Section 4 to deal with biinvariant plurisubharmonic functions on domains in Ol'shanskiĭ semigroups. In this section $\mathfrak{g}$ is a Lie algebra with compactly embedded Cartan subalgebra $\mathfrak{t}$ and maximal compactly embedded subalgebra $\mathfrak{k}$. Moreover G denotes a connected Lie group with Lie algebra $\mathfrak{g}$ and T and K the analytic subgroups corresponding to $\mathfrak{t}$ and $\mathfrak{k}$.

DEFINITION 2.1. — *Let $\Delta^+ \subseteq \Delta$ denote a positive system.*

(a) *For a $\mathfrak{g}$-module V and $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ we set $V^\lambda := \{v \in V : (\forall X \in \mathfrak{t}_{\mathbb{C}}) X.v = \lambda(X)v\}$. This space is called the weight space of weight λ and λ is called a weight of V if $V^\lambda \neq \{0\}$. We write $\mathcal{P}_V$ for the set of weights of V.*

(b) *Let V be a $\mathfrak{g}_{\mathbb{C}}$-module and $v \in V^\lambda$ a weight vector of weight λ. We say that v is a primitive element of V (with respect to Δ^+) if $v \neq 0$ and $\mathfrak{g}_{\mathbb{C}}^\alpha.v = \{0\}$ holds for all $\alpha \in \Delta^+$.*

(c) *A $\mathfrak{g}_{\mathbb{C}}$-module V is called a highest weight module with highest weight λ (with respect to Δ^+) if it is generated by a primitive element of weight λ.*

□

DEFINITION 2.2. — *Let $(\pi, \mathcal{H})$ be a unitary representation of the group G, i.e., $\pi: G \to U(\mathcal{H})$ is a continuous homomorphism into the unitary group $U(\mathcal{H})$ of the Hilbert space $\mathcal{H}$.*

(a) *We write $\mathcal{H}^\infty$ ($\mathcal{H}^\omega$) for the corresponding space of smooth (analytic) vectors, i.e., for the set of all those elements $v \in \mathcal{H}$ for which the mapping $G \to \mathcal{H}$, $g \mapsto \pi(g).v$ is smooth (real analytic). We write $d\pi$ for the derived representation of $\mathfrak{g}$ on $\mathcal{H}^\infty$ given by*

$$d\pi(X).v = \frac{d}{dt}\bigg|_{p=0} \pi(\exp tX).v$$

for $X \in \mathfrak{g}$ and $v \in \mathcal{H}^\infty$. We extend this representation to a representation of the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ on the complex vector space $\mathcal{H}^\infty$.

(b) *A vector $v \in \mathcal{H}$ is said to be K-finite if it is contained in a K-invariant finite dimensional subspace of $\mathcal{H}$. We write $\mathcal{H}^K$ for the set of K-finite vectors in $\mathcal{H}$. Note that the space $\mathcal{H}^{K,\infty}$ of K-finite smooth vectors is a $\mathfrak{g}_\mathbb{C}$-submodule of $\mathcal{H}^\infty$ (cf. [Ne94a, p.121]).*

(c) *A unitary representation $(\pi, \mathcal{H})$ of the connected Lie group G with $\mathbb{L}(G) = \mathfrak{g}$ is called a unitary highest weight representation if the $\mathfrak{g}_\mathbb{C}$-module $\mathcal{H}^{K,\infty}$ of smooth K-finite vectors is a highest weight module.* □

Let $\mathfrak{g}$ and $\mathfrak{t}$ be as above and suppose that Δ_p^+ is a $\mathfrak{k}$-adapted positive system. Let further $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{l}$ be as in Definition 1.1(c) and $C_{\max,s} = (i\Delta_{p,s}^+)^\star$. Then there exists a unique generating invariant closed convex cone $W_s \subseteq \mathfrak{l}$ with $W_s \cap \mathfrak{t}_\mathfrak{l} = C_{\max,s}$. Then $W_{\max,s} := \mathfrak{u} + W_s \subseteq \mathfrak{g}$ is a pointed generating invariant closed convex cone. The corresponding Ol'shanskiĭ semigroup $S_{\max,s} := \Gamma(\mathfrak{g}, W_{\max,s}^0)$ has the property that $G := \langle \mathrm{Exp}(\mathfrak{g}) \rangle$ is the simply connected group with Lie algebra $\mathfrak{g}$ and that the mapping

$$G \times W_{\max,s} \to \Gamma(\mathfrak{g}, W_{\max,s}), \quad (g, X) \mapsto g\,\mathrm{Exp}(iX)$$

is a homeomorphism and a diffeomorphism $G \times W_{\max,s}^0 \to S_{\max,s}$ ([HiNe93, Cor. 7.35]).

In the following we will mainly be interested in certain Ol'shanskiĭ semigroups for which the corresponding cone W is contained in $W_{\max,s}$. A particularly important subsemigroup is $S_{\max} = G\,\mathrm{Exp}(W_{\max}^0)$, where $W_{\max} \subseteq \mathfrak{g}$ is the unique generating invariant cone with $W_{\max} \cap \mathfrak{t} = C_{\max}$ (cf. Theorem 1.2). Note that $\mathfrak{u} \subseteq W_{\max}$ so that $W_{\max} = \mathfrak{u} + (W_{\max} \cap \mathfrak{l})$, and since $W_{\max}$ contains all the simple compact ideals of $\mathfrak{g}$, the cone $W_{\max} \cap \mathfrak{l}$ need not be pointed.

Definition 2.3. — *Let $B(\mathcal{H})$ denote the C^*-algebra of bounded operators on the Hilbert space $\mathcal{H}$ and S an open Ol'shanskiĭ semigroup. A holomorphic representation $(\pi, \mathcal{H})$ of S on $\mathcal{H}$ is a holomorphic semigroup* *homomorphism $\pi: S \to B(\mathcal{H})$ with $\pi(s^*) = \pi(s)^*$ for all $s \in S$. If $S = \Gamma(\mathfrak{g}, W^0, D)$, then one can think of holomorphic representations of S as analytic extensions of the corresponding unitary representation π of the group $G := \langle \operatorname{Exp} \mathfrak{g} \rangle \subseteq \Gamma(\mathfrak{g}, W, D)$ which is uniquely determined by $\pi(g) \circ \pi(s) = \pi(gs)$ for $g \in G$ and $s \in S$ (cf. [Ne98, Cor. 2.4.23]).* □

In the next section we will need the following result. We recall that a functional $\lambda \in i\mathfrak{t}^*$ is said to be *dominant integral* with respect to Δ_k^+ if $\lambda(\check{\alpha}) \in \mathbb{N}_0$ holds for all $\alpha \in \Delta_k^+$.

THEOREM 2.4. — *Let $\lambda \in i\mathfrak{t}^*$ be dominant integral and $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} (\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^{\alpha})\alpha$ such that $\lambda + \rho \in i \operatorname{int} C_{\min}^\star$. Then the following assertions hold:*

(i) *There exists a unitary highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of G with highest weight λ.*

(ii) *π_λ extends to a holomorphic representation π_λ of $S_{\max}$ on $\mathcal{H}_\lambda$.*

(iii) *$\pi_\lambda(S_{\max})$ consists of trace class operators, the corresponding character*

$$\Theta_\lambda: S_{\max} \to \mathbb{C}, s \mapsto \operatorname{tr} \pi_\lambda(s)$$

is a holomorphic function, and for $s = \operatorname{Exp} X$, $X \in iC_{\max}^0$, we have

$$\Theta_\lambda(s) = \frac{\Theta_\lambda^K(s)}{\prod_{\alpha \in \Delta_p^+} (1 - e^{-\alpha(X)})^{\dim \mathfrak{g}_{\mathbb{C}}^{\alpha}}},$$

where $\Theta_\lambda^K: K_{\mathbb{C}} \to \mathbb{C}$ is the holomorphic character of the irreducible holomorphic representation of $K_{\mathbb{C}}$ with highest weight λ.

(iv) *For $s = \operatorname{Exp} X$, $X \in iC_{\max}^0$, we have $\log \|\pi(s)\| = \sup\langle \lambda, \mathcal{W}_{\mathfrak{k}}.X \rangle$.*

Proof. — (i) [Ne96b, Th. 3.9].

(ii) [Ne95b, Th. 3.7].

(iii) The first assertion is [Ne96b, Th. 4.3], the holomorphy of the character is proved in [Ne94a, Cor. 4.7], and the explicit formula for the character on $\operatorname{Exp}(iC_{\max}^0)$ follows from the discussion following Theorem 4.5 in [Ne94c].

(iv) This is an easy consequence of [Ne96b, Prop. 4.2]. □

3. Calculations in low dimensional cases.

In this section we collect the calculations in low dimensional Lie algebras that will be needed to obtain the characterizations of the biinvariant (strictly) plurisubharmonic functions and the invariant geodesically convex functions in Section 4. The most relevant cases are $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(1,1)$, $\mathfrak{su}(2)$, and where $\mathfrak{g}$ is the four dimensional oscillator algebra.

3.1. The solvable type.

In this subsection $\mathfrak{g}$ denotes a solvable Lie algebra with compactly embedded Cartan subalgebra $\mathfrak{t}$. We assume that $\Delta^+ = \{\alpha\}$, so that there exist only two root spaces in $\mathfrak{g}_{\mathbb{C}}$. Then $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{l}$, where $\mathfrak{l} \subseteq \mathfrak{t}$ complements the center $\mathfrak{z}(\mathfrak{g})$ and $\mathfrak{u} = \mathfrak{z}(\mathfrak{g}) + \mathfrak{g}^{[\alpha]}$ is the nilradical. Note that $\mathfrak{u}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{\alpha} + \mathfrak{g}_{\mathbb{C}}^{-\alpha} + \mathfrak{z}(\mathfrak{g}_{\mathbb{C}})$ so that $[\mathfrak{u}_{\mathbb{C}}, \mathfrak{u}_{\mathbb{C}}] \subseteq [\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{-\alpha}] \subseteq \mathfrak{z}(\mathfrak{g}_{\mathbb{C}})$. We extend the root α to a functional on $\mathfrak{g}_{\mathbb{C}}$ that vanishes on $\mathfrak{u}_{\mathbb{C}}$. If $\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^{\alpha} = 1$, $[\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{-\alpha}] \neq \{0\}$ and $\dim \mathfrak{t} = 2$, then $\mathfrak{g}$ is called the *four dimensional oscillator algebra*. It has a basis (P, Q, Z, H) where the non-zero brackets are given by

$$[P,Q] = Z, \quad [H,P] = Q, \quad \text{and} \quad [H,Q] = -P.$$

Let $G_{\mathbb{C}}$ denote the simply connected group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and $G \subseteq G_{\mathbb{C}}$ the simply connected subgroup with Lie algebra $\mathfrak{g}$. Then $G_{\mathbb{C}} = G \exp(i\mathfrak{g})$, where the map

$$G \times i\mathfrak{g} \to G_{\mathbb{C}}, \quad (g, X) \mapsto g \exp(X)$$

is a diffeomorphism (cf. Lawson's Theorem in [HiNe93, Ch. 7]). For $s = g\exp(X)$, $X \in i\mathfrak{g}$, we have

$$s^* = \exp(X) g^{-1} = g^{-1} \exp\big(\mathrm{Ad}(g).X\big),$$

so that $s^* = s$ entails $g^2 = \mathbf{1}$, hence $g = \mathbf{1}$ since G is simply connected and solvable. For $s = s^* = \exp X$ we define $\log s := X$.

PROPOSITION 3.1. — *Let $s = \exp(Z_\alpha + Z_{-\alpha})\exp(Z)$ with $Z \in \mathfrak{l}$, $\alpha(Z) \neq 0$, $Z_{\pm\alpha} \in \mathfrak{g}_{\mathbb{C}}^{\pm\alpha}$, and suppose that $s^* = s$. Then there exists a $g \in G$ such that*

$$gsg^{-1} = \exp\Big(Z + \frac{1}{2}\coth\big(\frac{\alpha(Z)}{2}\big)[Z_\alpha, Z_{-\alpha}]\Big).$$

Proof. — We write $s = s_h s_l$ according to the semidirect decomposition $G_{\mathbb{C}} = U_{\mathbb{C}} \rtimes L_{\mathbb{C}}$. Then $s^* = s_l^* s_h^* \in U_{\mathbb{C}} s_l^*$ implies $s_l = s_l^*$ and $s_h^* = s_l^{-1} s_h s_l$. We conclude in particular that $\alpha(Z) \in \mathbb{R}$. We use $[\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{z}(\mathfrak{g})$ and $\exp(A)\exp(B) = \exp(A + B + \frac{1}{2}[A,B])$ for $A, B \in \mathfrak{u}_{\mathbb{C}}$ to obtain for $X \in \mathfrak{g}^{[\alpha]}$

$$\begin{aligned}
&\exp X s \exp(-X) \\
&= \exp X \exp(Z_\alpha + Z_{-\alpha}) \exp(-X) \cdot \exp(X)\exp(Z)\exp(-X) \\
&= \exp\left(e^{\operatorname{ad} X}.(Z_\alpha + Z_{-\alpha})\right) \exp(X) \exp(-e^{\operatorname{ad} Z}.X)\exp(Z) \\
&= \exp(Z_\alpha + Z_{-\alpha} + [X, Z_\alpha + Z_{-\alpha}]) \exp(X - e^{\operatorname{ad} Z}.X - \frac{1}{2}[X, e^{\operatorname{ad} Z}.X]) \exp(Z) \\
&= \exp(C)\exp(Z)
\end{aligned}$$

with

$$\begin{aligned}
C = Z_\alpha + Z_{-\alpha} + [X, Z_\alpha + Z_{-\alpha}] + X - e^{\operatorname{ad} Z}.X - \tfrac{1}{2}[X, e^{\operatorname{ad} Z}.X] \\
+ \tfrac{1}{2}[Z_\alpha + Z_{-\alpha}, X - e^{\operatorname{ad} Z}.X].
\end{aligned}$$

Then

$$C \in Z_\alpha + Z_{-\alpha} + X - e^{\operatorname{ad} Z}.X + \mathfrak{z}(\mathfrak{g}_{\mathbb{C}})$$

so that the requirement $C \in i\mathfrak{z}(\mathfrak{g})$ for $X = X_\alpha + \overline{X_\alpha}$ leads to $Z_\alpha + Z_{-\alpha} = -X + e^{\operatorname{ad} Z}.X$, i.e.,

$$Z_\alpha = \left(e^{\alpha(Z)} - 1\right) X_\alpha \quad \text{and} \quad Z_{-\alpha} = \left(e^{-\alpha(Z)} - 1\right)\overline{X_\alpha}.$$

Now

$$\begin{aligned}
C &= [X, Z_\alpha + Z_{-\alpha}] - \frac{1}{2}[X, -X + e^{\operatorname{ad} Z}.X] + \frac{1}{2}[Z_\alpha + Z_{-\alpha}, -Z_\alpha - Z_{-\alpha}] \\
&= [X, Z_\alpha + Z_{-\alpha}] - \frac{1}{2}[X, Z_\alpha + Z_{-\alpha}] \\
&= \frac{1}{2}[X, Z_\alpha + Z_{-\alpha}] = \frac{1}{2}[(e^{\alpha(Z)} - 1)^{-1} Z_\alpha + (e^{-\alpha(Z)} - 1)^{-1} Z_{-\alpha}, Z_\alpha + Z_{-\alpha}] \\
&= \frac{1}{2}\left(\frac{1}{e^{\alpha(Z)} - 1} - \frac{1}{e^{-\alpha(Z)} - 1}\right)[Z_\alpha, Z_{-\alpha}] \\
&= \frac{1}{2}\left(\frac{e^{-\frac{\alpha(Z)}{2}}}{e^{\frac{\alpha(Z)}{2}} - e^{-\frac{\alpha(Z)}{2}}} - \frac{e^{\frac{\alpha(Z)}{2}}}{e^{-\frac{\alpha(Z)}{2}} - e^{\frac{\alpha(Z)}{2}}}\right)[Z_\alpha, Z_{-\alpha}] \\
&= \frac{1}{2}\frac{e^{-\frac{\alpha(Z)}{2}} + e^{\frac{\alpha(Z)}{2}}}{e^{\frac{\alpha(Z)}{2}} - e^{-\frac{\alpha(Z)}{2}}}[Z_\alpha, Z_{-\alpha}] = \frac{1}{2}\coth\left(\frac{\alpha(Z)}{2}\right)[Z_\alpha, Z_{-\alpha}].
\end{aligned}$$

$\square$

COROLLARY 3.2. — *Let $X \in i\mathfrak{t}$ with $\alpha(X) \neq 0$. Then the following assertions hold:*

(i) *If $s = \exp(X_\alpha - \overline{X_\alpha})\exp(X)\exp(X_\alpha - \overline{X_\alpha})$, then there exists $g \in G$ with*

$$gsg^{-1} = \exp\left(X - 2\coth\left(\frac{\alpha(X)}{2}\right)[X_\alpha, \overline{X_\alpha}]\right).$$

(ii) *If $s = \exp(zX_\alpha)\exp(X)$, $z \in \mathbb{C}$, then there exist $g_1, g_2 \in G$ with*

$$g_1 s g_2 = \exp\left(X - \frac{|z|^2}{2(e^{2\alpha(X)} - 1)}[X_\alpha, \overline{X_\alpha}]\right).$$

Proof. — (i) In view of

$$[X_\alpha - \overline{X_\alpha}, e^{\alpha(X)}X_\alpha - e^{-\alpha(X)}.\overline{X_\alpha}] = (e^{\alpha(X)} - e^{-\alpha(X)})[X_\alpha, \overline{X_\alpha}],$$

we have

$$\begin{aligned} s &= \exp(X_\alpha - \overline{X_\alpha})\exp(e^{\alpha(X)}X_\alpha - e^{-\alpha(X)}.\overline{X_\alpha})\exp(X) \\ &= \exp\Big(\big(1 + e^{\alpha(X)}\big)X_\alpha - \big(1 + e^{-\alpha(X)}\big)\overline{X_\alpha} + \sinh\big(\alpha(X)\big)[X_\alpha, \overline{X_\alpha}]\Big)\exp(X). \end{aligned}$$

Therefore, in the notation of Proposition 3.1, we have

$$Z_\alpha = \big(1 + e^{\alpha(X)}\big)X_\alpha, \quad Z_{-\alpha} = -\big(1 + e^{-\alpha(X)}\big)\overline{X_\alpha}$$

and $Z = X + \sinh\big(\alpha(X)\big)[X_\alpha, \overline{X_\alpha}]$. Hence, according to Proposition 3.1, there exists a $g \in G$ with

$$\begin{aligned} \log(gsg^{-1}) &= X + \sinh\big(\alpha(X)\big)[X_\alpha, \overline{X_\alpha}] \\ &\qquad - \frac{1}{2}\coth\left(\frac{\alpha(X)}{2}\right)\big(1 + e^{\alpha(X)}\big)\big(1 + e^{-\alpha(X)}\big)[X_\alpha, \overline{X_\alpha}] \\ &= X + \sinh\big(\alpha(X)\big)[X_\alpha, \overline{X_\alpha}] - \coth\left(\frac{\alpha(X)}{2}\right)\big(1 + \cosh\alpha(X)\big)[X_\alpha, \overline{X_\alpha}]. \end{aligned}$$

Now we calculate

$$\begin{aligned} \sinh t - \big(\coth\frac{t}{2}\big)(1 + \cosh t) &= \frac{2\big(\sinh\frac{t}{2}\big)^2\cosh\frac{t}{2} - (\cosh\frac{t}{2})2(\cosh\frac{t}{2})^2}{\sinh\frac{t}{2}} \\ &= -\frac{2\cosh\frac{t}{2}}{\sinh\frac{t}{2}} = -2\coth\frac{t}{2} \end{aligned}$$

and obtain

$$\log(gsg^{-1}) = X - 2\coth\left(\frac{\alpha(X)}{2}\right)[X_\alpha, \overline{X_\alpha}].$$

(ii) In view of $[zX_\alpha, \overline{zX_\alpha}] = |z|^2[X_\alpha, \overline{X_\alpha}]$, we may w.l.o.g. assume that $z = 1$. Since, according to $\alpha(X) \neq 0$, s is contained in $S_{\max}$ or $S_{\max}^{-1}$, the existence of g_1 and g_2 follows from $S_{\max} \subseteq G\exp(i\mathfrak{t})G$. Let $g_1sg_2 = \exp(A)$ with $A \in i\mathfrak{t}$. Then $\exp(2A) = (g_1sg_2)^*g_1sg_2 = g_2^{-1}s^*sg_2$ and therefore $s^*s = g_2\exp(2A)g_2^{-1}$. So Proposition 3.1 applies to s^*s. In view of

$$\begin{aligned} s^*s &= \exp(X)\exp(-\overline{X_\alpha})\exp(X_\alpha)\exp(X) \\ &= \exp(X)\exp(X_\alpha - \overline{X_\alpha} + \tfrac{1}{2}[X_\alpha, \overline{X_\alpha}])\exp(X) \\ &= \exp(e^{\alpha(X)}X_\alpha - e^{-\alpha(X)}\overline{X_\alpha})\exp(2X + \tfrac{1}{2}[X_\alpha, \overline{X_\alpha}]), \end{aligned}$$

we have

$$Z = 2X + \frac{1}{2}[X_\alpha, \overline{X_\alpha}], \quad Z_\alpha = e^{\alpha(X)}X_\alpha, \quad \text{and} \quad Z_{-\alpha} = -e^{-\alpha(X)}\overline{X_\alpha}.$$

Therefore Proposition 3.1 implies that

$$\begin{aligned} A &= X + \frac{1}{4}[X_\alpha, \overline{X_\alpha}] + \frac{1}{4}\coth\Big(\frac{\alpha(Z)}{2}\Big)[Z_\alpha, Z_{-\alpha}] \\ &= X + \frac{1}{4}[X_\alpha, \overline{X_\alpha}] - \frac{1}{4}\coth\big(\alpha(X)\big)[X_\alpha, \overline{X_\alpha}] \\ &= X + \frac{1 - \coth\big(\alpha(X)\big)}{4}[X_\alpha, \overline{X_\alpha}] = X - \frac{1}{2(e^{2\alpha(X)} - 1)}[X_\alpha, \overline{X_\alpha}], \end{aligned}$$

because

$$1 - \coth t = \frac{\sinh t - \cosh t}{\sinh t} = -\frac{e^{-t}}{\sinh t} = -\frac{2}{e^{2t} - 1}. \qquad \square$$

3.2. The non-compact reductive case.

In this subsection $\mathfrak{g}$ denotes a reductive Lie algebra with compactly embedded Cartan subalgebra $\mathfrak{t}$ and the property that the commutator algebra is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(1,1)$. Then there exist only two root spaces in $\mathfrak{g}_{\mathbb{C}}$ and $\Delta^+ = \{\alpha\}$. We normalize $X_\alpha \in \mathfrak{g}^\alpha$ by the requirement that $[X_\alpha, \overline{X_\alpha}] = \check{\alpha}$. Let $G_{\mathbb{C}} \cong Z(G_{\mathbb{C}})_0 \times \mathrm{Sl}(2,\mathbb{C})$ denote the simply connected group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, $G \cong Z(G)_0 \times \mathrm{SU}(1,1) \subseteq G_{\mathbb{C}}$ the analytic subgroup with Lie algebra $\mathfrak{g}$, and $S_{\max} \subseteq G_{\mathbb{C}}$ the maximal Ol'shanskiĭ semigroup associated to Δ^+.

PROPOSITION 3.3. — *Let $X \in i\mathfrak{t}$ with $\alpha(X) > 0$. Then the following assertions hold:*

(i) *If $s = \exp\big(t(X_\alpha - \overline{X_\alpha})\big)\exp(X)\exp\big(t(X_\alpha - \overline{X_\alpha})\big)$ with $t \in \mathbb{R}$ and t is sufficiently small, then there exists $g \in G$ with $gsg^{-1} = \exp(X +$*

$f(t)[X_\alpha, \overline{X_\alpha}])$, *where*

$$f(t) = \operatorname{arcosh}\Big(\cos(2t)\cosh(\tfrac{1}{2}\alpha(X))\Big) - \frac{\alpha(X)}{2}.$$

(ii) *If* $s = \exp(zX_\alpha)\exp(X)$ *with* $z \in \mathbb{C}$ *and* z *is sufficiently small, then there exist* $g_1, g_2 \in G$ *with* $g_1 s g_2 = \exp\big(X + f(|z|^2)[X_\alpha, \overline{X_\alpha}]\big)$, *where*

$$f(r) = \frac{1}{2}\operatorname{arcosh}\Big(\cosh\big(\alpha(X)\big) - \frac{r}{2}e^{-\alpha(X)}\Big) - \frac{\alpha(X)}{2} \quad \textit{and} \quad f'(0) < 0.$$

Proof. — Since $G_\mathbb{C} \cong Z(G_\mathbb{C})_0 \times \mathrm{Sl}(2,\mathbb{C})$, it suffices to prove the assertions in the case where $\mathfrak{g} = \mathfrak{su}(1,1) \cong \mathfrak{sl}(2,\mathbb{R})$. We choose the basis

$$H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad Q = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Then $\mathfrak{t} = \mathbb{R}H$ is a compactly embedded Cartan subalgebra, and with the appropriate choice of a positive root,

$$X_\alpha := \frac{1}{2}(P - iQ) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \overline{X_\alpha} := \frac{1}{2}(P + iQ) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and $\check{\alpha} = [X_\alpha, \overline{X_\alpha}] = -iH = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Moreover $X = \mu\check{\alpha}$ with $\mu := \frac{\alpha(X)}{2}$ and therefore $\exp(X) = \begin{pmatrix} e^\mu & 0 \\ 0 & e^{-\mu} \end{pmatrix}$.

(i) Since $X_\alpha - \overline{X_\alpha} = -iQ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we have

$$\begin{aligned} s &= \exp\big(t(X_\alpha - \overline{X_\alpha})\big)\exp(X)\exp\big(t(X_\alpha - \overline{X_\alpha})\big) \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} e^\mu & 0 \\ 0 & e^{-\mu} \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \\ &= \begin{pmatrix} e^\mu\cos t & e^{-\mu}\sin t \\ -e^\mu\sin t & e^{-\mu}\cos t \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \\ &= \begin{pmatrix} e^\mu\cos^2 t - e^{-\mu}\sin^2 t & \cosh(\mu)\sin(2t) \\ -\cosh(\mu)\sin(2t) & e^{-\mu}\cos^2 t - e^\mu\sin^2 t \end{pmatrix}. \end{aligned}$$

Thus $\operatorname{tr} s = (\cos^2 t - \sin^2 t)(e^\mu + e^{-\mu}) = 2\cos(2t)\cosh(\mu)$. If $g \in G = \mathrm{SU}(1,1)$ is chosen such that $gsg^{-1} = \begin{pmatrix} e^{\mu'} & 0 \\ 0 & e^{-\mu'} \end{pmatrix}$ is diagonal, then $\operatorname{tr}(s) = \operatorname{tr}(gsg^{-1}) = 2\cosh(\mu')$ and so

$$\mu' = \operatorname{arcosh}\big(\cos(2t)\cosh(\mu)\big) = \operatorname{arcosh}\big(\cos(2t)\cosh(\tfrac{1}{2}\alpha(X))\big).$$

Then we only have to solve the equation $\alpha(X) + 2f(t) = 2\mu'$, and (i) follows.

(ii) Since the semigroup $S_{\max} \subseteq \mathrm{Sl}(2,\mathbb{C})$ is open, we find for sufficiently small z elements $g_1, g_2 \in G$ with $g_1 s g_2 = \exp(X + \widetilde{f}(z)[X_\alpha, \overline{X_\alpha}])$ for a certain number $\widetilde{f}(z) \in \mathbb{R}$. Let $2\mu' := \alpha(X) + 2\widetilde{f}(z)$. Then

$$s = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\mu} & 0 \\ 0 & e^{-\mu} \end{pmatrix} = \begin{pmatrix} e^{\mu} & ze^{-\mu} \\ 0 & e^{-\mu} \end{pmatrix}$$

and therefore

$$\begin{aligned} s^* s &= \begin{pmatrix} e^{\mu} & 0 \\ -\overline{z}e^{-\mu} & e^{-\mu} \end{pmatrix} \begin{pmatrix} e^{\mu} & ze^{-\mu} \\ 0 & e^{-\mu} \end{pmatrix} \\ &= \begin{pmatrix} e^{2\mu} & z \\ -\overline{z} & e^{-2\mu}(1-|z|^2) \end{pmatrix} = g_2^{-1} \begin{pmatrix} e^{2\mu'} & 0 \\ 0 & e^{-2\mu'} \end{pmatrix} g_2 \end{aligned}$$

for an element $g_2 \in G = \mathrm{SU}(1,1)$. So

$$2\cosh(2\mu') = \mathrm{tr}(s^* s) = e^{2\mu} + e^{-2\mu}(1 - |z|^2) = 2\cosh(2\mu) - e^{-2\mu}|z|^2.$$

Solving this equation we find $\mu' = \frac{1}{2}\operatorname{arcosh}\big(\cosh(2\mu) - \frac{1}{2}e^{-2\mu}|z|^2\big)$. Now the first assertion follows from solving the equation defining μ'.

To see that $f'(0) < 0$, in view of the chain rule, it suffices to check that

$$\operatorname{arcosh}'\Big(\cosh\big(\alpha(X)\big)\Big) > 0$$

which in turn follows from $\cosh'\big(\alpha(X)\big) = \sinh\big(\alpha(X)\big) > 0$ because $\alpha(X) > 0$. □

3.3. The compact case.

In this subsection $\mathfrak{g}$ denotes a compact Lie algebra with Cartan subalgebra $\mathfrak{t}$ and the property that the commutator algebra is isomorphic to $\mathfrak{su}(2)$. Then there exist only two root spaces in $\mathfrak{g}_{\mathbb{C}}$ and $\Delta^+ = \{\alpha\}$. We normalize $X_\alpha \in \mathfrak{g}^\alpha$ by the requirement that $[\overline{X_\alpha}, X_\alpha] = \check{\alpha}$. Let $G_{\mathbb{C}} \cong Z(G_{\mathbb{C}})_0 \times \mathrm{Sl}(2,\mathbb{C})$ denote the simply connected group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and $G \cong Z(G)_0 \times \mathrm{SU}(2) \subseteq G_{\mathbb{C}}$ the analytic subgroup with Lie algebra $\mathfrak{g}$.

PROPOSITION 3.4. — *Let $X \in i\mathfrak{t}$ with $\alpha(X) \neq 0$. Then the following assertions hold:*

(i) *If* $s = \exp\big(t(X_\alpha - \overline{X_\alpha})\big)\exp(X)\exp\big(t(X_\alpha - \overline{X_\alpha})\big)$ *with* $t \in \mathbb{R}$, *then, for* t *sufficiently small, there exists* $g \in G$ *with* $gsg^{-1} = \exp(X + f(t)[X_\alpha, \overline{X_\alpha}])$, *where*

$$f(t) = -\operatorname{arcosh}\Big(\cosh(2t)\cosh(\tfrac{1}{2}\alpha(X))\Big) + \frac{\alpha(X)}{2}.$$

(ii) *If* $s = \exp(zX_\alpha)\exp(X)$ *with* $z \in \mathbb{C}$ *and* z *is sufficiently small, then there exist* $g_1, g_2 \in G$ *with*

$$g_1 s g_2 = \exp\big(X + f(|z|^2)[X_\alpha, \overline{X_\alpha}]\big),$$

where

$$f(r) = -\frac{1}{2}\operatorname{arcosh}\Big(\cosh\big(\alpha(X)\big) + \frac{r}{2}e^{-\alpha(X)}\Big) + \frac{\alpha(X)}{2} \quad \text{and} \quad f'(0) < 0.$$

Proof. — Since $G_\mathbb{C} \cong Z(G_\mathbb{C})_0 \times \mathrm{Sl}(2,\mathbb{C})$, it suffices to prove the assertion in the case where $\mathfrak{g} = \mathfrak{su}(2)$. We choose the basis (H, iP, iQ) (cf. proof of Proposition 3.3). Here $\mathfrak{t} = \mathbb{R}H$ is a compactly embedded Cartan subalgebra, and with the appropriate choice of a positive root,

$$X_\alpha := \frac{1}{2}(P - iQ) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \overline{X_\alpha} = \frac{1}{2}(-P - iQ) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

so that $[\overline{X_\alpha}, X_\alpha] = -iH = \check{\alpha}$. Moreover $X = \mu\check{\alpha}$ with $\mu := \frac{\alpha(X)}{2} \neq 0$ and therefore $\exp(X) = \begin{pmatrix} e^\mu & 0 \\ 0 & e^{-\mu} \end{pmatrix}$.

(i) Since $X_\alpha - \overline{X_\alpha} = P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have

$$\begin{aligned} s &= \exp\big(t(X_\alpha - \overline{X_\alpha})\big)\exp X \exp\big(t(X_\alpha - \overline{X_\alpha})\big) \\ &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} e^\mu & 0 \\ 0 & e^{-\mu} \end{pmatrix}\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \\ &= \begin{pmatrix} e^\mu \cosh t & e^{-\mu}\sinh t \\ e^\mu \sinh t & e^{-\mu}\cosh t \end{pmatrix}\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \\ &= \begin{pmatrix} e^\mu \cosh^2 t + e^{-\mu}\sinh^2 t & \cosh(\mu)\sinh(2t) \\ \cosh(\mu)\sinh(2t) & e^{-\mu}\cosh^2 t + e^\mu \sinh^2 t \end{pmatrix}. \end{aligned}$$

Thus $\operatorname{tr} s = (\cosh^2 t + \sinh^2 t)(e^\mu + e^{-\mu}) = 2\cosh(2t)\cosh(\mu)$. If $g \in G = \mathrm{SU}(2)$ is chosen such that $gsg^{-1} = \begin{pmatrix} e^{\mu'} & 0 \\ 0 & e^{-\mu'} \end{pmatrix}$ is diagonal, then

$\operatorname{tr}(s) = \operatorname{tr}(gsg^{-1}) = 2\cosh(\mu')$. Hence

$$\mu' = \operatorname{arcosh}\big(\cosh(2t)\cosh(\mu)\big) = \operatorname{arcosh}\big(\cosh(2t)\cosh(\tfrac{1}{2}\alpha(X))\big).$$

Then we only have to solve the equation $\alpha(X) - 2f(t) = 2\mu'$ and (i) follows.

(ii) Since $S_{\max} = \mathrm{Sl}(2, \mathbb{C})$, we find $g_1, g_2 \in G$ with $g_1 s g_2 = \exp(X + \widetilde{f}(z)[X_\alpha, \overline{X_\alpha}])$ for a certain number $\widetilde{f}(z) \in \mathbb{R}$. Let $2\mu' := \alpha(X) - 2\widetilde{f}(z)$. Then

$$s = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\mu} & 0 \\ 0 & e^{-\mu} \end{pmatrix} = \begin{pmatrix} e^{\mu} & ze^{-\mu} \\ 0 & e^{-\mu} \end{pmatrix}$$

and therefore

$$\begin{aligned} s^* s &= \begin{pmatrix} e^{\mu} & 0 \\ \overline{z}e^{-\mu} & e^{-\mu} \end{pmatrix} \begin{pmatrix} e^{\mu} & ze^{-\mu} \\ 0 & e^{-\mu} \end{pmatrix} \\ &= \begin{pmatrix} e^{2\mu} & z \\ \overline{z} & e^{-2\mu}(1 + |z|^2) \end{pmatrix} = g_2^{-1} \begin{pmatrix} e^{2\mu'} & 0 \\ 0 & e^{-2\mu'} \end{pmatrix} g_2 \end{aligned}$$

for an element $g_2 \in G = \mathrm{SU}(2)$. So

$$2\cosh(2\mu') = \operatorname{tr}(s^* s) = e^{2\mu} + e^{-2\mu}(1 + |z|^2) = 2\cosh(2\mu) + e^{-2\mu}|z|^2.$$

Solving this equation we find $\mu' = \frac{1}{2}\operatorname{arcosh}\big(\cosh(2\mu) + \frac{1}{2}e^{-2\mu}|z|^2\big)$. Now the first assertion follows from solving the equation defining μ'. That $f'(0) < 0$ follows by the same argument as in the proof of Proposition 3.3. $\square$

4. Biinvariant plurisubharmonic functions.

We want to characterize the G-biinvariant plurisubharmonic functions on $S_{\max}$ or rather on open G-biinvariant subdomains $D = G\operatorname{Exp}(D_h)$ of this semigroup. Then $D_h \subseteq iW_{\max}$ is a G-invariant domain and the result will be that biinvariant plurisubharmonic functions on D correspond to locally convex invariant functions on the domain D_h. We will also obtain a strict version of this result showing that strictly plurisubharmonic functions correspond to stably convex functions. It is remarkable that the latter statement is far from being a direct consequence of the first one. The major difficulty is caused by the non-compactness of the G-orbits in $W_{\max}^0$. The proof of this characterization is divided into several parts. First we show that if φ is a biinvariant plurisubharmonic function on D, then the corresponding function ψ on D_h is locally convex. So far the arguments are purely geometric. To obtain the converse, we need some representation theoretic arguments supplying us with sufficiently many biinvariant plurisubharmonic functions. The proof for the strict version of the correspondence has the same structure. The first implication is purely geometric and for the converse we need some strictly plurisubharmonic biinvariant function on $S_{\max}$. Such a function need not always exist, but it does if and only if $\mathfrak{g}$ is admissible, or if and only if $W^0_{\max}$ permits stably convex invariant functions. Our construction of a strictly plurisubharmonic biinvariant function uses an injective holomorphic representation $\pi: S_{\max} \to B_2(\mathcal{H})$, where $B_2(\mathcal{H})$ is the space of Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$.

DEFINITION 4.1. — (a) *Let V be a complex vector space and $J: V \to V$ the associated complex structure. A skew symmetric real bilinear form $\omega: V \times V \to \mathbb{R}$ is said to be*

(1) *positive if $\omega(v, J.v) \geqslant 0$ for all $v \in V$.*

(2) *strictly positive if $\omega(v, J.v) > 0$ for $0 \neq v \in V$.*

(3) *a $(1,1)$-form if $\omega(J.v, J.w) = \omega(v, w)$ for $v, w \in V$. Note that ω is a $(1,1)$-form if and only if $h(v,w) := \omega(v, J.w)$ defines a real symmetric bilinear form on V.*

(b) *If M is a complex manifold, then a 2-form ω on M is called positive, strictly positive, or a $(1,1)$-form, if for each $x \in M$ the form $\omega(x)$ on $T_x(M)$ has this property.* □

DEFINITION 4.2. — (a) *Let $\Omega \subseteq \mathbb{C}$ be open. A function $\varphi: \Omega \to \mathbb{R} \cup \{-\infty\}$ is called subharmonic if*

(i) *φ is upper semicontinuous, i.e., for each $t \in \mathbb{R}$ the set $\{z \in \Omega: \varphi(z) < t\}$ is open.*

(ii) *For each open relatively compact disc $D \subseteq \Omega$ and each continuous function f on $\overline{D}$ which is harmonic in D, and which satisfies $\varphi|_{\partial D} \leqslant f|_{\partial D}$, we have $\varphi|_D \leqslant f$.*

A function $f \in C^2(\Omega)$ is called strictly subharmonic if Δf is a positive function.

(b) *Let M be a complex manifold. An upper semicontinuous function $\varphi: M \to \mathbb{R} \cup \{-\infty\}$ is called* plurisubharmonic *if for each holomorphic mapping $\gamma: \Omega \to M$, where $\Omega \subseteq \mathbb{C}$ is a domain, the composition $\varphi \circ \gamma$ is subharmonic. We write* $\mathrm{Psh}(M)$ *for the cone of plurisubharmonic functions on M. Note that this definition implies that if $h: X \to M$ is holomorphic and φ is plurisubharmonic on M, then $\varphi \circ h$ is plurisubharmonic on X. A* *function* $\varphi \in C^2(M)$ *is said to be* strictly plurisubharmonic *if the* $(1,1)$-*form* $\omega = dJd\varphi$ *on* M *is strictly positive.* □

Example 4.3. — (a) Let M be a complex manifold and J the associated complex structure. Then J acts on $\alpha \in T_x(M)^*$ by $J.\alpha(v) := \alpha(J^{-1}.v) = -\alpha(J.v)$. Then for each $f \in C^\infty(M)$ the 2-form $dJdf = 2i\partial\overline{\partial}f$ is a $(1,1)$-form. Locally this 2-form is given by

$$dJdf = 2i \sum_{j,k} \frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} dz_j \wedge d\overline{z}_k$$

and for $h_f(v,w) := dJdf(v,iw)$ we have $h_f = 2\sum_{j,k} \frac{\partial^2 f}{\partial z_j \partial \overline{z}_k}(dz_j \otimes d\overline{z}_k + d\overline{z}_k \otimes dz_j)$. If M is an open subset of $\mathbb{C}$ we obtain with $\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = \frac{1}{4}\Delta$ and

$$dz \otimes d\overline{z} + d\overline{z} \otimes dz = 2(dx \otimes dx + dy \otimes dy)$$

the formula $h_f = \Delta f \cdot (dx \otimes dx + dy \otimes dy)$. This shows that the definitions of strictly plurisubharmonic functions and strictly subharmonic functions are consistent.

(b) If $\mathcal{H}$ is a Hilbert space, then the function defined by $F(z) = \frac{1}{2}\|z\|^2$ is strictly plurisubharmonic. In fact, we have $dF(z)(v) = \mathrm{Re}\langle z, v\rangle$,

$$(JdF)(z)(v) = \mathrm{Re}\langle z, -iv\rangle = \mathrm{Re}\, i\langle z, v\rangle = -\mathrm{Im}\langle z, v\rangle = \mathrm{Im}\langle v, z\rangle,$$

and therefore $(dJdF)(z)(v,w) = \mathrm{Im}\langle w, v\rangle - \mathrm{Im}\langle v, w\rangle = 2\,\mathrm{Im}\langle w, v\rangle = -2\,\mathrm{Im}\langle v, w\rangle$. It follows in particular for $\omega = dJdF$ that $\omega(v, iv) = -2\,\mathrm{Im}\langle v, iv\rangle = 2\,\mathrm{Im}\, i\langle v, v\rangle = 2\|v\|^2 \geqslant 0$.

(c) Let $\widetilde{F}(z) := \log\|z\|$. We claim that $\widetilde{F}$ is plurisubharmonic on $\mathcal{H}$. In fact, it is upper semicontinuous. Let $(e_j)_{j\in J}$ be an orthonormal basis in $\mathcal{H}$. Then $\widetilde{F}$ is the supremum of the functions defined by

$$\widetilde{F}_I(z) = \frac{1}{2}\log\left(\sum_{j\in I} |\langle z, e_j\rangle|^2\right),$$

where $I \subseteq J$ is finite. That these functions are plurisubharmonic follows from [Hö73, Cor. 1.6.8] and the fact that the functions $z \mapsto \frac{1}{2}\log|\langle z, e_j\rangle|^2 = \log|\langle z, e_j\rangle|$ are plurisubharmonic because $z \mapsto \langle z, e_j\rangle$ is holomorphic ([Hö73, Cor. 1.6.6]). Therefore $\widetilde{F}$ is plurisubharmonic by [Hö73, Th. 1.6.2]. □

Remark 4.4. — With respect to the decomposition $S_{\max} = G\exp(iW^0_{\max})$, the G-double cosets in $S_{\max}$ can be written as $G(\exp X)G = G\exp(\mathcal{O}_X)$, where $X \in iC^0_{\max}$ is determined up to $\mathcal{W}_{\mathfrak{k}}$-conjugacy because

$$\mathcal{O}_X \cap i\mathfrak{t} = \mathrm{Ad}(G).X \cap i\mathfrak{t} = \mathcal{W}_{\mathfrak{k}}.X$$

(cf. [HNP94, Prop. 5.7]).

Hence the G-biinvariant subsets $D \subseteq S_{\max}$ are in one-to-one correspondence with the $\mathcal{W}_{\mathfrak{k}}$-invariant subsets $\mathcal{D} \subseteq iC^0_{\max}$ via $\mathcal{D} \mapsto D = G\exp(\mathcal{D})G$. Moreover, the G-biinvariant functions on D are in one-to-one correspondence with the $\mathcal{W}_{\mathfrak{k}}$-invariant functions on $\mathcal{D}$. We recall from Theorem 1.4 that if D is open, then this correspondence preserves continuity and smoothness of functions. □

To each $X \in i\,\mathrm{int}\, C_{\max}$ we associate the cone

$$C^X := \mathrm{cone}\{\alpha(X)[X_\alpha, \overline{X_\alpha}]: \alpha \in \Delta, X_\alpha \in \mathfrak{g}^\alpha_{\mathbb{C}}\} = iC_{\min} + C^{X,k}$$

with $C^{X,k} := -\,\mathrm{cone}\{\alpha(X)\check{\alpha}: \alpha \in \Delta_k\}$.

PROPOSITION 4.5. — *Let $D \subseteq S_{\max}$ be a G-biinvariant domain, $\widetilde{\varphi} \in \mathrm{Psh}(D)^{G\times G}$, and $\varphi := \widetilde{\varphi} \circ \mathrm{Exp}$ on $\mathcal{D} := iC_{\max} \cap \mathrm{Exp}^{-1}(D)$. Then φ is a locally convex function with $d\varphi(X)(C^X) \subseteq \mathbb{R}^-$ for all $X \in \mathcal{D}$.*

Proof. — Since the mapping $\mathrm{Exp}: \mathfrak{t} + \mathcal{D} \to D$ is holomorphic, the function φ is a $\mathfrak{t}$-invariant plurisubharmonic function on the tube domain $\mathfrak{t} + \mathcal{D}$, hence it is locally convex (cf. [AL92, p.369]).

Let $X_0 \in \mathcal{D}$, $\alpha \in \Delta$ with $\alpha(X_0) > 0$ and $0 \neq X_\alpha \in \mathfrak{g}^\alpha_{\mathbb{C}}$. We have to show that

$$d\varphi(X_0)([X_\alpha, \overline{X_\alpha}]) \leqslant 0.$$

Note that $d\varphi(X_0)$ makes sense as a sublinear functional on $\mathfrak{t}$ because φ is a convex function. The assertion will follow by showing that the functions $t \mapsto \varphi(X_0 + t[X_\alpha, \overline{X_\alpha}])$ are decreasing.

We consider the subalgebra $\mathfrak{g}_1 := \mathfrak{t} + \mathrm{span}\{X_\alpha + \overline{X_\alpha}, i(X_\alpha - \overline{X_\alpha})\}$ and note that it is one of the three types considered in Section 3. For $\alpha([X_\alpha, \overline{X_\alpha}]) > 0$ it is of the non-compact reductive type, for $\alpha([X_\alpha, \overline{X_\alpha}]) < 0$ it is compact and otherwise it is solvable (cf. [HiNe93, Th. 7.4]). Putting $W_1 := W_{\max} \cap \mathfrak{g}_1$, we now obtain a holomorphic morphism $S_1 := G_1 \,\mathrm{Exp}(iW^0_1) \to S_{\max}$ which is induced by the injection $\mathfrak{g}_1 \to \mathfrak{g}$.

Let $\gamma(z) = \exp(zX_\alpha)\exp(X_0)$. Then it follows from Corollary 3.2(ii), and Propositions 3.3(ii), 3.4(ii) that for sufficiently small $z \in C$ there exist $g_1, g_2 \in G_1$ with

$$g_1\gamma(z)g_2 = \exp\left(X_0 + f(|z|^2)[X_\alpha, \overline{X_\alpha}]\right),$$

and the function $r \mapsto f(r)$ is strictly decreasing for small values of r. This means in particular that

$$\widetilde{\varphi}(\gamma(z)) = \widetilde{\varphi}(g_1\gamma(z)g_2) = \varphi(X_0 + f(|z|^2)[X_\alpha, \overline{X_\alpha}]).$$

Now the plurisubharmonicity of $\widetilde{\varphi}$ and the holomorphy of γ imply that $\widetilde{\varphi}\circ\gamma$ is a subharmonic function, and therefore, for sufficiently small r, we obtain

$$\varphi(X_0) = \widetilde{\varphi}(\gamma(z)) \leqslant \int_0^1 \widetilde{\varphi}(\gamma(re^{2\pi it}))\, dt = \widetilde{\varphi}(\gamma(r)) = \varphi(X_0 + f(r^2)[X_\alpha, \overline{X_\alpha}]).$$

Now the fact that f is strictly decreasing for small values of r implies that the function $t \mapsto \varphi(X_0 + t[X_\alpha, \overline{X_\alpha}])$ is decreasing for small positive values of t. □

So far we have a necessary condition for functions on domains $\mathcal{D} \subseteq it$ to be restrictions of G-biinvariant plurisubharmonic functions on a G-biinvariant domain $D \subseteq S_{\max}$. According to Theorem 1.6, this is the same condition which characterizes the locally convex G-invariant functions on the domain $D_h = \mathrm{Ad}(G).\mathcal{D} \subseteq iW_{\max}^0$.

Next we show that the conditions described in Proposition 4.5 are also sufficient for a Weyl group invariant function on $\mathcal{D}$ to be the restriction of a plurisubharmonic G-biinvariant function on D. From that and Theorem 1.6 it will then follow that a G-biinvariant function $\widetilde{\varphi}$ on D is plurisubharmonic if and only if the function $\widetilde{\varphi} \circ \mathrm{Exp}|_{D_h}$ is locally convex.

4.1. Constructing plurisubharmonic functions.

Let X be a set. Then a function $K: X \times X \to \mathbb{C}$ is called *positive definite* if there exists a Hilbert space $\mathcal{H}_K \subseteq \mathbb{C}^X$ containing the functions $K_x: y \mapsto K(x, y)$, $x \in X$, such that $f(x) = \langle f, K_x\rangle$ holds for all $x \in X$ and $f \in \mathcal{H}_K$. If S is a semigroup endowed with an involutive antiautomorphism $s \mapsto s^*$, then a function $\varphi: S \to \mathbb{C}$ is called *positive definite* if the associated kernel $K(s,t) := \varphi(st^*)$ is positive definite. If M is a complex manifold, then we write $\overline{M}$ for the same manifold endowed with the opposite complex structure.

PROPOSITION 4.6. — *Let M be a complex manifold and $K: M \times \overline{M} \to \mathbb{C}$ a non-zero holomorphic positive definite kernel. Then the function $K: M \to \mathbb{R} \cup \{-\infty\}$, $z \mapsto \log K(z,z)$ is plurisubharmonic.*

Proof. — Let $\mathcal{H}_K \subseteq \mathrm{Hol}(M)$ denote the Hilbert space associated to the kernel K. Then we have a map $\eta: M \to \mathcal{H}_K^*$, given by $\eta(z)(f) = f(z) = \langle f, K_z \rangle$. For each $f \in \mathcal{H}_K$ the function $z \mapsto \langle f, K_z \rangle$ is holomorphic. Hence η is weakly holomorphic. Since, in view of $\|\eta(z)\|^2 = \langle K_z, K_z \rangle = K(z,z)$, it is also locally bounded, we see that η is holomorphic (cf. [HiNe93, Lemma 9.7(i)]). Therefore $\log K(z,z) = 2 \log \|\eta(z)\|$ is plurisubharmonic by Example 4.3(c). □

In the following we write $B_1(\mathcal{H})$ for the space of all trace class operators on the Hilbert space $\mathcal{H}$.

COROLLARY 4.7. — *Let S be an open Ol'shanskiĭ semigroup. Then the following assertions hold:*

(i) *If φ is a non-zero holomorphic positive definite function on S, then the mapping $S \to \mathbb{R}$, $s \mapsto \log \varphi(ss^*)$ is plurisubharmonic.*

(ii) *If $(\pi, \mathcal{H})$ is a holomorphic representations of S such that $\pi(S) \subseteq B_1(\mathcal{H})$, then the function*

$$\Theta_\pi : s \mapsto \mathrm{tr}\,(\pi(s))$$

is holomorphic, and $F(s) = \log \Theta_\pi(ss^*) = \log \|\pi(s)\|_{HS}^2$ is plurisubharmonic and G-biinvariant.

Proof. — (i) Let $K(s,t) := \varphi(st^*)$ denote the positive definite kernel corresponding to φ. Then the plurisubharmonicity of the function $s \mapsto \log \varphi(ss^*)$ follows from Proposition 4.6.

Suppose that $\varphi(ss^*) = 0$ for an $s \in S$. Then $\|K_s\|^2 = K(s,s) = 0$ implies that $K_s = 0$. Therefore $\varphi(Ss^*) = \{0\}$ and since Ss^* is an open subset of S (cf. [HiNe93, Th. 3.20]), the holomorphy of φ yields a contradiction to $\varphi \neq 0$.

(ii) The G-biinvariance of F is clear. To see that it is also plurisubharmonic, note that the left multiplication representation of S on $B_2(\mathcal{H})$ given by $s.A := \pi(s)A$ is a holomorphic representation and that Θ_π is an associated positive definite function (cf. [Ne94a, Cor. 4.7]). Now the assertion follows from (i). □

LEMMA 4.8. — *Let A be a C^*-algebra, $A_h \subseteq A$ the subspace of hermitian elements, and $U(A) := \{a \in A\colon aa^* = a^*a = \mathbf{1}\}$ the group of unitary elements. Then the following assertions hold:*

(i) *The function $\varphi\colon A \to \mathbb{R}, a \mapsto \log \|a\|$ is plurisubharmonic and $U(A)$-biinvariant.*

(ii) *The function $\varphi\colon A_h \to \mathbb{R}, X \mapsto \log \|e^X\|$ is convex and $U(A)$-invariant with respect to the conjugation action.*

Proof. — (i) The biinvariance with respect to $U(A)$ is clear. In view of Proposition 4.6, for every state $f\colon A \to \mathbb{C}$, i.e., for every positive functional f with $\|f\| = 1$, the function $\varphi_f\colon a \mapsto \log f(aa^*)$ is plurisubharmonic. Since $2\varphi(a) = \log \|aa^*\| = \sup_f \varphi_f(a)$, and φ is upper semicontinuous on A, the assertion follows from [Hö73, Th. 1.6.2].

(ii) The $U(A)$-invariance of φ follows from $\|e^{aXa^{-1}}\| = \|ae^X a^{-1}\| = \|e^X\|$ for $X \in A_h$ and $a \in U(A)$. We assume that $A \neq \{0\}$. To see that φ is convex, we note that

$$\log \|e^X\| = \sup \operatorname{Spec}(X) = \sup\{f(X)\colon f \in A_h', \|f\| = 1, f \geqslant 0\}.$$

Now the convexity of φ follows from the linearity of the functions $f \in A_h'$. □

PROPOSITION 4.9. — *Let S be an open Ol'shanskiĭ semigroup and $(\pi, \mathcal{H})$ a holomorphic representation. Then the function $s \mapsto \log \|\pi(s)\|$ is plurisubharmonic.*

Proof. — Since the mapping $\pi\colon S \to B(\mathcal{H})$ is holomorphic, the assertion follows from Lemma 4.8(i). □

Let $D \subseteq S_{\max}$ be a G-biinvariant domain, $D_h := \operatorname{Exp}^{-1}(D) \cap iW_{\max}$, and $\mathcal{D} := D_h \cap i\mathfrak{t}$. Then $\mathcal{D} \subseteq i\mathfrak{t}$ is a $\mathcal{W}_\mathfrak{k}$-invariant open subset. For a $\mathcal{W}_\mathfrak{k}$-invariant function f on $\mathcal{D}$ we write $\widetilde{f}$ for the corresponding G-biinvariant function on D defined by

$$\widetilde{f}(g_1 \operatorname{Exp}(X) g_2) = \widetilde{f}(g_1 g_2 \operatorname{Exp}(\operatorname{Ad}(g_2^{-1}).X)) := f(X).$$

In view of Theorem 1.4, the function $\widetilde{f}$ is continuous and smooth whenever f is so. We write $f^\sharp$ for the function defined by $f^\sharp(X) = \max\{f(\gamma.X)\colon \gamma \in \mathcal{W}_\mathfrak{k}\}$.

THEOREM 4.10. — *Let $D \subseteq S_{\max}$ be a G-biinvariant domain. Then the following assertions hold:*

(i) *A G-biinvariant function $\widetilde{\varphi}: D \to \mathbb{R}$ is plurisubharmonic if and only if the function $\varphi := \widetilde{\varphi} \circ \mathrm{Exp}: D_h \to \mathbb{R}$ is locally convex.*

(ii) *If, in addition, $\mathcal{D} \cap i\mathfrak{t}^+$ is convex, then each biinvariant plurisubharmonic function $\widetilde{\varphi}$ on D extends to a biinvariant plurisubharmonic function on the domain $G\,\mathrm{Exp}(\mathrm{conv}\, D_h)$.*

Proof. — (i) Let $\widetilde{\varphi}$ be a biinvariant plurisubharmonic function on D. According to Proposition 4.5, we know that the restriction of φ to $\mathcal{D}$ is locally convex and satisfies $d\varphi(X)(C^X) \subseteq \mathbb{R}^-$ for all $X \in \mathcal{D}$. Hence φ is locally convex by Theorem 1.6.

Suppose, conversely, that φ is locally convex. Since plurisubharmonicity is a local property, we may w.l.o.g. assume that $\mathcal{D} \cap i\mathfrak{t}^+$ is a convex set. Then we use Theorem 1.6 to extend φ to a convex invariant function ψ on the convex invariant set $\mathrm{conv}(D_h) \subseteq iW^0_{\max}$. In the same way we obtain a biinvariant extension of $\widetilde{\varphi}$ to a biinvariant function $\widetilde{\psi}$ on the biinvariant domain $G\,\mathrm{Exp}(\mathrm{conv}\, D_h) \subseteq S_{\max}$ by setting $\widetilde{\psi}(g\,\mathrm{Exp}\, X) := \psi(X)$. After these modifications, we may w.l.o.g. assume that the set D_h is convex and that φ is a convex function on this set.

Let $C := iC_{\min}$. Then the convexity of φ implies that the restriction of φ to $\mathcal{D}$ is convex, $\mathcal{W}_{\mathfrak{k}}$-invariant, and C-decreasing (cf. Theorem 1.6). We have to show that for each such function f on $\mathcal{D}$, the function $\widetilde{f}$ is plurisubharmonic.

Let f be a convex $\mathcal{W}_{\mathfrak{k}}$-invariant C-decreasing function on $\mathcal{D}$. To show that $\widetilde{f}$ is plurisubharmonic, we first note that, according to [Ne96a, Lemma 1.12], we have $f = \sup_{j\in J} f_j$, where the functions f_j are affine and C-decreasing. Since f is $\mathcal{W}_{\mathfrak{k}}$-invariant, we even have $f = \sup\limits_{j\in J} f_j^\sharp$. Then $\widetilde{f} = \sup\limits_{j\in J} \widetilde{f_j^\sharp}$. Hence it suffices to show that the functions $\widetilde{f_j^\sharp}$ are plurisubharmonic ([Hö73, Th. 1.6.2]). Thus we may w.l.o.g. assume that $f = h^\sharp$, where h is affine and C-decreasing. Since constant functions are trivially plurisubharmonic, we may even assume that h is linear, i.e., $h \in -C^\star = -(iC_{\min})^\star = iC^\star_{\min}$.

To understand the geometry of the situation, we recall the positive system Δ_k^+ of compact roots. Then the cone

$$C_1 := \{g \in -C^\star \colon (\forall \alpha \in \Delta^+) g(\check{\alpha}) \geqslant 0\}$$

of *dominant* functionals in $-C^\star$ is a closed convex cone which is a fundamental domain for the action of the Weyl group $\mathcal{W}_{\mathfrak{k}}$ on $-C^\star$. Since

$h^\sharp = (\gamma.h)^\sharp$ for all $\gamma \in \mathcal{W}_\mathfrak{k}$, we may w.l.o.g. assume that h is dominant. On the other hand $(\Delta_k^+)^\star$ is a fundamental domain for the $\mathcal{W}_\mathfrak{k}$-action on $i\mathfrak{t}$ and hence $(\Delta^+)^\star \subseteq (\Delta_p^+)^\star = iC_{\max}$ is a fundamental domain for the action of $\mathcal{W}_\mathfrak{k}$ on $iC_{\max}$. If X is contained in this cone, then $\mathcal{W}_k.X \subseteq X - \operatorname{cone}(\{\check{\alpha}: \alpha \in \Delta_k^+\})$ (cf. [Ne94d, Cor. 2.3]) so that

$$h^\sharp(X) = \max h(\mathcal{W}_\mathfrak{k}.X) = h(X).$$

This means that $h^\sharp$ is the unique function on $iC_{\max}$ which is $\mathcal{W}_\mathfrak{k}$-invariant and restricts to h on $(\Delta^+)^\star$.

Let

$$\mathcal{E} := \{h \in C_1: \widetilde{h}^\sharp \in \operatorname{Psh}(D)\}.$$

By the observations made above, the mapping $h \mapsto \widetilde{h}^\sharp$ is linear on the domain specified above. Hence the fact that $\operatorname{Psh}(D)$ is a convex cone implies that $\mathcal{E}$ is a convex cone. We show that this cone is closed. In fact, if $h_n \in \mathcal{E}$ is a sequence converging to h, then Lemma 1.3 shows that on each compact subset of $S_{\max}$, the functions $\widetilde{h}_n^\sharp$ converge uniformly to the function $\widetilde{h}^\sharp$ and hence $\widetilde{h}^\sharp$ is plurisubharmonic as follows easily from [Hö73, Th. 1.6.3].

Now let $\lambda \in i\mathfrak{t}^*$ be dominant integral such that $\lambda + \rho \in i \operatorname{int} C_{\min}^\star$. Then Theorem 2.4 implies that there exists a holomorphic representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $S_{\max}$ such that for $X \in iC_{\max}$ we have

$$\log \|\pi(\operatorname{Exp} X)\| = \sup\langle \lambda, \mathcal{W}_\mathfrak{k}.X\rangle = \lambda^\sharp(X).$$

Since the function $s \mapsto \log\|\pi(s)\|$ is plurisubharmonic by Proposition 4.9, we conclude that $\lambda \in \mathcal{E}$. Next Lemmas 5.9 and 5.12 in [Ne96b] show that $\mathbb{R}^+\mathcal{E}$ is dense in C_1. On the other hand, $\mathcal{E}$ is a closed convex cone and thus $\mathcal{E} = C_1$. □

4.2. Strictly plurisubharmonic functions.

In the preceding subsection we have seen that biinvariant plurisubharmonic functions correspond to invariant locally convex functions on $W_{\max}^0$. In this section we will show that this correspondence relates strictly plurisubharmonic functions to stably locally convex functions.

LEMMA 4.11. — *Let $r > 0$ and $f: \{z \in \mathbb{C}: |z| < r\} \to \mathbb{R}$ be a strictly subharmonic function of the form $f(z) = g(|z|^2)$ with $g \in C^2([0, \sqrt{r}[)$. Then $(\Delta f)(0) = 4g'(0) > 0$.*

Proof. — For the Laplace operator in polar coordinates we have $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$. Therefore $\Delta f(r) = h''(r) + \frac{1}{r}h'(r)$ for $h(r) = g(r^2)$. This leads to $h'(r) = 2rg'(r^2)$ and therefore to $h''(r) = 2g'(r^2) + 4r^2 g''(r^2)$ which entails $\Delta f(0) = 2g'(0) + 2g'(0) = 4g'(0)$. □

LEMMA 4.12. — *If $\mathfrak{g}$ is admissible, then there exists a strictly plurisubharmonic biinvariant function ψ on $S_{\max}$.*

Proof. — To get the function ψ, we first use [Ne96b, Lemma 5.10] and the admissibilty of $\mathfrak{g}$ to find $\lambda_1, \ldots, \lambda_n \in i\mathfrak{t}^*$ dominant integral with $\lambda_j + \rho \in i \operatorname{int} C^\star_{\min}$ such that

$$\{Y \in \mathfrak{t} \colon (\forall j)\lambda_j(Y) \in 2\pi\mathbb{Z}\} = \exp_G^{-1}(\mathbf{1}) \cap \mathfrak{t}.$$

Then Proposition 5.6 in [Ne96b] entails that for the corresponding highest weight representations π_{λ_j} (cf. Theorem 2.4) we have

$$\bigcap_{j=1}^{n} \ker \pi_{\lambda_j} = \{\mathbf{1}\},$$

so that the representation $\pi := \bigoplus_{j=1}^{n} \pi_j$ yields an injective holomorphic mapping

$$\pi \colon S_{\max} \to B_2(\mathcal{H})$$

([Ne95b, Th. 3.7]). In view of Corollary 4.7, the function $\psi(s) := \operatorname{tr} \pi(ss^*) = \|\pi(s)\|_2^2$ is plurisubharmonic because the function $f(A) := \|A\|_2^2$ on $B_2(\mathcal{H})$ is strictly plurisubharmonic, i.e., $dJdf$ is a strictly positive $(1,1)$-form on the complex Hilbert space $B_2(\mathcal{H})$ (cf. Example 4.3(b)). From $\psi = f \circ \pi = \pi^* f$ we conclude that $dJd\psi = dJd\pi^* f = \pi^*(dJdf)$. The differential of the map $\pi \colon S_{\max} \to B_2(\mathcal{H})$ is given by $d\pi(s)d\lambda_s(\mathbf{1})X = \pi(s)d\pi(X)$ (cf. proof of [Ne94a, Cor. 4.7]). Since we can write each $s \in S_{\max}$ as $s = g\operatorname{Exp}(iX)$, the spectral theory of selfadjoint operators implies that the operators $\pi(s) = \pi(g)e^{id\pi(X)}$ are injective. Combining this with the fact that $\ker \pi$ is trivial, we see that the differentials $d\pi(s)$ are injective. Therefore the $(1,1)$-form $\pi^*(dJdf)$ is strictly positive, i.e., ψ is strictly plurisubharmonic. □

THEOREM 4.13. — *Let $D \subseteq S_{\max}$ be a G-biinvariant domain. Then a G-biinvariant smooth function $\widetilde{\varphi}$ on D is strictly plurisubharmonic if and only if the function $\varphi := \widetilde{\varphi} \circ \operatorname{Exp}|_{D_{\mathfrak{h}}}$ is stably locally convex.*

Proof. — First we assume that $\widetilde{\varphi}$ is strictly plurisubharmonic. Since the mapping $\operatorname{Exp}\colon \mathfrak{t} + \mathcal{D} \to D$ is holomorphic, the function $\widetilde{\varphi} \circ \operatorname{Exp}$ is a $\mathfrak{t}$-invariant strictly plurisubharmonic function on the tube domain $\mathfrak{t} + \mathcal{D}$, hence $\varphi\,|_{\mathcal{D}}$ is stably locally convex as follows directly by applying Lemma 4.11 to affine line segments in $\mathcal{D}$.

Let $X_0 \in \mathcal{D}$, $\alpha \in \Delta$ with $\alpha(X_0) > 0$ and $0 \neq X_\alpha \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$. By the same reduction argument as in the proof of Proposition 4.5, we see that for $\gamma(z) = \exp(zX_\alpha)\exp(X_0)$ we have

$$\widetilde{\varphi}\big(\gamma(z)\big) = \varphi\big(X_0 + f(|z|^2)[X_\alpha, \overline{X_\alpha}]\big),$$

where $f'(0) < 0$ (cf. Corollary 4.2(ii), Propositions 4.3(ii), 4.4(ii)).

Now $\gamma'(0) \neq 0$ implies that γ defines a local embedding of an open disc about 0 into D, hence that $\widetilde{\varphi} \circ \gamma$ is strictly subharmonic because $\widetilde{\varphi}$ is strictly plurisubharmonic. Therefore Lemma 4.11 entails that

$$0 < f'(0)d\varphi(X_0)([X_\alpha, \overline{X_\alpha}]),$$

so that $d\varphi(X_0)([X_\alpha, \overline{X_\alpha}]) < 0$. It follows in particular that $[X_\alpha, \overline{X_\alpha}] \neq 0$, i.e., that $\mathfrak{g}$ has cone potential. Therefore Theorem 1.6 implies that the function φ on D_h is stably convex.

To prove the converse, suppose that $\varphi\colon D_h \to \mathbb{R}$ is stably convex, i.e., $\varphi_{\mathfrak{t}} := \varphi\,|_{\mathcal{D}}$ is stably locally convex, $d\varphi_{\mathfrak{t}}(X) \in -\operatorname{int}(C^X)^\star$ for all $X \in \mathcal{D}$, and $\mathfrak{g}$ is admissible (Theorem 1.6).

Let $X \in \mathcal{D}$ and U be a relatively compact open convex set containing X which is invariant under the stabilizer $\mathcal{W}_{\mathfrak{k}}^X$ of X and satisfies $\overline{U} \subseteq \mathcal{D}$. Further we require that $\alpha(Y) \neq 0$ for $Y \in U$ whenever $\alpha(X) \neq 0$. Let $\widetilde{\varphi}_1$ be the strictly plurisubharmonic function from Lemma 4.12. In view of the compactness of $\overline{U}$, there exists an $\varepsilon > 0$ such that $\varphi_{2,\mathfrak{t}} := \varphi_{\mathfrak{t}} - \varepsilon\varphi_{1,\mathfrak{t}}$ is convex and satisfies $d\varphi_{2,\mathfrak{t}}(Y) \in -\operatorname{int}(C^X)^\star$ for all $Y \in U$. In view of Theorem 1.6, this also implies that $d\varphi_{2,\mathfrak{t}}(Y) \in -(C^Y)^\star$ for all $Y \in \mathcal{W}_{\mathfrak{k}}.U$. Now Theorem 1.6 shows that the function φ_2 is locally convex on $\operatorname{Ad}(G).U$. Using Theorem 4.10, we conclude that the function $\widetilde{\varphi}_2$ is plurisubharmonic on the G-biinvariant open subset $G(\exp U)G \subseteq S_{\max}$. Now $\widetilde{\varphi} = \widetilde{\varphi}_2 + \varepsilon\widetilde{\varphi}_1$ and the fact that $\widetilde{\varphi}_1$ is strictly plurisubharmonic imply that $\widetilde{\varphi}$ is strictly plurisubharmonic on $G(\exp U)G$. Since X was arbitrary in $\mathcal{D}$, it follows that $\widetilde{\varphi}$ is strictly plurisubharmonic on D. $\square$

4.2. Invariant geodesically convex functions.

Let $M = G/H$ be a symmetric space, i.e., there exists an involutive automorphism τ of the connected Lie group G such that H is an open subgroup of $G^\tau := \{g \in G: \tau(g) = g\}$. Accordingly we have a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ of the Lie algebra of $\mathfrak{g}$ with

$$\mathfrak{h} := \{X \in \mathfrak{g}: d\tau(\mathbf{1}).X = X\} \quad \text{and} \quad \mathfrak{q} := \{X \in \mathfrak{g}: d\tau(\mathbf{1}).X = -X\}.$$

The mapping $\operatorname{Exp}: \mathfrak{q} \to M, X \mapsto \exp(X)H$ is called the *exponential function* of M. We call a curve segment $\gamma:]a,b[\to M$ a *geodesic segment* if there exists $g \in G$ and $X \in \mathfrak{q}$ with $\gamma(t) = g\operatorname{Exp}(tX)$ for all $t \in]a,b[$. Now a function $f: \Omega \to \mathbb{R}$ on an open subset Ω of M is said to be *geodesically convex* if $f \circ \gamma:]a,b[\to \mathbb{R}$ is a convex function for all geodesic segments γ with $\gamma(]a,b[) \subseteq \Omega$.

We consider the maximal Ol'shanskiĭ semigroup $S_{\max} = G\operatorname{Exp}(iW_{\max}^0) \subseteq G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is simply connected. Let $\pi: G_{\mathbb{C}} \to G_{\mathbb{C}}/G$ denote the quotient map of the corresponding symmetric space and $\operatorname{Exp}: i\mathfrak{g} \to G_{\mathbb{C}}/G$ the corresponding exponential function.

LEMMA 4.14. — *If φ is a right-invariant plurisubharmonic function on a right-invariant domain $D \subseteq S_{\max}$, then the function $\psi: \pi(d) \mapsto \varphi(d)$ on $\pi(D)$ is geodesically convex.*

Proof. — Let $d \in D$. Then the geodesics through $\pi(d)$ are given by $\gamma(t) = d.\operatorname{Exp}(tX)$ with $X \in i\mathfrak{g}$. Let $I \subseteq \mathbb{R}$ be an interval with $\gamma(t) \in \pi(D)$ for $t \in I$. Then the function

$$F: I + i\mathbb{R} \to \mathbb{R}, \quad z \mapsto \varphi\big(d\exp(zX)\big)$$

is plurisubharmonic and $i\mathbb{R}$-invariant, hence convex. This proves in particular that the function

$$I \to \mathbb{R}, \quad t \mapsto \varphi\big(d\exp(tX)\big)$$

is convex. This means that ψ is geodesically convex. □

As an example due to J.-J. Loeb shows (cf. [Fe94]), a geodesically convex function on $\mathrm{Sl}(2,\mathbb{C})/\mathrm{SU}(2)$ need not correspond to a right-$\mathrm{SU}(2)$-invariant plurisubharmonic function on $\mathrm{Sl}(2,\mathbb{C})$. In this subsection we will show that for a biinvariant function on a biinvariant domain $D \subseteq S_{\max}$ geodesic convexity and the plurisubharmonicity are equivalent conditions. In view of Lemma 4.14 and Theorem 4.10, it suffices to show that for a geodesically convex function ψ on $\pi(D) \subseteq \pi(S) \subseteq G_{\mathbb{C}}/G$, the corresponding function $\varphi\colon D_h \to \mathbb{R}$, $X \mapsto \psi(\operatorname{Exp} X)$ is locally convex.

Let $q\colon G_{\mathbb{C}}/G \to G_{\mathbb{C}}, gG \mapsto gg^* = g\overline{g}^{-1}$ denote the quadratic representation of $G_{\mathbb{C}}/G$ in $G_{\mathbb{C}}$. Then $q(\operatorname{Exp} X) = \exp(2X)$ for $X \in i\mathfrak{g}$ and q maps the geodesic $t \mapsto g.\operatorname{Exp}(tX)$ to the curve

$$t \mapsto g \operatorname{Exp}(2tX) g^*$$

in $G_{\mathbb{C}}$. So we have to show that if ψ, considered as a function on $\exp(D_h)$, is convex on the curves $t \mapsto g\operatorname{Exp}(2tX)g^*$, then the corresponding function φ on D_h is locally convex.

Proposition 4.15. — *Let $D = \operatorname{Exp}(D_h) \subseteq \operatorname{Exp}(iW^0_{\max}) \subseteq G_{\mathbb{C}}/G$ be a G-invariant domain and ψ a G-invariant function on D. Then ψ is geodesically convex if and only if the function $\varphi := \psi \circ \operatorname{Exp}\colon D_h \to \mathbb{R}$ is locally convex.*

Proof. — If φ is locally convex, then the function

$$\widetilde{\varphi}\colon G\exp(D_h) \to \mathbb{R}, g\exp(X) \mapsto \varphi(X)$$

is G-biinvariant and plurisubharmonic (Theorem 4.10), hence Lemma 4.14 implies that ψ is geodesically convex.

Conversely, we assume that ψ is geodesically convex. Then it is clear that $\varphi\mid_{D_h\cap i\mathfrak{t}}$ is a locally convex function. Let $X \in \mathfrak{t} \cap D_h$ and $\alpha \in \Delta^+$ with $\alpha(X) \neq 0$. In view of Theorem 1.6, we only have to show that $d\varphi(X)\big(\alpha(X)[X_\alpha, \overline{X_\alpha}]\big) \leqslant 0$ holds for all $\alpha \in \Delta$ and $X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$.

The fact that the function ψ is geodesically convex implies that for $X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$ the function

$$t \mapsto \psi\big(\gamma(t)\big) \quad \text{with} \quad \gamma(t) = \exp(X).\operatorname{Exp}\big(t(X_\alpha - \overline{X_\alpha})\big)$$

is convex. We have to show that φ is decreasing in the direction of $\alpha(X)[X_\alpha, \overline{X_\alpha}]$. We may w.l.o.g. assume that $\alpha(X) \neq 0$. According to Corollary 3.2(i) and Propositions 3.3(i), 3.4(i), we find for sufficiently small t and

$$s_t := q(x_t) = \exp\big(t(X_\alpha - \overline{X_\alpha})\big)\exp(2X)\exp\big(t(X_\alpha - \overline{X_\alpha})\big)$$

an element $g \in G$ with $q(g.x_t) = gs_tg^{-1} = \exp\big(2X + 2f(t)[X_\alpha, \overline{X_\alpha}]\big)$, where $t \mapsto f(t)$ is a symmetric function which is strictly decreasing for small positive values of t. Hence $g.x_t = \operatorname{Exp}\big(X + f(t)[X_\alpha, \overline{X_\alpha}]\big)$ and therefore

$$\psi(x_t) = \psi(g.x_t) = \varphi\big(X + f(t)[X_\alpha, \overline{X_\alpha}]\big).$$

Now the convexity and the symmetry of the function $t \mapsto \psi(x_t)$ imply that it is increasing for small values of t, thus $d\varphi(X)([X_\alpha, \overline{X_\alpha}]) \leqslant 0$.

This shows that $\varphi|_{D_h \cap \mathfrak{t}}$ is a convex function with $d\varphi(X)(C^X) \subseteq \mathbb{R}^-$ for all $X \in D_h \cap \mathfrak{t}$. Now Theorem 1.6 implies that the function φ on D_h is locally convex. □

5. The Stein property of Ol'shanskiĭ semigroups.

An Ol'shanskiĭ semigroup $S = \Gamma(\mathfrak{g}, W^0, D)$ is a group if and only if $W = \mathfrak{g}$. In this case $S = G_{\mathbb{C}}/D$, where $G_{\mathbb{C}}$ is the simply connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and $D \subseteq G_{\mathbb{C}}$ is a discrete central subgroup. As the simple example $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ shows, not all such groups are Stein. In the following theorem we recall the main facts on Stein groups.

THEOREM 5.1. — *Let G be a connected complex Lie group. Then the following are equivalent:*

(1) *G is Stein.*

(2) *G is holomorphically separable.*

(3) *$Z(G)_0 \cong \mathbb{C}^n \times (\mathbb{C}^*)^m$.*

Proof. — [MaMo60, pp. 146, 147]. □

We note that (3) in the preceding theorem is the property which is rather easy to check in concrete situations.

COROLLARY 5.2. — (a) *Linear complex groups are Stein.*

(b) *Simply connected complex groups are Stein.* □

The following result will be crucial in the remainder of this section ([MaMo60, Th. 4]):

THEOREM 5.3. — *If $P \to B$ is a holomorphic principal bundle such that the fiber G and the base B are Stein, then P is Stein.* □

Note that we don't have to assume in Theorem 5.3 that the group G is connected. Hence it implies in particular the classical result of Stein for the case where G is discrete and $P \to B$ is a covering.

In many applications the following observation is also useful ([He93, Prop. 1(iv)]).

PROPOSITION 5.4. — *If G is a connected real Lie group, then the universal complexification $G_{\mathbb{C}}$ of G is Stein.* □

DEFINITION 5.5. — (a) *Let $W \subseteq \mathfrak{g}$ be a generating invariant convex cone, $G_{\mathbb{C}}$ the simply connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, $S_W := \langle \exp(\mathfrak{g} + iW^0) \rangle \subseteq G_{\mathbb{C}}$, $\Gamma(\mathfrak{g}, W^0)$ the simply connected covering of S_W, and $D \subseteq \Gamma(\mathfrak{g}, W^0)$ a discrete central subgroup. Then $\Gamma(\mathfrak{g}, W, D) := \Gamma(\mathfrak{g}, W)/D$ is called the Ol'shanskiĭ semigroup associated to the data $(\mathfrak{g}, W, D)$. If, in addition, D is invariant under the involution $s \mapsto s^*$ induced by the involution $g \mapsto \overline{g}^{-1}$ on $G_{\mathbb{C}}$, then $\Gamma(\mathfrak{g}, W, D)$ is said to be involutive. This means that on $\Gamma(\mathfrak{g}, W, D)$ we have an involutive antiautomorphism $s \mapsto s^*$ which is holomorphic on the interior and induces $X \mapsto -\overline{X}$ on $\mathfrak{g} + iW$.*

(b) *In the following we write $\mathfrak{h}_W := W \cap (-W)$ for the edge of W which is an ideal in $\mathfrak{g}$. We recall from [Ne95a, Th. 1.5(iii)] that the group $H(\Gamma(\mathfrak{g}, W))$ of units of $\Gamma(\mathfrak{g}, W)$ is the simply connected real Lie group with Lie algebra $\mathfrak{g} + i\mathfrak{h}_W$. Since $(\mathfrak{h}_W)_{\mathbb{C}}$ is an ideal in this Lie algebra, the subgroup corresponding to $(\mathfrak{h}_W)_{\mathbb{C}}$ is simply connected, hence isomorphic to the group $H := \langle \exp(\mathfrak{h}_W)_{\mathbb{C}} \rangle \subseteq G_{\mathbb{C}}$.* □

As a simple example with $G = \mathfrak{g} = \mathbb{R}^2$, $D = \mathbb{Z}^2$, and $\mathfrak{h} = \mathbb{R}$ shows, it is possible that the image of the group H in $\Gamma(\mathfrak{g}, W, D)$ is not closed. E.g. it may happen that $D \subseteq G$, G/D is a torus, $H \cong \mathbb{C}$, and $H \cap (G/D)$ is a dense wind (cf. Example 5.19(b)).

LEMMA 5.6. — *If $S = \Gamma(\mathfrak{g}, W^0, D)$ is holomorphically separable, then the image H_1 of H in $\Gamma(\mathfrak{g}, W, D)$ is Stein.*

Proof. — Let $s \in S$. Then the fact that the left multiplication map $\lambda_s : H_1 \to S$ is holomorphic and injective (cf. [HiNe93, Th. 3.20]) implies that H_1 is holomorphically separable, hence Stein (Theorem 5.1). □

DEFINITION 5.7. — *Let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ an ideal, $\mathfrak{a} := \mathfrak{g} + i\mathfrak{h} \subseteq \mathfrak{g}_{\mathbb{C}}$, and A a connected Lie group with Lie algebra $\mathfrak{a}$. Then A is a so called* CR Lie group. *If B is a complex group and $\alpha : A \to B$ a morphism of Lie groups, then we say that α is* partially holomorphic *if the restriction of α to the subgroup generated by $\exp \mathfrak{h}_{\mathbb{C}}$ is holomorphic, or, equivalently, if $d\alpha|_{\mathfrak{h}_{\mathbb{C}}}$ is complex linear.*

A partially holomorphic morphism $\eta_A : A \to A_{p,\mathbb{C}}$ in a connected complex Lie group $A_{p,\mathbb{C}}$ is called a universal partial complexification *if* *for each partially holomorphic morphism $\alpha: A \to B$ there exists a unique holomorphic morphism $\beta: A_{p,\mathbb{C}} \to B$ with $\beta \circ \eta_A = \alpha$.*

The uniqueness of such an object up to partially holomorphic isomorphism follows from the universal property. For the existence let $G_{\mathbb{C}}$ denote the simply connected group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and $\alpha: \widetilde{A} \to G_{\mathbb{C}}$ the canonical morphism. Then α is partially holomorphic. Let $N \subseteq G_{\mathbb{C}}$ denote the smallest closed complex normal subgroup of $G_{\mathbb{C}}$ containing the image of $\pi_1(A)$ under α. Then we define $A_{p,\mathbb{C}} := G_{\mathbb{C}}/N$ and $\eta_A(a) := \alpha(a)N$. Then $\eta_A: A \to A_{p,\mathbb{C}}$ is well defined and partially holomorphic.

To verify that $(\eta_A, A_{p,\mathbb{C}})$ has the universal property, let $\beta: A \to B$ be partially holomorphic. Then there exists a unique holomorphic homomorphism $\gamma: G_{\mathbb{C}} \to B$ with $d\gamma\,|_{\mathfrak{g}} = d\beta\,|_{\mathfrak{g}}$. From that it follows that $\gamma \circ \alpha = \beta \circ q$, where $q: \widetilde{A} \to A$ is the universal covering. Hence γ contains $\alpha\big(\pi_1(H)\big)$ in its kernel. Since $\ker \gamma$ is a closed complex normal subgroup of $G_{\mathbb{C}}$, γ factors to a holomorphic homomorphism $\widetilde{\gamma}: A_{p,\mathbb{C}} \to B$. This proves the existence.

We note that if $\mathfrak{a} = \mathfrak{g}_{\mathbb{C}}$, then A is a complex group, hence $A_{p,\mathbb{C}} = A$. This shows in particular that the groups $A_{p,\mathbb{C}}$ need not always be Stein. □

PROPOSITION 5.8. — *Let $S = \Gamma(\mathfrak{g}, W, D)$ be a closed Ol'shanskiĭ semigroup and $\eta_{H(S)}: H(S) \to H(S)_{p,\mathbb{C}}$ the universal partial complexification of its group of units $H(S)$. Then $\eta_{H(S)}$ extends to a continuous morphism $\eta_S: S \to H(S)_{p,\mathbb{C}}$ which is holomorphic on the interior and which has the universal property of the free complex group on S, i.e., each morphism of S into a complex group which is holomorphic on S^0 factors over η_S.*

Proof. — First we recall from [Ne95a, Th. I,5] that $H(S)$ is connected with fundamental group D. We claim that $\eta_{H(S)}: H(S) \to H(S)_{p,\mathbb{C}}$ extends to S. In fact, let $G_{\mathbb{C}}$ denote the simply connected complex group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, $\gamma: \widetilde{S} \to S_W \subseteq G_{\mathbb{C}}$ the universal covering morphism of S_W, and $q: G_{\mathbb{C}} \to H(S)_{p,\mathbb{C}}$ the canonical quotient morphism. Then $q \circ \gamma\,|_{H(\widetilde{S})} = \eta_{H(\widetilde{S})}$ implies that $D = \pi_1\big(H(S)\big) \subseteq \ker(q \circ \gamma\,|_{H(\widetilde{S})})$ (cf. Definition 5.5(b)), and it follows that $q \circ \gamma$ factors to a continuous morphism $\eta_S: S \to H(S)_{p,\mathbb{C}}$ which is holomorphic on S^0.

Now suppose that $\alpha: S \to B$ is a continuous morphism which is holomorphic on S^0 mapping S into a complex group B. Then $\alpha\,|_{H(S)}$ is partially holomorphic and we find a holomorphic morphism $\beta: H(S)_{p,\mathbb{C}} \to B$ with $\beta \circ \eta_{H(S)} = \alpha\,|_{H(S)}$. Then $\beta \circ \eta_S = \alpha$ follows by analytic extension.

This proves that $\eta_S \colon S \to H(S)_{p,\mathbb{C}}$ has the universal property of the free complex group on S. □

LEMMA 5.9. — *If W is pointed, then $H(S)_{p,\mathbb{C}} = H(S)_{\mathbb{C}}$.*

Proof. — If W is pointed, then $\widetilde{S} = \Gamma(\mathfrak{g}, W) = H(\widetilde{S})\operatorname{Exp}(iW) \cong H(\widetilde{S}) \times iW$ (cf. [HiNe93, Cor. 7.35]) and $H(\widetilde{S}) = H(S)\tilde{}$. Hence $S = H(S)\operatorname{Exp}(iW)$ with $H(S) = \langle \operatorname{Exp} \mathfrak{g} \rangle$. This proves that $H(S)_{p,\mathbb{C}} = H(S)_{\mathbb{C}}$. □

LEMMA 5.10. — *If the subgroup $H_1 := \langle \operatorname{Exp}_S(\mathfrak{h}_W)_{\mathbb{C}} \rangle \subseteq S = \Gamma(\mathfrak{g}, W, D)$ is closed, then the quotient morphism $q \colon S \to S/H_1$ defines the structure of a complex H_1-principal bundle on S and S/H_1 is also a complex Ol'shanskiĭ semigroup.*

Proof. — We consider the natural morphism $q \colon \Gamma(\mathfrak{g}, W) \to \Gamma(\mathfrak{g}_1, W_1)$, where $\mathfrak{g}_1 := \mathfrak{g}/\mathfrak{h}_W$ and $W_1 := W/\mathfrak{h}_W$ is a pointed generating invariant cone in $\mathfrak{g}_1$. According to our assumption that H_1 is closed, the subgroup $H \cdot D$ of $\Gamma(\mathfrak{g}, W)$ is closed.

We claim that the mapping q defines the structure of an H-principal bundle on $\Gamma(\mathfrak{g}, W)$. We consider the commutative diagram

$$\begin{array}{ccc} \Gamma(\mathfrak{g}, W) & \xrightarrow{\quad p \quad} & S_W \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle q_1} \\ \Gamma(\mathfrak{g}_1, W_1) & \xrightarrow{\quad p_1 \quad} & S_{W_1}, \end{array}$$

where $q_1 \colon G_{\mathbb{C}} \to G_{\mathbb{C}}/H$ is the quotient map. Then Lemma 3.2 in [Ne94b] implies that the map $q_1 \circ p \colon \Gamma(\mathfrak{g}, W) \to S_{W_1}$ defines the structure of a smooth principal bundle with group $\widetilde{H}\pi_1(S_W) \cong H\pi_1(S_W)$.

We conclude that $\Gamma(\mathfrak{g}, W^0)/H$ carries the structure of a complex manifold such that the quotient map $\Gamma(\mathfrak{g}, W^0) \to \Gamma(\mathfrak{g}, W^0)/H$ is the projection of an H-principal bundle. Since H is simply connected, the exact homotopy sequence of this bundle yields that $\Gamma(\mathfrak{g}, W^0)/H$ is simply connected and that the action of the discrete group $\pi_1(S_W)H/H$ on $\Gamma(\mathfrak{g}, W^0)/H$ defines a simply connected covering $\Gamma(\mathfrak{g}, W^0)/H \to S_{W_1}$. We conclude that $\Gamma(\mathfrak{g}, W^0)/H \cong \Gamma(\mathfrak{g}_1, W_1^0) = \widetilde{S_{W_1}}$ and hence that the natural map $q \colon \Gamma(\mathfrak{g}, W) \to \Gamma(\mathfrak{g}_1, W_1)$ is an H-principal bundle, holomorphic on the interior.

Now the closedness of $HD \subseteq \Gamma(\mathfrak{g}, W)$ implies that $D_1 := q(HD) = q(D)$ is closed, and hence a discrete central subgroup of $\Gamma(\mathfrak{g}_1, W_1)$. Let $S_1 := \Gamma(\mathfrak{g}_1, W_1^0, D_1)$. Then the natural map $\Gamma(\mathfrak{g}, W^0) \to S_1$ factors to a map $\gamma: S \to S_1$ such that the following diagram is commutative:

$$\begin{tikzcd}
\Gamma(\mathfrak{g}, W^0) \arrow[r, "\beta"] \arrow[d, "q"] & S \arrow[d, "\gamma"] \\
\Gamma(\mathfrak{g}_1, W_1^0) \arrow[r, "\beta_1"] & S_1.
\end{tikzcd}$$

The fibers of γ are the cosets of the normal subgroup $H_1 := \beta(HD)$ of $\Gamma(\mathfrak{g}, W, D)$. Using a right-invariant Riemannian metric on S, it is easy to see that with respect to the action of H_1, γ is the projection of a holomorphic H_1-principal bundle. □

LEMMA 5.11. — *Let $S = \Gamma(\mathfrak{g}, W, D)$ be a closed Ol'shanskiĭ semigroup such that $D \subseteq \langle \operatorname{Exp}_{\widetilde{S}} \mathfrak{g} \rangle$ holds in $\widetilde{S}$ and put $G := \langle \operatorname{Exp}_S \mathfrak{g} \rangle \subseteq S$. Then there exists a G-biinvariant smooth positive plurisubharmonic function φ on S^0 such that*

$$\lim_{s_n \to s} \varphi(s_n) = \infty$$

holds for all $s \in S \setminus S^0$.

Proof. — Let $q: \widetilde{S} \to S$ denote the universal covering. Since $D \subseteq \widetilde{G} := \langle \operatorname{Exp}_{\widetilde{S}} \mathfrak{g} \rangle$, the G-biinvariant plurisubharmonic functions are in one-to-one correspondence with the $\widetilde{G}$-biinvariant plurisubharmonic functions on $\widetilde{S}^0$. Moreover, since every sequence $s_n \to s \in S \setminus S^0$ has a lift to a sequence $\widetilde{s}_n \to \widetilde{s} \in \widetilde{S} \setminus \widetilde{S}^0$, it suffices to prove the assertion under the additional assumption that S is simply connected, i.e., $S = \Gamma(\mathfrak{g}, W)$.

Then $H = \langle \exp(\mathfrak{h}_W)_{\mathbb{C}} \rangle$ is a normal subgroup of the simply connected group $H(S)$ (cf. Definition 5.5(b)), hence H is simply connected and closed. In view of Lemma 5.10, the quotient map $p: S = \Gamma(\mathfrak{g}, W) \to S_1 := S/H = \Gamma(\mathfrak{g}/\mathfrak{h}_W, W/\mathfrak{h}_W)$ defines on S^0 the structure of a holomorphic H-principal bundle. Thus the pullback of a plurisubharmonic G_1-biinvariant plurisubharmonic function on S_1^0 will be a G-biinvariant plurisubharmonic funtion on S. This reduces our problem further to the case where W is pointed. We assume this.

In this case $S = G \operatorname{Exp}(iW)$ is topologically a product decomposition. Moreover, since W is pointed, $\mathfrak{g}$ contains a compactly embedded Cartan subalgebra $\mathfrak{t}$ such that $W \subseteq W_{\max}$. Using [Ne96a, Lemma 3.11], we find a smooth positive convex G-invariant function ψ on W^0 with

$\lim_{X \to X_0} \psi(X) = \infty$ for all $X_0 \in \partial W$. We put $\varphi\big(g\,\mathrm{Exp}(iX)\big) := \psi(X)$. Then φ is G-biinvariant and, in view of Theorem 4.10, it is plurisubharmonic.

Let $s_n \to s \in S \setminus S^0$. Then $s = g\,\mathrm{Exp}(iX)$ with $X \in \partial W$. If $s_n = g_n\,\mathrm{Exp}(iX_n)$, then we have $X_n \to X$ and therefore $\varphi(s_n) = \psi(X_n) \to \infty$. This completes the proof. □

After these preparations we can turn to some results showing that many classes of Ol'shanskiĭ semigroups consist of Stein manifolds.

PROPOSITION 5.12. — *Let $S = \Gamma(\mathfrak{g}, W, D)$ be a closed Ol'shanskiĭ semigroup with $D \subseteq \langle \mathrm{Exp}_{\widetilde{S}}\,\mathfrak{g}\rangle$. If $\eta_S \colon S \to \eta_S(S) \subseteq H(S)_{p,\mathbb{C}}$ is a covering with $\ker \eta_S \subseteq G := \langle \mathrm{Exp}_S\,\mathfrak{g}\rangle$, and $H(S)_{p,\mathbb{C}}$ is Stein, then S^0 is Stein.*

Proof. — In view of Theorem 5.3, it suffices to show that the image $\eta_S(S^0) \subseteq H(S)_{p,\mathbb{C}}$ is Stein. Hence we may w.l.o.g. assume that $S \subseteq H(S)_{p,\mathbb{C}}$. Note that the assumption of Lemma 5.11 is satisfied because $D = \pi_1(S) \subseteq \langle \mathrm{Exp}_{\widetilde{S}}\,\mathfrak{g}\rangle$.

Let φ_1 be a strictly plurisubharmonic exhaustion function of $H(S)_{p,\mathbb{C}}$, i.e., the sets $\varphi_1^{-1}(]-\infty, c])$ are all compact (cf. [Hö73, Th. 5.2.10]) and φ_2 a G-biinvariant smooth positive plurisubharmonic function on S^0 which has the property that $\varphi_2(s_n) \to \infty$ whenever $s_n \to s \in S \setminus S^0$ (Lemma 5.11). Then $\varphi := \varphi_1 + \varphi_2$ is an exhaustion function on S. In fact, it is strictly plurisubharmonic, and for $c \in \mathbb{R}$ the set $\varphi^{-1}(]-\infty, c])$ is on the one hand relatively compact in $H(S)_{p,\mathbb{C}}$ and also closed, hence compact. Thus we have found an exhaustion function for the open subset S^0 of the Stein manifold $H(S)_{p,\mathbb{C}}$. This proves that S^0 is Stein ([Hö73, Th. 5.2.10]). □

THEOREM 5.13. — *Let $S = \Gamma(\mathfrak{g}, W^0, D)$ be an Ol'shanskiĭ semigroup such that*

(1) *the image H_1 of H in $\Gamma(\mathfrak{g}, W, D)$ is closed and Stein, and*

(2) *for $S_1 := \Gamma(\mathfrak{g}, W, D)/H_1$ the mapping η_{S_1} is a covering onto its image.*

Then S is Stein.

Proof. — In view of Lemma 5.10, the quotient map $S \to S_1 := S/H_1$ defines the structure of a principal H_1-bundle on S. Since H_1 is Stein by assumption, it therefore suffices to show that $S_1 = \Gamma(\mathfrak{g}_1, W_1^0, D_1)$ is Stein (Theorem 5.3).

Since W_1 is pointed, $H(S_1)_{p,\mathbb{C}} = H(S_1)_{\mathbb{C}}$ (Lemma 5.9) is Stein (Proposition 5.4). Moreover, all other assumptions made in Proposition 5.12 follow from the pointedness of W_1. Hence Proposition 5.12 and (2) imply that S_1 is Stein. □

The following corollary and Theorem 5.18 are the main results on the Stein property of Ol'shanskiĭ semigroups.

COROLLARY 5.14. — *Every simply connected open Ol'shanskiĭ semigroup $\Gamma(\mathfrak{g}, W^0)$ is Stein.*

Proof. — Let $S := \Gamma(\mathfrak{g}, W)$. In this case the subgroup $H = H_1$ is normal in the simply connected group of units $H(S)$ and therefore closed and Stein (Corollary 5.2(b)). Moreover, the quotient S/H is again simply connected and therefore of the form $S_1 := \Gamma(\mathfrak{g}_1, W_1)$ with W_1 pointed. Then $H(S_1)_{p,\mathbb{C}} = H(S_1)_{\mathbb{C}}$ (Lemma 5.9) and $\eta_{S_1}: S_1 \to S_{W_1}$ is a covering. Hence the assumptions of Theorem 5.13 are satisfied and therefore S is Stein. □

DEFINITION 5.15. — *We call $S = \Gamma(\mathfrak{g}, W, D)$ regular if W is weakly elliptic, i.e., if $\operatorname{Spec}(\operatorname{ad} X) \subseteq i\mathbb{R}$ for all $X \in W$, and if $D \subseteq \langle \operatorname{Exp}_{\widetilde{S}} \mathfrak{g} \rangle$. Note that in this case we have $S = G \operatorname{Exp}(iW) \cong G \times iW$, where $G = \langle \operatorname{Exp}_S \mathfrak{g} \rangle$ (cf. Lawson's Theorem, [HiNe93, Th. 7.34, Cor. 7.35]). So we also write $S = \Gamma(G, W)$.* □

LEMMA 5.16. — *If $S = \Gamma(G, W)$ is regular, then $H(S)_{p,\mathbb{C}} = G_{\mathbb{C}}$.*

Proof. — We have $H(S) = G \operatorname{Exp}(i\mathfrak{h}_W)$ and $H_1 = H_W \exp(i\mathfrak{h}_W)$, where $H_W = \langle \exp \mathfrak{h}_W \rangle$ is a normal subgroup of G. So $H(S)^{\sim} = \widetilde{G} \operatorname{Exp}(i\mathfrak{h}_W)$ and $\pi_1(H(S)) = \pi_1(G) \subseteq \widetilde{G}$. Hence the smallest closed complex subgroup of $\widetilde{G}_{\mathbb{C}}$ containing the image of $\pi_1(G)$ under the canonical morphism $\widetilde{G} \to G_{\mathbb{C}}$ coincides with the image of $\pi_1(H(S))$ under the canonical morphism $H(S)^{\sim} \to \widetilde{G}_{\mathbb{C}}$. Therefore the construction of $H(S)_{p,\mathbb{C}}$ (cf. Definition 5.7) implies that it coincides with $G_{\mathbb{C}}$. □

LEMMA 5.17. — *If $S = \Gamma(G, W)$ is regular and $\eta_G: G \to G_{\mathbb{C}}$ has discrete kernel, then S^0 is Stein.*

Proof. — According to Lemma 5.16 and Proposition 5.4, the complex group $H(S)_{p,\mathbb{C}}$ is Stein. Moreover, $\ker \eta_S = \ker \eta_G$ is discrete, so that η_S is a covering onto its image. Therefore Proposition 5.12 implies that S^0 is Stein. □

The following theorem is one of the central results of this paper. We are grateful to Bernhard Krötz for contributing a key idea which made the proof work in full generality.

THEOREM 5.18. — *The interior S^0 of a regular Ol'shanskiĭ semigroup $S = \Gamma(G, W)$ is Stein.*

Proof. — Let $\eta_G : G \to G_{\mathbb{C}}$ be the universal complexification, $\mathfrak{a} := \ker d\eta_G(\mathbf{1})$ and $D := \pi_1(G)$. The group $G_{\mathbb{C}}$ is constructed as follows. If $\widetilde{G}_{\mathbb{C}}$ is the simply connected complex group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, then we have a canonical morphism $\eta_{\widetilde{G}} : \widetilde{G} \to \widetilde{G}_{\mathbb{C}}$ of simply connected real Lie groups which is a universal complexification of $\widetilde{G}$. Let $B \subseteq \widetilde{G}_{\mathbb{C}}$ be the smallest closed complex subgroup containing $\eta_{\widetilde{G}}(D)$. Since $\eta_{\widetilde{G}}(D)$ is central in $\widetilde{G}_{\mathbb{C}}$, we see that $B \subseteq Z(\widetilde{G}_{\mathbb{C}})$. Now $G_{\mathbb{C}} = \widetilde{G}_{\mathbb{C}}/B$ and therefore $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{g}$, where $\mathfrak{b}$ denotes the Lie algebra of B. It follows in particular that $\mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{g})$.

We consider the subalgebra $\mathfrak{h} := \mathfrak{g} + i\mathfrak{a}$ of $\mathfrak{g}_{\mathbb{C}}$ and write $\widetilde{H}$ for the associated simply connected group which, in view of the fact that $\mathfrak{a}$ is central in $\mathfrak{g}$, can be written as $\widetilde{H} \cong \widetilde{G} \times i\mathfrak{a}$, so that $H := \widetilde{H}/(D \times \{0\}) \cong G \times i\mathfrak{a}$. As a normal abelian subgroup of $\widetilde{H}$, the group $\widetilde{A} := \exp \mathfrak{a}_{\mathbb{C}} \subseteq \widetilde{H}$ is simply connected, hence isomorphic to $\mathfrak{a}_{\mathbb{C}}$. Therefore the subgroup $A := \exp_H \mathfrak{a}_{\mathbb{C}} \cong \mathfrak{a}_{\mathbb{C}}/(D \cap \widetilde{A})$ is isomorphic to $\mathbb{C}^n/\Gamma$, where $\Gamma \subseteq \mathbb{R}^n$ is some discrete subgroup. Thus $A \cong \mathbb{C}^n \times (\mathbb{C}^*)^m$ for some $n, m \in \mathbb{N}_0$, and so A is Stein. Furthermore the fact that $\exp_G \mathfrak{a} = (\ker \eta_G)_0$ is closed in G implies that A is a closed subgroup of H.

Let $\widehat{W} := \overline{\mathfrak{a} + W} \subseteq \mathfrak{g}$. Since the morphism $\widetilde{H} \to \Gamma(\mathfrak{g}, \widehat{W})$ is injective (cf. Definition 5.5(b)), we can define $\widehat{S} := \Gamma(\mathfrak{g}, \widehat{W}, D)$. Then we have the following commutative diagram of morphisms of Ol'shanskiĭ semigroups, where $\alpha : S \to \widehat{S}$ is a covering of the open subsemigroup $\alpha(S)$ of $\widehat{S}$ and $H \subseteq \widehat{S}$ is a closed subgroup (cf. [Ne95a, Lemma 1.11]):

$$\begin{array}{ccc}
\Gamma(\mathfrak{g}, W) & \xrightarrow{\quad\gamma\quad} & \Gamma(\mathfrak{g}, \widehat{W}) \\
\downarrow{\scriptstyle q} & & \downarrow \\
S = \Gamma(G, W) = \Gamma(\mathfrak{g}, W, D) & \xrightarrow{\quad\alpha\quad} & \widehat{S} = \Gamma(G, \widehat{W}) = \Gamma(\mathfrak{g}, \widehat{W}, D).
\end{array}$$

Now the fact that A is Stein implies that $\widehat{S}^0$ is Stein if $\widehat{S}^0/A$ is Stein (cf. Theorem 5.3). On the other hand the natural map $\eta_{\widehat{S}/A} : \widehat{S}/A \to G_{\mathbb{C}}$ is a covering of the Ol'shanskiĭ semigroup $\eta_{\widehat{S}/A}(\widehat{S}/A) \subseteq G_{\mathbb{C}}$. Since $G_{\mathbb{C}}$ is Stein (Proposition 5.4), the open Ol'shanskiĭ semigroup $\eta_{\widehat{S}/A}(\widehat{S}^0/A)$ is Stein because $\widehat{W}/\mathfrak{a}$ is a weakly elliptic cone (Lemma 5.17), and consequently

$\widehat{S}^0/A$ is Stein by Theorem 5.3. We conclude that $\widehat{S}^0$ is Stein. Therefore $\alpha(S^0) \subseteq \widehat{S}^0$ is Stein (Lemma 5.11), and the fact that $\alpha: S^0 \to \alpha(S^0)$ is a covering entails that S^0 is Stein (Theorem 5.3). □

Example 5.19. — (a) We construct an interesting example of a regular Ol'shanskiĭ semigroup S such that η_S is not discrete.

Let $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R})$, $\widetilde{G} = \mathbb{R} \times \mathrm{Sl}(2,\mathbb{R})^\sim$ and recall that $Z(\widetilde{G})$ decomposes accordingly as $\mathbb{R} \times \mathbb{Z}$. Let $D := (\mathbb{Z} \times \{0\}) + \mathbb{Z}(\sqrt{2}, 2)$ and $G := \widetilde{G}/D$. Then G contains $\mathrm{Sl}(2,\mathbb{R})^\sim$ as a dense normal subgroup, $\widetilde{G}_{\mathbb{C}} \cong \mathbb{C} \times \mathrm{Sl}(2,\mathbb{C})$, and the kernel of the mapping $\eta_{\widetilde{G}}$ is $D_1 = \{0\} \times 2\mathbb{Z}$. Thus $D \ker \eta_{\widetilde{G}} = (\mathbb{Z} + \sqrt{2}\mathbb{Z}) \times 2\mathbb{Z}$. The smallest closed complex subgroup of $\widetilde{G}_{\mathbb{C}}$ containing its image in $\widetilde{G}_{\mathbb{C}}$ is $\mathbb{C} \times 2\mathbb{Z}$. Hence $G_{\mathbb{C}} \cong \mathrm{Sl}(2,\mathbb{C})$ and therefore η_G does not have discrete kernel. If $W \subseteq \mathfrak{g}$ is a regular generating invariant closed convex cone, then $S^0 := \Gamma(G, W^0)$ is Stein (Theorem 5.18) even though $\ker \eta_G$ is not discrete.

(b) We construct an Ol'shanskiĭ semigroup, where H_1 is not closed but nevertheless S^0 is Stein.

Let $\mathfrak{g} = \mathbb{R}^2 = \widetilde{G}$, $D = \mathbb{Z} \times \mathbb{Z}$ and $W = (\mathbb{R}^+ \times \{0\}) + \mathfrak{h}_W$, where $\mathfrak{h}_W = \mathbb{R}(1, \sqrt{2})$. Then $G \cong \mathbb{T}^2$ is a two-dimensional torus, $\widetilde{G}_{\mathbb{C}} \cong \mathbb{C}^2$, and $G_{\mathbb{C}} \cong \mathbb{C}^* \times \mathbb{C}^*$. The interior of the Ol'shanskiĭ semigroup $S := \Gamma(G, W)$ is a logarithmically convex Reinhardt domain in $G_{\mathbb{C}}$, hence Stein. But $H_1 = \exp(\mathfrak{h}_W)_{\mathbb{C}} \subseteq G_{\mathbb{C}}$ is not closed. It is isomorphic to $\mathbb{C}$, where $\exp \mathfrak{h}_W \subseteq G$ is a dense wind. □

6. Biinvariant domains of holomorphy.

In Theorem 5.18 we have shown that all regular open Ol'shanskiĭ semigroups are Stein manifolds. This applies in particular to the semigroup $S_{\max} = G\,\mathrm{Exp}(iW^0_{\max})$, where G is any connected Lie group with Lie algebra $\mathfrak{g}$. In this section we turn to biinvariant domains $D = G\,\mathrm{Exp}(D_h)$ in $S_{\max}$ and show that they are Stein, i.e., domains of holomorphy if and only if D_h is convex. In the next section we will even be able to calculate the envelope of holomorphy of a biinvariant domain in $S_{\max}$.

THEOREM 6.1. — *A biinvariant domain $D = G\,\mathrm{Exp}(D_h) \subseteq S_{\max} = G\,\mathrm{Exp}(iW^0_{\max})$ is Stein if and only if D_h is convex.*

Proof. — First we assume that D_h is convex. In view of [Ne96a, Lemma 3.11], there exists a smooth convex invariant function φ on D_h satisfying $\varphi(X_n) \to \infty$ for $X_n \to X \in \partial D_h$.

Let ψ be a strictly plurisubharmonic exhaustion function of $S_{\max}$ and $\widetilde{\varphi}$ the G-biinvariant function on D defined by $\widetilde{\varphi}(g \operatorname{Exp} X) := \varphi(X)$. Then $\lim \widetilde{\varphi}(x_n) \to \infty$ holds for $x_n \to x \in \partial D$. Thus $\psi + \widetilde{\varphi}$ is a strictly plurisubharmonic exhaustion function on D and [Hö73, Th. 5.2.10] implies that D is Stein because $S_{\max}$ is Stein (Theorem 5.18).

Now we assume that D is Stein. To show that D_h is convex we may w.l.o.g. assume that the group G is simply connected because the simply connected covering $\widetilde{D} = \widetilde{G} \operatorname{Exp}(D_h)$ is also Stein (cf. Theorem 5.3). Let $\mathcal{D} := D_h \cap i\mathfrak{t} \subseteq iC^0_{\max}$.

First we consider the closed submanifold $T \operatorname{Exp}(\mathcal{D}) \subseteq D$. Then $T \operatorname{Exp}(\mathcal{D})$ must be Stein, and hence the covering $\widetilde{T} \operatorname{Exp}(\mathcal{D}) \cong \mathfrak{t} + \mathcal{D} \subseteq \mathfrak{t}_{\mathbb{C}}$ is Stein. Thus all connected components of $\mathcal{D}$ are convex subsets of $i\mathfrak{t}$. Since D is connected, the intersection $\mathcal{D}^+ := \mathcal{D} \cap \mathfrak{t}^+$ must be connected because otherwise D would be disconnected (Theorem 1.4). Next we show that $\mathcal{W}_{\mathfrak{k}}$ leaves the connected components of $\mathcal{D}$ invariant which in turn shows that $\mathcal{D}$ is connected and therefore convex.

Let $(K,K) \subseteq K$ denote the commutator subgroup. Then (K,K) is semisimple and therefore compact. Since the domain D is (K,K)-biinvariant and Stein, there exists a strictly plurisubharmonic exhaustion function $\widetilde{\varphi}$ on D which is (K,K)-biinvariant (cf. [Fe94, Lemma 4.11]).

Let $X \in \mathcal{D}^+$ and write it as $X = X_0 + X_1$ with $X_0 \in i\mathfrak{z}(\mathfrak{k})$ and $X_1 \in i[\mathfrak{k},\mathfrak{k}]$. Put $D_h^X := \{Y \in i[\mathfrak{k},\mathfrak{k}] \colon X_0 + Y \in D_h\}$, $D^X := (K,K)\exp(D_h^X) \subseteq (K,K)_{\mathbb{C}}$, and consider the holomorphic map

$$\eta \colon D^X \to D, k \operatorname{Exp}(Y) \mapsto k \operatorname{Exp}(Y) \operatorname{Exp}(X_0) = k \operatorname{Exp}(X_0 + Y).$$

Then η is a closed embedding, hence $\widetilde{\varphi} \circ \eta$ is a (K,K)-biinvariant strictly plurisubharmonic exhaustion function on D^X. Define the function φ on D_h^X by $\widetilde{\varphi}(k \operatorname{Exp} Y) = \varphi(Y)$. Then φ is a locally convex invariant function on D_h^X (Theorem 4.10). Since the set $\mathcal{D}^+$ is convex, the same holds for its intersection with $X + [\mathfrak{k},\mathfrak{k}]$, hence for the set $\mathcal{D}^{X,+}$. Therefore Theorem 1.6 shows that φ has a convex invariant extension to the convex invariant set $\operatorname{conv}(D_h^X) \subseteq i[\mathfrak{k},\mathfrak{k}]$. Using the fact that $\widetilde{\varphi}$ is an exhaustion function, we see that the function φ tends to infinity at all boundary points of D_h^X. Therefore $\operatorname{conv}(D_h^X)$ contains no boundary points, i.e., D_h^X is convex. Since

$\mathcal{W}_{\mathfrak{k}}.X \subseteq X + D_h^X$, it is contained in the connected component of X in D_h. This proves that $\mathcal{D}$ is convex.

The remaining argument is a rank-1-reduction. Let $X \in \mathcal{D}$, $\alpha \in \Delta_p^+$ and $X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$. We claim that $X + \mathbb{R}^+[X_\alpha, \overline{X_\alpha}] \subseteq \mathcal{D}$. Of course we may w.l.o.g. assume that $[X_\alpha, \overline{X_\alpha}] \neq 0$.

Let first $\alpha \in \Delta_r^+$. Then the subalgebra $\mathfrak{g}_1 := \operatorname{span}\{X_0, i[X_\alpha, \overline{X_\alpha}], X_\alpha + \overline{X_\alpha}, i(X_\alpha - \overline{X_\alpha})\}$ is isomorphic to the four dimensional oscillator algebra (cf. first part of Section 3), $\mathfrak{t}_1 := \mathfrak{g}_1 \cap \mathfrak{t}$ is a compactly embedded Cartan subalgebra, and $C_{\max,1} := C_{\max} \cap \mathfrak{t}_1$ is the corresponding maximal cone. Hence the simple connectedness of G shows that we obtain a holomorphic closed embedding $S_{\max,1} = G_1 \operatorname{Exp}(iW^0_{\max,1}) \to S_{\max}$ which is induced by the injection $\mathfrak{g}_1 \to \mathfrak{g}$. Then $D \cap S_{\max,1}$ is a G_1-biinvariant domain of holomorphy in $S_{\max,1}$, where $\mathcal{D}_1 = \mathcal{D} \cap i\mathfrak{t}_1$ is convex. Therefore, for $\alpha \in \Delta_r^+$, we may w.l.o.g. assume that $\mathfrak{g} = \mathfrak{g}_1$.

We put $a = \operatorname{Exp} X \in D$ and consider the domain $\Gamma := U_{\mathbb{C}} a \cap D$, where $U = (G, G) \subseteq G$ is the three-dimensional Heisenberg group. Then Γ is a domain of holomorphy in the three dimensional complex manifold $U_{\mathbb{C}} a \cong \mathfrak{u}_{\mathbb{C}} \cong \mathbb{C}^3$ which is invariant under conjugation with elements in G and U-biinvariant, where the latter property follows from the normality of U in G. We claim that

$$\operatorname{Exp}(i\mathfrak{u}) \operatorname{Exp}(X) = \operatorname{Exp}(X + i\mathfrak{u}).$$

In fact, consider the quotient morphism $\pi: G_{\mathbb{C}} \to G_{\mathbb{C}}/U_{\mathbb{C}}$. Then

$$\pi\big(g \operatorname{Exp}(X)\big) = \pi(g) \operatorname{Exp}(d\pi(X))$$

implies that the inverse image of $\pi\big(\operatorname{Exp}(X)\big)$ is given on the one hand by $U_{\mathbb{C}} \operatorname{Exp}(X)$ and on the other hand by $U \operatorname{Exp}(i\mathfrak{u} + X)$. We conclude that $\operatorname{Exp}(i\mathfrak{u} + X) = \operatorname{Exp}(i\mathfrak{u}) \operatorname{Exp}(X)$ and therefore that $U_{\mathbb{C}} a = U \operatorname{Exp}(X + i\mathfrak{u})$. Thus $\Gamma = U \operatorname{Exp}(\Gamma_h)$, where $\Gamma_h \subseteq X + i\mathfrak{u}$ is a G-invariant domain. Therefore

$$\Gamma_h = \operatorname{Ad}(G).(\Gamma_h \cap i\mathfrak{t}) \subseteq \operatorname{Ad}(G).(X + i\mathfrak{z}).$$

The convexity of $\mathcal{D}$ implies that $\mathcal{D} \cap (X + i\mathfrak{z})$ is a convex set and therefore either a line, a half-line, or a line segment. Suppose that $X + \mathbb{R}^+[X_\alpha, \overline{X_\alpha}]$ is not contained in $\mathcal{D}$ and let $s_0 := \sup\{s : X + s[X_\alpha, \overline{X_\alpha}] \in \mathcal{D}\}$. For $Z \in \mathfrak{z}$ we have

$$\begin{aligned}
\operatorname{Ad}(G).(X + Z) &= Z + \operatorname{Ad}(G).X = Z + e^{\operatorname{ad} \mathfrak{g}^{[\alpha]}}.X \\
&= Z + \{e^{\operatorname{ad}(X_\alpha + \overline{X_\alpha})}.X : X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha\} \\
&= Z + X - \alpha(X)\{(X_\alpha - \overline{X_\alpha}) + [\overline{X_\alpha}, X_\alpha] : X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha\}.
\end{aligned}$$

It is clear that this orbit is a smooth hypersurface in the three dimensional space $X + iu$. Let $\varphi\colon W^0_{\max} \to \mathbb{R}$ be a G-invariant stably convex invariant function and $\widetilde{\varphi}$ the corresponding biinvariant smooth plurisubharmonic function on $S_{\max}$. Then $\widetilde{\varphi}|_\Gamma$ is a smooth U-biinvariant G-invariant strictly plurisubharmonic function. Hence

$$\Gamma \subseteq \{p \in U_{\mathbb{C}}a\colon \widetilde{\varphi}(p) \geqslant \varphi(X + s_0[X_\alpha, \overline{X_\alpha}])\}$$

because φ is decreasing along the line $s \mapsto X + s[X_\alpha, \overline{X_\alpha}]$. Thus the smooth boundary $\partial\Gamma$ contains the point $a' := \operatorname{Exp}(X + s_0[X_\alpha, \overline{X_\alpha}])$ and since $\widetilde{\varphi}$ is strictly plurisubharmonic, the Levi condition cannot be satisfied in a' (cf. [Ra86, Th. 2.3, Th. 2.11]). This contradiction shows that s_0 cannot exist, i.e., that $X + \mathbb{R}^+[X_\alpha, \overline{X_\alpha}] \subseteq \mathcal{D}$.

Next we assume that $\alpha \in \Delta^+_{p,s}$. Then the subalgebra

$$\mathfrak{g}_0 := \operatorname{span}\{i[X_\alpha, \overline{X_\alpha}], X_\alpha + \overline{X_\alpha}, i(X_\alpha - \overline{X_\alpha})\}$$

is isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{su}(1,1)$. If X is not contained in $i\mathfrak{g}_0$, we put $\mathfrak{g}_1 := \mathfrak{g}_0 + \mathbb{R}iX \cong \mathfrak{u}(1,1)$. Otherwise we set $\mathfrak{g}_1 := \mathfrak{g}_0$ and $\mathfrak{t}_1 := \mathfrak{g}_1 \cap \mathfrak{t}$. Let $W := W_{\max} \cap \mathfrak{g}_1$ and $S_1 := \Gamma(\mathfrak{g}_1, W^0)$. In view of the simple connectedness of G, we obtain a closed embedding $\eta\colon S_1 \to S_{\max}$ which is induced by the injection $\mathfrak{g}_1 \to \mathfrak{g}$. Again, $D \cap S_{\max,1}$ is a G_1-biinvariant domain of holomorphy in S_1, where $\mathcal{D}_1 = \mathcal{D} \cap i\mathfrak{t}_1$ is convex. Therefore, for $\alpha \in \Delta^+_p$, we may w.l.o.g. assume that $\mathfrak{g} = \mathfrak{g}_1$. Then $\mathfrak{z} \subseteq W_{\max,1}$ implies $W_{\max,1} = \mathfrak{z} \oplus W_{\max,0}$, where $W_{\max,0} \subseteq [\mathfrak{g}, \mathfrak{g}] \cong \mathfrak{sl}(2, \mathbb{R})$ is a pointed generating invariant cone.

We put $a = \operatorname{Exp} X \in D$ and let $X = X_0 + X_1$ with $X_0 \in \mathfrak{z}(\mathfrak{g})$ and $a_0 = \operatorname{Exp} X_0$. We consider the domain $\Gamma := (G, G)\operatorname{Exp}(X_0 + i[\mathfrak{g}, \mathfrak{g}]) \cap D$, where $(G, G) \subseteq G$ is a three-dimensional simple group. Then Γ is a domain of holomorphy in the three dimensional complex manifold $S_{\max,0}a_0$ which is (G, G)-biinvariant. Thus $\Gamma = (G, G)\operatorname{Exp}(\Gamma_h)$, where $\Gamma_h \subseteq X_0 + i[\mathfrak{g}, \mathfrak{g}]$ is a G-invariant domain. Therefore

$$\Gamma_h = \operatorname{Ad}(G).(\Gamma_h \cap i\mathfrak{t}) \subseteq \operatorname{Ad}(G).(X + i\mathfrak{z}).$$

Again we assume that $X + \mathbb{R}^+[X_\alpha, \overline{X_\alpha}]$ is not contained in $\mathcal{D}$ and put $s_0 := \sup\{s\colon X + s[X_\alpha, \overline{X_\alpha}] \in \mathcal{D}\}$. Let $\varphi\colon W^0_{\max} \to \mathbb{R}$ be a G-invariant stably convex invariant function and $\widetilde{\varphi}$ the corresponding biinvariant smooth plurisubharmonic function on $S_{\max}$. Then $\widetilde{\varphi}|_\Gamma$ is a smooth (G, G)-biinvariant G-invariant strictly plurisubharmonic function. Hence

$$\Gamma \subseteq \{p \in S_{\max,0}a_0\colon \widetilde{\varphi}(p) \geqslant \varphi(X + s_0[X_\alpha, \overline{X_\alpha}])\}$$

because φ is decreasing along the line $s \mapsto X + s[X_\alpha, \overline{X_\alpha}]$. Thus the fact that the boundary $\partial\Gamma$ is smooth leads, exactly as for $\alpha \in \Delta_r^+$, to the contradiction that the Levi condition cannot be satisfied in $a' := \operatorname{Exp}(X + s_0[X_\alpha, \overline{X_\alpha}])$ (cf. [Ra86, Th. 2.3, Th. 2.11]). Hence $X + \mathbb{R}^+[X_\alpha, \overline{X_\alpha}] \subseteq \mathcal{D}$.

The preceding two rank-1-reductions imply that $[X_\alpha, \overline{X_\alpha}] \in \lim \mathcal{D}$ whenever $\alpha \in \Delta_p^+$ and $X_\alpha \in \mathfrak{g}_\mathbb{C}^\alpha$ (cf. Definition 1.5(d)). Since $\lim(\mathcal{D}) = \lim(\overline{\mathcal{D}})$ is closed (cf. [Ne98, Prop. V.1.5] and also [HNP94, Prop. 1.1]), we conclude that $C_{\min} + \mathcal{D} \subseteq \mathcal{D}$. From that it follows that D_h is convex (Proposition 1.7). □

7. Envelopes of holomorphy.

First we recall some basic facts on Stein manifolds (cf. [Ro63]). Let M be a Stein manifold. Then $\operatorname{Hol}(M)$ is a Fréchet algebra and one can identify M with the *structure space*

$$S\big(\operatorname{Hol}(M)\big) := \operatorname{Hom}_\mathbb{C}\big(\operatorname{Hol}(M), \mathbb{C}\big)$$

of all continuous $\mathbb{C}$-algebra homomorphisms from $\operatorname{Hol}(M)$ to $\mathbb{C}$ endowed with the corresponding weak-$*$-topology ([Ro63, Th. 2.6]). The corresponding mapping $\eta: M \to S\big(\operatorname{Hol}(M)\big)$ is given by point evaluations $\eta(x)(f) := f(x)$.

Let $D \subseteq M$ be a domain lying in M. Then we have a canonical restriction homomorphism

$$\operatorname{Hol}(M) \to \operatorname{Hol}(D)$$

which is continuous. According to [Ro73, Th. 4.6], the space $\widehat{D} := S\big(\operatorname{Hol}(D)\big)$ carries the structure of a Stein manifold such that the canonical map

$$q: \widehat{D} \to S\big(\operatorname{Hol}(M)\big) = M$$

defines on $\widehat{D}$ the structure of a Riemannian domain over M, i.e., q is a local homeomorphism. The space $\widehat{D}$ is called the *envelope of holomorphy of D*. We note that $\operatorname{Hol}(D)$ also carries a Fréchet topology obtained by identifying it with $\operatorname{Hol}(\widehat{D})$. That these two topologies coincide follows from the open mapping theorem.

The objective of this section is to show that if $M = S_{\max}$ and $D \subseteq S_{\max}$ is a biinvariant domain, then $q: \widehat{D} \to M$ is *schlicht*, i.e., an open embedding. From that it follows that $\widehat{D}$ is simply the largest biinvariant Stein domain in $S_{\max}$ containing D (cf. Section 6).

LEMMA 7.1. — *Let H be a Lie group acting on D in such a way that the mapping $H \times D \to D$ is smooth and H acts by holomorphic mappings. Then the following assertions hold:*

(i) *The action of H on* $\mathrm{Hol}(D)$ *is continuous in the sense that the mapping $H \times \mathrm{Hol}(D) \to \mathrm{Hol}(D)$ is continuous.*

(ii) *The action of H on D lifts to an action on $\widehat{D}$ with the same properties.*

Proof. — (i) Since each $g \in H$ acts on $\mathrm{Hol}(D)$ by a continuous automorphism, it suffices to prove continuity in $\{\mathbf{1}\} \times \mathrm{Hol}(D)$. Let $f_0 \in \mathrm{Hol}(D)$, $K \subseteq D$ be compact, and $p_K(f) := \sup|f(K)|$ for $f \in \mathrm{Hol}(D)$. Suppose that $g_n \to \mathbf{1}$ and $f_n \to f_0$. We choose $K' \subseteq D$ compact with $g_n.K \subseteq K'$ for all $n \in \mathbb{N}$. For $x \in K$ we then have

$$\begin{aligned} p_K(g_n.f_n - f) &\leqslant p_K(g_n.f_n - g_n.f) + p_K(g_n.f - f) \\ &= p_{g_n^{-1}.K}(f_n - f) + p_K(g_n.f - f) \\ &\leqslant p_{K'}(f_n - f) + p_K(g_n.f - f). \end{aligned}$$

Since $p_{K'}(f_n - f) \to 0$, it suffices to show that $p_K(g_n.f - f) \to 0$. But this follows from the uniform continuity of f on the compact set K'. Thus we have shown that the action $H \times \mathrm{Hol}(D) \to \mathrm{Hol}(D)$ is continuous.

(ii) For $h \in H$ let $\mu_h : D \to D$ denote the corresponding holomorphic diffeomorphism of D. Then $\mu_h^* : f \mapsto f \circ \mu_h^{-1}$ is a continuous automorphism of $\mathrm{Hol}(D)$, hence yields a holomorphic diffeomorphism of $\widehat{D}$. Thus we obtain an action of H on $\widehat{D}$ by holomorphic diffeomorphisms.

We claim that this action is continuous. Since we know already that H acts by homeomorphisms on $\widehat{D}$, we have to show that $g_n \to \mathbf{1}$ and $\chi_n \to \chi$ implies that $g_n.\chi_n \to \chi$, i.e., that for all $f \in \mathrm{Hol}(D)$ we have

$$f(g_n.\chi_n) = \langle \chi_n, g_n^{-1}.f\rangle \to \langle \chi, f\rangle.$$

Let $K \subseteq \widehat{D}$ be a compact set containing χ and the sequence $\{\chi_n : n \in \mathbb{N}\}$. Since the topology on $\mathrm{Hol}(D)$ coincides with the one obtained by the identification with $\mathrm{Hol}(\widehat{D})$, convergence in $\mathrm{Hol}(D)$ implies uniform convergence on K. Therefore (i) implies that $g_n^{-1}.f \to f$, and hence that $\lim_{n\to\infty}\langle \chi_n, g_n^{-1}.f - f\rangle = 0$. Therefore

$$\lim_{n\to\infty}\langle \chi_n, g_n^{-1}.f\rangle = \lim_{n\to\infty}\langle \chi_n, f\rangle = \langle \chi, f\rangle$$

and the action of H on $\widehat{D}$ is continuous. That the action is in fact smooth now follows easily from the fact that $q: \widehat{D} \to M$ is a local diffeomorphism and that the action on M is smooth. □

Our main technique to show that in the aforementioned case the domain $\widehat{D} \to D$ is schlicht, will be by reductions to rather simple cases. The following observation will be crucial.

LEMMA 7.2. — *Let $\widehat{D}_1$ be the envelope of holomorphy of a domain D_1 in another Stein manifold M_1 and suppose that we have given a holomorphic map $\gamma: D_1 \to D$. Then there exists a unique holomorphic map*

$$\widehat{\gamma}: \widehat{D_1} \to \widehat{D}$$

with $\widehat{\gamma} \circ \eta_D = \eta_{D_1}$, where $\eta_D: D \to \widehat{D}$ and $\eta_{D_1}: D_1 \to \widehat{D_1}$ are the natural embeddings.

Moreover, if γ is equivariant with respect to the action of a Lie group H, then $\widehat{\gamma}$ is equivariant with respect to the natural actions of H induced on $\widehat{D}$ and $\widehat{D_1}$.

Proof. — First we note that γ induces a continuous homomorphism of Fréchet algebras $\mathrm{Hol}(D) \to \mathrm{Hol}(D_1)$, and hence a holomorphic map $\widehat{\gamma}: \widehat{D_1} \to \widehat{D}$.

The second statement follows from Lemma 7.1(ii) and the functoriality of the extension of the H-action from D to $\widehat{D}$. □

Now we turn to our more concrete situation. Let $D = G\,\mathrm{Exp}(D_h)$ and $\mathcal{D} = D_h \cap it \subseteq C^0_{\max}$. In view of Lemma 7.1, the group $G \times G$ acts smoothly on $\widehat{D}$ such that $q: \widehat{D} \to M$ is equivariant. Before we can determine the envelope of holomorphy of D, we have to describe the smallest biinvariant Stein domain in $S_{\max}$ containing D.

LEMMA 7.3. — *Let $\widetilde{D} = G\,\mathrm{Exp}(\widetilde{D}_h) \subseteq S_{\max}$ be the smallest biinvariant Stein domain containing D and $\widetilde{\mathcal{D}} = \widetilde{D}_h \cap it$. Then $\widetilde{\mathcal{D}} = \mathrm{conv}(\mathcal{D}) + iC_{\min}$. The set $\widetilde{\mathcal{D}}$ can also be characterized as the smallest convex subset of it containing $\mathcal{D}$ and satisfying $\mathbb{R}^+[X_\alpha, \overline{X_\alpha}] \subseteq \lim(\widetilde{\mathcal{D}})$ for $X_\alpha \in \mathfrak{g}^\alpha_{\mathbb{C}}$, $\alpha \in \Delta^+_p$.*

Proof. — In view of Theorem 6.1, the domain $\widetilde{D}$ is determined by $\widetilde{D}_h = \mathrm{conv}(D_h)$. Note that since the interior of a convex set is open, the convex hull of the open set D_h is open.

We apply Proposition 1.7 with $\widetilde{C} = -iD_h \underline{\subseteq} W^0_{\max}$ to see that $\widetilde{\mathcal{D}} + iC_{\min} \subseteq \overline{\widetilde{\mathcal{D}}}$. Since $\mathcal{D}$ is open, $\widetilde{\mathcal{D}} + iC_{\min} \subseteq \operatorname{int} \overline{\widetilde{\mathcal{D}}} = \widetilde{\mathcal{D}}$ follows. Thus $\operatorname{conv}(\mathcal{D}) + iC_{\min} \subseteq \widetilde{\mathcal{D}}$.

On the other hand $\mathcal{D}_1 := \operatorname{conv}(\mathcal{D}) + iC_{\min}$ is an open subset of $iC^0_{\max}$ which is $\mathcal{W}_{\mathfrak{k}}$-invariant. Hence $\operatorname{Ad}(G).\mathcal{D}_1$ is open and, by Proposition 1.7, it is also convex.

To prove the characterization of $\widetilde{\mathcal{D}}$, we first observe that $\widetilde{\mathcal{D}}$ is convex and that for $X_\alpha \in \mathfrak{g}^\alpha_{\mathbb{C}}$, $\alpha \in \Delta^+_p$ we have

$$\mathbb{R}^+[X_\alpha, \overline{X_\alpha}] \subseteq iC_{\min} \subseteq \lim(\widetilde{\mathcal{D}}).$$

If, on the other hand, $\mathcal{D}_1$ is a convex subset of $i\mathfrak{t}$ containing $\mathcal{D}$ and with $\mathbb{R}^+[X_\alpha, \overline{X_\alpha}] \subseteq \lim(\mathcal{D}_1)$ whenever $X_\alpha \in \mathfrak{g}^\alpha_{\mathbb{C}}$, $\alpha \in \Delta^+_p$, then the definition of $C_{\min}$ implies that $iC_{\min} \subseteq \lim(\overline{\mathcal{D}_1})$. Hence $\widetilde{\mathcal{D}} = \operatorname{conv}(\mathcal{D}) + iC_{\min} \subseteq \overline{\mathcal{D}_1}$ together with the fact that the set on the left hand side is open imply that $\widetilde{\mathcal{D}} \subseteq \mathcal{D}_1$. □

In view of the preceding lemma, we know that D is in fact a domain in the Stein manifold $\widetilde{D}$ and hence that $\widehat{D}$ must be a Riemannian domain over the smaller space $\widetilde{D}$. Our main result will be that $\widehat{D} \to \widetilde{D}$ is schlicht and surjective, hence that $\widehat{D} = \widetilde{D}$ (Theorem 7.9).

For the following lemma we recall the Weyl chamber $i\mathfrak{t}^+ = \{X \in i\mathfrak{t} \colon (\forall \alpha \in \Delta^+_k)\alpha(X) \geqslant 0\}$ which is a fundamental domain for the action of $\mathcal{W}_{\mathfrak{k}}$ on $i\mathfrak{t}$.

LEMMA 7.4. — *Let $D \subseteq S_{\max}$ be a biinvariant domain, X a $(G \times G)$-space and $\gamma \colon D \to X$ a $(G \times G)$-equivariant map. Then γ is continuous if and only if $\gamma \circ \operatorname{Exp}|_{\mathcal{D}\cap i\mathfrak{t}^+}$ is continuous, where $\mathfrak{t}^+$ is a Weyl chamber in $\mathfrak{t}$.*

Proof. — Suppose that $\gamma \circ \operatorname{Exp} |_{\mathcal{D}\cap i\mathfrak{t}^+}$ is continuous. Since $D = G\operatorname{Exp}(D_h)$ is a direct product decomposition, it suffices to show that $\gamma\circ\operatorname{Exp}$ is continuous on D_h. In view of Lemma 1.3, it suffices to show that γ is continuous on $\operatorname{Exp}(\operatorname{Ad}(K).iC^0_{\max})$. Suppose that $Y_n = \operatorname{Ad}(k_n).X_n \to Y = \operatorname{Ad}(k).X$, where X_n is the representative of $\mathcal{O}_{Y_n}$ in the Weyl chamber $i\mathfrak{t}^+$. Then the proof of [Ne96a, Lemma 3.5] implies that $X_n \to X$. Moreover, since $\operatorname{Ad}(K)|_{\mathfrak{k}}$ is a compact group, we may w.l.o.g. assume that $\operatorname{Ad}(k_n)|_{\mathfrak{k}}$ converges to $\operatorname{Ad}(k)|_{\mathfrak{k}}$ and therefore choose the sequence $k_n \in K$ in such a way that $k_n \to k$. Then

$$\begin{aligned}\gamma(\operatorname{Exp} Y_n) &= k_n\gamma(\operatorname{Exp} X_n)k_n^{-1} \to k\gamma(\operatorname{Exp} X)k^{-1}\\ &= \gamma\big(\operatorname{Exp}\operatorname{Ad}(k).X\big) = \gamma(\operatorname{Exp} Y)\end{aligned}$$

implies that γ is continuous. □

LEMMA 7.5. — *Let $X \in i\mathfrak{t}$ and $p \in \widehat{D}$ with $q(p) = \operatorname{Exp} X$. Then $gpg^{-1} = p$ holds for all $g \in G^X = \{g \in G: \operatorname{Ad}(g).X = X\}$.*

Proof. — First we note that since $iX \in \mathfrak{g}$ is a compact element and $G^X = G^{iX}$, it follows from [Ne94e, Th. 1.14] that G^X is a connected subgroup of G. Let $Y \in \mathfrak{g}^X = \mathbb{L}(G^X)$. We write $\mathcal{Y}$ and $\widehat{\mathcal{Y}}$ for the vector fields on D and $\widehat{D}$ corresponding to Y and the action of G given by $g.x := gxg^{-1}$. Then the equivariance of q implies that

$$0 = \mathcal{Y}(\operatorname{Exp} X) = dq(p)\widehat{\mathcal{Y}}(p),$$

and therefore that $\widehat{\mathcal{Y}}(p) = 0$ because q is a local diffeomorphism, i.e., $dq(p)$ is a bijection. This shows that p is fixed by all one-parameter subgroups of G^X, hence by all of G^X because this group is connected. □

The following proposition will be the crucial tool in the proof of the schlichtness of the envelopes of holomorphy of biinvariant domains.

PROPOSITION 7.6. — *Let $\mathcal{D}_1 \subseteq iC^0_{\max}$ be such that $\mathcal{D}_1 \cap i\mathfrak{t}^+$ is connected and suppose that we have a map*

$$\gamma: \operatorname{Exp}(\mathcal{D}_1) \to \widehat{D}$$

with $q \circ \gamma = \operatorname{id}_{\operatorname{Exp}(\mathcal{D}_1)}$ and the property that for $X, X' \in \mathcal{D}_1$ with $X' \in \mathcal{W}_{\mathfrak{k}}.X$ there exists $k \in N_K(T)$ with $X' = \operatorname{Ad}(k).X$ and $\gamma(\operatorname{Exp} X') = k\gamma(\operatorname{Exp} X)k^{-1}$. Then there exists a unique $(G\times G)$-equivariant holomorphic map

$$\widehat{\gamma}: D_1 := G\operatorname{Exp}(\mathcal{D}_1)G \to \widehat{D}$$

extending γ.

Proof. — The uniqueness of $\widehat{\gamma}_1$ follows from the equivariance requirement and the fact that $D_1 = G\operatorname{Exp}(\mathcal{D}_1)G$. To prove the existence, suppose that $d = g_1 \operatorname{Exp}(X) g_2 = g_1' \operatorname{Exp}(X') g_2'$ with $X, X' \in \mathcal{D}_1$. Then

$$g_1 g_2 \operatorname{Exp}\big(\operatorname{Ad}(g_2)^{-1}.X\big) = g_1' g_2' \operatorname{Exp}\big(\operatorname{Ad}(g_2')^{-1}.X'\big)$$

implies that $g_1 g_2 = g_1' g_2'$ and $\operatorname{Ad}(g_2' g_2^{-1}).X = X'$. Hence $\mathcal{O}_X = \mathcal{O}_{X'}$ and therefore $X' \in \mathcal{W}_{\mathfrak{k}}.X$ (cf. Remark 4.4). According to our consistency assumption, there exists $k \in N_K(T)$ with $X' = \operatorname{Ad}(k).X$ and $\gamma(\operatorname{Exp} X') = k\gamma(\operatorname{Exp} X)k^{-1}$.

In view of Lemma 7.5, $\operatorname{Ad}(k^{-1} g_2' g_2^{-1}).X = X$ implies that

$$k^{-1} g_2' g_2^{-1} \gamma(\operatorname{Exp} X) g_2 (g_2')^{-1} k = \gamma\big(\operatorname{Exp}(X)\big),$$

i.e.,

$$g_2' g_2^{-1} \gamma(\operatorname{Exp} X) g_2 (g_2')^{-1} = k\gamma\big(\operatorname{Exp}(X)\big)k^{-1} = \gamma(\operatorname{Exp} X').$$

Therefore

$$\begin{aligned} g_1\gamma(\operatorname{Exp} X)g_2 = g_1 g_2 g_2^{-1}\gamma(\operatorname{Exp} X)g_2 &= g_1' g_2'(g_2')^{-1}\gamma(\operatorname{Exp} X')g_2' \\ &= g_1'\gamma(\operatorname{Exp} X')g_2'. \end{aligned}$$

So we may define a map

$$\widehat{\gamma}: D_1 \to \widehat{D} \quad \text{by} \quad g_1 \operatorname{Exp}(X) g_2 \mapsto g_1 \gamma(\operatorname{Exp} X) g_2.$$

The map $\widehat{\gamma}$ is $G \times G$-equivariant by construction. It remains to prove that it is holomorphic. To apply Lemma 7.4, we have to show that γ is continuous on $\mathcal{W}_{\mathfrak{k}}(\mathcal{D}_1)$. The continuity on $\mathcal{D}_1$ is clear. For $k \in N_K(T)$ the continuity on $\operatorname{Ad}(k).\mathcal{D}_1$ follows from the equivariance which guarantees that $\gamma\big(\operatorname{Exp}(\operatorname{Ad}(k).X)\big) = k\gamma(\operatorname{Exp} X)k^{-1}$ holds for $k \in N_K(T)$. Hence Lemma 7.4 shows that $\widehat{\gamma}$ is continuous.

Also from the equivariance and $q \circ \gamma = \mathrm{id}_{\operatorname{Exp}(\mathcal{D}_1)}$ we obtain that $q \circ \widehat{\gamma} = \mathrm{id}_{D_1}$. Let $X \in D_1$ and U be a connected open neighborhood of $\gamma(X)$ such that $q\,|_U: U \to q(U)$ is biholomorphic. Let further V be a neighborhood of X with $\widehat{\gamma}(V) \subseteq U$. Then $V \to q(U), Y \mapsto q\big(\widehat{\gamma}(Y)\big) = Y$ is a holomorphic map. Composing this with $(q\,|_U)^{-1}$ we get $\widehat{\gamma}\,|_V$. Hence $\widehat{\gamma}$ is holomorphic. □

The proof of the main result will consist of several reductions. The following general lemma on connected components of Weyl group invariant sets will be used for the reduction to the case where $\mathcal{D}$ is connected.

LEMMA 7.7. — *Let $\mathcal{D} \subseteq \mathfrak{t}$ be an open $\mathcal{W}_{\mathfrak{k}}$-invariant subset such that $\mathcal{D}^+ = \mathcal{D} \cap \mathfrak{t}^+$ is connected, $\Upsilon \subseteq \Delta_k^+$ be a basis of the system of positive compact roots, $\Upsilon^0 = \{\alpha \in \Upsilon : (\exists X \in \mathcal{D}^+)\alpha(X) = 0\}$, and $\mathcal{W}_{\mathfrak{k}}^0 = \langle\{s_\alpha : \alpha \in \Upsilon^0\}\rangle$. Then the following assertions hold:*

(i) *$\mathcal{D}_1 := \mathcal{W}_{\mathfrak{k}}^0.\mathcal{D}^+$ is the connected component of $\mathcal{D}$ containing $\mathcal{D}^+$.*

(ii) *A Weyl chamber $\gamma(\mathfrak{t}^+)$ intersects $\operatorname{conv}(\mathcal{D}_1)$ if and only if it intersects $\mathcal{D}_1$, i.e., if and only if $\gamma \in \mathcal{W}_{\mathfrak{k}}^0$.*

(iii) *$\operatorname{conv}(\mathcal{D}_1)$ is a connected component of $\mathcal{W}_{\mathfrak{k}}.\big(\operatorname{conv}(\mathcal{D}_1)\big)$.*

(iv) *Let $\alpha \in \Delta_k^+$ with $s_\alpha \notin \mathcal{W}_{\mathfrak{k}}^0$. Then $\bigcup_{X \in \mathcal{D}_1} [X, s_\alpha(X)] = \operatorname{conv}\big(\mathcal{D}_1 \cup s_\alpha(\mathcal{D}_1)\big)$.*

Proof. — (i) Let $\mathcal{D}_1 \subseteq \mathcal{D}$ denote the connected component of $\mathcal{D}$ containing $\mathcal{D}^+$. For $\alpha \in \Upsilon^0$ we have $s_\alpha(\mathcal{D}^+) \cap \mathcal{D}^+ \neq \emptyset$ and therefore $s_\alpha(\mathcal{D}_1) \subseteq \mathcal{D}_1$ because the elements of the Weyl group permute the connected components of $\mathcal{D}$. Since the stabilizer of $\mathcal{D}_1$ in $\mathcal{W}_{\mathfrak{k}}$ is a subgroup, it follows that $\mathcal{D}^1 := \mathcal{W}_{\mathfrak{k}}^0(\mathcal{D}^+) \subseteq \mathcal{D}_1$.

Next we show that $\mathcal{D}^1$ is open in $\mathcal{D}$. We have to show that with each element X the set $\mathcal{D}^1$ contains a whole neighborhood of X. In view of $\mathcal{D}^1 = \mathcal{W}_{\mathfrak{k}}^0(\mathcal{D}^+)$, we may w.l.o.g. assume that $X \in \mathcal{D}^+$. So let $U \subseteq \mathfrak{t}$ be a convex neighborhood of X such that $U \cap \mathfrak{t}^+ \subseteq \mathcal{D}^+$, U is invariant under the subgroup $\mathcal{W}_{\mathfrak{k}}^X$ of $\mathcal{W}_{\mathfrak{k}}$, and if $\alpha(X) \neq 0$, then $\ker \alpha \cap U = \emptyset$ for $\alpha \in \Delta_k$. Since $\mathcal{W}_{\mathfrak{k}}^X$ is generated by $\{s_\alpha : \alpha \in \Upsilon \cap X^\perp\}$, we have $\mathcal{W}_{\mathfrak{k}}^X \subseteq \mathcal{W}_{\mathfrak{k}}^0$. On the other hand $U = \mathcal{W}_{\mathfrak{k}}^X.(U \cap \mathfrak{t}^+)$ follows from the fact that $\mathfrak{t}^+$ is a fundamental domain for the action of $\mathcal{W}_{\mathfrak{k}}^X$ on the cone

$$\{Y \in \mathfrak{t} : (\forall \alpha \in \Delta^+ \setminus X^\perp) i\alpha(Y) \geqslant 0\}$$

which contains U. Hence $U = \mathcal{W}_{\mathfrak{k}}^X.(U \cap \mathfrak{t}^+) \subseteq \mathcal{W}_{\mathfrak{k}}^0.\mathcal{D}^+ = \mathcal{D}^1$. This proves that $\mathcal{D}^1$ is open in $\mathfrak{t}$.

Now let $\gamma \in \mathcal{W}_{\mathfrak{k}}$ with $\gamma.\mathcal{D}^1 \cap \mathcal{D}^1 \neq \emptyset$. We claim that $\gamma \in \mathcal{W}_{\mathfrak{k}}^0$. Thus, after replacing γ by a suitable element in the $\mathcal{W}_{\mathfrak{k}}^0$-double coset of γ, we may w.l.o.g. assume that $\gamma(\mathcal{D}^+) \cap \mathcal{D}^+ \neq \emptyset$. Let $X \in \mathcal{D}^+$ with $\gamma(X) \in \mathcal{D}^+$. Then the fact that $\mathfrak{t}^+$ is a fundamental domain for the action of $\mathcal{W}_{\mathfrak{k}}$ on $\mathfrak{t}$ implies that $\gamma(X) = X$, i.e., $\gamma \in \mathcal{W}_{\mathfrak{k}}^X$. As we have seen above, $\mathcal{W}_{\mathfrak{k}}^X \subseteq \mathcal{W}_{\mathfrak{k}}^0$, and our claim follows.

Therefore the connected components of $\mathcal{D}$ are given by the sets $\gamma.\mathcal{D}^1$ which are mutually disjoint and $\gamma(\mathcal{D}^1) = \gamma'(\mathcal{D}^1)$ holds if and only if $\gamma^{-1}\gamma' \in \mathcal{W}_{\mathfrak{k}}^0$. This proves (i).

(ii) Let $X \in \mathfrak{t}^+$ such that $\Upsilon^0 = \Upsilon \cap X^\perp$. Then the set $\Delta_X^+ := \{\alpha \in \Delta_k^+ : i\alpha(X) > 0\}$ is invariant under the group $\mathcal{W}_{\mathfrak{k}}^0 = \mathcal{W}_{\mathfrak{k}}^X$ and

$$\mathcal{D}^+ \subseteq C := \{Y \in \mathfrak{t} : (\forall \alpha \in \Delta_X^+) i\alpha(Y) > 0\}$$

implies that

$$\operatorname{conv}(\mathcal{D}_1) = \operatorname{conv}(\mathcal{W}_{\mathfrak{k}}^0.\mathcal{D}^+) \subseteq C.$$

Since $\{Y \in \mathfrak{t} : (\forall \alpha \in \Upsilon \cap X^\perp) i\alpha(Y) \geqslant 0\}$ is a fundamental domain for the action of $\mathcal{W}_{\mathfrak{k}}^0$ on $\mathfrak{t}$, we see that each element in $\overline{C}$ is conjugate under $\mathcal{W}_{\mathfrak{k}}^0$ to an element in $\mathfrak{t}^+$. Hence a Weyl chamber $\gamma(\mathfrak{t}^+)$ is contained in C if and only if $\gamma \in \mathcal{W}_{\mathfrak{k}}^0$.

(iii) follows immediately from the inclusion $\operatorname{conv}(\mathcal{D}_1) \subseteq C$ proved in (ii).

(iv) The inclusion "$\subseteq$" holds trivially. So we have to show that the set $\mathcal{D}_2$ on the left hand side is convex, i.e., that for two points $Y, Z \in \mathcal{D}_2$ the line segment $[Y, Z]$ is contained in $\mathcal{D}_2$. This is clear if both points are contained in $\mathcal{D}_1$ or $s_\alpha(\mathcal{D}_1)$. Thus we may w.l.o.g. assume that $Y \in \mathcal{D}_1$ and $Z \in s_\alpha(\mathcal{D}_1)$. Then the line segments between pairs of the type $tY + (1-t)s_\alpha(Z)$ and $ts_\alpha(Y) + (1-t)Z$ for $t \in [0,1]$ are contained in $\mathcal{D}_2$. Now the fact that $\{Y, Z, s_\alpha(Y), s_\alpha(Z)\}$ is contained in an affine two dimensional plane invariant under s_α shows that $\mathcal{D}_2$ contains the line segment $[Y, Z]$, hence is convex. □

LEMMA 7.8. — *Let $\alpha \in \Delta_p^+$, $X \in \mathcal{D}$, $X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$, $Y = [X_\alpha, \overline{X_\alpha}]$, and $\varepsilon > 0$ such that $X + \varepsilon Y \in \mathcal{D}$. Then the curve $[0, \varepsilon] \to D, s \mapsto \operatorname{Exp}(X + sY)$ extends to a continuous mapping*

$$\gamma_{X,Y} : \mathbb{R}^+ \to \widehat{D}.$$

Proof. — Since $q: \widehat{D} \to MD$ is a local diffeomorphism, lifts are unique, and one can always extend a lift on a closed interval $[0, s_1] \to \widehat{D}$ to a slightly bigger interval. Hence there exists a maximal lift $\gamma_{X,Y}: [0, s_0[\to \widehat{D}$.

As in the proof of Theorem 6.1, we distinguish two cases. Suppose first that $\alpha \in \Delta_r^+$. Since we may w.l.o.g. assume that $Y \neq 0$, the subalgebra $\mathfrak{g}_1 := \operatorname{span}\{X_0, i[X_\alpha, \overline{X_\alpha}], X_\alpha + \overline{X_\alpha}, i(X_\alpha - \overline{X_\alpha})\}$ is isomorphic to the four dimensional oscillator algebra (cf. first part of Section 3). By the same arguments as in the proof of Theorem 6.1, we may w.l.o.g. assume that $\mathfrak{g} = \mathfrak{g}_1$ because the corresponding domain D_1 embeds into D and $q_1: \widehat{D_1} \to D_1$ lifts to a map $\eta: \widehat{D_1} \to \widehat{D}$ (Lemma 7.2).

We put $a = \operatorname{Exp} X \in D$ and consider the domain $\Gamma := U_{\mathbb{C}} a \cap D$, where $U = (G, G) \subseteq G$ is the three-dimensional Heisenberg group. Then Γ is a domain in the three dimensional complex manifold $U_{\mathbb{C}} a \cong \mathfrak{u}_{\mathbb{C}} \cong \mathbb{C}^3$ which is invariant under conjugation with elements in G and U-biinvariant, where the latter property follows from the normality of U in G. As in the proof of Theorem 6.1, we obtain $\Gamma = U \operatorname{Exp}(\Gamma_h)$, where $\Gamma_h \subseteq \operatorname{Ad}(G).(X + i\mathfrak{z})$. By passing to the connected component of $\mathcal{D} \cap (X + i\mathfrak{z})$ containing X, we may w.l.o.g. assume that this set is connected.

Let $q': \widehat{\Gamma} \to \Gamma$ denote the envelope of holomorphy of Γ. Since we have a canonical map $\widehat{\Gamma} \to \widehat{D}$, it suffices to show that the ray $\operatorname{Exp}(X + \mathbb{R}^+ Y)$ lifts to $\widehat{\Gamma}$. Suppose that the maximal lift with respect to γ is defined on the line segment $X + [0, s_0[Y$. Let $\mathcal{D}' := \{X + sY \in \mathcal{D} : s < s_0\}$ and

$\Gamma' := U\operatorname{Exp}\big(\operatorname{Ad}(G).\mathcal{D}'\big)$. Since the Weyl group of $\mathfrak{g}$ is trivial, a slight modification of Proposition 7.6 yields a lift

$$\Gamma' \to \widehat{\Gamma}$$

so that we may w.l.o.g. assume that $\Gamma = \Gamma'$. As we have seen in the proof of Theorem 6.1, the smooth boundary $\partial\Gamma$ contains the point $a' := \operatorname{Exp}(X + s_0 Y)$ and the Levi-condition is not satisfied in a'. As is shown in [Ra86, Proof of Th. 2.11, p.57], this implies that there exists a Hartogs figure $H \subseteq \Gamma$ such that $a \in \widehat{H} \not\subseteq \Gamma$. Hence each holomorphic function on Γ extends uniquely to a holomorphic function on the domain $\widehat{H} \cup \Gamma$. It follows in particular that we can slightly extend the lift of the segment $X + [0, s_0[Y$ to a bigger segment $X + [0, s_0 + \delta]Y$ with $\delta > 0$. This contradicts the definition of s_0 and hence proves that there exists a lift on $X + \mathbb{R}^+ Y$.

Next we assume that $\alpha \in \Delta^+_{p,s}$. Then the subalgebra

$$\mathfrak{g}_0 := \operatorname{span}\{i[X_\alpha, \overline{X_\alpha}], X_\alpha + \overline{X_\alpha}, i(X_\alpha - \overline{X_\alpha})\}$$

is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(1,1)$. By the same reductions as in the proof of Theorem 6.1, we may w.l.o.g. assume that $\mathfrak{g} = \mathfrak{g}_0$. Let s_0 be maximal such that the ray $X + [0, s_0[Y$ lifts to $\widehat{D}$. With the same argument as in the solvable case we see that we may w.l.o.g. assume that $X + [0, s_0[Y \subseteq \mathcal{D}$. Now the smooth boundary ∂D contains the point $a' := \operatorname{Exp}(X + s_0 Y)$ and the Levi-condition is not satisfied in a'. So the same argument as above shows that there exists a lift of the ray $X + \mathbb{R}^+ Y$. □

THEOREM 7.9. — *For each biinvariant domain $D \subseteq S_{\max}$ the mapping $\widehat{D} \to \widetilde{D}$ is schlicht, i.e., the smallest biinvariant Stein domain in $S_{\max}$ containing D is the envelope of holomorphy of D.*

Proof. — Let $\mathfrak{t}^+ \subseteq \mathfrak{t}$ denote the Weyl chamber associated to Δ^+_k. Then $\mathcal{D}^+ := \mathcal{D} \cap i\mathfrak{t}^+$ is connected because D is connected (cf. [Ne96a, Lemma 3.5]). We write $\mathcal{D}_1 \subseteq \mathcal{D}$ for the connected component of $\mathcal{D}$ containing $\mathcal{D}^+$. We put $D_1 := T\operatorname{Exp}(\mathcal{D}_1)$ and note that the envelope of holomorphy of D_1 is given by $\widehat{D_1} = T\operatorname{Exp}(\operatorname{conv}\mathcal{D}_1)$ (cf. [Hö73, Th. 2.5.10]). Therefore we obtain an embedding $\gamma\colon \widehat{D_1} \to \widehat{D}$ with $q\circ\gamma = \operatorname{id}_{\widehat{D_1}}$. To check the consistency condition of Proposition 7.6, we first note that, according to Lemma 7.7(ii), $\operatorname{conv}(\mathcal{D}_1) \subseteq \mathcal{W}^0_{\mathfrak{k}}.(i\mathfrak{t}^+)$. Since the mapping $\mathcal{D}_1 \to \mathcal{D}$ is equivariant with respect to the action of the group $\mathcal{W}^0_{\mathfrak{k}}$, it follows from the second part of Lemma 7.2 that for $\gamma = \operatorname{Ad}(k)\,|_{\mathfrak{t}} \in \mathcal{W}^0_{\mathfrak{k}}$ with $k \in N_K(\mathfrak{t})$ we have

$$\gamma\big(\operatorname{Exp}\operatorname{Ad}(k).X\big) = k\gamma(\operatorname{Exp} X)k^{-1}$$

for each $X \in \operatorname{conv}(\mathcal{D}_1)$. Hence the consistency condition of Proposition 7.6 is satisfied and we obtain an extension

$$\widehat{\gamma}\colon D_2 := G \operatorname{Exp}\big(\operatorname{conv}(\mathcal{D}_1)\big)G \to \widehat{D}.$$

Since $D \subseteq D_2$ is a subdomain which lifts to $\widehat{D}$, we see that $\widehat{D} = \widehat{D}_2$. Therefore, in view of Lemma 7.7(iii), we may w.l.o.g. assume that the connected components of the set $\mathcal{D}$ are convex. We note that, according to Lemma 7.7(i), the connected component $\mathcal{D}_1$ of $\mathcal{D}$ containing $\mathcal{D}^+$ is invariant under the group $\mathcal{W}_{\mathfrak{k}}^0$. It follows in particular that for $X \in \mathcal{D}_1$ and $s_\alpha \in \mathcal{W}_{\mathfrak{k}}^0$ the line segment $[X, s_\alpha(X)] = \operatorname{conv}\{X, s_\alpha(X)\}$ is contained in $\mathcal{D}_1$. If $\mathcal{W}_{\mathfrak{k}} = \mathcal{W}_{\mathfrak{k}}^0$ this implies that $\mathcal{D}_1 = \mathcal{D}$, hence that $\mathcal{D}$ is convex. Suppose that this is not the case and that $\alpha \in \Delta_k^+$ satisfies $s_\alpha \notin \mathcal{W}_{\mathfrak{k}}^0$. Then $s_\alpha(\mathcal{D}_1) \cap \mathcal{D}_1 = \emptyset$ and Lemma 7.7(iv) implies that

$$\mathcal{D}_2 := \bigcup_{X \in \mathcal{D}_1} [X, s_\alpha(X)] = \operatorname{conv}\big(\mathcal{D}_1 \cup s_\alpha(\mathcal{D}_1)\big).$$

Now let $X \in \mathcal{D}_1$ and put $Y := \frac{1}{2}\big(X + s_\alpha(X)\big)$. Let $\mathcal{D}_\alpha := \mathbb{R}\check{\alpha} \cap (\mathcal{D}_1 - Y)$ and note that this set is a line segment in $\mathbb{R}\check{\alpha}$. Let $G^\alpha \subseteq G$ denote the subgroup with Lie algebra $\mathfrak{g}^\alpha := \mathfrak{g}^{[\alpha]} + i\mathbb{R}\check{\alpha} \cong \mathfrak{su}(2)$. Hence

$$D_\alpha := G^\alpha \operatorname{Exp}(\mathcal{D}_\alpha) G^\alpha \subseteq G_{\mathbb{C}}^\alpha$$

is a G^α-biinvariant domain. Then it follows from [Fe94, Kor. 3.8] that $\widehat{D_\alpha} \to G_{\mathbb{C}}^\alpha$ is schlicht, i.e., that $\widehat{D_\alpha} \subseteq G_{\mathbb{C}}^\alpha$. Moreover $\widehat{D_\alpha} = G^\alpha \operatorname{Exp}(\widehat{\mathcal{D}_\alpha}) G^\alpha$ with $\widehat{\mathcal{D}_\alpha} = \operatorname{conv}(\mathcal{D}_\alpha)$. Hence the inclusion

$$D_\alpha \to D, \quad \operatorname{Exp}(Z) \mapsto \operatorname{Exp}(Y + Z) = \operatorname{Exp}(Y)\operatorname{Exp}(Z)$$

extends to a mapping $\gamma_Y\colon \widehat{D_\alpha} \to \widehat{D}$. Since $\mathcal{D}_2$ is covered by the lines $Y + \mathbb{R}\check{\alpha}$, $Y \in \ker \alpha$, we therefore obtain a lift $\gamma\colon \operatorname{Exp}(\mathcal{D}_2) \to \widehat{D}$, $Y + s\check{\alpha} \mapsto \gamma_Y(\operatorname{Exp} s\check{\alpha})$.

We claim that γ is continuous. In fact, let $X \in \mathcal{D}_1$ and Y as above. Let further U be a neighborhood of Y in $\ker \alpha$ and $\mathcal{D}_\alpha^0 \subseteq \mathcal{D}_\alpha$ with $\mathcal{D}_\alpha^0 + U \subseteq \mathcal{D}_1$. Then the mapping

$$\operatorname{Exp}(\mathcal{D}_\alpha^0 + U) \to \operatorname{Exp}(\mathcal{D}_\alpha^0) \subseteq D_\alpha^0, \quad \operatorname{Exp}(Z + Y) \mapsto \operatorname{Exp}(Z)$$

is continuous. Let $f \in \operatorname{Hol}(\widehat{D})$. Then, to see that $Y \mapsto f \circ \gamma_Y \mid_{\widehat{D_\alpha^0}}$ is continuous, it suffices to check that $Y \mapsto f \circ \gamma_Y \mid_{D_\alpha^0}$ is continuous, which trivially follows from $f(\gamma_Y(k)) = f(\operatorname{Exp}(Y)k)$. We see in particular that for a given $Z \in \operatorname{conv}(\mathcal{D}_\alpha^0)$ the mapping

$$Y \mapsto \gamma_Y(\operatorname{Exp} Z) = \gamma(\operatorname{Exp}(Y + Z))$$

depends continuously on Y. Therefore the fact that $q\colon \widehat{D} \to D$ is a local homeomorphism and $q \circ \gamma = \mathrm{id}_{\operatorname{Exp}(\mathcal{D}_2)}$ implies that γ is continuous.

Next we check the consistency condition. Let $k_\alpha \in G^\alpha$ with $\operatorname{Ad}(k_\alpha)\mid_{\mathfrak{t}} = s_\alpha$. Then the $(G^\alpha \times G^\alpha)$-equivariance of the mappings γ_Y implies that

$$\begin{aligned}\gamma(\operatorname{Exp}(\operatorname{Ad}(k_\alpha).(Y+Z))) &= \gamma(\operatorname{Exp}(Y + \operatorname{Ad}(k_\alpha).Z))\\ &= \gamma_Y(\operatorname{Exp}(s_\alpha.Z)) = k_\alpha\gamma_Y(\operatorname{Exp} Z)k_\alpha^{-1} = k_\alpha\gamma(\operatorname{Exp}(Y+Z))k_\alpha^{-1}.\end{aligned}$$

Now Proposition 7.6 implies that γ extends to a $(G \times G)$-equivariant lift

$$\widehat{\gamma}\colon D_2 := G\operatorname{Exp}(\mathcal{D}_2))G \to \widehat{D}.$$

As before, we therefore can assume that $D = D_2$, hence that $s_\alpha \in \mathcal{W}_{\mathfrak{k}}^0$. After repeating this process at most finitely many times, we see that we may w.l.o.g. assume that $\mathcal{W}_{\mathfrak{k}}^0 = \mathcal{W}_{\mathfrak{k}}$, hence that $\mathcal{D}$ is a convex subset of $i\mathfrak{t}$.

Now let $\alpha \in \Delta_{p,s}^+$ and put $\mathcal{D}_1 := \mathcal{D} + \mathbb{R}^+\check{\alpha}$. We want to obtain a lift $\operatorname{Exp}(\mathcal{D}_1) \to \widehat{D}$. So let $Y = [X_\alpha, \overline{X_\alpha}]$ with $\alpha \in \Delta_p^+$. Let $\widehat{\mathcal{Y}}$ denote the vector field on $\widehat{D}$ which is the unique lift of the vector field on D given by

$$\mathcal{Y}(p) := \frac{d}{dt}\Big|_{t=0} \operatorname{Exp}(tY).$$

Then, according to Lemma 7.8, this vector field is forward complete on a subset of $\widehat{D}$ containing $\operatorname{Exp}(\mathcal{D}) \subseteq D \subseteq \widehat{D}$. Hence the mapping $\mathcal{D}\times\mathbb{R}^+ \to \widehat{D}$, $(X,s) \mapsto \gamma_{X,Y}(s)$ is continuous. Thus we obtain a lift

$$\gamma\colon \operatorname{Exp}(\mathcal{D} + \mathbb{R}^+Y) \to \widehat{D}, \quad \operatorname{Exp}(X + sY) \mapsto \gamma_{X,Y}(s)$$

which is continuous.

Suppose that $Y \in i\mathfrak{t}^+$. Then $\mathcal{D}_1^+ := \mathcal{D}+\mathbb{R}^+Y \subseteq i\mathfrak{t}^+$. Now Proposition 7.6 provides a $(G \times G)$-equivariant extension of the mapping $\gamma\mid_{\operatorname{Exp}(\mathcal{D}_1^+)}$ to the domain $D_1 := G\operatorname{Exp}(\mathcal{D}_1^+)G$. Hence we may w.l.o.g. assume that $\mathcal{D}^+ + \mathbb{R}^+Y \subseteq \mathcal{D}^+$. In view of what we have shown so far, we may also w.l.o.g assume that $\mathcal{D}$ is convex, hence that $\mathcal{D} + \mathbb{R}^+Y \subseteq \mathcal{D}$. Since $\mathcal{D}$ is also invariant under the Weyl group $\mathcal{W}_{\mathfrak{k}}$, it follows automatically that

$\operatorname{cone}(\mathcal{W}_{\mathfrak{k}}.Y) \subseteq \lim(\mathcal{D})$. This shows that the assumption $Y \in i\mathfrak{t}^+$ was no limitation of generality. Let

$$C_{\min,s} := \operatorname{cone}\{i[\overline{X_\alpha}, X_\alpha]: X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha, \alpha \in \Delta_{p,s}^+\}.$$

Then the cone $C_{\min,s}$ is generated by finitely many rays, hence, after repeating the above steps finitely many times, we may w.l.o.g. assume that $\mathcal{D} + C_{\min,s} \subseteq \mathcal{D}$.

Next we consider the case when $\alpha \in \Delta_r^+$. Let $(Y_n)_{n\in\mathbb{N}}$ be a countable set of generators of the cone $iC_{\min,z}$, where

$$C_{\min,z} := \operatorname{cone}\{i[\overline{X_\alpha}, X_\alpha]: X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha, \alpha \in \Delta_r^+\}$$

and $Y_n = [X_\alpha^n, \overline{X_\alpha^n}]$ with $X_\alpha^n \in \mathfrak{g}_{\mathbb{C}}^\alpha$. Inductively we define

$$\mathcal{D}_0 := \mathcal{D} \quad \text{and} \quad \mathcal{D}_{n+1} := \mathcal{D}_n + \mathbb{R}^+ Y_n.$$

Then the same argument as above shows that the embedding $\operatorname{Exp}(\mathcal{D}_n) \to \widehat{D}$ extends to an embedding $\operatorname{Exp}(\mathcal{D}_{n+1}) \to \widehat{D}$. Note that at each step the fact that Y_n is central implies that $\mathcal{D}_n$ is invariant under the Weyl group. Since the cone $\lim(\mathcal{D}')$ for the open convex set $\mathcal{D}' := \bigcup_{n\in\mathbb{N}} \mathcal{D}_n$ is closed, it follows that it contains $iC_{\min}$. Thus $\mathcal{D}' = \widetilde{\mathcal{D}}$ follows from Lemma 7.3.

At each step the consistency condition follows from the fact that Y_n is fixed by $\mathcal{W}_{\mathfrak{k}}$ and the uniqueness of the extension. Hence Proposition 7.6 yields a lift

$$\gamma: \widetilde{D} := G \operatorname{Exp}(\widetilde{\mathcal{D}}) G \to \widehat{D}.$$

Then the connectedness of $\widehat{D}$ and the fact that $\widehat{D}$ is a Riemannian domain over $\widetilde{D}$ implies that $\gamma(\widetilde{D}) = \widehat{D}$, so that $\widehat{D} \to D$ is schlicht, and finally that $\widetilde{D} = \widehat{D}$. Thus we have shown that each holomorphic function on D extends uniquely to a holomorphic function on $\widetilde{D}$. □

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