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THE GRAPH POLYNOMIAL AND THE NUMBER OF PROPER VERTEX COLORINGS

by Michael TARSI

1. Introduction.

Let $G = (V, E)$ be a graph, $V = \{1, 2, \dots, n\}$, let

$$P_G = \prod_{ij \in E, i < j} (x_j - x_i)$$

be its graph polynomial, and let $\overline{P^k}_G$ be the remainder of this polynomial modulo the ideal generated by the polynomials $x_i^k - 1$, $1 \leq i \leq n$. Put $Z_k^n = \{0, 1, \dots, k-1\}^n$. For a polynomial

$$P = P(x_1, \dots, x_n) = \sum_{v \in Z_k^n} a_v \prod_{i=1}^n x_i^{v_i}.$$

Define $\|P\|_2^2 = \sum_{v \in Z_k^n} |a_v|^2$.

In a recent joint work with Noga Alon, [2] we proved the following result, in which $C_k(G)$ is the set of all proper colorings c of a graph G by the k colors $\{0, \dots, k-1\}$.

THEOREM 1.1. — *In the above notation*

$$(1) \quad \|\overline{P^k}_G\|_2^2 = \frac{4^{|E|}}{k^n} \sum_{c \in C_k(G)} \prod_{ij \in E, i < j} \sin^2 \left[\frac{\pi(c(i) - c(j))}{k} \right].$$

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Theorem 1.1 clearly provides a lower bound for the number of proper k -colorings of a graph G in terms of $\|\overline{P^k}_G\|_2^2$. For the special case $k = 3$ this is an equality, showing that the precise number $|C_3(G)|$ of proper 3-colorings satisfies

$$(2) \quad \|\overline{P^3}_G\|_2^2 = 3^{|E|-n}|C_3(G)|.$$

The following notation is used in the sequel: A *Partial orientation* of a graph is obtained when some edges are assigned with an orientation while the other edges remain undirected.

Given a partial orientation and a vertex x , the number of oriented edges with their tails, (heads) at x is denoted by $d^+(x)$, ($d^-(x)$).

A $k, 1$ -flow in a graph $G = (V, E)$ is a partial orientation of G , where for every vertex $x \in V$, $d^+(x) - d^-(x) \equiv 0 \pmod k$.

$F_k^1(G)$ stands for the set of all $k, 1$ -flows of G .

The support $\sigma(f)$ of a flow $f \in F_k^1(G)$ is the set of edges of G which are oriented in f .

The main result of this paper is the following equality, which gives another combinatorial interpretation to the left hand side of 1:

THEOREM 1.2.

$$(3) \quad \|\overline{P^k}_G\|_2^2 = (-1)^{|E|} \sum_{f \in F_k^1(G)} (-2)^{|E|-|\sigma(f)|}.$$

Note that $F_k^1(G)$ corresponds to the subset of $F_k(G)$ (the set of all Z_k -flows in G) where the permitted flow values are 1, -1 and 0. Since $Z_3 = \{0, 1, -1\}$, in the special case of $k = 3$ we obtain

$$(4) \quad \|\overline{P^3}_G\|_2^2 = (-1)^{|E|} \sum_{f \in F_3(G)} (-2)^{|E|-|\sigma(f)|}$$

and by 2

$$(5) \quad 3^{|E|-n}|C_3(G)| = (-1)^{|E|} \sum_{f \in F_3(G)} (-2)^{|E|-|\sigma(f)|}.$$

Considering the parity of the right hand side, all summands are clearly even, except those corresponding to flows f for which $\sigma(f) = E$. These flows are the 3- *Nowhere Zero Flows* (3- *NZF*) of G . This implies:

THEOREM 1.3. — *The set of 3–Nowhere zero flows ($NZF_3(G)$) and the set of proper 3-colorings of a graph G are of the same parity, or more precisely*

$$(6) \qquad |C_3(G)| \equiv |NZF_3(G)| \pmod{4}.$$

Remark. — Clearly $|C_3(G)|$ is always divisible by 6 (permutations of the 3 colors) and flows come in pairs (obtained by reversing the entire orientation). It is a common convention, however, to refer to permutations of the colors as the same coloring (e.g. phrases like “uniquely k -colorable” etc.). Accordingly, when talking of an “odd” $|C_3(G)|$ we actually refer to 2 modulo 4 and the same holds to the parity of $NZF_3(G)$.

One can easily observe that in 4-regular graphs, Eulerian orientations are the only 3–Nowhere zero flows. The following is then a direct consequence of Theorem 1.3:

THEOREM 1.4. — *The set of proper 3-colorings of a 4-regular graph G and the set of Eulerian orientations of G have the same parity.*

Fleischner and Stiebitz [3] proved that in every ‘*Cycle and triangles*’ graph there is an odd number of Eulerian orientations. They used this result and the ‘*Graph polynomial method*’ presented in [1], to prove that such graphs are 3-colorable. Recently Sachs [4] proved, by means of direct induction, that the number of 3-colorings of every *Cycle and triangles* graph is odd. Theorem 1.4 shows that Sachs result can be derived from the Fleischner and Stiebitz Lemma. This observation provides an affirmative answer to a question posted by Jack Edmonds.

2. Proving the main result.

As observed in [1], each term in the expansion of P_G to the sum of $2^{|E|}$ monomials, corresponds to an orientation of G . Similar terms, that is monomials with the same degree for every variable, correspond to orientations which share the same outdegree sequence $\{d^+(x) | x \in V\}$.

When computing $\overline{P^k}_G$, two monomials are similar if the corresponding degrees are congruent modulo k . The outdegree sequences corresponding to such two monomials are clearly, term by term, congruent modulo k . We refer to such orientations as k -equivalent.

Let ω_1 and ω_2 be two orientations of a graph $G = (V, E)$, and let $\omega_1 \Delta \omega_2$ denote the set of edges whose orientation is reversed when going from ω_1 to ω_2 . For ω_1 and ω_2 to be k -equivalent, $\omega_1 \Delta \omega_2$ should satisfy (as a subgraph of any of the two orientations) $d^+(x) \equiv d^-(x) \pmod k$ for every $x \in V$. In other words, $\omega_1 \Delta \omega_2$ should be the support of a $k, 1$ -flow in G .

Furthermore, selecting either x or y as the head of an edge (x, y) , corresponds to the selection of either x or $-y$ from the term $(x - y)$, in the expansion of P_G (and of $\overline{P^k}_G$ as well). Two k -equivalent orientations ω_1 and ω_2 then yield similar monomials, of the same sign, if $|\omega_1 \Delta \omega_2|$ is even, or of different signs, if it is odd.

Let us agree that an edge (x_i, x_j) with $i < j$ is *oriented backwards* if its head is at x_i , and define $\text{parity}(\omega) = 0$ if the number of backwards oriented edges is even and $\text{parity}(\omega) = 1$ if this number is odd.

For $v \in Z_k^n$ let S_v be the k -equivalence class of orientations of G with $d^+(x_i) \equiv v_i \pmod k$, $1 \leq i \leq n$, and let S_v^0 , respectively S_v^1 denote the subsets of even, respectively odd orientations in that class.

For any $\omega \in S_v$, let $(F_k^1)^0(\omega)$, respectively $(F_k^1)^1(\omega)$ be the sets of $k, 1$ -flows, the supports of which are even, respectively odd oriented subgraphs of ω .

Following the discussion above we obtain

$$(7) \quad a_v = |S_v^0| - |S_v^1|$$

and for $i = 0, 1$ ($\bar{0} = 1, \bar{1} = 0$) and any $\omega \in S_v^i$:

$$(8) \quad |(F_k^1)^i(\omega)| = |S_v^0| \quad \text{and} \quad |(F_k^1)^{\bar{i}}(\omega)| = |S_v^1|.$$

Consider now the quantity

$$\sum_{\omega \in S_v} (|(F_k^1)^0(\omega)| - |(F_k^1)^1(\omega)|).$$

By 8 it can be rewritten as:

$$\sum_{\omega \in S_v^0} |S_v^0| - |S_v^1| + \sum_{\omega \in S_v^1} |S_v^1| - |S_v^0|.$$

By 7, this equals:

$$\sum_{\omega \in S_v^0} a_v - \sum_{\omega \in S_v^1} a_v = (|S_v^0| - |S_v^1|)a_v = a_v^2.$$

Summing up the above over all equivalent classes we obtain

$$(9) \quad \|P_G^k\|_2^2 = \sum_{\omega \in D(G)} (|(F_k^1)^0(\omega)| - |(F_k^1)^1(\omega)|)$$

where $D(G)$ is the set of all orientations of G .

Let us now change the order of summation in the right hand side of 9 by taking $k, 1$ -flows, instead of orientation, as the leading index:

$$(10) \quad \|P_G^k\|_2^2 = \sum_{f \in F_k^1(G)} (-1)^{|\sigma(f)|} |D(f)|.$$

Here $D(f) = \{\omega | f \in F_k^1(\omega)\}$ is the set of all expansions of f to (complete) orientations (of all the edges) of G .

An expansion of a partial orientation is constructed by selecting an orientation for each undirected edge and hence $|D(f)| = 2^{|E| - |\sigma(f)|}$ which completes the proof of Theorem 1.2.

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