



# ANNALES

DE

# L'INSTITUT FOURIER

Assaf GOLDBERGER & Ehud DE SHALIT

**Tamely ramified Hida theory**

Tome 52, n° 1 (2002), p. 1-45.

[http://aif.cedram.org/item?id=AIF\\_2002\\_\\_52\\_1\\_1\\_0](http://aif.cedram.org/item?id=AIF_2002__52_1_1_0)

© Association des Annales de l'institut Fourier, 2002, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (<http://aif.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://aif.cedram.org/legal/>). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

*Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>*

## TAMELY RAMIFIED HIDA THEORY

by A. GOLDBERGER and E. de SHALIT

---

Let  $N$  be a natural number, and  $p$  a prime not dividing  $N$ . Let  $X_1(N, p)$  denote the modular curve associated with the group  $\Gamma_1(N, p) = \Gamma_1(N) \cap \Gamma_0(p)$ , and  $X_1(Np^\nu)$  the curve associated with  $\Gamma_1(Np^\nu)$ . Several authors (Hida, Mazur, Tilouine, Wiles ...; see [H], [MW1], [W]) studied the tower obtained from the transition maps between these curves, which correspond to the inclusions  $\Gamma_1(N, p) \supset \Gamma_1(Np^\nu) \supset \Gamma_1(Np^{\nu+1})$ . The Hecke ring acts as a ring of correspondences on this tower. In particular, the diamond operators induce an action of  $\mathbb{Z}_p^\times$ , and  $p$ -adic modules built from the tower can be studied as modules over the Iwasawa algebra  $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ . In this way the geometry of modular curves is connected with deformation theory of  $p$ -adic Galois representations.

Any module over  $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$  breaks up as a direct sum according to the characters of  $\mathbb{F}_p^\times \subset \mathbb{Z}_p^\times$ . Thus the first step of the tower, the covering  $X_1(Np) \rightarrow X_1(N, p)$ , is exploited in a rather mild way, and is not the source of a true deformation. Around the mid 80's it was noticed by Mazur and Tate that various results and conjectures in the  $p$ -adic theory of modular curves (or cyclotomic fields) had finite, "refined" versions, which could probably be linked - in the situation described here - to a deformation theory with group  $\mathbb{F}_p^\times$  [MT]. In [dS1] and [dS2] one of us proposed to use the kernel of  $p - 1$  on the Jacobian as a source of such "finite deformation theory". This was applied there to a conjecture of Mazur and Tate [MT]

---

*Keywords:* Modular curve –  $p$ -adic periods – Hecke operators.  
*Math. classification:* 11F85.

on the  $p$ -adic periods of elliptic curves with split multiplicative reduction, much in the style of Greenberg and Stevens [GrSt], who used Hida theory to prove the conjecture of Mazur Tate and Teitelbaum [MTT], which predated the Mazur-Tate conjecture.

For non-trivial technical reasons, the articles [dS1] and [dS2] dealt with the case  $N = 1$  (prime conductor). The purpose of this work is to return to the general set-up, and study the deformation theory which is obtained from the  $p - 1$  torsion in the Jacobians of  $X_1(N, p)$  and  $X_1(Np)$ . We call this “tame” Hida theory because the bad prime, where most of the action takes place, is still  $p$ . Clearly, what we do below is only a “weight 2” theory. The higher weights, which in Hida’s theory are subsumed by the weight 2 theory, have to await further developments.

The following is a brief summary. In Section 1 we collect preliminary results, and introduce notation and conventions to be used throughout. Section 2 is devoted to a detailed study of  $J_1(Np)[p - 1]$ , as a deformation of its part fixed by the group of  $p$ -diamond operators, isomorphic to  $\mathbb{F}_p^\times$ . As we may treat the  $l$ -Sylow subgroup separately for each prime  $l$  dividing  $p - 1$ , we fix  $l$  (different from 2) and study  $J_1(Np)[r]$  where  $r$  is the highest power of  $l$  dividing  $p - 1$ . We address the structure over the group ring of  $\mathbb{F}_p^\times$ , Hecke structure, Galois action, and in particular the filtration for the action of the decomposition group at  $p$ . The results obtained in Section 2 are summarized in Theorem 2.8. Section 3 is devoted to questions of breaking the deformation into components, and in particular distinguishing the  $p$ -new components from the  $p$ -old ones or from other new components. Section 4 finally treats one such  $p$ -new component, and shows how *the infinitesimal variation of the  $U_p$  operator in the deformation* is related to the  *$p$ -adic period matrix* of the abelian variety. Theorem 4.6 may be considered the main result.

To illustrate the main result, let us consider an elliptic curve  $A$  over  $\mathbb{Q}$  of conductor  $Np$ , with split multiplicative reduction at  $p$ . We normalize  $A$  within its  $\mathbb{Q}$ -isogeny class by requiring that it appears as a factor of the Jacobian of the modular curve  $X_1(N, p)$ . (This is a slight deviation from the more traditional point of view, as a factor of the Jacobian of  $X_0(Np)$ .) Let  $q_A$  be the  $p$ -adic Tate period of  $A$  and write

$$q_A = up^\nu$$

with  $u \in \mathbb{Z}_p^\times$  and  $\nu \in \mathbb{Z}$ .

Pick an odd prime  $l$  dividing  $p - 1$ , let  $r$  be the highest power of  $l$  dividing  $p - 1$ , and  $R = \mathbb{Z}/r\mathbb{Z}$ . Let

$$u_R = u \otimes 1 \in \mathbb{Z}_p^\times \otimes R = \mathbb{F}_p^\times \otimes R = \mu_r \text{ and } \nu_R = \nu \otimes 1 \in R.$$

The ratio  $u_R : \nu_R$  is the  $l$ -primary part of the *refined  $\mathcal{L}$ -invariant* of  $A$ . Recall that the ( $p$ -adic)  $\mathcal{L}$ -invariant of Mazur-Tate-Teitelbaum is the ratio  $\log_p(u) : \nu$ , which is blind to the  $p - 1$  root of unity in  $q_A$ . The refined  $\mathcal{L}$ -invariant is made precisely to capture that bit of information.

We make two technical assumptions. The first, that  $A$  is not  $l$ -Eisenstein (see [M] and Section 1.6 below for the precise meaning of this). The second, that the degree of the modular parametrization  $X_1(N, p) \rightarrow A$  is prime to  $l$ . (This guarantees condition (A2) in Section 3.3.) Let us also assume, for simplicity, that  $l$  does not divide  $N$ .

Consider the Jacobians  $J_0$  of  $X_1(N, p)$  and  $J_1$  of  $X_1(Np)$ . Because of our assumptions,  $A[r] \cong R^2$  is a direct summand, as a Hecke and Galois module, of  $J_0[r]$ . On the other hand  $J_1[r]$  is also a  $\Lambda$ -module, where  $\Lambda = R[\mu_r]$ , and  $\mu_r \subset \mathbb{F}_p^\times$  act via the  $p$ -diamond operators. We show that there is a unique direct summand (Hecke and Galois stable)  $\mathfrak{e}J_1[r] \cong \Lambda^2$  of  $J_1[r]$  which is a deformation of  $A[r]$  (Proposition 3.4).

We now examine the action of the Hecke operator  $U_p^{-2}$  on both pieces. On  $A[r]$  it acts trivially. On the deformation module  $\mathfrak{e}J_1[r]$  it acts via multiplication by  $1 + \lambda$  with  $\lambda$  in the augmentation ideal  $I$  of  $\Lambda$ . Identifying  $I/I^2$  with  $\mu_r$ , we finally arrive at the desired relation

$$u_R = \nu_R \lambda \text{ mod } I^2.$$

Thus, up to a factor of  $-1/2$ , it may be said that the ratio  $u_R : \nu_R$  is the *derivative* of the  $U_p$  operator in the deformation of  $A[r]$  which lives inside the Jacobian of  $X_1(Np)$ .

## 1. Preliminaries and notation.

### 1.1. Abstract diamond and Hecke operators.

Let  $N \geq 4$  be an integer, and let  $p \geq 7$  be a prime which does not divide  $N$ . Let  $l$  be an odd prime dividing  $p - 1$ , and  $r = l^n$  the maximal power of  $l$  dividing  $p - 1$ . Write  $p - 1 = rr'$ , where  $(r', r) = 1$ , so that

$\mathbb{F}_p^\times = \mu_r \times \mu_{r'}$ . Let  $R = \mathbb{Z}/r\mathbb{Z}$  and

$$(1.1) \quad \underline{\Lambda} = R[\mathbb{F}_p^\times] = \Lambda \otimes \Lambda',$$

where  $\Lambda = R[\mu_r]$  is local, and  $\Lambda' = R[\mu_{r'}]$  is étale over  $R$ . We write a typical element of  $\underline{\Lambda}$  as  $\sum n_a \langle a \rangle$ . Let  $I$  (resp.  $I'$ , resp.  $\underline{I}$ ) be the augmentation ideal in  $\Lambda$  (resp.  $\Lambda'$ , resp.  $\underline{\Lambda}$ ). We have  $\underline{I} = I \otimes \Lambda' + \Lambda \otimes I'$ . Note that  $I$  is nilpotent, that the maximal ideal of  $\Lambda$  is  $(l) + I$ , and that  $\langle -1 \rangle - \langle 1 \rangle$  belongs to  $I'$ .

Consider the commutative ring  $\mathcal{H}$  generated over  $\mathbb{Z}$  by the symbols  $T_q$  ( $q$  a prime,  $q \nmid Np$ ),  $U_q$  ( $q|Np$ ),  $\langle t \rangle_N$  for  $t \in (\mathbb{Z}/N\mathbb{Z})^\times$ , and  $\langle a \rangle_p$ , for  $a \in \mathbb{F}_p^\times$ . We call  $\mathcal{H}$  the *abstract Hecke ring*, and the  $\langle a \rangle_p$  the  *$p$ -diamond operators*. All our Hecke rings will be quotients of  $\mathcal{H}$ . If  $M$  is a module over  $\mathcal{H}$  we denote by  $\mathcal{H}(M)$  the image of  $\mathcal{H}$  in  $\text{End}(M)$ , and we call it the Hecke ring *cut out by  $M$* . If  $M'$  is a submodule of  $M$ , stable under  $\mathcal{H}$ , both  $\mathcal{H}(M')$  and  $\mathcal{H}(M/M')$  are quotients of  $\mathcal{H}(M)$  in a natural way.

We shall write  $M^*$  for  $\text{Hom}(M, R)$ , and let  $T \in \mathcal{H}$  act on it via  $(T\mu)(m) = \mu(Tm)$  ( $m \in M$ ,  $\mu \in M^*$ ).

If  $M$  is a module over a ring  $R$ ,  $\mathfrak{a} \subset R$  is an ideal, and  $a \in \mathfrak{a}$ , we denote by  $M[a] = \text{Ker}(a|_M)$  the kernel of multiplication by  $a$ , and by  $M[\mathfrak{a}] = \bigcap_{a \in \mathfrak{a}} M[a]$ .

In many cases the  $\mathcal{H}$ -module  $M$  is killed by  $r$ . Then  $\mathcal{H}(M)$  becomes a  $\underline{\Lambda}$ -algebra, in which  $\mathbb{F}_p^\times$  act by the  $p$ -diamond operators.

## 1.2. Modular curves and Jacobians.

Let  $\mathfrak{H}$  denote the upper half plane, and  $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$  (with the usual topology). If  $\Gamma$  is a congruence group in  $SL_2(\mathbb{Z})$ , we let  $X_\Gamma$  denote the corresponding complex modular curve, isomorphic as a Riemann surface to  $\Gamma \backslash \mathfrak{H}^*$ . If  $\Gamma = \Gamma_0 = \Gamma_1(N, p) = \Gamma_1(N) \cap \Gamma_0(p)$  we shall denote by  $X_0 = X_1(N, p)$  the well-known model of  $X_\Gamma$  over  $\mathbb{Q}$ , whose non-cuspidal points parametrize elliptic curves with a point of order  $N$  and a cyclic subgroup of order  $p$ . If  $\Gamma = \Gamma_1 = \Gamma_1(Np)$  we shall likewise denote by  $X_1$  the model of  $X_\Gamma$  over  $\mathbb{Q}$  parametrizing elliptic curves with a point of order  $Np$ . The covering  $X_1 \rightarrow X_0$  is defined over  $\mathbb{Q}$ , unramified, and is *Galois* with Galois group  $\mathbb{F}_p^\times$ . Note that our assumption  $N \geq 4$  insures that  $\Gamma_0$  is torsion-free, so acts faithfully on  $\mathfrak{H}$ , and has no elliptic elements. We shall denote by  $Y_\Gamma$  the open modular curve, and by  $C_\Gamma = X_\Gamma - Y_\Gamma$  the (reduced, 0-dimensional) subscheme of the cusps. In the two examples mentioned above,  $Y_0$  and  $C_0$ , as well as  $Y_1$  and  $C_1$ , are defined over  $\mathbb{Q}$ .

We shall denote by  $J_\Gamma = \text{Pic}^0(X_\Gamma)$  the Jacobian variety of  $X_\Gamma$ , and by  $J_\Gamma^\#$  the *generalized* Jacobian with respect to the modulus  $C_\Gamma$ . It classifies isomorphism classes of line bundles  $\mathcal{L}$  of degree 0 on  $X_\Gamma$ , together with a trivialization of  $\mathcal{L}|_{C_\Gamma}$ . Alternatively, it classifies divisors of degree 0 on  $X_\Gamma$  supported away from the cusps, modulo principal divisors of functions which are constant ( $\neq 0, \infty$ ) along the cusps. Thus  $J_\Gamma^\#$  is an extension of  $J_\Gamma$  by a  $(\#C_\Gamma - 1)$ -dimensional torus:

$$(1.2) \quad 0 \rightarrow \text{Hom}(\mathbb{Z}[C_\Gamma]_0, \mathbb{G}_m) \rightarrow J_\Gamma^\# \rightarrow J_\Gamma \rightarrow 0,$$

where

$$(1.3) \quad \mathbb{Z}[C_\Gamma]_0 = \text{Ker}(\text{deg} : \mathbb{Z}[C_\Gamma] \rightarrow \mathbb{Z}).$$

A semi-abelian variety is, in general, not isogenous to the product of its toric part with an abelian variety. But in our case the *Manin-Drinfel'd theorem*, which asserts that divisors of degree 0 supported on  $C_\Gamma$  represent torsion points in  $J_\Gamma$ , precisely guarantees this. Indeed, applying the functor  $\text{Hom}(J_\Gamma, -)$  to the short exact sequence above we get an exact sequence

$$(1.4) \quad 0 \rightarrow \text{Hom}(J_\Gamma, J_\Gamma^\#) \rightarrow \text{Hom}(J_\Gamma, J_\Gamma) \rightarrow \text{Hom}(\mathbb{Z}[C_\Gamma]_0, J_\Gamma),$$

where we have used the principal polarization  $\text{Ext}^1(J_\Gamma, \mathbb{G}_m) = J_\Gamma$ . The identity map  $\text{id} \in \text{Hom}(J_\Gamma, J_\Gamma)$  gets mapped here to the *divisor class homomorphism*  $\delta \in \text{Hom}(\mathbb{Z}[C_\Gamma]_0, J_\Gamma)$ , which maps an element of  $\mathbb{Z}[C_\Gamma]_0$  to its divisor class in  $J_\Gamma$ . By the Manin-Drinfel'd theorem  $\delta$  is killed by some integer  $m$ , so  $m$  times the identity map of  $J_\Gamma$  lifts to a map  $J_\Gamma \rightarrow J_\Gamma^\#$ , proving that the extension (1.2) splits up to isogeny.

Following our example, we retain the symbols  $J_0, J_0^\#, J_1$  and  $J_1^\#$  for the models of the corresponding  $J_\Gamma$  and  $J_\Gamma^\#$  over  $\mathbb{Q}$ . We therefore have an exact sequence of group schemes over  $\mathbb{Q}$ :

$$(1.5) \quad 0 \rightarrow \text{Hom}(\mathbb{Z}[C_0]_0, \mathbb{G}_m) \rightarrow J_0^\# \rightarrow J_0 \rightarrow 0$$

(and similarly for  $J_1^\#$ ), where as before

$$(1.6) \quad \mathbb{Z}[C_0]_0 = \text{Ker}(\text{deg} : \mathbb{Z}[C_0] \rightarrow \mathbb{Z})$$

and  $\text{Hom}(\mathbb{Z}[C_0]_0, \mathbb{G}_m)$  is the torus over  $\mathbb{Q}$  whose character group is  $\mathbb{Z}[C_0]_0$  (with the Galois action arising from the action on the cusps).

As a rule, we regard Jacobians as Picard schemes, so that correspondences on  $X_\Gamma$  induce endomorphisms of  $J_\Gamma$  by Picard (*contravariant*) functoriality. In particular we have maps  $J_0 \rightarrow J_1$  and  $J_0^\# \rightarrow J_1^\#$  deduced from the covering  $(X_1, C_1) \rightarrow (X_0, C_0)$ , and these are defined over  $\mathbb{Q}$ .

Let  $J_{0,\text{old}}$  be the  $p$ -old subvariety<sup>1</sup> of  $J_0$ . It is the sum of the images of  $J_1(N)$  in  $J_0$ , under the maps induced from the two degeneracy maps  $X_0 \rightarrow X_1(N)$ . Let

$$(1.7) \quad J_0^{\text{new}} = J_0/J_{0,\text{old}}$$

be the  $p$ -new quotient of  $J_0$ . There is a unique abelian subvariety  $J_{0,\text{new}}$  of  $J_0$  which is complementary to  $J_{0,\text{old}}$  in the sense that their intersection is finite, and their sum is the whole  $J_0$ . We call it the  $p$ -new subvariety of  $J_0$ , and  $J_0^{\text{old}} = J_0/J_{0,\text{new}}$  we call the  $p$ -old quotient. From now on we omit the prefix  $p$ - when referring to these abelian varieties. The new/old subvarieties/quotients are defined over  $\mathbb{Q}$ , and the obvious maps  $J_{0,\text{new}} \rightarrow J_0^{\text{new}}$  and  $J_{0,\text{old}} \rightarrow J_0^{\text{old}}$  are isogenies.

### 1.3. The Hecke and diamond action on the modular curves and their Jacobians.

We denote by  $T_q$  ( $q \nmid Np$ ),  $U_q$  ( $q|Np$ ),  $\langle t \rangle_N$  and  $\langle a \rangle_p$  (the latter being trivial in the case of  $X_0$ ) the correspondences of  $X_0$  and  $X_1$  described, for example, in [MW], Chapter 2, Section 5. They induce endomorphisms of the Jacobians (or generalized Jacobians), defined over  $\mathbb{Q}$ , which we denote by the same letter. (Caution:  $T = T^*$  is the endomorphism defined by Picard functoriality, while [MW] choose to work with the endomorphisms defined by Albanese functoriality. One should take care of the  $U'_q$ s in particular. Not only are  $U_{q^*}$  and  $U_q^*$  different - they do not commute). These endomorphisms all commute with each other, so we get an embedding

$$(1.8) \quad \mathcal{H}(J_0) \hookrightarrow \text{End}(J_0/\mathbb{Q}),$$

and similarly in other circumstances. Needless to say, the action of the  $p$ -diamond operators on  $J_0$  or  $J_0^\#$  is trivial.

We shall have an occasion to use the following fact. Write  $N = N_l N'_l$  where  $N_l$  is the  $l$ -part of  $N$  and  $(l, N'_l) = 1$ . Let  $\mathcal{H}_l(J_0) = \mathcal{H}(J_0) \otimes \mathbb{Z}_l$ , a finite flat  $\mathbb{Z}_l$ -algebra, so that

$$\mathcal{H}'_l(J_0) = \text{End}(J_0) \otimes \mathbb{Z}_l \cap \mathcal{H}(J_0) \otimes \mathbb{Q}_l$$

---

<sup>1</sup> As the referee pointed out, it is more common to use superscripts for the subvarieties, and subscripts for the quotient varieties. We hope that the reader will bear with our convention.

is its *saturation* in the Endomorphism ring. The group  $\mathcal{H}'_l(J_0)/\mathcal{H}_l(J_0)$  is a finite  $l$ -group. To each maximal ideal  $\mathfrak{M}$  in the semi-local ring  $\mathcal{H}_l(J_0)$  there is associated an idempotent  $e_{\mathfrak{M}}$  in the ring, projecting onto its completion at  $\mathfrak{M}$ . If  $l$  divides  $N$  we call  $\mathfrak{M}$  *ordinary* if  $U_l$  does not belong to  $\mathfrak{M}$ .

PROPOSITION 1.1<sup>2</sup>. — *Let  $\mathfrak{M}$  be a maximal ideal of  $\mathcal{H}_l(J_0)$ . If either (i)  $N_l = 1$  or (ii)  $N_l = l$  and  $\mathfrak{M}$  is ordinary, then*

$$(1.9) \quad e_{\mathfrak{M}}\mathcal{H}_l(J_0) = e_{\mathfrak{M}}\mathcal{H}'_l(J_0).$$

*Proof.* — We follow [M], Proposition (9.5) on p. 95. The elements of

$$\mathcal{H}'(J_0) = \text{End}(J_0) \cap \mathcal{H}(J_0) \otimes \mathbb{Q}$$

act on the Néron model of  $J_0$  over  $R = \mathbb{Z}_l[\zeta_N]$ , hence on its identity component, which is  $\text{Pic}_{\mathfrak{X}/R}^0$  by Raynaud's theorem ([BLR], 9.5/4), hence  $\mathcal{H}'_l(J_0)$  acts on its Lie algebra, which is  $H^1(\mathfrak{X}, \mathcal{O})$  ([BLR], 8.4/1). Here by  $\mathfrak{X}/R$  we mean the proper and flat scheme which is the compactification of the scheme representing the moduli problem ( $[\text{bal.}\Gamma_1(N)]^{\text{can}}, \Gamma_0(p)$ ) over  $R$ . Its special fiber is *reduced*, smooth if  $N_l = 1$ , and, if  $N_l = l$ , consists of 2 smooth complete curves intersecting transversally at the supersingular points. See [KM], Chapter 3, for the terminology and for the precise structure of this moduli problem. Note that we chose to work with  $[\text{bal.}\Gamma_1(N)]^{\text{can}}$  over  $R$  and not with  $[\Gamma_1(N)]$  over  $\mathbb{Z}_l$ . The moduli scheme for the second moduli problem has a non-reduced special fiber. (See the discussion of the fiber at  $p$  in Section 2.5 below.) In particular the special fiber is a local complete intersection, hence Gorenstein and Cohen-Macaulay. We let  $\Omega$  be the sheaf of regular differentials on  $\mathfrak{X}/R$ . See [M], Section 3, p. 67, and [DeRap]. Since  $\mathfrak{X}/R$  is Gorenstein, this sheaf is invertible, and in fact is given by the recipe of [DeRap], Section 2.3. As in [M], Section 3, one proves that  $H^1(\mathfrak{X}, \mathcal{O})$  and  $H^0(\mathfrak{X}, \Omega)$  are free over  $R$  and *dual to each other* as  $R$ -modules. Their formation commutes with passing to the special fiber. For these facts it is important to know that the special fiber of  $\mathfrak{X}$  is reduced, because the proof of Lemma 3.3 in [M] uses the fact that the global functions on the special fiber are just the constants from the residue field of  $R$ .

Let  $\mathfrak{M}$  be any maximal ideal of  $\mathcal{H}_l(J_0)$ , and let  $e_{\mathfrak{M}}$  be the corresponding idempotent. We claim that  $e_{\mathfrak{M}}H^1(\mathfrak{X}, \mathcal{O})$  is free of rank 1 over  $e_{\mathfrak{M}}\mathcal{H}_l(J_0)$ .

---

<sup>2</sup> We would like to thank K. Ribet and A. Agashe for help with the proof of this proposition.

By Nakayama's lemma, it is enough to show that  $H^1(\mathfrak{X}, \mathcal{O})/\mathfrak{M}H^1(\mathfrak{X}, \mathcal{O})$  is a 1-dimensional vector space over  $\mathcal{H}_l(J_0)/\mathfrak{M}\mathcal{H}_l(J_0)$ . Dualizing, we deduce the desired result from the fact that  $H^0(\mathfrak{X}, \Omega)[\mathfrak{M}]$  is one-dimensional (see [W], Lemma 2.2). This "multiplicity one" result is a consequence of the  $q$ -expansion principle, and it is here that we need the assumption that  $\mathfrak{M}$  is ordinary, if  $l|N$ .

Write  $\mathbf{T}' = e_{\mathfrak{M}}\mathcal{H}'_l(J_0)$  and  $\mathbf{T} = e_{\mathfrak{M}}\mathcal{H}_l(J_0)$ . Then

$$\mathbf{T} \subset \mathbf{T}' \hookrightarrow \text{End}_{\mathbf{T}}(e_{\mathfrak{M}}H^1(\mathfrak{X}, \mathcal{O})) = \mathbf{T},$$

so  $\mathbf{T} = \mathbf{T}'$ , and the proposition is proved.  $\square$

#### 1.4. The Atkin-Lehner "involutions".

There are important automorphisms of the curves  $X_0$  and  $X_1$ , denoted by  $w_M = w_{M, \zeta}$ , for every  $M|Np$  such that  $(M, Np/M) = 1$ , which are defined over  $\mathbb{Q}(\zeta_M)$ , and which are involutions up to diamond operators. Their definition depends on a choice of a root of unity. Fix an  $Np$ -th root of unity  $\zeta = \zeta_{Np}$  and let  $\zeta_M = \zeta_{Np}^{Np/M}$  for  $M|Np$ . We shall define  $w_{M, \zeta}$  on  $X_1$ . The definition then descends to  $X_0$ . The curve  $X_1$  is a fine moduli space over  $\mathbb{Q}$  for (isomorphism classes of) triples  $(E, P_M, P_{Np/M})$  where  $E$  is an elliptic curve and  $P_M$  (resp.  $P_{Np/M}$ ) is a point of exact order  $M$  (resp.  $Np/M$ ) on  $E$ . We denote the isomorphism class of a triple  $(-)$  by  $[-]$ . Let

$$(1.10) \quad w_{M, \zeta}([E, P_M, P_{Np/M}]) = [E', P'_M, P'_{Np/M}],$$

where  $E' = E/\langle P_M \rangle$ ,  $P'_M = \tilde{P}_M \bmod \langle P_M \rangle$ , with  $\tilde{P}_M$  any point of exact order  $M$ , such that for the Weil  $e_M^E$ -pairing on  $E[M]$ ,

$$(1.11) \quad e_M^E(P_M, \tilde{P}_M) = \zeta_M,$$

and  $P'_{Np/M} = P_{Np/M} \bmod \langle P_M \rangle$ . We denote by  $w_{M, \zeta}$  also the corresponding automorphisms of  $J_0$ ,  $J_0^\#$ ,  $J_1$  and  $J_1^\#$ , induced by Picard functoriality.

The following relations between the  $w$ -operators and the Hecke operators, and between the  $w$ -operators and themselves (as endomorphisms of the Jacobians, or generalized Jacobians) hold. We shall be interested in  $w_N$ ,  $w_p$ , and  $w_{Np}$ . The corresponding relations in the ring of correspondences on the curve are obtained by reversing the order of composition. All are easy to check from the modular interpretation. See also [ALi].

- (1)  $w_{Np} = \langle N \rangle_p^{-1} \circ w_N \circ w_p = \langle p \rangle_N^{-1} \circ w_p \circ w_N$ .

- (2)  $w_{Np}^2 = 1$ ,  $w_p^2 = \langle -p \rangle_N$ ,  $w_N^2 = \langle -N \rangle_p$ .
- (3)  $\langle a \rangle_p \circ w_p = w_p \circ \langle a \rangle_p^{-1}$ ,  $\langle a \rangle_p \circ w_N = w_N \circ \langle a \rangle_p$ .
- (4)  $T_q \circ w_p = \langle q \rangle_p \circ w_p \circ T_q$ ,  $T_q \circ w_N = \langle q \rangle_N \circ w_N \circ T_q$  (for  $q \nmid Np$ ),
- (5)  $U_q \circ w_p = \langle q \rangle_p \circ w_p \circ U_q$  (for  $q|N$ ),  $U_p \circ w_N = \langle p \rangle_N \circ w_N \circ U_p$ .

Note that there is no simple relation between  $w_p$  and  $U_p$ . The two do not commute, not even up to diamond operators.

### 1.5. The action on the new part.

Over  $\mathbb{C}$  we may identify  $S_2(\Gamma)$  with the cotangent space to  $J_\Gamma$ , and since we are in characteristic 0, the Hecke ring cut out by  $J_\Gamma$  is the same as the one cut out by the space of cuspforms. The decomposition

$$(1.12) \quad S_2(\Gamma_0) = S_2^{\text{old}}(\Gamma_0) \oplus S_2^{\text{new}}(\Gamma_0)$$

is stable under  $\mathcal{H}$ . It follows that the same is true about the old/new subvariety/quotient of  $J_0$ , and that

$$(1.13) \quad \mathcal{H}(J_{0,\text{new}}) = \mathcal{H}(J_0^{\text{new}}) = \mathcal{H}(S_2^{\text{new}}(\Gamma_0)).$$

It is known, and easily checked from the modular interpretation, that  $U_p + w_p$  sends  $J_0$  into the old subvariety. It follows that on  $J_{0,\text{new}}$  we have the relation:

- (6)  $U_p = -w_p$ .

It is also known that  $J_{0,\text{new}}$  is stable under all the Atkin-Lehner involutions. See [ALe], Theorem 3.

### 1.6. Eisenstein modules.

The exact sequence

$$(1.14) \quad 0 \rightarrow \text{Hom}(\mathbb{Z}[C_1]_0, \mathbb{G}_m) \rightarrow J_1^\# \rightarrow J_1 \rightarrow 0$$

induces a surjection  $\mathcal{H}(J_1^\#) \rightarrow \mathcal{H}(\text{Hom}(\mathbb{Z}[C_1]_0, \mathbb{G}_m))$ . Let  $\mathfrak{I}$  be its kernel, i.e. the annihilator of  $\mathbb{Z}[C_1]_0$ . It is called *the Eisenstein ideal*. Any prime ideal of  $\mathcal{H}(J_1^\#)$  containing  $\mathfrak{I}$  (i.e. in the support of  $\mathcal{H}(J_1^\#)/\mathfrak{I}$ ) is called an *Eisenstein prime*, and any prime ideal containing  $(l, \mathfrak{I})$  is called an *l-Eisenstein prime*. If  $h$  is any quotient of  $\mathcal{H}(J_1^\#)$  we call  $h$  *purely Eisenstein* if its support (as an  $\mathcal{H}(J_1^\#)$ -module) consists of Eisenstein primes, and

*non-Eisenstein* if its support is disjoint from the Eisenstein primes. Finally if  $Z$  is a Hecke-stable subquotient of  $J_1^\#$  we call  $Z$  purely Eisenstein or non-Eisenstein if  $\mathcal{H}(Z)$  is. Thus  $Z$  is non-Eisenstein precisely when the image of  $\mathfrak{J}$  under the map  $\mathcal{H}(J_1^\#) \rightarrow \mathcal{H}(Z)$  is the full  $\mathcal{H}(Z)$ .

A Hecke-stable subvariety  $A$  of  $J_1$  is called *non  $l$ -Eisenstein* if  $A[l^n]$  is non Eisenstein for all  $n$ , or, equivalently, if the support of  $\mathcal{H}(A)$  contains no  $l$ -Eisenstein primes. Suppose that this is the case, and consider  $\mathcal{H}(J_1^\#[l^n])$ , which is an Artinian semi-local ring. Let  $e$  be the idempotent in this ring which is the projection onto the non-Eisenstein components. Then  $e$  kills  $\text{Hom}(\mathbb{Z}[C_1]_0, \mu_{l^n})$  and therefore  $eJ_1^\#[l^n] = eJ_1[l^n]$ . On the other hand  $eA[l^n] = A[l^n]$ , and we may view  $A[l^n]$  as a submodule of  $J_1^\#[l^n]$ , even without having to find a map from  $A$  to  $J_1^\#$ .

Suppose that  $A$  (assumed now non  $l$ -Eisenstein), has a Hecke-stable subvariety  $A^\perp$  complementary to it, in the sense that their intersection is finite, while their sum is the whole  $J_1$ . Assume, in addition, that this  $A^\perp$  satisfies

$$(1.15) \quad A[l^n] \cap A^\perp[l^n] = (0).$$

Then for  $A'$ , the pre-image of  $A^\perp$  in  $J_1^\#$ , we have the same thing:  $A[l^n] \cap A'[l^n] = (0)$  (apply  $e$  to prove it!), and therefore

$$(1.16) \quad J_1^\#[l^n] = A[l^n] \oplus A'[l^n],$$

a direct sum which is of course Hecke and Galois-stable. These remarks will be used in Chapter 3.

### 1.7. Néron models.

Quite generally, if  $F$  is a finite extension of  $\mathbb{Q}_p$  and  $A$  an abelian variety over  $F$ , we denote by  $\mathcal{N}(A, \mathcal{O}_F)$  the Néron model of  $A$  over  $\mathcal{O}_F$ , by  $\mathcal{N}(A, \mathcal{O}_F)^0$  its identity component, and by  $\Phi(A, \mathcal{O}_F) = \mathcal{N}(A, \mathcal{O}_F) / \mathcal{N}(A, \mathcal{O}_F)^0$  the group of connected components, which is a finite étale group over the residue field  $\kappa(F)$  of  $F$ . We shall need the following result.

PROPOSITION 1.2. — *For any  $F$ , the group  $\Phi(J_0, \mathcal{O}_F)$  is new. More precisely, the exact sequence of abelian varieties over  $F$ :*

$$(1.17) \quad 0 \rightarrow J_{0,\text{old}} \rightarrow J_0 \rightarrow J_0^{\text{new}} \rightarrow 0$$

induces by functoriality maps between Néron models, hence maps between groups of connected components, and the map

$$(1.18) \quad \Phi(J_0, \mathcal{O}_F) \rightarrow \Phi(J_0^{\text{new}}, \mathcal{O}_F)$$

is an embedding.

*Proof.* — Since  $\mathbb{Z}_p$  is absolutely unramified and  $J_0$  has semi-stable reduction, Theorem 7.5/4(ii) of [BLR] gives the exactness of Néron models over  $\mathbb{Z}_p$  (we denote here, for brevity  $\mathcal{J}_0 = \mathcal{N}(J_0, \mathbb{Z}_p)$  etc.)

$$(1.19) \quad 0 \rightarrow \mathcal{J}_{0,\text{old}} \rightarrow \mathcal{J}_0 \rightarrow \mathcal{J}_0^{\text{new}}$$

and therefore the exactness of

$$(1.20) \quad 0 \rightarrow \mathcal{J}_{0,\text{old}} \rightarrow (\mathcal{J}_0)^0 \rightarrow (\mathcal{J}_0^{\text{new}})^0 \rightarrow 0.$$

Since for abelian varieties with semi-stable reduction the formation of the identity component of the Néron model commutes with finite flat base change ([BLR], 7.4/4), the sequence

$$(1.21) \quad 0 \rightarrow \mathcal{N}(J_{0,\text{old}}, \mathcal{O}_F) \rightarrow \mathcal{N}(J_0, \mathcal{O}_F)^0 \rightarrow \mathcal{N}(J_0^{\text{new}}, \mathcal{O}_F)^0 \rightarrow 0$$

is exact. Consider the diagram

$$(1.22) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathcal{N}(J_{0,\text{old}}, \mathcal{O}_F) & \rightarrow & \mathcal{N}(J_0, \mathcal{O}_F)^0 & \rightarrow & \mathcal{N}(J_0^{\text{new}}, \mathcal{O}_F)^0 \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{N}(J_{0,\text{old}}, \mathcal{O}_F) & \rightarrow & \mathcal{N}(J_0, \mathcal{O}_F) & \rightarrow & \mathcal{N}(J_0^{\text{new}}, \mathcal{O}_F). \end{array}$$

The vertical arrows are injective, hence so is the first arrow in the bottom row. Since  $J_0$  has semi-stable reduction over  $F$ , the *proof* of Theorem 7.5/4(ii) in [BLR] shows that the bottom row is exact too. (Note that the assumption (\*) there about the absolute index of ramification of  $F$  was necessary only to get (i), but not to deduce (ii) from (i)). The lemma now follows from the snake lemma. □

**COROLLARY 1.3.** — *On  $\Phi(J_0, \mathcal{O}_F)$  we have the relation  $U_p = -w_p$ .*

The generalized Jacobians that we shall meet are all semi-abelian varieties (extensions of abelian varieties by tori). As such they have Néron (lft-)models too, which are constructed for example in [BLR], Chapter 10. We shall denote the Néron lft-model of a semi-abelian variety  $A^\#$  by the same notation  $\mathcal{N}(A^\#, \mathcal{O}_F)$ .

### 1.8. Two algebraic lemmas.

LEMMA 1.4. — *Let  $W$  be an abelian group, and  $\alpha, \beta \in \text{End}(W)$  two endomorphisms such that  $\alpha + \beta = 1$ , and  $\alpha\beta$  is nilpotent. Then*

$$(1.23) \quad W = W_\alpha \oplus W_\beta$$

where  $W_\alpha$  is the subgroup of all the elements of  $W$  annihilated by some power of  $\alpha$ , and similarly for  $W_\beta$ .

*Proof.* — Elementary. □

LEMMA 1.5. — *Let  $\Lambda$  be a commutative ring,  $I$  a nilpotent ideal, and  $R = \Lambda/I$ . Let  $W$  be a  $\Lambda$ -module, and  $\bar{W} = W/IW$ . Let  $H \subset \text{End}_\Lambda(W)$  be a commutative subring. Suppose  $e_0 \in H$  is such that  $\bar{e}_0 \in \text{End}_R(\bar{W})$  is an idempotent. Then there exists a unique idempotent  $e \in H$  lifting  $\bar{e}_0$ , i.e.  $\bar{e} = \bar{e}_0$ .*

*Proof.* — For  $e, f \in \text{End}_\Lambda(W)$  write  $e \equiv f \pmod{I^m}$  if  $(e - f)W \subset I^m W$ . We shall define inductively a sequence  $e_m \in H$  with the property that

$$(1.24) \quad e_m^2 \equiv e_m \pmod{I^{m+1}}, \quad \text{and} \quad e_{m+1} \equiv e_m \pmod{I^{m+1}}.$$

Simply let  $e_{m+1} = e_m + (e_m^2 - e_m)(1 - 2e_m)$ . A straightforward computation yields

$$(1.25) \quad e_{m+1}^2 - e_{m+1} = (e_m^2 - e_m)^2(4e_m^2 - 4e_m - 3),$$

so the induction is carried on. For large enough  $m$ ,  $e = e_m$  is an idempotent.

To prove the uniqueness, suppose that  $e$  and  $e'$  are both idempotents lifting  $\bar{e}_0$ . Writing  $e' = e + j$ , and assuming  $j \equiv 0 \pmod{I^{m+1}}$ , we get from

$$e + j = (e + j)^2 = e + 2ej + j^2$$

that  $j(1 - 2e) = j^2$ , hence  $j = j^2(1 - 2e) \equiv 0 \pmod{I^{m+2}}$ , and so  $j = 0$ . □

## 2. The deformation modules.

In this section we introduce the main objects which we wish to study. They are finite groups built from the  $p - 1$  torsion in the (generalized)

Jacobians of  $X_0$  and  $X_1$ . It is clearly sufficient to treat their  $l$ -primary part for each  $l$  separately, and in doing so we shall exclude  $l = 2$ . Recall that  $r$  is the highest power of  $l$  dividing  $p - 1$ , and  $R = \mathbb{Z}/r\mathbb{Z}$ .

**2.1. The deformation.**

Write  $J_1^\#[r, I']$  for  $J_1^\#[r][I']$ .

PROPOSITION 2.1. — Consider the map  $J_0^\#[r] \rightarrow J_1^\#[r, I']$ , as a map of finite free  $R$ -modules. Then

- (i) This map is injective.
- (ii) It identifies  $J_0^\#[r]$  with  $J_1^\#[r, I'][I]$ .
- (iii) Consider the surjection

$$(2.1) \quad J_1^\#[r, I']^* \rightarrow J_0^\#[r]^*$$

where  $(-)^* = \text{Hom}(-, R)$ . Then  $J_0^\#[r]^*$  is identified with  $J_1^\#[r, I']^* \otimes_\Lambda \Lambda/I$ .

- (iv) As a  $\Lambda$ -module  $J_1^\#[r, I']^*$  is isomorphic to a direct sum of a free  $\Lambda$ -module with one copy of  $\Lambda/I$ .

Proof. — (i) We have canonical identifications  $J_0^\#[r] = H_{\text{ét},c}^1(Y_0/\mathbb{Q}, \mu_r)$  and  $J_0^\#[r]^* = H_{\text{ét}}^1(Y_0/\mathbb{Q}, R)$ . Artin’s comparison theorem allows us to identify these with the singular cohomology of  $Y_0(\mathbb{C})$  (with or without compact supports, and with coefficients in the same groups). The latter can be computed as a group cohomology. We thus have

$$(2.2) \quad J_0^\#[r]^* = H_{\text{ét}}^1(Y_0/\mathbb{Q}, R) = H_{\text{sing}}^1(Y_0(\mathbb{C}), R) = H^1(\Gamma_0, R),$$

where we have taken into account the fact that  $\Gamma_0$  acts faithfully and freely on  $\mathfrak{H}$ .

Similarly we have

$$(2.3) \quad J_1^\#[r]^* = H_{\text{ét}}^1(Y_1/\mathbb{Q}, R) = H_{\text{sing}}^1(Y_1(\mathbb{C}), R) = H^1(\Gamma_1, R) = H^1(\Gamma_0, \underline{\Lambda}).$$

The last equality is a consequence of Shapiro’s lemma in group cohomology. Here  $\Gamma_0$  acts on  $\underline{\Lambda}$  via  $\Gamma_0/\Gamma_1 \simeq \mathbb{F}_p^\times$ . To fix ideas we choose the isomorphism sending a matrix  $\gamma \in \Gamma_0$  to  $a_\gamma \text{ mod } p$  (we could have chosen  $d_\gamma \text{ mod } p$  too).

It can be checked that the diamond action on  $J_1^\#[r]^*$  is translated to the diamond action on the coefficients in  $H^1(\Gamma_0, \underline{\Lambda})$ , so

$$(2.4) \quad J_1^\#[r, I']^* = H^1(\Gamma_0, \Lambda).$$

Next, the map  $J_1^\#[r, I']^* \rightarrow J_0^\#[r]^*$  is the map  $H^1(\Gamma_0, \Lambda) \rightarrow H^1(\Gamma_0, R)$  coming from the augmentation map  $\Lambda \rightarrow R$ . This map is surjective, since  $\Gamma_0$  is free.

Dualizing, we get (i). Assertions (ii) and (iii) are again dual to each other. Their proof can be found in [dS1], Proposition 1.3. (The assumption  $N = 1$  is not used there.)

Assertion (iv) was proved in [dS1], Proposition 2.8, Step 1, under the assumption that  $l \geq 5$ . Once again, the assumption  $N = 1$  was not used there in the proof, and the proof carries over to the case  $l = 3$ , since  $\Gamma_0$  has no 3-torsion.  $\square$

The injectivity in (i) is one place where it is advantageous to work with the generalized Jacobians. For the ordinary Jacobians, the map is not injective. Its kernel consists of the  $r$ -torsion in the Shimura subgroup.

*Notation.* — Write

$$(2.5) \quad V = J_0[r]^*, \quad V^\# = J_0^\#[r]^*$$

$$(2.6) \quad \mathbf{V} = J_1[r, I']^*, \quad \mathbf{V}^\# = J_1^\#[r, I']^*.$$

These are modules over  $\mathcal{H}$ , equipped with a commuting action of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ .

## 2.2. The local filtration on $V$ and $V^\#$ .

The restriction of  $V$  or  $V^\#$  to the decomposition group at  $p$  admits a 2-step filtration. It is defined geometrically, and its graded pieces are unramified. These facts are not new. At least in the case of  $V$ , and in the  $l$ -adic setting, they are used in [MW] and in papers of Wiles from the same period. For completeness we review them.

Recall that  $X_0$  has a model  $\mathfrak{X}_0$  over  $\mathbb{Z}_p$ , an open subset of which,  $\mathfrak{Y}_0$ , is the fine moduli space for the moduli problem  $([\Gamma_1(N)], [\Gamma_0(p)])$ . For every  $\mathbb{Z}_p$ -algebra  $T$ , the points of  $\mathfrak{Y}_0(T)$  are isomorphism classes  $[E, P_N, H]$  of triples consisting of an elliptic curve  $E$  over  $T$ , a point  $P_N \in E[N](T)$  “of exact order  $N$ ” (see [KM], p. 99), and a cyclic finite flat subgroup scheme  $H \subset E[p]$  of rank  $p$  (loc.cit. p. 100). The *cuspidal subscheme*  $\mathfrak{C}_0 = \mathfrak{X}_0 - \mathfrak{Y}_0$  is finite étale over  $\mathbb{Z}_p$  ([KM], Theorem 10.12.2).

The scheme  $\mathfrak{X}_0$  is flat and proper over  $\mathbb{Z}_p$ , and regular as a 2-dimensional scheme ([KM], Theorem 6.6.2, p. 167). Via the map “forget the  $\Gamma_0(p)$  structure”, it is finite and flat of rank  $p + 1$  over  $X_1(N)/\mathbb{Z}_p$ .

The special fiber  $\mathfrak{X}_{0/\mathbb{F}_p}$  is reduced and consists of 2 irreducible smooth components  $X_0^\mu$  and  $X_0^{\text{ét}}$ , which are the curves  $\mathfrak{M}(\Gamma_1(N), (1, 0))$  and  $\mathfrak{M}(\Gamma_1(N), (0, 1))$  of [KM], Proposition 13.4.4, p. 408. The map  $X_0^\mu \rightarrow X_1(N)_{/\mathbb{F}_p}$  is an isomorphism. The map  $X_0^{\text{ét}} \rightarrow X_1(N)_{/\mathbb{F}_p}$  is purely inseparable of degree  $p$ . Denoting by  $Y_0^\mu$  and  $Y_0^{\text{ét}}$  the intersections of  $X_0^\mu$  and  $X_0^{\text{ét}}$  with  $\mathfrak{Y}_0$ , for any  $\mathbb{F}_p$ -algebra  $T$ , points of  $Y_0^\mu(T)$  are isomorphism classes of triples  $[E, P_N, \text{Ker } F]$ , where  $E$  is an elliptic curve over  $T$ ,  $P_N \in E[N](T)$  is a point of exact order  $N$ , and  $F$  is the Frobenius homomorphism  $F : E \rightarrow E^{(p)}$  over  $T$ . The curve  $Y_0^{\text{ét}}$  classifies isomorphism classes of triples  $[E, P_N, H]$ , where  $E$  and  $P_N$  are as before, and  $H$  is a cyclic finite flat subgroup scheme of rank  $p$  such that if  $\pi : E \rightarrow E/H$  is the canonical isogeny,  $E \simeq (E/H)^{(p)}$ , and  $\text{Ker}({}^t\pi) = \text{Ker } F$ . Over a perfect base we have then  $H = \text{Ker}(V : E \rightarrow E^{(p^{-1})})$ , where  $V$  is the Verschiebung homomorphism.

The two components of the special fiber intersect transversally at the set  $S \subset X_1(N)(\overline{\mathbb{F}}_p)$  of the super-singular points. The supersingular point  $[E, P_N, \text{Ker } F] \in X_0^\mu(\overline{\mathbb{F}}_p)$  is glued to  $[E, P_N, \text{Ker } V] \in X_0^{\text{ét}}(\overline{\mathbb{F}}_p)$  because for a supersingular curve  $\text{Ker } F = \text{Ker } V$  ([KM], Theorem 13.4.7).

If  $L$  is a finite extension of  $\mathbb{Q}_p$ , and  $x = [E, P_N, H] \in \mathfrak{Y}_0(\mathcal{O}_L)$  is a point with ordinary reduction, then  $x$  meets the special fiber in  $X_0^\mu$  if and only if the reduction of  $H$  is connected, and it meets the special fiber in  $X_0^{\text{ét}}$  if and only if the reduction of  $H$  is étale.

The cusps  $C_0$  reduce injectively in the special fiber (since the cuspidal scheme is étale) and break up as the union of  $C_0^\mu$  and  $C_0^{\text{ét}}$ , the cusps in  $X_0^\mu$  and  $X_0^{\text{ét}}$ .

The automorphism  $w_p$  extends to  $\mathfrak{X}_0$  and interchanges the two components of the special fiber. On the set  $S$  it has the same affect as the absolute Frobenius  $\sigma \in \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ . Whenever necessary we shall identify  $X_0^{\text{ét}}$  with  $X_1(N)$  using  $w_p$ , followed by the isomorphism  $X_0^\mu \simeq X_1(N)$  obtained from the map “forget level  $p$ ”.

Let  $M = \mathbb{Z}[S]$  be the free abelian group on the set  $S$  of super-singular points in  $X_1(N)(\overline{\mathbb{F}}_p)$ . Let  $M_0$  the kernel of the degree homomorphism from  $M$  to  $\mathbb{Z}$ . Let  $\text{Hom}(M_0, \mathbb{G}_m)$  denote the (non-split) torus over  $\mathbb{F}_p$  whose character group is  $M_0$  (with the Galois action coming from its action on  $S$ ).<sup>3</sup>

---

<sup>3</sup> A more canonical description of  $M_0$  is as the dual graph of  $\mathfrak{X}_{0/\mathbb{F}_p}$ . Thus, the action of  $w_p$  on  $M_0$  is via  $s \mapsto -\sigma(s)$  ( $s \in S$ ), although on  $S$  it acts like  $\sigma$ , because  $w_p$  interchanges the two components of the special fiber. One can check that this action

The semi-abelian variety  $J_0^\#$  has a Néron (lft-)model  $\mathcal{J}_0^\#$  over  $\mathbb{Z}_p$  (see [BLR], 10.2/2). Similarly  $J_0$  has a Néron model  $\mathcal{J}_0$  over  $\mathbb{Z}_p$ . The following proposition describes the structure of the connected components of the special fibers of these Néron models.

PROPOSITION 2.2. — (i) *Let  $\mathcal{T}$  be the (lft-)Néron model of the torus  $\mathrm{Hom}(\mathbb{Z}[C_0]_0, \mathbb{G}_m)$ . Then the sequence*

$$(2.7) \quad 0 \rightarrow \mathcal{T}^0 \rightarrow \mathcal{J}_0^{\#0} \rightarrow \mathcal{J}_0^0 \rightarrow 0$$

is an exact sequence of group schemes over  $\mathbb{Z}_p$ .

(ii) *There are exact sequences of group schemes over  $\mathbb{F}_p$ :*

$$(2.8) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathrm{Hom}(M, \mathbb{G}_m) & \rightarrow & \mathcal{J}_{0/\mathbb{F}_p}^{\#0} & \rightarrow & J_0^{\mathrm{ét}\#} \times J_0^{\mu\#} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathrm{Hom}(M_0, \mathbb{G}_m) & \rightarrow & \mathcal{J}_{0/\mathbb{F}_p}^0 & \rightarrow & J_0^{\mathrm{ét}} \times J_0^\mu \rightarrow 0 \end{array}$$

where  $J_0^\mu$  (resp.  $J_0^{\mathrm{ét}}$ ) is the Jacobian of  $X_0^\mu$  (resp.  $X_0^{\mathrm{ét}}$ ), and similarly  $J_0^{\mu\#}$  (resp.  $J_0^{\mathrm{ét}\#}$ ) is the generalized Jacobian with respect to the cusps  $C_0^\mu$  (resp.  $C_0^{\mathrm{ét}}$ ).

*Proof.* — (i) Since  $\mathfrak{C}_0$  is an étale scheme over  $\mathbb{Z}_p$ , the torus  $\mathrm{Hom}(\mathbb{Z}[C_0]_0, \mathbb{G}_m)$  is split over  $\mathbb{Q}_p^{nr}$ . It follows that the connected component of the special fiber of its Néron (lft-)model is simply  $\mathrm{Hom}(\mathbb{Z}[C_0]_0, \mathbb{G}_m/\mathbb{F}_p)$ , and to prove the exactness of the sequence in (i) we may pass to an unramified extension, and assume therefore that our torus is split. In this case the claim follows from the proof of Proposition 7 in [BLR], Section 10.1.

(ii) According to Raynaud's theorem, and its generalization to the generalized Jacobian with respect to  $C_0$  (which is valid since  $C_0$  extends to an étale subscheme over  $\mathbb{Z}_p$ , see [dS3]),  $\mathcal{J}_{0/\mathbb{F}_p}^0$  (resp.  $\mathcal{J}_{0/\mathbb{F}_p}^{\#0}$ ) represents  $\mathrm{Pic}_{\mathfrak{x}_0/\mathbb{F}_p}^0$  (resp. the identity component of the relative Picard functor with trivialization along  $C_0$ ). Any (trivialized) line bundle whose class belongs to  $\mathrm{Pic}_{\mathfrak{x}_0/\mathbb{F}_p}^0$  gives upon restriction to the two components points in the corresponding (generalized) Jacobians. The kernel of this map consists of line bundles which are trivial on each of the two components. Choosing trivializations and comparing them on  $S$  we get the homomorphism from  $M$  to  $\mathbb{G}_m$ . If the trivializations are not given, they can be modified by a

---

is compatible with the inclusion of  $\mathrm{Hom}(M_0, \mathbb{G}_m)$  in the connected component of the special fiber of the Néron model of  $J_0$  over  $\mathbb{Z}_p$  (see Proposition 2.2 below), and with the action of  $w_p$  induced on that connected component from the (Picard) action on  $J_0$ .

scalar factor on each component, and the resulting homomorphism is well-defined only on  $M_0$ . This establishes the exactness of the two rows in (ii), and the commutativity of the diagram is clear.  $\square$

From the proposition we obtain the following exact sequences:

$$(2.9) \quad \begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}(M, \mu_r) & \rightarrow & \mathcal{J}_0^{\#0}(\bar{\mathbb{F}}_p)[r] & \rightarrow & J_0^{\text{ét}\#}[r] \times J_0^{\mu\#}[r] \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \text{Hom}(M_0, \mu_r) & \rightarrow & \mathcal{J}_0^0(\bar{\mathbb{F}}_p)[r] & \rightarrow & J_0^{\text{ét}}[r] \times J_0^\mu[r] \rightarrow 0. \end{array}$$

Let us also remark that the two arrows in

$$(2.10) \quad J_0(\mathbb{Q}_p^{nr})[r] \longleftarrow \mathcal{J}_0(\mathbb{Z}_p^{nr})[r] \supset J_0^0(\mathbb{Z}_p^{nr})[r] \rightarrow \mathcal{J}_0^0(\bar{\mathbb{F}}_p)[r]$$

are isomorphisms (and similarly for the generalized Jacobians). For the one on the left this follows from the universal property of Néron models. For the one on the right note that  $\mathcal{J}_0^0[r]$  is an étale group scheme, because  $r$  is relatively prime to  $p$ .

DEFINITION 2.1. — Let  $J_0[r]^{\text{sub}}$  (resp.  $J_0^\#[r]^{\text{sub}}$ ) denote the subgroup of  $\mathcal{J}_0^0(\mathbb{Z}_p^{nr})[r]$  (resp.  $\mathcal{J}_0^{\#0}(\mathbb{Z}_p^{nr})[r]$ ) consisting of the elements which map to 0 in  $J_0^\mu$  (resp. in  $J_0^{\mu\#}$ ). Let  $J_0[r]^{\text{quot}} = J_0[r]/J_0[r]^{\text{sub}}$  and similarly for the generalized Jacobians. Dualizing, we obtain exact sequences

$$(2.11) \quad \begin{array}{ccccccc} 0 & \rightarrow & U & \rightarrow & V & \rightarrow & W \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & U^\# & \rightarrow & V^\# & \rightarrow & W^\# \rightarrow 0 \end{array}$$

where  $U = (J_0[r]^{\text{quot}})^*$ ,  $W = (J_0[r]^{\text{sub}})^*$ , and similarly for the generalized Jacobians.

The vertical arrows are all injections. This is because  $J_0^\#[r]^{\text{sub}}$  maps surjectively to  $J_0[r]^{\text{sub}}$ . The cokernels of the vertical maps fit into the exact sequence

$$(2.12) \quad 0 \rightarrow \mathbb{Z}[C_0^\mu]_0 \otimes \mu_r^{-1} \rightarrow \mathbb{Z}[C_0]_0 \otimes \mu_r^{-1} \rightarrow \mathbb{Z}[C_0^{\text{ét}}] \otimes \mu_r^{-1} \rightarrow 0.$$

The modules  $U, V, W, U^\#, V^\#$  and  $W^\#$  are stable under the local Galois group at  $p$ . We shall now see that they are also *stable under  $\mathcal{H}$* . We shall check it for the operator  $U_p$ . Similar (and easier) computations apply to the other Hecke operators. Let  $x = [E, P_N, H] \in \mathfrak{Y}_0(\mathcal{O}_L)$  for some finite extension  $L$  of  $\mathbb{Q}_p$ . Then in  $\text{Div}(\mathfrak{Y}_0(\mathcal{O}_L))$  we have

$$(2.13) \quad U_p^*(x) = \sum [E_i, P_{N,i}, H_i]$$

where (enlarging  $L$  if necessary) each  $x_i = (E_i, P_{N,i}, H_i)$  is defined over  $\mathcal{O}_L$  and there is a cyclic isogeny  $\lambda_i$  of degree  $p$  defined over  $\mathcal{O}_L$  carrying  $(E_i, P_{N,i}, H_i)$  to  $(E, P_N, H)$ . In particular, the scheme theoretic intersection of  $H_i$  and  $\text{Ker}\lambda_i$  is 0, and  $H = \lambda_i(H_i) = \lambda_i(E_i[p])$ . If the reduction  $\bar{x}$  of  $x$  is ordinary, and  $\bar{x}$  belongs to  $X_0^{\text{ét}}(\overline{\mathbb{F}}_p)$ , then each  $H_i$  must have an étale reduction, and so  $\bar{x}_i$  belongs to  $X_0^{\text{ét}}(\overline{\mathbb{F}}_p)$ . On the other hand if  $\bar{x}$  belongs to  $S$ , clearly every  $\bar{x}_i$  belongs to  $S$ . It follows that if

$$\delta = \sum n_i x_i \in \text{Div}^0(\mathfrak{Y}_0(\mathcal{O}_L))$$

is a divisor of degree 0 supported away from the cuspidal scheme, whose reduction is supported on  $X_0^{\text{ét}}$ , then so is  $U_p^* \delta$ . Any point in  $J_0[r]^{\text{sub}}$  is the class of (the generic fiber of) a divisor of this form, and vice versa. Thus  $U_p = U_p^*$  preserves  $J_0[r]^{\text{sub}}$ . (Note that if we used the Albanese functoriality we should have defined  $J_0[r]^{\text{sub}}$  using  $J_0^\mu$  instead of  $J_0^{\text{ét}}$ .) The proof for the generalized Jacobian is identical. Compare the arguments here with [MW], chapter 2, Section 9, in particular Propositions 2 and 3 there.

### 2.3. The local filtration on $V$ and $V^\#$ .

At the  $\Gamma_1$  level the geometry is a little more complicated, and one has to go to  $\mathbb{Q}_p(\zeta_p)$ , or at least to its subfield  $K$  of degree  $r$  over  $\mathbb{Q}_p$ , to get a good understanding of the picture. It turns out that there is a similar filtration there, which is again stable under the local Galois group and Hecke. It is even stable under the full  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ , but the graded pieces are not unramified anymore. One of them inherits a “geometric inertia action” of  $\text{Gal}(K^{nr}/\mathbb{Q}_p^{nr})$  via the  $\mu_r$ -diamond operators.

It is well-known that  $J_1$  acquires semi-stable reduction over  $\mathbb{Q}_p(\zeta_p)$ . However, we want to be able to work over a field of absolute index of ramification strictly less than  $p - 1$ . This is possible because  $J_1[r, I']$  is in fact the  $r$ -torsion of an abelian variety which acquires semi-stable reduction over the field  $K$  mentioned above. We now explain this point.

Let  $X_1 \rightarrow X_2 \rightarrow X_0$  be the intermediate cover (defined over  $\mathbb{Q}$ ) with  $\text{Gal}(X_2/X_0)$  cyclic of order  $r$ . Let  $J_2$  be its Jacobian variety. In

$$(2.14) \quad J_0 \rightarrow J_2 \rightarrow J_1$$

the kernel of  $J_2 \rightarrow J_1$  is a  $\mu$ -type subgroup, of order  $(p - 1)/r$ . Since  $r$  and  $(p - 1)/r$  are relatively prime,  $J_2[r] \subset J_1[r]$ , and in fact we claim that

$$(2.15) \quad J_2[r] = J_1[r, I'],$$

so  $\mathbf{V} = J_2[r]^*$ . Indeed, quite generally, if  $X' \rightarrow X$  is an unramified cyclic covering of curves over an algebraically closed field  $k$  of characteristic 0, or at least relatively prime to the order of  $\text{Gal}(X'/X) = \Gamma$ , and  $J$  and  $J'$  are their Jacobians, we have an exact sequence

$$(2.16) \quad 0 \rightarrow \text{Hom}(\Gamma, k^\times) \rightarrow J(k) \rightarrow J'(k)^\Gamma \rightarrow 0.$$

This is proved using Galois cohomology of cyclic groups.

LEMMA 2.3. — *The abelian variety  $J_2$  admits semi-stable reduction over the unique subfield  $K$  of  $\mathbb{Q}_p(\zeta_p)$  of degree  $r$  over  $\mathbb{Q}_p$ .*

*Proof.* — Let  $q$  be any prime not dividing  $p(p-1)/r$  (for example, we could take  $q = l$ ). Then by the discussion above we have for the rational Tate modules

$$(2.17) \quad V_q J_2 = V_q J_1[I'].$$

Let  $L = \mathbb{Q}_p(\zeta_p)$  and let  $\mathcal{I}_L$  be the inertia subgroup of  $\text{Gal}(\bar{L}/L)$ . Then since  $J_1$  acquires semistable reduction over  $L$ ,  $V = V_q J_1$  has a subspace  $W = V^{\mathcal{I}_L}$  such that  $\mathcal{I}_L = \text{Gal}(\bar{L}/L^{nr})$  acts trivially on  $W$  and  $V/W$ . Since  $L$  is normal over  $\mathbb{Q}_p$ ,  $W$  is stable under the full inertia group at  $p$ , and it can be computed that the action of  $\text{Gal}(\bar{L}/\mathbb{Q}_p^{nr})$  on  $W$  and on  $V/W$  factors through the diamond operators, i.e. through the canonical isomorphism of  $\text{Gal}(L^{nr}/\mathbb{Q}_p^{nr}) \simeq \text{Gal}(L/\mathbb{Q}_p) \simeq \mathbb{F}_p^\times$  with the group of diamond operators. It follows that  $\text{Gal}(\bar{K}/K^{nr})$  acts trivially on the subspace  $W[I'] \subset V[I'] = V_q J_2$  and on the quotient  $V_q J_2/W[I']$ , hence by the criterion for semi-stable reduction ([BLR], 7.4/6),  $J_2$  acquires semi-stable reduction over  $K$ .  $\square$

Alternatively, the above lemma follows from the semi-stable, regular model of  $X_2$  over  $\mathcal{O}_K$  which we describe ahead, and Raynaud's theorem.

Recall that for any finite extension  $F$  of  $\mathbb{Q}_p$  and any abelian variety  $A$  over  $F$  we denote by  $\mathcal{N}(A, \mathcal{O}_F)$  the Néron model of  $A$  over  $\mathcal{O}_F$ . Thus, for example,  $\mathcal{J}_0 = \mathcal{N}(J_0, \mathbb{Z}_p)$ . Like in the case of  $X_0$ , the curve  $X_{2/K}$  has a regular, flat and proper model  $\mathfrak{X}_{2/\mathcal{O}_K}$ , the special fiber of which is reduced and consists of two smooth irreducible components  $X_{2/\mathbb{F}_p}^{\text{ét}}$  and  $X_{2/\mathbb{F}_p}^\mu$  (*Igusa curves*) which intersect transversally at the supersingular points  $S$ . In the language of [KM],  $\mathfrak{X}_2$  contains an open set  $\mathfrak{Y}_2$  which is the fine moduli space for the moduli problem  $([\Gamma_1(N)], ([\text{bal.}\Gamma_1(p)]/\mu_{r'} \times \mu_{r'})^{\text{can}})$  (see [KM], Theorem 10.12.2, p. 326 and the remark following it). The scheme of cusps  $\mathfrak{C}_2 = \mathfrak{X}_2 - \mathfrak{Y}_2$  is finite étale over  $\mathcal{O}_K$ .

The map  $X_2 \rightarrow X_0$ , which is Galois unramified of degree  $r$ , extends over  $\mathcal{O}_K$  to a map  $\mathfrak{X}_2 \rightarrow \mathfrak{X}_0/\mathcal{O}_K$ , which on the special fiber induces maps  $X_2^{\acute{e}t} \rightarrow X_0^{\acute{e}t}$  and  $X_2^\mu \rightarrow X_0^\mu$ . These two maps are totally ramified over the super-singular points, but are étale over the ordinary locus. They are interchanged by the “involution”  $w_p$ .

Let  $J_2^{\acute{e}t}$  and  $J_2^\mu$  be the Jacobians of the curves  $X_{2/\mathbb{F}_p}^{\acute{e}t}$  and  $X_{2/\mathbb{F}_p}^\mu$ . Then by Raynaud’s theorem we have for any field  $F$  containing  $K$  a diagram of group schemes over the residue field  $\kappa_F$ :

$$(2.18) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathrm{Hom}(M_0, \mathbb{G}_m) & \rightarrow & \mathcal{N}(J_0, \mathcal{O}_F)_{/\kappa_F}^0 & \rightarrow & J_0^{\acute{e}t} \times J_0^\mu \rightarrow 0 \\ & & \parallel & & \cap & & \cap \\ 0 & \rightarrow & \mathrm{Hom}(M_0, \mathbb{G}_m) & \rightarrow & \mathcal{N}(J_2, \mathcal{O}_F)_{/\kappa_F}^0 & \rightarrow & J_2^{\acute{e}t} \times J_2^\mu \rightarrow 0 \end{array}$$

with exact rows. The injectivity of the middle vertical arrow is deduced from the injectivity of the right vertical arrow, which is a consequence of the fact that the coverings  $X_2^{\acute{e}t} \rightarrow X_0^{\acute{e}t}$  and  $X_2^\mu \rightarrow X_0^\mu$  inducing the maps between the Jacobians are totally ramified over  $S$ .

A similar analysis applies to the generalized Jacobians (always with respect to the cusps). Thanks to the fact that  $\mathfrak{C}_2$  is finite étale over  $\mathcal{O}_K$ , the proposition analogous to 2.2 is valid. Note that there are precisely  $r$  cusps in  $C_2$  above any given cusp of  $C_0$ . Thus we have exact sequences

$$(2.19) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathrm{Hom}(M, \mu_r) & \rightarrow & \mathcal{J}_2^{\#0}(\overline{\mathbb{F}}_p)[r] & \rightarrow & J_2^{\acute{e}t\#}[r] \times J_2^{\mu\#}[r] \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathrm{Hom}(M_0, \mu_r) & \rightarrow & \mathcal{J}_2^0(\overline{\mathbb{F}}_p)[r] & \rightarrow & J_2^{\acute{e}t}[r] \times J_2^\mu[r] \rightarrow 0 \end{array}$$

where we have denoted by  $\mathcal{J}_2 = \mathcal{N}(J_2, \mathcal{O}_K)$  etc. We can now define the filtration on  $\mathbf{V}$  and  $\mathbf{V}^\#$ .

**DEFINITION 2.2.** — *Let  $J_2[r]^{\mathrm{sub}}$  (resp.  $J_2^\#[r]^{\mathrm{sub}}$ ) denote the subgroup of  $\mathcal{J}_2^0(\mathcal{O}_K^{nr})[r]$  (resp.  $\mathcal{J}_2^{\#0}(\mathcal{O}_K^{nr})[r]$ ) consisting of the elements which map to 0 in  $J_2^\mu$  (resp. in  $J_2^{\mu\#}$ ). Let  $J_2[r]^{\mathrm{quot}} = J_2[r]/J_2[r]^{\mathrm{sub}}$  and similarly for the generalized Jacobians. Dualizing, we obtain exact sequences*

$$(2.20) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathbf{U} & \rightarrow & \mathbf{V} & \rightarrow & \mathbf{W} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathbf{U}^\# & \rightarrow & \mathbf{V}^\# & \rightarrow & \mathbf{W}^\# \rightarrow 0 \end{array}$$

where  $\mathbf{U} = (J_2[r]^{\mathrm{quot}})^*$ ,  $\mathbf{W} = (J_2[r]^{\mathrm{sub}})^*$ ,  $\mathbf{U}^\# = (J_2^\#[r]^{\mathrm{quot}})^*$ ,  $\mathbf{W}^\# = (J_2^\#[r]^{\mathrm{sub}})^*$ .

As in the case of  $J_0$ , the vertical maps are injections, and the six modules are stable under the local Galois group of  $K$ ,  $\mathrm{Gal}(\bar{K}/K)$ , and

under the Hecke algebra  $\mathcal{H}$ . As we shall see later, they are even stable under  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ .

**2.4. Structure of the filtered deformation.**

It will be important to analyze the *filtered deformation*  $V^\# \rightarrow V^\#$  in the way it was done, without the filtration, in Proposition 2.1. We shall show that this surjective map induces *surjective* maps on the sub/quotient modules, and that the structure as  $\Lambda$ -modules is as “clean” as possible, given Proposition 2.1(iv). These tasks turn out to be somewhat delicate, and we shall break them to several steps. We shall then recollect all the information we have gathered in one theorem.

Assume that  $F$  is a field containing  $K$ , and that  $J_0[r]$  and  $J_2[r]$  are all  $F$ -rational. If  $\Sigma$  denotes the Shimura subgroup of  $J_0$ , then  $\Sigma[r] = \text{Ker}(J_0 \rightarrow J_2)$ . Consider the diagram

$$(2.21) \quad \begin{array}{ccccccc} 0 & \rightarrow & 0 & \rightarrow & \mathcal{N}(J_0, \mathcal{O}_F)^0(\mathcal{O}_F)[r] & \rightarrow & \mathcal{N}(J_2, \mathcal{O}_F)^0(\mathcal{O}_F)[r] \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \Sigma[r] & \rightarrow & J_0[r] & \rightarrow & J_2[r]. \end{array}$$

with exact rows.

Let

$$(2.22) \quad \Psi = (J_0[r]/\Sigma[r] \cap \mathcal{N}(J_2, \mathcal{O}_F)^0(\mathcal{O}_F)[r]) / \mathcal{N}(J_0, \mathcal{O}_F)^0(\mathcal{O}_F)[r].$$

Then  $\Psi$  embeds in  $\mathcal{N}(J_2, \mathcal{O}_F)^0_{/\kappa_F} / \mathcal{N}(J_0, \mathcal{O}_F)^0_{/\kappa_F}$  (note that  $r$  torsion in the Néron model maps injectively under the specialization to the special fiber because  $p$  does not divide  $r$ ). On the other hand  $\Psi$  is a submodule of  $J_0[r] / (\Sigma[r] + \mathcal{N}(J_0, \mathcal{O}_F)^0(\mathcal{O}_F)[r])$ , hence

$$(2.23) \quad \Psi \subset \Phi(J_0, \mathcal{O}_F) / \Sigma[r]$$

and in particular we have the identity  $U_p = -w_p$  on  $\Psi$ .

Let  $H$  be the subgroup of  $\Psi$  which maps to  $J_2^{\text{ét}}/J_0^{\text{ét}} \times \{0\}$ . Note that  $H$  is stable under  $U_p$  because  $J_2^{\text{ét}}/J_0^{\text{ét}} \times \{0\}$  is stable under  $U_p$ , but  $H$  can not be stable under  $w_p$  because it interchanges  $J_2^{\text{ét}}/J_0^{\text{ét}}$  with  $J_2^\mu/J_0^\mu$ . However, we have noticed that  $U_p = -w_p$  on  $H$ . This proves that  $H = 0$ .

PROPOSITION 2.4. — *Inside  $J_1[r]$ , we have*

$$(2.24) \quad \text{Im } J_0[r] \cap J_1[r, I']^{\text{sub}} = \text{Im } J_0[r]^{\text{sub}}.$$

*Proof.* — As noted before  $J_1[r, I'] = J_2[r]$ . Let  $F$  be any field containing  $K$  so that  $J_2[r]$  and  $J_0[r]$  are rational over  $F$ . Let  $x_F \in J_2[r]^{\text{sub}}$  be the image of  $y_F \in J_0[r]$ . Extend  $x_F$  and  $y_F$  to sections  $x$  and  $y$  of the Néron models over  $\mathcal{O}_F$ . Then  $x \in \mathcal{N}(J_2, \mathcal{O}_F)^0(\mathcal{O}_F)$  maps to  $J_2^{\text{ét}} \times \{0\}$  so by the last remark before the proposition, the image of  $y$  in  $\Psi$  lies in  $H = 0$ , hence we can change  $y$  by an element of  $\Sigma[r]$  to bring it into the connected component. We may therefore assume that  $y \in \mathcal{N}(J_0, \mathcal{O}_F)^0(\mathcal{O}_F)$ . It is now clear that if the intersection of  $x$  with the special fiber projects to  $J_2^{\text{ét}}$ , then so does the intersection of  $y$  with the special fiber.  $\square$

PROPOSITION 2.5. — *In the following diagram:*

$$(2.25) \quad \begin{array}{ccccccc} 0 & \rightarrow & J_0^\# [r]^{\text{sub}} & \rightarrow & J_0^\# [r] & \rightarrow & J_0^\# [r]^{\text{quot}} & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & J_1^\# [r, I']^{\text{sub}} & \rightarrow & J_1^\# [r, I'] & \rightarrow & J_1^\# [r, I']^{\text{quot}} & \rightarrow & 0 \end{array}$$

*all the vertical arrows are injective.*

*Proof* (Compare [dS1], Lemma 2.7, p. 85-86). — We only have to prove that

$$J_0^\# [r] \cap J_1^\# [r, I']^{\text{sub}} \subset J_0^\# [r]^{\text{sub}}.$$

Let  $x \in J_0^\# [r] \cap J_1^\# [r, I']^{\text{sub}}$ . By the previous proposition, and after modifying  $x$  by an element of  $J_0^\# [r]^{\text{sub}}$ , we may assume that  $x$  lies in the kernel of the map to  $J_1[r, I']^{\text{sub}}$ , i.e.,

$$(2.26) \quad x \in \text{Hom}(\mathbb{Z}[C_2^{\text{ét}}], \mu_r).$$

Here we made use of the diagram (over  $\mathbb{F}_p$ )

$$(2.27) \quad \begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}(\mathbb{Z}[C_2]_0, \mu_r) & \rightarrow & \mathcal{J}_2^\# [r]^0 & \rightarrow & \mathcal{J}_2[r]^0 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \text{Hom}(\mathbb{Z}[C_2^{\text{ét}}]_0 \times \mathbb{Z}[C_2^\mu]_0, \mu_r) & \rightarrow & J_2^{\text{ét}\#} [r] \times J_2^{\mu\#} [r] & \rightarrow & J_2^{\text{ét}} [r] \times J_2^\mu [r] & \rightarrow & 0 \end{array}$$

and the exact sequence

$$(2.28) \quad 0 \rightarrow \text{Hom}(\mathbb{Z}[C_2^{\text{ét}}], \mu_r) \rightarrow \text{Hom}(\mathbb{Z}[C_2]_0, \mu_r) \rightarrow \text{Hom}(\mathbb{Z}[C_2^\mu]_0, \mu_r) \rightarrow 0.$$

Consider now the diagram

$$\begin{array}{ccccccc}
 & & 0 & \rightarrow & 0 & \rightarrow & \Sigma[r] & \dashrightarrow & \dashrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 (2.29) & 0 & \rightarrow & \text{Hom}(\mathbb{Z}[C_0]_0, \mu_r) & \rightarrow & J_0^\#[r] & \rightarrow & J_0[r] & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \rightarrow & \text{Hom}(\mathbb{Z}[C_2]_0, \mu_r) & \rightarrow & J_2^\#[r] & \rightarrow & J_2[r] & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 \dashrightarrow & \text{Hom}(\mathcal{M}_0, \mu_r) & \rightarrow & X & \rightarrow & J_2[r]/\text{Im } J_0[r] & \rightarrow & 0
 \end{array}$$

(obtained from the snake lemma) with  $\mathcal{M}_0 = \text{Ker}(\mathbb{Z}[C_2]_0 \rightarrow \mathbb{Z}[C_0]_0)$ . The image of  $\Sigma[r]$  in  $\text{Hom}(\mathcal{M}_0, \mu_r)$  can be computed explicitly (use the action of the diamond operators  $\Delta_r$  on  $\Sigma[r] = \text{Hom}(\Delta_r, \mu_r)$ ). But in any case it is invariant under  $w_p$ , because  $\Sigma$  is invariant under it. In particular, it intersects trivially  $\text{Hom}(\mathcal{M}^{\text{ét}}, \mu_r) \subset \text{Hom}(\mathcal{M}_0, \mu_r)$ , where

$$(2.30) \quad \mathcal{M}^{\text{ét}} = \text{Ker}(\mathbb{Z}[C_2^{\text{ét}}] \rightarrow \mathbb{Z}[C_0^{\text{ét}}]),$$

because  $w_p$  interchanges the ét and the  $\mu$  cusps. Similarly we have another diagram

$$\begin{array}{ccccccc}
 & & 0 & \rightarrow & 0 & \rightarrow & 0 & \dashrightarrow & \dashrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 (2.31) & 0 & \rightarrow & \text{Hom}(\mathbb{Z}[C_0^{\text{ét}}], \mu_r) & \rightarrow & J_0^\#[r]^{\text{sub}} & \rightarrow & J_0[r]^{\text{sub}} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \rightarrow & \text{Hom}(\mathbb{Z}[C_2^{\text{ét}}], \mu_r) & \rightarrow & J_2^\#[r]^{\text{sub}} & \rightarrow & J_2[r]^{\text{sub}} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 \dashrightarrow & \text{Hom}(\mathcal{M}^{\text{ét}}, \mu_r) & \rightarrow & Y & \rightarrow & J_2[r]^{\text{sub}}/J_0[r]^{\text{sub}} & \rightarrow & 0.
 \end{array}$$

Now  $x \in \text{Hom}(\mathbb{Z}[C_2^{\text{ét}}], \mu_r) \cap J_0^\#[r]$ . Chasing the diagrams we see that its image in  $\text{Hom}(\mathcal{M}_0, \mu_r)$  lies in

$$(2.32) \quad \text{Hom}(\mathcal{M}^{\text{ét}}, \mu_r) \cap \Sigma[r] = \{0\}$$

and therefore  $x \in \text{Hom}(\mathbb{Z}[C_0^{\text{ét}}], \mu_r) \subset J_0^\#[r]^{\text{sub}}$ . □

### 2.5. The action of the full decomposition group.

We wish to show that  $J_1^\#[r, I']^{\text{sub}}$  and  $J_1[r, I']^{\text{sub}}$  are stable under  $\text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)$ , and to compute its action. It will follow that the six modules in Definition 2.2 are acted upon by the full decomposition group, and not only by  $\text{Gal}(\bar{K}/K)$ . For that purpose consider the model  $\mathfrak{X}'_{2/\mathbb{Z}_p}$  of  $X_2$  which is associated with the moduli problem  $([\Gamma_1(N)], [\Gamma_1(p)]/\mu_{r'})$ . See

[KM], Theorem 5.5.1 for the representability.  $\mathfrak{X}'_2$  is proper and flat over  $\mathbb{Z}_p$ , regular as a 2-dimensional scheme, but unlike the model  $\mathfrak{X}_2$  over the ring  $\mathcal{O}_K$ , its special fiber is non-reduced. In fact, by [KM], Theorem 13.5.4 on p. 416,  $\mathfrak{X}'_{2/\mathbb{F}_p}$  is the union of two irreducible components  $X'_2{}^\mu$  and  $X'_2{}^{\text{ét}}$ , intersecting transversally at the set  $S$  of supersingular points. The curve  $X'_2{}^\mu$  is non-reduced, and its multiplicity as an abstract curve is  $r$ . When we reduce it,  $(X'_2{}^\mu)^{\text{red}}$  becomes equal to  $X_1(N)$ . The curve  $X'_2{}^{\text{ét}}$  is reduced, and it is an Igusa curve. In fact it is isomorphic to  $X_2^{\text{ét}}$ .

The Néron model  $\mathcal{N}(J_2, \mathbb{Z}_p)$  can be computed by Raynaud's theorem from this model. As  $\mathfrak{X}'_2$  is not semi-stable, the connected component of the special fiber of  $\mathcal{N}(J_2, \mathbb{Z}_p)$  has a unipotent part. In fact  $\mathcal{N}(J_2, \mathbb{Z}_p)_{/\mathbb{F}_p}^0$  will be an extension of  $J_2^{\text{ét}} \times J_2^\mu$  by an affine algebraic group, which is itself the extension of the torus  $\text{Hom}(M_0, \mathbb{G}_m)$  by a unipotent group. This unipotent part will contribute nothing to the  $r$ -torsion. From the map

$$(2.33) \quad \mathcal{N}(J_2, \mathbb{Z}_p) \times_{\mathbb{Z}_p} \mathcal{O}_K \rightarrow \mathcal{N}(J_2, \mathcal{O}_K) = \mathcal{J}_2$$

(obtained by the universal property of Néron models) we shall get a map

$$(2.33) \quad \mathcal{N}(J_2, \mathbb{Z}_p)_{/\mathbb{F}_p}^0 \rightarrow \mathcal{J}_2_{/\mathbb{F}_p}^0,$$

and the pre-image of  $J_2^{\text{ét}}[r] \times \{0\} = J_2^{\text{ét}}[r] \times \{0\}$  under the map

$$(2.34) \quad \mathcal{N}(J_2, \mathbb{Z}_p)_{/\mathbb{F}_p}^0[r] \rightarrow J_2^{\text{ét}}[r] \times J_2^\mu[r]$$

will map isomorphically onto  $J_2[r]^{\text{sub}}$ , both being extensions of  $J_2^{\text{ét}}[r]$  by  $\text{Hom}(M_0, \mu_r)$ . Since this pre-image is a subgroup of  $\mathcal{N}(J_2, \mathbb{Z}_p)_{/\mathbb{F}_p}^0(\mathbb{Z}_p^{nr})$  which is stable under the full decomposition group, this proves our claim, and at the same time shows that  $J_1[r, I']^{\text{sub}}$  is *unramified for the full decomposition group*. The same holds for the generalized Jacobian.

To compute the action of  $\sigma$ , the arithmetic Frobenius at  $p$ , let us start with a non-cuspidal point  $x = [E, P_N, P_p] \in X_1^{\text{ét}}(\overline{\mathbb{F}}_p)$  where  $P_N$  is a point of exact order  $N$ , and  $P_p$  a point of exact order  $p$  which generates  $\text{Ker}(V : E \rightarrow \sigma^{-1}E)$  as a Cartier divisor. See the discussion in Section 2.2 about the moduli problem represented by  $X_0^{\text{ét}}$ . The difference between  $X_1^{\text{ét}}$  and  $X_2^{\text{ét}}$  is that in the latter  $P_p$  is only taken modulo the action of  $\mu_{r'}$ . To simplify the notation, we shall do the computations at the level of  $X_1$ . Now

$$(2.35) \quad U_p^*(x) = \sum_{i=1}^p [E/\langle P_p \rangle, \langle p \rangle_N^{-1} P_N \text{ mod } \langle P_p \rangle, Q_{p,i} \text{ mod } \langle P_p \rangle]$$

where  $Q_{p,i} \in E(\overline{\mathbb{F}}_p)$  is a solution of  $pQ_{p,i} = P_p$ . There are  $p^2$  solutions of this equation (taken with multiplicities) and modulo  $\langle P_p \rangle$  there are  $p$ . If  $x$  was an ordinary point then  $\langle P_p \rangle$  is a reduced subgroup, and all the  $Q_{p,i}$  are equal modulo  $\langle P_p \rangle$ . If  $x$  was in  $S$  then  $\langle P_p \rangle = \text{Ker} F = \text{Ker} V$  and of course all the  $Q_{p,i}$  are 0. Thus in any case the sum on the right hand side consists of  $p$  equal terms. Applying  $\sigma$  to any of them yields

$$(2.36) \quad [E/E[p], \langle p \rangle_N^{-1} P_N \bmod E[p], Q_{p,i} \bmod E[p]] = [E, P_N, P_p] = x.$$

Since a point of  $J_1[r, I']^{\text{sub}}$  or  $J_1^\# [r, I']^{\text{sub}}$  is represented by a divisor of degree 0 with support on  $x$ 's as above, and since the  $U_p$  action is by Picard functoriality we arrive at the formula

$$(2.37) \quad U_p(x) = p\sigma^{-1}(x).$$

**PROPOSITION 2.6** (Compare [dS1], Proposition 3.2). — *The modules  $\mathbf{U}^\#$  and  $\mathbf{W}^\#$  are stable under  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ . The action of the decomposition group on  $\mathbf{W}^\#$  is through the unramified character  $\phi$  sending the arithmetic Frobenius  $\sigma$  to  $\phi(\sigma) = U_p$  (and therefore  $U_p$  acts invertibly on  $\mathbf{W}^\#$ ).*

*Proof.* — This follows from the above discussion, the observation that  $p \equiv 1 \bmod r$ , and the conventions about Galois and Hecke actions after dualizing. □

The next proposition shows that the action of the decomposition group on  $\mathbf{U}^\#$  is ramified.

**PROPOSITION 2.7.** — *Let  $\omega : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{F}_p^\times \subset \mathcal{H}$  be the character which associates to  $\tau$  the diamond operator*

$$(2.38) \quad \omega(\tau) = \langle \chi(\tau) \rangle_{Np}$$

where  $\chi(\tau)$  is determined by  $\tau(\zeta) = \zeta^{\chi(\tau)}$  for  $\zeta \in \mu_{Np}(\overline{\mathbb{Q}}_p)$ . Then  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  acts on  $\mathbf{U}^\#$  via the character  $\omega\phi^{-1}$ .

*Proof.* — A proof can be modeled on the proof of Proposition 3.7 in [dS1], paying attention to the  $\Gamma_1(N)$  level structure, which was missing there. Alternatively (for  $\mathbf{U}$  at least), we can use the *twisted Weil pairing* of [MW2], p. 243, defined by

$$(2.39) \quad [x, y] = \langle x, w_{Np}y \rangle$$

for  $x, y \in J_1[r, I']$ . (Here  $w_{Np} = w_{Np}^* = w_{Np^*}$  because  $w_{Np}^2 = 1$ .) This twisted pairing has two advantages over the original one: (a) The transpose

of  $U_p^*$  (or, for that matter, any  $T \in \mathcal{H}$ ) under it, is  $U_p^*$  itself (resp.  $T$  itself), and not  $U_{p^*}$ , which may not be in  $\mathcal{H}$ . (b)  $J_1[r, I']^{\text{sub}}$  is maximal isotropic for it, and therefore set in perfect duality with  $J_1[r, I']^{\text{quot}}$ . Indeed, by Grothendieck's "Théorème d'orthogonalité" ([SGA7], exposé IX, Thm. 2.4) and by the fact that  $w_{Np}$  preserves  $\mathcal{J}_1^0[r, I']$ , the restriction of  $[\cdot, \cdot]$  to  $\mathcal{J}_1^0[r, I']$  factors through the projection to  $J_2^{\text{ét}}[r] \times J_2^\mu[r]$ , but since  $w_{Np}$  interchanges the two components of the special fiber, if both  $x$  and  $y$  project to  $J_2^{\text{ét}}[r]$  (i.e. belong to  $J_1[r, I']^{\text{sub}}$ ) then  $[x, y] = 0$ .

We can now compute for  $x \in J_1[r, I']^{\text{sub}}$  and  $y \in J_1[r, I']^{\text{quot}}$  and any  $\tau$  in the local Galois group

$$(2.40) \quad \begin{aligned} [x, y] &= \tau[x, y] = \langle \tau x, \tau w_{Np} y \rangle = \langle \tau x, w_{Np} \langle \chi(\tau) \rangle_{Np} \tau y \rangle \\ &= [\phi^{-1}(\tau)x, \omega(\tau)\tau y] = [x, \phi^{-1}\omega(\tau)\tau y] \end{aligned}$$

so by non-degeneracy,  $\tau y = \omega^{-1}\phi(\tau)y$ . Dualizing we see that  $\tau$  acts on  $\mathbf{U}$  via  $\omega\phi^{-1}(\tau)$ . Observe that implicit in the definition of  $w_{Np}$  there is a choice of an  $Np$ -th root of unity  $\zeta$ , hence  $\tau$  does not commute with  $w_{Np}$ , but rather satisfies  $\tau \circ w_{Np} = w_{Np} \circ \langle \chi(\tau) \rangle_{Np}^* \circ \tau$ .  $\square$

## 2.6. Structure over $\Lambda$ .

Recall the identifications

$$(2.41) \quad \begin{array}{ccc} J_1^\# [r, I']^* & = & H^1(\Gamma_1, R)_{I'} = H^1(\Gamma_0, \Lambda) \\ \downarrow & & \downarrow \text{Cor} \quad \downarrow \text{can.} \\ J_0^\# [r]^* & = & H^1(\Gamma_0, R) = H^1(\Gamma_0, R) \end{array}$$

from Section 2.1, where Cor is the corestriction map, and can the canonical map induced by the projection  $\Lambda \rightarrow R$ . Let  $\gamma_0, \gamma_1, \dots, \gamma_{2m}$  be a set of generators of  $\Gamma_0$  as a free group, such that  $\gamma_i \in \Gamma_1$  for  $i \geq 1$ . (The rank of  $\Gamma_0$  is odd. In fact

$$(2.42) \quad 2m + 1 = 2\text{genus}(X_0) + \#C_0 - 1$$

and  $\#C_0$  is even.) Then  $\gamma_0$  projects to a generator of the cyclic group  $\Gamma_0/\Gamma_1 \simeq \mathbb{F}_p^\times$ . Using the  $\gamma_i$  we write

$$(2.43) \quad H^1(\Gamma_0, R) = \text{Hom}(\Gamma_0, R) \simeq R^{2m+1}, \quad h \mapsto (h(\gamma_0), \dots, h(\gamma_{2m})).$$

Similarly, since  $\Gamma_0$  is free,

$$(2.44) \quad Z^1(\Gamma_0, \Lambda) \simeq \Lambda^{2m+1}, \quad z \mapsto (z(\gamma_0), \dots, z(\gamma_{2m})),$$

and under this isomorphism  $B^1(\Gamma_0, \Lambda) \simeq I \times (0) \times \cdots \times (0)$ . This gives a realization of the isomorphism encountered in (2.1)(iv), which depends on the choice of the  $\gamma_i$ . Let  $v_0, \dots, v_{2m}$  denote the elements of  $J_1^\#[r, I]^* = H^1(\Gamma_0, \Lambda) \simeq (\Lambda/I) \oplus \Lambda \oplus \cdots \oplus \Lambda$  which correspond under the isomorphism to  $(1, 0, \dots, 0), \dots, (0, \dots, 1)$ . Let  $u_i$  denote the images of  $v_i$  in  $J_0^\#[r]^* = H^1(\Gamma_0, R) \simeq R^{2m+1}$ .

Note that while the line  $Rv_0$  depends on the choice of the  $\gamma_i$ , the line  $Ru_0$  does not, because under any change of the  $\gamma_i$ ,  $v_0$  goes over to  $\sum \lambda_i v_i$  with  $\lambda_0 \in R^\times$  and  $\lambda_i \in \Lambda[I] \subset I$  for  $i \geq 1$ . This ‘‘canonical line’’  $Ru_0 \subset J_0^\#[r]^* = H^1(\Gamma_0, R)$  simply consists of homomorphism from  $\Gamma_0$  to  $R$  factoring through  $\Gamma_0/\Gamma_1$ . Now giving  $u \in J_0^\#[r]^*$  is the same as giving a cyclic covering  $X \rightarrow X_0$  defined over  $\mathbb{Q}$ , unramified outside the cusps, of degree dividing  $r$ , together with a distinguished generator of  $\text{Gal}(X/X_0)$ . Such a  $u$  comes from  $J_0[r]^* \subset J_0^\#[r]^*$  if and only if it is also unramified at the cusps. Consider in particular the covering  $X_2 \rightarrow X_0$  which is of the above type. It corresponds to the line  $Ru_0$ , and since it is unramified also at the cusps, we conclude that  $Ru_0 \subset J_0[r]^*$ .

We claim that  $Ru_0$  is mapped injectively into  $W = (J_0[r]^{\text{sub}})^*$ , hence also into  $W^\#$ . A fortiori,  $Rv_0$  will map injectively into  $W^\#$ . This was proved in the case of prime conductor ([dS1], p. 87), by an argument on Mumford curves. Here we need a different argument. What we have to show is that  $u_0(J_0[r]^{\text{sub}})$  is of order  $r$ . By the non-degeneracy of the Weil pairing on  $J_0[r]$  there is a unique  $\tilde{u}_0 \in J_0[r]$  such that

$$(2.45) \quad u_0(x) = \langle \tilde{u}_0, x \rangle$$

for every  $x \in J_0[r]$ , where  $\langle \cdot, \cdot \rangle$  denotes the Weil  $e_r$ -pairing, followed by a fixed isomorphism  $\mu_r \simeq R$ . Unwinding the definitions we see the following. The homomorphism  $u_0$  corresponds to the cyclic covering  $X_2 \rightarrow X_0$  and a generator  $\sigma$  of  $\text{Gal}(X_2/X_0)$ . Let  $\zeta$  be the primitive  $r$ th root of unity corresponding to 1 under the fixed isomorphism  $\mu_r \simeq R$ . Kummer theory then establishes the existence of a unique  $g \in \bar{\mathbb{Q}}(X_0)^\times / \bar{\mathbb{Q}}(X_0)^{\times r}$  such that  $\bar{\mathbb{Q}}(X_2) = \bar{\mathbb{Q}}(X_0)(g^{1/r})$ , and such that  $\sigma(g) = \zeta g$ . Since the covering is everywhere unramified the divisor  $\text{div}(g) = rD$  for some divisor of degree 0  $D$ , and  $\tilde{u}_0 = [D]$  is the class of this  $D$  in  $\text{Pic}^0(X_0) = J_0$ . It follows from this interpretation that  $\tilde{u}_0$  lies in the Shimura subgroup, and we have noticed before that the Shimura subgroup intersects the connected component of the Néron model trivially, and therefore maps injectively into  $\Phi(J_0, \mathbb{Z}_p)$ . We now invoke Grothendieck’s ‘‘Théorème d’orthogonalité’’

([SGA7], exposé IX, Thm. 2.4), which implies that  $\text{Hom}(M_0, \mu_r)$  is orthogonal under the Weil pairing to  $\mathcal{J}_0^0(\bar{\mathbb{F}}_p)[r]$ , and is therefore dual (by counting) to  $J_0[r]/\mathcal{J}_0^0(\bar{\mathbb{F}}_p)[r]$ . In particular, it follows that for a suitable  $x \in \text{Hom}(M_0, \mu_r) \subset J_0[r]^{\text{sub}}$ ,  $\langle \tilde{u}_0, x \rangle$  is of order  $r$ , as desired.

Observe that

$$(2.46) \quad \text{genus}(X_0) = 2\text{genus}(X_0^{\text{ét}}) + \#(S) - 1$$

so that

$$(2.47) \quad m + 1 = 2\text{genus}(X_0^{\text{ét}}) + \#(C_0^{\text{ét}}) + \#(S) - 1 = \text{rank}_R(J_0^{\#}[r]^{\text{sub}})^*.$$

We choose our generators of  $\Gamma_0$  in such a way so that  $u_0, u_1, \dots, u_m$  project to a basis of  $W^{\#}$  and  $u_{m+1}, \dots, u_{2m}$  form a basis of  $U^{\#}$ . The same counting argument as in [dS1], Proposition 2.8, Step 3 (p. 87-88), now shows that

$$(2.48) \quad \mathbf{W}^{\#} \simeq \Lambda/I \oplus \Lambda^m, \quad \mathbf{U}^{\#} \simeq \Lambda^m,$$

and the sequence  $0 \rightarrow \mathbf{U}^{\#} \rightarrow \mathbf{V}^{\#} \rightarrow \mathbf{W}^{\#} \rightarrow 0$  splits as a sequence of  $\Lambda$ -modules. We summarize the discussion of this chapter in the following theorem.

**THEOREM 2.8.** — *In the commutative diagram*

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbf{U}^{\#} & \rightarrow & \mathbf{V}^{\#} & \rightarrow & \mathbf{W}^{\#} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & U^{\#} & \rightarrow & V^{\#} & \rightarrow & W^{\#} \rightarrow 0, \end{array}$$

where  $\mathbf{U}^{\#} = (J_1^{\#}[r, I']^{\text{quot}})^*$ ,  $\mathbf{V}^{\#} = J_1^{\#}[r, I']^*$ ,  $\mathbf{W}^{\#} = (J_1^{\#}[r, I']^{\text{sub}})^*$  and  $U^{\#} = (J_0^{\#}[r]^{\text{quot}})^*$ ,  $V^{\#} = J_0^{\#}[r]^*$ ,  $W^{\#} = (J_0^{\#}[r]^{\text{sub}})^*$ :

- *The rows are exact and the vertical arrows are surjective.*
- *All six modules are stable under  $\mathcal{H}$  and under  $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ . The modules  $U^{\#}, W^{\#}$ , and  $\mathbf{W}^{\#}$  (but not  $\mathbf{U}^{\#}$ ) are unramified for the action of the decomposition group. In fact  $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$  acts on  $\mathbf{W}^{\#}$  through the character  $\phi$ , and on  $\mathbf{U}^{\#}$  through  $\omega\phi^{-1}$ , where  $\phi(\sigma) = U_p$  if  $\sigma$  is the arithmetic Frobenius in  $\text{Gal}(\mathbb{Q}_p^{nr}/\mathbb{Q}_p)$ , and  $\omega(\tau) = \langle \chi(\tau) \rangle_{N_p}$  is the Teichmüller character.*
- *As a short exact sequence  $\Lambda$ -modules, the top row splits, and*

$$(2.50) \quad \mathbf{W}^{\#} \simeq \Lambda/I \oplus \Lambda^m, \quad \mathbf{U}^{\#} \simeq \Lambda^m.$$

*The bottom row is identified with (the top row)  $\otimes_{\Lambda} R$ .*

- The map  $\mathbf{W}^\# \rightarrow W^\#$  is dual to the inclusion  $J_0^\#[r]^{\text{sub}} \hookrightarrow J_1^\#[r, I']^{\text{sub}}$ . This inclusion fits in the following exact sequence:

$$(2.51) \quad \begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}(M, \mu_r) & \rightarrow & J_0^\#[r]^{\text{sub}} & \rightarrow & J_0^{\text{ét}\#}[r] \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & \text{Hom}(M, \mu_r) & \rightarrow & J_1^\#[r, I']^{\text{sub}} & \rightarrow & J_2^{\text{ét}\#}[r] \rightarrow 0. \end{array}$$

In fact  $J_0^\#[r]^{\text{sub}}$  is the  $r$ -torsion in the generalized Jacobian of  $X_0^{\text{ét}} = X_1(N)_{/\mathbb{F}_p}$  with respect to the reduced modulus  $C_0^{\text{ét}} \cup S$  of the cusps and supersingular points, and the same can be said for  $J_1^\#[r, I']^{\text{sub}}$  and the Igusa curve  $X_2^{\text{ét}}$ . The degree  $r$  covering  $X_2^{\text{ét}} \rightarrow X_0^{\text{ét}}$  is totally ramified over  $S$  and étale elsewhere (including over the cusps).

We complement the theorem with the following fact concerning  $\mathbf{V}$  (as opposed to  $\mathbf{V}^\#$ ).

PROPOSITION 2.9. — As  $\mathcal{H}$ -modules,  $\mathbf{W}$  is isomorphic to  $\text{Hom}(\mathbf{U}, R) = \mathbf{U}^*$ .

*Proof.* — This has already been noticed in the proof given to Proposition 2.7, using the twisted Weil pairing. □

Note the difference between Hida’s theory of “ordinary”  $p$ -adic deformations, and our “tame Hida theory”. While Hida had to restrict to the ordinary part in order to have  $U_p$  acting invertibly, in our case this is automatic (of course, it is acting now on  $r$ -torsion!). Another difference is that in Hida’s situation the  $p$ -divisible groups had a 2-step filtration for the decomposition group at  $p$ , where the quotient was unramified, while in our case the submodule  $J_1^\#[r, I']^{\text{sub}}$  is the unramified piece, and the source of the ramification in  $J_1^\#[r, I']^{\text{quot}}$  is different, the so-called “geometric inertia group action”. Yet the formal structure of our theorem is very similar to Hida’s. In particular, the (almost) freeness of the deformation over  $\Lambda$ , and the fact that the  $\Gamma_0$  structures are the  $I$ -coinvariants of their deformations is an analogue of Hida’s control theorem.

### 3. The $p$ -old and $p$ -new parts of the deformation.

#### 3.1. A study of the new subvariety.

The following results are due to Ribet. Let  $i$  and  $j = i \circ w_p$  be the two degeneracy maps  $X_0 \rightarrow X_1(N)$ , which correspond, in terms of the moduli

problem, to the maps

$$(3.1) \quad i : [E, P_N, H] \mapsto [E, P_N], \quad j : [E, P_N, H] \mapsto [E/H, P_N \bmod H].$$

Then

$$(3.2) \quad J_{0,\text{old}} = \text{Im}(i^*, j^*) : J_1(N)^2 \rightarrow J_0.$$

Here we write  $J_1(N)^2$  as column vectors of length 2, and  $i^*$  acts on the first entry,  $j^*$  on the second. If  $i_*$  and  $j_*$  are the maps  $J_0 \rightarrow J_1(N)$  induced by Albanese functoriality, then

$$(3.3) \quad J_{0,\text{new}} = \text{Ker}({}^t(i_*, j_*) : J_0 \rightarrow J_1(N)^2)^0.$$

LEMMA 3.1. — *The endomorphism  $\alpha = {}^t(i_*, j_*) \circ (i^*, j^*) \in \text{End}(J_1(N)^2)$  is an isogeny, represented by the matrix*

$$(3.4) \quad \begin{pmatrix} p+1 & T_p^* \\ T_{p*} & p+1 \end{pmatrix},$$

and  $T_{p*} = (\langle p \rangle_N^{-1})^* \circ T_p^*$ .

*Proof.* — Clearly  $i_* \circ i^* = \text{deg}(i) = p+1 = \text{deg}(j) = j_* \circ j^*$ . An easy computation shows that  $i_* \circ j^* = T_p^*$ ,  $j_* \circ i^* = T_{p*}$ , and that  $T_{p*} = \langle p \rangle_{N*} \circ T_p^* = (\langle p \rangle_N^{-1})^* \circ T_p^*$  (see also [MW], p. 236).  $\square$

Jacobians are self-dual abelian varieties, and  ${}^t(i_*, j_*)$  is the dual of  $(i^*, j^*)$ . Now it is a theorem of Ribet, based on results of Ihara, that  $(i^*, j^*) : J_1(N)^2 \rightarrow J_0$  is *injective* (see [Ri], Theorem 4.1). Denoting by  $(-)^{\vee}$  the dual abelian variety, we get, upon dualizing the isomorphism

$$(3.5) \quad J_1(N)^2 \xrightarrow{(i^*, j^*)} J_{0,\text{old}} \hookrightarrow J_0,$$

that the map  ${}^t(i_*, j_*)$  factors as

$$(3.6) \quad J_0 = J_0^{\vee} \rightarrow J_{0,\text{old}}^{\vee} \simeq J_1(N)^2,$$

and has a connected kernel (so in (3.3) we may dispose of the  $(-)^0$ ).

We conclude that  $J_0^{\text{old}} = J_{0,\text{old}}^{\vee}$  and likewise, dualizing

$$0 \rightarrow J_{0,\text{old}} \rightarrow J_0 \rightarrow J_0^{\text{new}} \rightarrow 0$$

we get  $J_0^{\text{new}} = J_{0,\text{new}}^{\vee}$ , and the exact sequence

$$0 \rightarrow J_{0,\text{new}} \rightarrow J_0 \rightarrow J_0^{\text{old}} \rightarrow 0.$$

The two isogenies  $J_{0,\text{new}} \rightarrow J_0^{\text{new}}$  and  $\tilde{\alpha} : J_{0,\text{old}} \rightarrow J_0^{\text{old}}$  are therefore polarizations, being of the form  $\varphi^\vee \circ \varphi$ .

The matrix  $\alpha$  breaks as the composition of three maps

$$(3.7) \quad \alpha : J_1(N)^2 \xrightarrow{\alpha_0} J_{0,\text{old}} \xrightarrow{\tilde{\alpha}} J_0^{\text{old}} \xrightarrow{\alpha_0^\vee} J_1(N)^2.$$

As  $\alpha_0$  is an isomorphism,

$$(3.8) \quad \deg(\alpha) = \#(J_{0,\text{old}} \cap J_{0,\text{new}}).$$

Let  $f_1, \dots, f_g$  ( $g = \text{genus}(X_1(N))$ ) be a basis of  $S_2(\Gamma_1(N))$  consisting of eigenforms of  $T_p$ . Let  $a_{p,i}$  be the eigenvalue of  $T_p$  on  $f_i$ . Then by the Eichler-Shimura isomorphism we can diagonalize the action of  $T_p^*$  on  $V_l J_1(N) \otimes \bar{\mathbb{Q}}_l$  ( $l$  any prime) with  $a_{p,1}, \dots, a_{p,g}, \bar{a}_{p,1}, \dots, \bar{a}_{p,g}$  on the diagonal. In the same basis the diagonal matrix representing  $T_{p^*}$  will be  $\bar{a}_{p,1}, \dots, \bar{a}_{p,g}, a_{p,1}, \dots, a_{p,g}$  (because  $T_{p^*}$  and  $T_p^*$  are dual to each other under the Hermitian Petersson inner product). We conclude that

$$(3.9) \quad \deg(\alpha) = \left\{ \prod_{i=1}^g ((p+1)^2 - a_{p,i} \bar{a}_{p,i}) \right\}^2.$$

The Weil estimates  $|a_{p,i}| \leq 2\sqrt{p}$  give the Archimedean estimate

$$(p-1)^{4g} \leq \deg(\alpha) \leq (p+1)^{4g}.$$

On the other hand, we will be interested in the  $l$ -part of  $\deg(\alpha)$ . Since  $l|p-1$  we have  $\deg(\alpha) \equiv \{\prod_{i=1}^g (4 - a_{p,i} \bar{a}_{p,i})\}^2 \pmod{l}$ .

### 3.2. The Néron models.

Let  $\mathcal{J}_{0,\text{new}}$  (resp.  $\mathcal{J}_0^{\text{new}}$ ) be the Néron model of  $J_{0,\text{new}}$  (resp.  $J_0^{\text{new}}$ ) over  $\mathbb{Z}_p$ . From [BLR], Theorem 7.5/4 we deduce the exactness of the sequences

$$(3.10) \quad 0 \rightarrow \mathcal{J}_{0,\text{new}} \rightarrow \mathcal{J}_0 \rightarrow J_0^{\text{old}} \rightarrow 0,$$

$$(3.11) \quad 0 \rightarrow J_{0,\text{old}} \rightarrow \mathcal{J}_0 \rightarrow \mathcal{J}_0^{\text{new}},$$

from which we get exact sequences for the identity components

$$(3.12) \quad 0 \rightarrow \mathcal{J}_{0,\text{new}} \cap \mathcal{J}_0^0 \rightarrow \mathcal{J}_0^0 \rightarrow J_0^{\text{old}} \rightarrow 0,$$

$$(3.13) \quad 0 \rightarrow J_{0,\text{old}} \rightarrow \mathcal{J}_0^0 \rightarrow (\mathcal{J}_0^{\text{new}})^0 \rightarrow 0.$$

Recall the exact sequence

$$0 \rightarrow \text{Hom}(M_0, \mathbb{G}_m) \rightarrow \mathcal{J}_{0/\mathbb{F}_p}^0 \rightarrow J_0^{\text{ét}} \times J_0^\mu \rightarrow 0$$

for the connected component of the special fiber, which came from Raynaud's theorem. In the next proposition we put the three last exact sequences into one diagram.

PROPOSITION 3.2. — *Consider the following diagram of group schemes over  $\mathbb{F}_p$  :*

$$(3.15) \quad \begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & \mathcal{J}_{0,\text{new}} \cap \mathcal{J}_0^0 & \supset & \mathcal{J}_{0,\text{new}}^0 \\ & \searrow & & & \downarrow & & \\ & \text{Hom}(M_0, \mathbb{G}_m) & \subset & & \mathcal{J}_0^0 & \longrightarrow & (\mathcal{J}_0^{\text{new}})^0 \longrightarrow 0 \\ 0 & \longrightarrow & J_{0,\text{old}} & \xrightarrow{\iota} & \downarrow \alpha & & \\ & & & & \pi \downarrow & \swarrow \psi & \\ & & & & J_0^{\text{old}} & \xleftarrow{\tilde{\gamma}} & J_1(N)^2 \\ & & & & \downarrow & & \searrow \\ & & & & 0 & & 0 \end{array}$$

where we have used the following notation. The map  $\iota$  (resp.  $\pi$ ) is the inclusion of (resp. projection onto) the old subvariety (resp. the old quotient). Then  $\tilde{\alpha} = \pi \circ \iota$  is the isogeny denoted by this name before. The map  $\psi$  is the one obtained from Raynaud's theorem, restricting a line bundle in  $\mathcal{J}_{0/\mathbb{F}_p}^0 = \text{Pic}_{X_0/\mathbb{F}_p}^0$  to  $X_0^{\text{ét}}$  and  $X_0^\mu$ . The target of  $\psi$  is  $J_0^{\text{ét}} \times J_0^\mu$ . This is identified with  $J_1(N)^2$  via the isomorphism

$$(3.16) \quad \theta^* : J_1(N)^2 \simeq J_0^{\text{ét}} \times J_0^\mu$$

which is derived from

$$(3.17) \quad \begin{aligned} \theta : X_0^{\text{ét}} \times X_0^\mu &\simeq X_1(N)^2, \theta([E', P'_N, H'], [E, P_N, H]) \\ &= ([E'/H', P'_N \bmod H'], [E, P_N]). \end{aligned}$$

Note that  $\theta^*$  sends the first copy of  $J_1(N)$  to  $J_0^{\text{ét}}$ , and the second to  $J_0^\mu$ . The map  $\tilde{\gamma}$  exists since  $\text{Ker}(\psi) \subset \text{Ker}(\pi)$  (as it is obviously new and connected). Then

- $\text{Hom}(M_0, \mathbb{G}_m) = \mathcal{J}_{0,\text{new}}^0$ .
- Let  $\gamma \in \text{End}(J_1(N)^2)$  be the map  $\tilde{\gamma}$  followed by the canonical isomorphism  $\alpha_0^\vee : J_0^{\text{old}} \rightarrow J_1(N)^2$ . Let  $\beta = \psi \circ \iota \circ \alpha_0 \in \text{End}(J_1(N)^2)$ .

Then as matrices

$$(3.18) \quad \beta = \begin{pmatrix} \langle p \rangle_N^{-1} V & 1 \\ 1 & V \end{pmatrix}, \quad \gamma = \begin{pmatrix} \langle p \rangle_N F & 1 \\ 1 & F \end{pmatrix}$$

where  $F$  and  $V$  are the Frobenius and Verschiebung endomorphisms of  $J_1(N)_{/\mathbb{F}_p}$ . Consequently  $\beta$  and  $\gamma$  are dual to each other, and  $\alpha = \gamma \circ \beta$ .

- $\deg(\gamma) = [\mathcal{J}_0^0 \cap \mathcal{J}_{0,\text{new}} : \mathcal{J}_{0,\text{new}}^0] = \deg(\beta)$  and

$$(3.19) \quad [\mathcal{J}_0^0 \cap \mathcal{J}_{0,\text{new}} : \mathcal{J}_{0,\text{new}}^0]^2 = \#(J_{0,\text{old}} \cap J_{0,\text{new}}).$$

*Proof.* — The first assertion is obvious, as  $\mathcal{J}_{0,\text{new}}^0/\mathbb{F}_p$  is a torus, but  $\text{Hom}(M_0, \mathbb{G}_m)$  is the maximal torus in  $\mathcal{J}_0^0$ . Let  $\gamma = \alpha_0^\vee \circ \tilde{\gamma}$  as above. Consider a point

$$(3.20) \quad {}^t(x, 0) = {}^t \left( \text{cl} \left\{ \sum m_i [E_i, P_{N,i}] \right\}, 0 \right) \in J_1(N)(\overline{\mathbb{F}}_p)^2,$$

( $\sum m_i = 0$ ,  $\text{cl}$  denotes the divisor class in  $\text{Pic}_{X_1(N)/\overline{\mathbb{F}}_p}^0$ ), with all the  $E_i$  ordinary elliptic curves over  $\overline{\mathbb{F}}_p$ , and represent  $x$  by the corresponding  $\sum m_i [E'_i, P'_{N,i}, H'_i] \in J_0^{\text{ét}}(\overline{\mathbb{F}}_p)$ , where  $H'_i = \text{Ker}(\text{Ver}_{E'_i})$ ,  $E_i = E'_i/H'_i$  and  $P_{N,i} = P'_{N,i} \bmod H'_i$ , so that the point  $[E'_i, P'_{N,i}, H'_i]$  lies on  $X_0^{\text{ét}}$  (but is not in  $S$ ). Here we denote by  $\text{Frob}_E : E \rightarrow E^{(p)}$  and  $\text{Ver}_E : E \rightarrow E^{(p^{-1})}$  the Frobenius and Verschiebung of an elliptic curve over  $\overline{\mathbb{F}}_p$  (to distinguish them from  $F$  and  $V$ , which are the corresponding endomorphisms of  $J_1(N)$ ). Now  ${}^t(x, 0) = \psi(\text{cl}\{\sum m_i [E'_i, P'_{N,i}, H'_i]\})$ , where now we consider the divisor as a divisor on  $\mathfrak{X}_{0/\overline{\mathbb{F}}_p}$ , which is of degree 0 on each irreducible component, and therefore represents a point in  $\mathcal{J}_0^0(\overline{\mathbb{F}}_p)$ . Thus

$$(3.21) \quad \begin{aligned} \gamma({}^t(x, 0)) &= \alpha_0^\vee \circ \pi(\text{cl}\{\sum m_i [E'_i, P'_{N,i}, H'_i]\}) \\ &= {}^t(i_*, j_*)(\text{cl}\{\sum m_i [E'_i, P'_{N,i}, H'_i]\}). \end{aligned}$$

It is easily checked that

$$(3.22) \quad i_*([E'_i, P'_{N,i}, H'_i]) = [E'_i, P'_{N,i}] = \langle p \rangle_N^* F([E_i, P_{N,i}])$$

$$(3.23) \quad \text{and } j_*([E'_i, P'_{N,i}, H'_i]) = [E_i, P_{N,i}].$$

This gives the first column in the matrix of  $\gamma$ . Similar computations give the second column, and the matrix of  $\beta$ . But

$$(3.24) \quad \gamma \circ \beta = \alpha_0^\vee \circ \tilde{\gamma} \circ \psi \circ \iota \circ \alpha_0 = \alpha_0^\vee \circ \pi \circ \iota \circ \alpha_0 = \alpha_0^\vee \circ \tilde{\alpha} \circ \alpha_0 = \alpha.$$

Notice that we have re-proved the Eichler-Shimura congruence relations

$$(3.25) \quad T_p^* = \langle p \rangle_N^* F + V, \quad T_{p^*} = F + \langle p \rangle_{N^*} V.$$

Finally the degrees of  $\gamma$  and  $\beta$  are easily computed from the big diagram and are each equal to  $\sqrt{\deg(\alpha)}$ , which was computed before. This concludes the proof.  $\square$

### 3.3. Breaking the deformation into components.

Although the deformation  $\mathbf{V}^\# \rightarrow V^\#$  can be studied as one piece, for applications (such as for elliptic curves uniformized by  $X_0$ ), it is often necessary to break it into “components”. We shall now make certain assumptions which we keep until the end of this work.

Fix an orthogonal decomposition

$$(3.26) \quad S_2(\Gamma_0) = S \oplus S^\perp$$

(with respect to the Petersson inner product) where the two subspaces are rational over  $\mathbb{Q}$  (with respect to the  $\mathbb{Q}$ -structure given by the  $q$ -expansions) and  $\mathcal{H}$ -stable. It is well-known that  $S$  and  $S^\perp$  are the cotangent spaces of abelian subvarieties  $A, A^\perp \subset J_0$ , defined over  $\mathbb{Q}$ , stable under  $\mathcal{H}$ , whose intersection is finite, and whose sum is  $J_0$ . We shall assume the following three conditions:

- (A1)  $A$  is not  $l$ -Eisenstein (see Section 1.6).
- (A2)  $(\#A \cap A^\perp, l) = 1$ .
- (A3)  $A$  is isogenous to a subvariety of  $J_0(Np)$ , and if  $l|N$ , then  $N_l = l$  and  $A$  is  $l$ -new.

The second assumption means that  $J_0[r] = A[r] \oplus A^\perp[r]$ , and (A1) and (A2) together imply that

$$(3.27) \quad J_0^\#[r] = A[r] \oplus A'[r],$$

where we have denoted by  $A'$  the inverse image of  $A^\perp$  in  $J_0^\#$  (see Section 1.6). Proposition 1.1 and Lemma 1.5 will allow us to lift this decomposition uniquely in  $J_1^\#[r, I']$ .

In the category of abelian varieties up to isogeny (defined over  $\mathbb{Q}$ ) we have

$$(3.28) \quad \mathcal{H}(J_0) \otimes \mathbb{Q} \subset \text{End}_{\mathcal{H}}(J_0 \otimes \mathbb{Q}) \subset \text{End}_{\mathcal{H}}(S_2(\Gamma_0, \mathbb{Q})) = \mathcal{H}(J_0) \otimes \mathbb{Q},$$

which shows that  $\text{End}_{\mathcal{H}}(J_0 \otimes \mathbb{Q}) = \mathcal{H}(J_0) \otimes \mathbb{Q}$ . Since  $A$  is  $\mathcal{H}$ -stable, the projector onto  $A \otimes \mathbb{Q}$  in the category of abelian varieties up to isogeny commutes with  $\mathcal{H}$ , hence is a (rational) Hecke operator, which is an idempotent. We call it  $e_A \in \mathcal{H}(J_0) \otimes \mathbb{Q}$ .

Let  $d$  be an integer which is divisible by  $\#(A \cap A^\perp)$ , and satisfies  $d \equiv 1 \pmod r$  (its existence follows from (A2)). Note that  $de_A \in \text{End}(J_0)$ , because it is the endomorphism which is equal to  $d_A$  on  $A$  and to 0 on  $A^\perp$ . As some rational multiple of  $de_A$  is a Hecke operator,  $de_A \in \mathcal{H}'(J_0)$ . We are in a position to apply Proposition 1.1, because if  $l|N$ , condition (A3) implies that  $U_l$  is invertible on  $A$ . Let  $\mathfrak{M}$  be any maximal ideal of  $\mathcal{H}_l(J_0)$ , which is in the support of  $\mathcal{H}_l(A)$ . If  $l|N$ ,  $\mathfrak{M}$  is ordinary. We conclude from Proposition 1.1 that

$$(3.29) \quad e_{\mathfrak{M}}de_A \in e_{\mathfrak{M}}\mathcal{H}'_l(J_0) = e_{\mathfrak{M}}\mathcal{H}_l(J_0) \subset \mathcal{H}_l(J_0).$$

Summing over all the maximal  $\mathfrak{M}$  in the support of  $\mathcal{H}_l(A)$  we see that  $de_A \in \mathcal{H}_l(J_0)$ . Its image in  $\mathcal{H}_l(J_0[r])$  is an idempotent, projecting to  $A[r]$ , because  $d \equiv 1 \pmod r$ . Since  $A[r]$  is not Eisenstein, we arrive at the following conclusion.

LEMMA 3.3. — *There exists a unique idempotent  $e \in \mathcal{H}(J_0^\#[r])$ , which projects onto  $A[r]$ , and kills  $A'[r]$ .*

By Lemma 1.5, this idempotent lifts uniquely to an idempotent  $e \in \mathcal{H}(J_1^\#[r, I'])$ . We shall be interested in the deformation

$$(3.30) \quad e\mathbf{V}^\# \rightarrow eV^\# = A[r]^*.$$

PROPOSITION 3.4. — *Under the assumptions (A1) and (A2) made above,  $e\mathbf{V}^\# = e\mathbf{V}$  and  $eV^\# = eV$ . As a  $\Lambda$ -module  $e\mathbf{V} \simeq \Lambda^{2g}$  and  $eV = e\mathbf{V} \otimes_\Lambda R \simeq R^{2g}$  where  $g$  is the dimension of  $A$ . Each of  $e\mathbf{U}^\# = e\mathbf{U}$  and  $e\mathbf{W}^\# = e\mathbf{W}$  is isomorphic to  $\Lambda^g$ , and as  $\mathcal{H}$ -modules they are dual to each other. The decomposition group at  $p$  acts on  $e\mathbf{U}$  and  $e\mathbf{W}$  by the characters  $\omega\phi^{-1}$  and  $\phi$ , as described in Theorem 2.8.*

*Proof.* — By assumption (A1) the idempotent  $e$  is supported at non-Eisenstein primes. It follows from the uniqueness of the lifting of  $e$  to  $e$  that  $e$  too is supported at non-Eisenstein primes, hence  $e\mathbf{V}^\# = e\mathbf{V}$  etc.

Consider the structure of  $J_1^\#[r, I']^* = \mathbf{V}^\#$  over  $\Lambda$  as discussed in Section 2.6. The line  $Rv_0 \simeq \Lambda/I \subset \mathbf{V}^\#$  is not canonical, but as explained in Section 2.6 its image  $Ru_0$  in  $J_0^\#[r]^* = V^\#$  is canonical. Furthermore, it

was computed there that  $u_0 \in J_0[r]^*$  and is represented, under the Weil pairing, by a unique  $\tilde{u}_0 \in J_0[r]$ , which belongs to the Shimura subgroup. The line  $R\tilde{u}_0$  maps *injectively* into the group of connected components  $\Phi(J_0, \mathbb{Z}_p)$ . All these identifications respect the Hecke action. However, a theorem of Ribet and Edixhoven (see [E]) asserts that the Hecke action on  $\Phi(J_0, \mathbb{Z}_p)$  is Eisenstein. We conclude that the Hecke action on  $Ru_0$  is Eisenstein too, so  $(1 - e)u_0 = u_0$ . It follows that in the decomposition  $\mathbf{V}^\# = \mathbf{eV}^\# \oplus (1 - \mathbf{e})\mathbf{V}^\#$ , the single copy of  $\Lambda/I$  survives in  $(1 - \mathbf{e})\mathbf{V}^\#$ , and  $\mathbf{eV}^\#$  is free over  $\Lambda$ . The rest follows from Theorem 2.8 and Proposition 2.9. In particular the equality of  $\Lambda$ -ranks of  $\mathbf{eU}$  and  $\mathbf{eW}$  follows from the fact that

$$(3.31) \quad \mathbf{eW} \simeq \text{Hom}(\mathbf{eU}, R)$$

and from the fact that both  $\mathbf{eU}$  and  $\mathbf{eW}$  are free over  $\Lambda$ . □

#### 4. The infinitesimal deformation of a $p$ -new component, and its relationship with the $p$ -adic period matrix.

##### 4.1. The infinitesimal deformation.

Let  $\tilde{\Lambda} = \Lambda/I^2 \simeq R[\varepsilon]$  ( $\varepsilon^2 = 0$ ), the ring of dual numbers over  $R$ . To make the isomorphism concrete fix once and for all a primitive  $r$ -th root of unity  $\rho \in \mathbb{F}_p^\times$  and identify  $\rho - 1 \pmod{I^2}$  with  $\varepsilon$ , so that  $\rho^k - 1 \equiv k\varepsilon \pmod{I^2}$ . For any  $\Lambda$ -module  $M$  we denote by  $\tilde{M}$  the module  $M/I^2M$ . For any  $R = \Lambda/I$  module  $N$  we denote by  $N(1)$  the module  $N \otimes_R I/I^2$ . Since  $I/I^2 = \mu_r \simeq R$  non-canonically,  $N$  and  $N(1)$  are non-canonically isomorphic, but we can use our choice of  $\rho$  to identify them if we map  $\varepsilon$  to 1. For future reference note also that the identification of  $I/I^2$  with  $\mu_r$  is via

$$(4.1) \quad \langle u \rangle_p - 1 \pmod{I^2} \mapsto u_R,$$

where  $u_R$  is the projection of  $u$  to  $\mu_r$ .

The basic diagram

$$(4.2) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathbf{eU} & \rightarrow & \mathbf{eV} & \rightarrow & \mathbf{eW} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & eU & \rightarrow & eV & \rightarrow & eW \rightarrow 0 \end{array}$$

describing the deformation of  $eV = A[r]^*$ , yields a diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & eU(1) & \rightarrow & eV(1) & \rightarrow & eW(1) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 (4.3) & & 0 & \rightarrow & e\tilde{U} & \rightarrow & e\tilde{V} & \rightarrow & e\tilde{W} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 0 & \rightarrow & eU & \rightarrow & eV & \rightarrow & eW & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0. & & & & 
 \end{array}$$

We now twist the local Galois action by the character  $\phi^{-1}$ , thereby trivializing the action on the right column of the diagram, and take the long exact sequences in cohomology. Consider the following portion of what is obtained:

$$\begin{array}{ccc}
 H^0(\mathbb{Q}_p, e\tilde{W}(\phi^{-1})) & \rightarrow & H^1(\mathbb{Q}_p, e\tilde{U}(\phi^{-1})) \\
 \downarrow & & \downarrow \\
 H^0(\mathbb{Q}_p, eW(\phi^{-1})) & \xrightarrow{\delta_0} & H^1(\mathbb{Q}_p, eU(\phi^{-1})) \\
 & & \downarrow \delta_1 \\
 & & H^2(\mathbb{Q}_p, eU(1)(\phi^{-1})).
 \end{array}$$

Since the vertical arrow between the  $H^0$  is surjective, and since  $H^0(\mathbb{Q}_p, eW(\phi^{-1})) = eW$ , we obtain the following result.

PROPOSITION 4.1. — *In the situation described above, the map*

$$(4.5) \quad \delta_1 \circ \delta_0 : eW \rightarrow H^2(\mathbb{Q}_p, eU(1)(\phi^{-1}))$$

is 0.

The rest of this work will be devoted to analyzing the relation  $\delta_1 \circ \delta_0 = 0$ .

### 4.2. The case of a $p$ -old subvariety.

Suppose that  $A$  is  $p$ -old. In this case we have nothing interesting, because the short exact sequence  $0 \rightarrow eU \rightarrow eV \rightarrow eW \rightarrow 0$  is *split*. Indeed, the fact that  $A$  is  $p$ -old implies that  $A[r] \subset \mathcal{J}_0^0[r]$ . In addition,  $J_{0,\text{new}} \subset A^\perp$  so  $\text{Hom}(M_0, \mu_r) \subset A^\perp[r]$ . Since by our assumption  $A[r]$  and  $A^\perp[r]$  intersect trivially,  $A[r] \hookrightarrow J_0^{\text{ét}}[r] \times J_0^\mu[r]$ . It is now clear that the filtration on  $A[r]$

corresponds to this decomposition and therefore splits, and dualizing, the filtration on  $eV$  splits. It follows that if  $A$  is  $p$ -old, already  $\delta_0 = 0$ .

### 4.3. The case of a $p$ -new subvariety.

Assume from now on that  $A$  is  $p$ -new. Since  $\mathcal{H}(A)$  is a quotient of  $\mathcal{H}(J_{0,\text{new}})$ , we get the relation  $U_p = -w_p$  on  $A$ , and therefore

$$(4.6) \quad U_p^2 = \langle -p \rangle_N.$$

By (A3), the Hecke operators  $\langle t \rangle_N$  act trivially on  $A$ , and therefore  $U_p^2 = 1$ . Furthermore, if  $A_0$  is the abelian subvariety of  $J_0(Np)$  isogenous to  $A$ , the isogeny  $A_0 \rightarrow A$  induces an isomorphism  $A_0[r] \simeq A[r]$ , because the kernel of  $J_0(Np) \rightarrow J_0$  is *purely Eisenstein* (see the proof of Proposition 2.5, in particular diagram (2.29) and replace  $p$  by  $N$  in the arguments there. See also [M], Proposition 11.7), and  $A$  satisfies (A1). We may therefore regard  $A[r]$  as a submodule of  $J_0(Np)$ .

### 4.4. The $p$ -adic period pairing.

Let  $\mathbb{Q}_p(S)$  be the unramified extension of  $\mathbb{Q}_p$  whose residue field  $\mathbb{F}_p(S)$  is the minimal field of definition for all the supersingular points  $S$  of  $X_1(N)/\mathbb{F}_p$ . Recall that we denoted by  $M_0 = \mathbb{Z}[S]_0$  the augmentation subgroup of  $\mathbb{Z}[S]$ . The  $p$ -adic uniformization of  $J_{0,\text{new}}$  over  $\mathbb{Q}_p(S)$  is obtained as follows (see [B]). The character group of the torus  $\mathcal{J}_{0,\text{new}}^0/\mathbb{F}_p$  is  $M_0$  (see Proposition 3.2). Let  $L_0$  be the character group of the torus  $(\mathcal{J}_{0/\mathbb{F}_p}^{\text{new}})^0$ , and recall that the abelian varieties  $J_{0,\text{new}}$  and  $J_0^{\text{new}}$  are dual to each other. Then there is a canonical isomorphism of rigid-analytic groups over  $\mathbb{Q}_p(S)$

$$(4.7) \quad J_{0,\text{new}} = \text{Hom}(M_0, \mathbb{G}_m^{\text{an}})/L_0$$

where  $L_0$  injects as a discrete cocompact lattice in  $\text{Hom}(M_0, \mathbb{Q}_p(S)^\times)$ . The pairing

$$(4.8) \quad Q : M_0 \times L_0 \rightarrow \mathbb{Q}_p(S)^\times$$

is called the  *$p$ -adic period pairing of  $J_{0,\text{new}}$* . The dual abelian variety is canonically uniformized as

$$(4.9) \quad J_0^{\text{new}} = J_{0,\text{new}}^\vee = \text{Hom}(L_0, \mathbb{G}_m^{\text{an}})/M_0.$$

The principal polarization of  $J_0$  induces the polarization map  $J_{0,\text{new}} \rightarrow J_0^{\text{new}}$ . The corresponding isogeny of connected components of special fibers of Néron models is

$$(4.10) \quad \text{Hom}(M_0, \mathbb{G}_m) \rightarrow \text{Hom}(L_0, \mathbb{G}_m)$$

and it therefore induces an inclusion  $h : L_0 \hookrightarrow M_0$ . The pairing

$$(4.11) \quad Q_h : (l, l') \mapsto Q(h(l), l'), \quad (l, l' \in L_0)$$

is symmetric and if  $\text{ord}$  denotes the valuation of  $\mathbb{Q}_p(S)$ ,  $\text{ord} \circ Q_h$  is positive definite. Note that  $h$  identifies  $L_0$  as the subgroup of  $M_0$  annihilating  $J_{0,\text{old}} \cap \mathcal{J}_{0,\text{new}}^0$ . The polarization map  $J_{0,\text{new}} \rightarrow J_0^{\text{new}}$  is now induced from the map  $h$  via the uniformizations of these two abelian varieties. Its degree is  $[M_0 : h(L_0)]^2$ .

The uniformization of  $A$  is obtained as follows. Let  $M_A$  be the character group of the connected component of the special fiber of the Néron model of  $A$ . Then  $M_A$  is a quotient of  $M_0$ . Let  $L_A = L_0 \cap \text{Hom}(M_A, \mathbb{Q}_p(S)^\times)$ . Then

$$(4.12) \quad A = \text{Hom}(M_A, \mathbb{G}_m^{\text{an}}) / L_A.$$

We denote by

$$(4.13) \quad Q_A : M_A \times L_A \rightarrow \mathbb{Q}_p(S)^\times$$

the corresponding pairing.

The Galois group  $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$  acts on  $A(\bar{\mathbb{Q}}_p)$ . It also acts on  $M_A$  and  $L_A$  via the unramified quotient  $\text{Gal}(\mathbb{Q}_p(S)/\mathbb{Q}_p)$ , and the  $p$ -adic uniformization as well as the  $p$ -adic period pairing  $Q_A$  are compatible with this action. The endomorphism  $w_p$  acts on  $M_A$  and  $L_A$  via  $-\sigma$ , where  $\sigma$  is the Frobenius automorphism of  $\text{Gal}(\mathbb{Q}_p(S)/\mathbb{Q}_p)$  (see footnote preceding Proposition 2.2).

LEMMA 4.2. — *The values of the period pairing  $Q_A$  lie in  $\mathbb{Q}_p^\times$ .*

*Proof.* — This is a consequence of assumption (A3). Apply  $\sigma$  and compute:

$$(4.14) \quad \sigma Q_A(m, l) = Q_A(\sigma m, \sigma l) = Q_A(w_p m, w_p l) = Q_A(w_p^2 m, l) = Q_A(m, l).$$

Alternatively, one can “twist” the abelian variety  $A$  by the character

$$\phi : \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow \text{Aut}(A)$$

(see Proposition 2.6, and recall that  $U_p$  is invertible). The twisted abelian variety has *split* multiplicative reduction over  $\mathbb{Q}_p$ , because the Galois action on  $M_0$  has been trivialized. However, it has the same period pairing as  $A$ .  $\square$

The principal polarization of  $J_0$  induces the polarization

$$(4.15) \quad A \subset J_{0,\text{new}} \rightarrow J_0^{\text{new}} \rightarrow A^\vee$$

on  $A$ , or the inclusion  $h_A : L_A \hookrightarrow M_A$ . By assumption (A2) the degree of this polarization is prime to  $l$ , and therefore

$$(4.16) \quad h_A : L_A \otimes \mathbb{Z}_l \simeq M_A \otimes \mathbb{Z}_l$$

is an isomorphism.

LEMMA 4.3. — *The short exact sequence  $0 \rightarrow eU \rightarrow eV \rightarrow eW \rightarrow 0$  is dual to the short exact sequence*

$$(4.17) \quad 0 \rightarrow \text{Hom}(M_A, \mu_r) \rightarrow A[r] \rightarrow L_A/rL_A \rightarrow 0$$

*which is obtained from the  $p$ -adic uniformization.*

*Proof.* — We know that  $eJ_0[r] = A[r]$ . It is enough to prove that

$$eJ_0[r]^{\text{sub}} = \text{Hom}(M_A, \mu_r).$$

Pick  $e' \in \mathcal{H}(J_0)$  projecting onto  $e$  in  $\mathcal{H}(J_0[r])$ , which maps  $J_0$  onto  $A$ . Let  $\mathcal{A}$  be the Néron model of  $A$  over  $\mathbb{Z}_p$ . Then  $e'$  maps  $\mathcal{J}_{0/\mathbb{F}_p}^0$  onto  $\mathcal{A}_{/\mathbb{F}_p}^0$ , so we have

$$(4.18) \quad eJ_0[r]^{\text{sub}} \subseteq e\mathcal{J}_{0/\mathbb{F}_p}^0[r] \subseteq \mathcal{A}_{/\mathbb{F}_p}^0[r] \\ = \text{Hom}(M_A, \mu_r) \subseteq e\text{Hom}(M_0, \mu_r) \subseteq eJ_0[r]^{\text{sub}}.$$

This proves the lemma.  $\square$

#### 4.5. The map $\delta_0$ .

We shall now analyze the map  $\delta_0 : eW \rightarrow H^1(\mathbb{Q}_p, eU(\phi^{-1}))$ . Since, by the last lemma,

$$(4.19) \quad eW = M_A/rM_A(-1), \quad eU = \text{Hom}(L_A, R), \quad \text{hence also} \\ H^1(\mathbb{Q}_p, eU(\phi^{-1})) = H^1(\mathbb{Q}_p, \text{Hom}(L_A, R)(\phi^{-1})) = \text{Hom}(L_A, H^1(\mathbb{Q}_p, R))$$

(note that the twist by  $\phi^{-1}$  has the affect of trivializing the quadratic, unramified Galois action on  $L_A$ ), the map  $\delta_0$  is a map from  $M_A/rM_A(-1)$  to the group  $\text{Hom}(L_A, H^1(\mathbb{Q}_p, R))$ , or, tensoring with  $\mu_r$  (which has a trivial Galois action), it is equivalent to a map

$$(4.20) \quad \delta_0(1) : M_A \otimes R \rightarrow \text{Hom}(L_A, \mathbb{Q}_p^\times \otimes R).$$

Here we have used Hilbert’s theorem 90, saying that  $H^1(\mathbb{Q}_p, \mu_r) = \mathbb{Q}_p^\times \otimes R$ .

PROPOSITION 4.4. — *The map  $\delta_0(1)$  is the map obtained from the pairing  $Q_A \otimes R$ .*

*Proof.* — This is an easy application of Kummer theory. See [dS1], Lemma 3.4. □

### 4.6. The map $\delta_1$ .

A similar analysis holds for  $\delta_1(1)$ . Its domain is

$$(4.21) \quad \text{Hom}(L_A, \mathbb{Q}_p^\times \otimes R) = \text{Hom}(L_A, R) \otimes \mathbb{Q}_p^\times,$$

and its range is the group

$$(4.22) \quad H^2(\mathbb{Q}_p, \text{Hom}(L_A, \mu_r)(\phi^{-1})) \otimes \mu_r = \text{Hom}(L_A, \mu_r),$$

where we have used local class field theory to identify  $H^2(\mathbb{Q}_p, \mu_r)$  with  $R$ .

For every  $\sigma \in \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$  consider  $\sigma - 1$  acting on  $e\tilde{U}(\phi^{-1})$ . Since  $\phi^2 = 1$  on  $eU = \text{Hom}(L_A, R)$  (or on  $eU(1) = \text{Hom}(L_A, \mu_r)$ ),  $\sigma - 1$  induces a map

$$(4.23) \quad P(\sigma) : \text{Hom}(L_A, R) \rightarrow \text{Hom}(L_A, \mu_r),$$

and the map  $\sigma \mapsto P(\sigma)$  is a homomorphism. (With respect to suitable  $R$ -bases the matrix of  $\sigma$  is

$$(4.24) \quad \begin{pmatrix} 1 & P(\sigma) \\ 0 & 1 \end{pmatrix}.)$$

It therefore factors through  $\text{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ .

PROPOSITION 4.5. — *The map*

$$(4.25) \quad \delta_1(1) : \text{Hom}(L_A, R) \otimes \mathbb{Q}_p^\times \rightarrow \text{Hom}(L_A, \mu_r)$$

is described as follows. Let  $t \in \mathbb{Q}_p^\times$ , and denote by  $\sigma_t \in \text{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$  the Artin symbol of  $t$ . Then

$$(4.26) \quad \delta_1(1)(x \otimes t) = P(\sigma_t)(x).$$

*Proof* (Compare [GS], Theorem 3.11). — Abbreviate  $G = \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ . Consider Tate's local duality pairing

$$(4.27) \quad \text{Hom}(G, R) \times (\mathbb{Q}_p^\times \otimes R) = H^1(\mathbb{Q}_p, R) \times H^1(\mathbb{Q}_p, \mu_r) \xrightarrow{\cup} H^2(\mathbb{Q}_p, \mu_r) = R,$$

which takes the pair  $(h, t \otimes 1)$  to  $h(\sigma_t)$ . (Although  $R$  and  $\mu_r$  are non-canonically isomorphic as Galois modules, we distinguish between them because they play dual roles). It may be “fattened” to a perfect pairing

$$(4.28) \quad (L_A \otimes \text{Hom}(G, R)) \times \text{Hom}(L_A, \mathbb{Q}_p^\times \otimes R) \rightarrow R,$$

and here  $L_A \otimes \text{Hom}(G, R) = H^1(\mathbb{Q}_p, eU(\phi^{-1})(1)^\vee)$  and  $\text{Hom}(L_A, \mathbb{Q}_p^\times \otimes R) = H^1(\mathbb{Q}_p, eU(\phi^{-1})(1))$ , where we have denoted by  $(-)^\vee = \text{Hom}(-, \mu_r)$  the Cartier dual.

Similarly, Tate's local duality gives the obvious pairing

$$(4.29) \quad R(-1) \times \mu_r = H^0(\mathbb{Q}_p, R(-1)) \times H^2(\mathbb{Q}_p, \mu_r(1)) \xrightarrow{\cup} H^2(\mathbb{Q}_p, \mu_r) = R,$$

which may be “fattened” to a perfect pairing

$$(4.30) \quad (L_A \otimes R(-1)) \times \text{Hom}(L_A, \mu_r) \rightarrow R.$$

Here

$$L_A \otimes R(-1) = H^0(\mathbb{Q}_p, eU(\phi^{-1})(2)^\vee)$$

and

$$\text{Hom}(L_A, \mu_r) = H^2(\mathbb{Q}_p, eU(\phi^{-1})(2)).$$

With respect to these two pairings the dual  $\delta_1(1)^*$  of  $\delta_1(1)$  is the connecting homomorphism

$$(4.31) \quad \delta_1(1)^* : H^0(\mathbb{Q}_p, eU(\phi^{-1})(2)^\vee) \rightarrow H^1(\mathbb{Q}_p, eU(\phi^{-1})(1)^\vee)$$

associated to the short exact sequence

$$(4.32) \quad 0 \rightarrow eU(\phi^{-1})(1)^\vee \rightarrow \mathbf{e}\tilde{\mathbf{U}}(\phi^{-1})(1)^\vee \rightarrow eU(\phi^{-1})(2)^\vee \rightarrow 0.$$

What we have to prove (dualized) is that

$$(4.33) \quad \delta_1(1)^*(l)(\sigma) = P(\sigma)^*(l), \quad l \in L_A \otimes R(-1), \quad \sigma \in G.$$

This is obvious from the definition of  $P(\sigma)$  (perhaps up to a sign, depending on how one normalizes connecting homomorphisms in long exact sequences). □

**4.7. The meaning of the relation  $\delta_1 \circ \delta_0 = 0$ .**

THEOREM 4.6. — *Let  $A$  be an abelian subvariety of  $J_0$  satisfying (A1), (A2), and (A3) above. Let  $M_A$  and  $L_A$  be the lattices defined above, and*

$$(4.34) \quad q_{A,R} : M_A \otimes R \rightarrow \text{Hom}(L_A, R) \otimes \mathbb{Q}_p^\times$$

the map obtained from the  $p$ -adic period pairing of  $A$ , taken modulo  $r$ . Let

$$(4.35) \quad \delta : \text{Hom}(L_A, R) \otimes \mathbb{Q}_p^\times \rightarrow \text{Hom}(L_A, \mu_r)$$

be the map  $\delta(l' \otimes t) = \{(\omega\phi^{-2})(\sigma_t) - 1\}(\tilde{l}')$  where  $\tilde{l}' \in \mathbf{e}\tilde{U}$  lifts  $l' \in eU = \text{Hom}(L_A, R)$ . Then  $\delta \circ q_{A,R} = 0$ .

*Proof.* — As we have seen, the maps  $q_{A,R}$  and  $\delta$  are none others but  $\delta_0(1)$  and  $\delta_1(1)$ , respectively. The characterization of the second relies also on our knowledge of the Galois action on  $\mathbf{e}\tilde{U}(\phi^{-1})$ , which is via the character  $\omega\phi^{-2}$  (see Proposition 3.4). □

To write the last theorem in a completely explicit form, decompose every  $t \in \mathbb{Q}_p^\times$  as

$$(4.36) \quad t = u(t)p^{\nu(t)},$$

where  $u(t)$  is a unit, and  $\nu(t) \in \mathbb{Z}$ . Recall that  $\phi(\sigma_t) = U_p^{\nu(t)}$  and  $\omega(\sigma_t) = \langle p \rangle_N^{\nu(t)} \langle u(t)^{-1} \rangle_p$ . Here we must recall that the  $p$ -adic and  $N$ -adic cyclotomic characters satisfy  $\chi_p(\sigma_t) = u(t)^{-1}$  and  $\chi_N(\sigma_t) = p^{\nu(t)}$ . Note that even under assumption (A3) we do not know that in the deformation  $\mathbf{e}\tilde{U}$  the operators  $\langle t \rangle_N$  act trivially, because the deformation is taken within  $J_1(Np)$ .

Now, if  $\tilde{l}' \in \mathbf{e}\tilde{U}$ ,

$$(4.37) \quad \{(\omega\phi^{-2})(\sigma_t) - 1\}(\tilde{l}') = (\nu_R(t)(U_p^{-2}\langle p \rangle_N - 1) - u_R(t))\tilde{l}',$$

where  $\nu_R(t) = \nu(t) \otimes 1 \in \mathbb{Z} \otimes R = R$  and  $u_R(t) = u(t) \otimes 1 \in \mathbb{Z}_p^\times \otimes R = \mu_R$ .

Let  $m_1, \dots, m_g$  be a basis of  $M_A$ , and  $l_1, \dots, l_g$  a basis of  $L_A$  ( $g = \dim A$ ). Let  $l'_1, \dots, l'_g$  be the dual basis in  $\text{Hom}(L_A, R)$ . Then

$$(4.38) \quad q_{A,R}(m_i \otimes 1) = \sum_j l'_j \otimes q_{ij}, \quad q_{ij} = Q_A(m_i, l_j).$$

Applying  $\delta$  we obtain from the relation  $\delta \circ q_{A,R}(m_i \otimes 1) = 0$  that

$$(4.39) \quad \sum_j \nu_R(q_{ij})(U_p^{-2}\langle p \rangle_N - 1)(\tilde{l}'_j) = \sum u_R(q_{ij})\tilde{l}'_j.$$

In terms of matrices, if  $[U_p^{-2}\langle p \rangle_N - 1]$  is the matrix representing  $U_p^{-2}\langle p \rangle_N - 1$  in a basis  $\{\tilde{l}'_j\}$  of  $\mathbf{e}\tilde{\mathbf{U}}$  lifting the basis  $\{l'_j\}$  of  $\text{Hom}(L_A, R)$ , we have the relation

$$(4.40) \quad [\nu_R(q_{ij})] [U_p^{-2}\langle p \rangle_N - 1] = [u_R(q_{ij})].$$

Let us consider the case of an *elliptic curve*  $A$  satisfying (A1)-(A3). Then there is only one period  $q$ , the Tate period of  $A$ . In this case  $eU = R$ ,  $\mathbf{e}\mathbf{U} = \Lambda$ , and the action of  $U_p^{-2}\langle p \rangle_N$  on  $\mathbf{e}\mathbf{U}$  is given by  $1 + \lambda$ , with  $\lambda \in I$ . The last formula reads then

$$(4.41) \quad \nu_R(q)\lambda \bmod I^2 = u_R(q).$$

Notice that in case  $(l, \nu_R(q)) = 1$ , the quantity  $u_R(q)/\nu_R(q)$  (we write  $\mu_r$  additively!) may be called *the refined  $\mathcal{L}$ -invariant* of  $A$  (similar to  $\log_p(q)/\nu(q)$ ). Our main result therefore relates it to the “derivative” of  $U_p^{-2}\langle p \rangle_N$  in the deformation  $\mathbf{e}\mathbf{U}$  of  $eU$ .

## BIBLIOGRAPHY

- [ALe] A.O.L. ATKIN, J. LEHNER, Hecke operators on  $\Gamma_0(m)$ , *Math. Annalen*, 185 (1970), 134-160.
- [ALi] A.O.L. ATKIN, W. LI, Twists of newforms and pseudo-eigenvalues of W-operators, *Inv. Math.*, 43 (1978), 221-244.
- [B] S. BOSCH, Abelian varieties from the rigid-analytic viewpoint, *Barsotti Symposium in Algebraic Geometry*, Academic Press (1994), 51-63.
- [BLR] S. BOSCH, W. LÜTKEBOHMERT; M. RAYNAUD, *Néron models*, *Ergebnisse der Math.*, 3. folge, Bd. 21, Springer (1990).
- [DR] P. DELIGNE, M. RAPOPORT, Schémas de modules de courbes elliptiques, in *LNМ* 349, Springer (1973).
- [dS1] E. de SHALIT, On certain Galois representations related to the modular curve  $X_1(p)$ , *Compositio Math.*, 95 (1995), 69-100.
- [dS2] E. de SHALIT,  $p$ -adic periods and modular symbols of elliptic curves of prime conductor, *Inv. Math.*, 121 (1995), 225-255.

- [dS3] E. de SHALIT, Néron models and  $p$ -adic uniformization of generalized Jacobians, in preparation.
- [E] B. EDIXHOVEN, L'action de l'algebre de Hecke sur les groupes de composantes des jacobiniennes des courbes modulaires est "Eisenstein". Astérisque, 196-197 (1991), 59-70.
- [GS] R. GREENBERG, G. STEVENS,  $p$ -adic  $L$ -functions and  $p$ -adic periods of modular forms, Inv. Math., 111 (1993), 407-447.
- [H] H. HIDA, Galois representations into  $GL_2(\mathbb{Z}_p[[X]])$  attached to ordinary cusp forms, Inv. Math., 85 (1986), 545-613.
- [KM] N. KATZ, B. MAZUR, Arithmetic moduli of elliptic curves, Ann. Math. Studies 108, Princeton (1985).
- [M] B. MAZUR, Modular curves and the Eisenstein ideal, Publ. Math. I.H.E.S., 47 (1977), 33-186.
- [MT] B. MAZUR, J. TATE, Refined conjectures of "Birch and Swinnerton-Dyer type", Duke Math. J., 54 (1987), 711-750.
- [MTT] B. MAZUR, J. TATE, J. TEITELBAUM, On  $p$ -adic analogues of the conjectures of Birch and Swinerton-Dyer, Inv. Math., 84 (1986), 1-48.
- [MW1] B. MAZUR, A. WILES, Class fields of abelian extensions of  $\mathbb{Q}$ , Inv. Math., 76 (1984), 179-330.
- [MW2] B. MAZUR, A. WILES On  $p$ -adic analytic families of Galois representations, Compositio Math., 59 (1986), 231-264.
- [Ri] K. RIBET, Congruence relations between modular forms, Proc. International Congress of Math., 17 (1983), 503-514.
- [SGA7] A. GROTHENDIECK, Modèles de Néron et monodromie, SGA 7I, exposé IX, LNM 288, Springer (1972).
- [W] A. WILES, Modular elliptic curves and Fermat's last theorem, Ann. of Math., 141 (1995), 443-551.

Manuscrit reçu le 14 mai 2001,  
 accepté le 12 juillet 2001.

Assaf GOLDBERGER,  
 University of Massachusetts  
 Department of Mathematics  
 Amherst MA (USA).  
 goldberger@math.umass.edu  
 &  
 Ehud de SHALIT,  
 Hebrew University  
 Institute of Mathematics  
 Giv'at-Ram  
 91904 Jerusalem (Israel).  
 deshalit@math.huji.ac.il