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TAME SEMIFLOWS FOR PIECEWISE LINEAR VECTOR FIELDS

by Daniel PANAZZOLO

1. Introduction.

Piecewise linear vector fields often appear in modeling physical and chemical phenomena. They are also of interest in Control Theory and a source of examples of so-called *chaotic dynamics* (see e.g. [ACT], [S]).

Given a decomposition $\mathbb{R}^n = \bigcup_{i\in I} \mathfrak{C}_i$ into pairwise disjoint subsets (called *cells*), we shall say that a vector field $X \in \mathcal{X}(\mathbb{R}^n)$ is a *piecewise linear vector field* on the decomposition $\mathcal{E} = \{\mathfrak{C}_i\}_{i\in I}$ if

$$X_{|\mathfrak{C}_i} = A_i x + b_i, \quad \text{for each } i \in I,$$

for some $n \times n$ real matrix A_i and a vector $b_i \in \mathbb{R}^n$. We let $\mathrm{PL}^n(\mathcal{E})$ denote the set of all such vector fields.

Intuitively, it is quite clear that a *flow* for a vector field $X \in \mathrm{PL}^n(\mathcal{E})$ should be defined as a suitable composition of the following *exponential maps*:

$$\mathbb{R} \times \mathbb{R}^n \ni (t,x) \longmapsto \mathrm{e}^{tA_i}x + \int_0^s \mathrm{e}^{(t-s)A_i} b_i \, \mathrm{d}s \in \mathbb{R}^n, \quad \text{for } i \in I,$$

which correspond to the flow of each individual vector field $X_i(x) = A_i x + b_i$.

Therefore, in view of the recent results concerning the o-minimality of structures including the exponential function (see e.g. [W], [DMM], [LR]),

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a natural question which arises is wherever the flow of a piecewise linear vector field can be definable in one of such o-minimal structures.

In particular, this fact would imply the definability of Poincaré first return maps in the same structure. In dimension two, an immediate consequence would be the non-accumulation of limit cycles in planar polycycles (in the same spirit of the work [MR]).

However, it is easy to convince ourselves that such flow is not always definable in an o-minimal structure.

Examples 1.1.

(i) Let $\Phi(t,x,y)$ be the flow of linear center $\dot{x} = -y, \ \dot{y} = x$. Any structure $\mathcal{S}$ which contains the graph of $\Phi(t,x,y)$ also contains the graph of $\sin(t)$, and obviously cannot be o-minimal.

(ii) Consider the one-dimensional vector field X defined as $X(x) = 1$ if $x \in \mathbb{Q}$ and $X(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. Then, a *flow* associated to X is clearly non-definable in any o-minimal structure (for instance, because the set of equilibrium points is not a finite union of intervals and points).

Our goal is to prove that the above two phenomena are the only sources of *non definability.* That is, if we remove the infinite *spiraling*, assume that the cells are in finite number (i.e. $\#I < \infty$) and that each cell $\mathfrak{C}_i$ is definable in some sufficiently large o-minimal structure, then one is able to prove some definability result.

It is important to remark that even the definition of a *flow* for a discontinuous vector fields is a subtle problem (see e.g. [Fi], [Ha]). That is the reason for introducing the notion of *weak-semiflow* in Section 3. Roughly speaking, this is a class of semiflows which is stable by the operations of *composition* and *restriction.*

Using this notion, we will be able to associate a weak-semiflow $\mathbf{\Phi}_{X,\xi}$ to each vector field $X \in \mathrm{PL}^n(\mathcal{E})$, when restricted to some finite list of *composable* cells $\xi = (\mathfrak{C}_{i_1}, \dots, \mathfrak{C}_{i_s})$ (we shall not define precisely this notion here). In this context, we prove the following result:

THEOREM 1.2. — *Let $\mathcal{E}$ be a cell decomposition of $\mathbb{R}^n$, definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$, and suppose that a vector field $X \in \mathrm{PL}^n(\mathcal{E})$ has bounded spiraling on a composable cell-list ξ. Then, ξ-restricted weak-semiflow $\mathbf{\Phi}_{X,\xi}$ is definable in $\mathbb{R}_{\text{an,exp}}$.*

In the case where $X \in \mathrm{PL}^n(\mathcal{E})$ is continuous (and hence globally Lipschitz), it will be an immediate consequence of the constructions that $\Phi_{X,\xi}$ coincides with the restriction of the usual flow Φ_X to the union of the cells $\mathfrak{C}_{i_1}, \ldots, \mathfrak{C}_{i_s}$.

In the last section, we shall introduce the notion of *definable polycycle* for a continuous piecewise linear vector field. Under suitable conditions, we shall prove the definability of the Poincaré first return map on a transversal section to such polycycles.

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2. Basic notions.

In this section, we fix some basic definitions and notations from the theory of o-minimal structures. For this, we follow closely the book of van den Dries [D].

DEFINITION 2.1. — *An* ***o-minimal structure*** *on* $\mathbb{R}$ *is a sequence* $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ *such that for each* $m \geq 0$*:*

1) $\mathcal{S}_n$ *is a boolean algebra of subsets of* $\mathbb{R}^n$;

2) $A \in \mathcal{S}_n \Rightarrow A \times \mathbb{R} \in \mathcal{S}_{n+1}$ *and* $\mathbb{R} \times A \in \mathcal{S}_{n+1}$;

3) $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j\} \in \mathcal{S}^n$, *for* $1 \leq i < j \leq n$;

4) $A \in \mathcal{S}_{n+1} \Rightarrow \pi(A) \in \mathcal{S}_n$, *where* $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ *is the projection map*;

5) $\{r\} \in \mathcal{S}_1$ *for each* $r \in \mathbb{R}$, *and* $\{(x,y) \in \mathbb{R}^2 \mid x < y\} \in \mathcal{S}_2$;

6) *the only sets in* $\mathcal{S}_1$ *are finite unions of intervals and points.*

We shall say that a set $A \subset \mathbb{R}^n$ is *definable* if $A \in \mathcal{S}_n$. A map $f : A \to \mathbb{R}^m$ is *definable* if its graph $\Gamma(f) \subset \mathbb{R}^{m+n}$ is definable.

2.1. Cell decomposition.

Let us fix once and for all some o-minimal structure $\mathcal{S}$ over the field of real numbers. For each definable set A in $\mathbb{R}^m$, we put

$$C^1(A) := \{f : A \to \mathbb{R} \mid f \text{ is definable and } C^1\},$$

$$C^1_\infty(A) := C^1(A) \cup \{-\infty, +\infty\},$$

where $-\infty$ and $+\infty$ are seen as constant functions on A. If A is not an open set, to say that f is C^1 means that there exists a definable open set $U \subset \mathbb{R}^m$ containing A and a definable C^1-function $F : U \to \mathbb{R}$ such that $F_{|A} = f$.

For f, g in $C^1_\infty(A)$, we write $f < g$ if $f(x) < g(x)$ for all $x \in A$. In this case, we define

$$(f,g)_A := \big\{(x,r) \in A \times \mathbb{R} \mid f(x) < r < g(x)\big\}.$$

Definition 2.2. — *Let $(i_1, \dots, i_m)$ be a sequence of zeros and ones of length m. An $(i_1, \dots, i_m)$-**cell** is a definable subset of $\mathbb{R}^m$ obtained by induction on m as follows:*

(i) *A (0)-cell is a one-element set $\{r\} \subset \mathbb{R}$, a (1)-cell is an interval $(a,b) \subset \mathbb{R}$.*

(ii) *Suppose $(i_1, \dots, i_m)$-cells are already defined. Then*

a $(i_1, \dots, i_m, 0)$-cell is the graph $\Gamma(f)$ of a function $f \in C^1(A)$, where A is an $(i_1, \dots, i_m)$-cell;

a $(i_1, \dots, i_m, 1)$-cell is a set $(f,g)_A$, where A is an $(i_1, \dots, i_m)$-cell and $f, g \in C^1_\infty(A)$, $f < g$.

*A **cell** in $\mathbb{R}^m$ is a $(i_1, \dots, i_m)$-cell, for some sequence $(i_1, \dots, i_m)$.*

A *decomposition* of $\mathbb{R}^m$ is a special kind of partition of $\mathbb{R}^m$ into finitely many cells. The definition is by induction on m:

(i) A decomposition of $\mathbb{R}^1 = \mathbb{R}$ is a collection
$$\big\{(-\infty, a_1), (a_1, a_2), \dots, (a_k, +\infty), \{a_1\}, \dots, \{a_k\}\big\}$$
where $a_1 < a_2 < \dots < a_k$ are points in $\mathbb{R}$.

(ii) A decomposition of $\mathbb{R}^{m+1}$ is a finite partition of $\mathbb{R}^{m+1}$ into cells A such that the set of projections $\pi(A)$ is a decomposition of $\mathbb{R}^m$. (Here $\pi : \mathbb{R}^{m+1} \to \mathbb{R}^m$ is the usual projection map.)

A decomposition $\mathcal{D}$ of $\mathbb{R}^m$ is said to *partition* a set $S \subset \mathbb{R}^m$ if S is a union of cells in $\mathcal{D}$.

Theorem 2.3 (C^1-cell decomposition; see [D], Chap. 7, Section 3.2).

($\mathrm{I_m}$) *For any definable set $A_1, \dots, A_k \subset \mathbb{R}^m$ there exists a decomposition of $\mathbb{R}^m$ into cells partitioning $A_1, \dots, A_k$.*

($\mathrm{II_m}$) *For every definable function $f : A \to \mathbb{R}$, $A \subset \mathbb{R}^m$, there exists a decomposition of $\mathbb{R}^m$ into cells, partitioning A, such that each restriction $f_{|C} : C \to \mathbb{R}$ is C^1, for each cell $C \subset A$ of the decomposition.*

2.2. The logical notation.

The logical formulas provide a synthetic way to show that a set belongs to some o-minimal structures. We shall use them quite often, adopting the usual conventions stated in [D].

It will be also convenient to adopt the following notation: given a subset $V \subset \{(y,x) \in \mathbb{R}^m \times \mathbb{R}^n\}$ and a point $x \in \mathbb{R}^n$, we let

$$V_x := V \cap \left(\mathbb{R}^m \times \{x\}\right)$$

denote the *fiber* of V over the point x.

3. Weak-semiflows.

DEFINITION 3.1. — *A* ***weak-semiflow*** *(or shortly, a w-semiflow) on $\mathbb{R}^n$ is a pair $\mathbf{\Phi} = (\mathcal{U}, \Phi)$, where $\mathcal{U} \subset \mathbb{R}^+ \times \mathbb{R}^n$ is a subset such that*

$$\forall x \in \mathbb{R}^n, \quad \mathcal{U}_x \textit{ is either empty or an interval } [0, t_+)$$

for some $t_+ \in \mathbb{R}^+_* \cup \{\infty\}$, and $\Phi : \mathcal{U} \to \mathbb{R}^n$ is a map with the following properties:

(a) *For all $x \in \mathbb{R}^n$ such that $\mathcal{U}_x \neq \emptyset$, we have $\Phi(0,x) = x$ and the curve*

$$o_x \ : \ t \in \mathcal{U}_x \longmapsto \Phi(t,x)$$

is continuous. We call it the orbit of $\mathbf{\Phi}$ through x. When $\mathcal{U}_x = \emptyset$, we say that the orbit o_x is empty.

(b) *For all $t \in \mathcal{U}_x$, if we consider the point $x_t := \Phi(t,x)$, then*

(i) *$\mathcal{U}_{x_t} = \{s \in \mathbb{R}^+ \mid t + s \in \mathcal{U}_x\}$, and*

(ii) *$\Phi(t+s, x) = \Phi(s, x_t)$, for all $t + s \in \mathcal{U}_x$.*

The ***support*** *of a a w-semiflow $\mathbf{\Phi} = (\mathcal{U}, \Phi)$ is the set*

$$\operatorname{supp} \mathbf{\Phi} = \left\{x \in \mathbb{R}^n \mid \mathcal{U}_x \neq \emptyset\right\}. \tag{1}$$

Below, we shall need to drop the condition (b.i) in some special cases, and replace it by the weaker condition

$$\text{(b.i)}' \quad \mathcal{U}_{x_t} \supset \left\{s \in \mathbb{R}^+ \mid t + s \in \mathcal{U}_x\right\}. \tag{2}$$

In this case, we shall say that the pair $\mathbf{\Phi} = (\mathcal{U}, \Phi)$ is an ***incomplete semiflow*** (or shortly, an i-semiflow).

Let $o_x(t) = \Phi(t,x)$ be the orbit of $\mathbf{\Phi}$ through x. We say that such orbit has *open end* if one of the following two conditions hold:

- $\mathcal{U}_x = [0,\infty)$, or
- $\mathcal{U}_x = [0,t_+)$ for some $t_+ \in \mathbb{R}^+_*$ and if there exists a $x_+ \in \mathbb{R}^n$ such

that

$$x_+ = \lim_{0 \le t < t_+} \Phi(t,x)$$

then we require that $\mathcal{U}_{x_+} = \emptyset$.

We shall say that the w-semiflow $\mathbf{\Phi}$ is ***definable*** on some o-minimal structure $\mathcal{S}$ when the graph $\Gamma(\Phi)$ of Φ is a definable set in $\mathbb{R}^+ \times \mathbb{R}^{2n}$.

Remark 3.2. — If we assume in Definition 3.1 that $\mathcal{U} = \mathbb{R}^+ \times \mathbb{R}^n$ and that Φ is continuous, we obtain the usual concept of a *semiflow*. This concept is too restrictive for our purposes, since is not invariant by the operations described below (see Remark 3.5).

3.1. Composing W-semiflows.

Let us consider two w-semiflows $\mathbf{\Phi}_1 = (\mathcal{U}^1, \Phi^1), \mathbf{\Phi}_2 = (\mathcal{U}^2, \Psi^2)$. We shall say that $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are *composable* if

$$\text{(3)} \qquad \text{for all } x \in \mathbb{R}^n, \text{ either } \begin{cases} \mathcal{U}^1_x \cap \mathcal{U}^2_x = \emptyset, \text{ or} \\ \mathcal{U}^1_x = \mathcal{U}^2_x,\ o^1_x \equiv o^2_x \text{ and } o^2_x \text{ has open end,} \end{cases}$$

where o^i_x denote the orbit of $\mathbf{\Phi}^i$ through x (for $i = 1,2$). In this case, we define the *composed w-semiflow* by considering the following map Ψ:

$$\text{(4)} \quad z = \Psi(t,x) \Longleftrightarrow \begin{cases} z = \Phi^1(t,x) & \text{if } (t,x) \in \mathcal{U}^1, \\ z = \Phi^2(t - t_+, x_+) & \text{if } \exists\, t_+ \in \mathbb{R}^+_* \mid \mathcal{U}^1_x = [0,t_+) \\ & \text{and } \exists\, x_+ = \lim_{0 \le t < t_+} \Phi^1(t,x) \\ & \text{and } t - t_+ \in \mathcal{U}^2_{x_+}, \\ z = \Phi^2(t,x) & \text{if } (t,x) \in \mathcal{U}^2. \end{cases}$$

Clearly, the map $z = \Psi(t,x)$ has its domain on some subset $\mathcal{V} \subset \mathbb{R}^+ \times \mathbb{R}^n$. Notice that assumption (3) implies that if $(t,x) \in \mathcal{U}^1 \cap \mathcal{U}^2$, there is no ambiguity in defining $\Psi(t,x) = \Phi^1(t,x)$ or $\Psi(t,x) = \Phi^2(t,x)$.

Lemma 3.3. — *Let $\mathbf{\Phi}^1$ and $\mathbf{\Phi}^2$ be composable w-semiflows. Then, the pair $\mathbf{\Psi} = (\mathcal{V}, \Psi)$ is a w-semiflow on $\mathbb{R}^n$.*

Proof. — All properties of Definition 3.1 are immediate to verify, except for the semigroup properties on item (b). Let us fix a point $x \in \mathbb{R}^n$ and a time $t \in \mathcal{V}_x$. We need to prove that for all $s \in \mathcal{V}_{x_t}$, we have

(i) $s + t \in \mathcal{V}_x$, and

(ii) $\Psi(t+s, x) = \Psi(s, x_t)$.

First of all, if $\mathcal{U}_x^1 = \emptyset$ then necessarily $\mathcal{V}_x = \mathcal{U}_x^2$ (because in this case only the third option of (4) can be applied to defined the positive orbit through x). We claim that

$$\mathcal{V}_{x_t} = \mathcal{U}_{x_t}^2.$$

Indeed, if $\mathcal{U}_{x_t}^1 = \emptyset$, we are done. Otherwise, Assumption (3) implies that $\mathcal{U}_{x_t}^1 = \mathcal{U}_{x_t}^2$ and the orbit of $\mathbf{\Phi}^1$ and $\mathbf{\Phi}^2$ through x_t coincide, for all t.

A problem would appear if the orbit of $\mathbf{\Phi}^1$ through x_t could be extended using the second option of (4), since this would imply that $\mathcal{V}_{x_t}$ is larger than $\mathcal{U}_{x_t}^2$. But, since the orbits of $\mathbf{\Phi}^1$ and $\mathbf{\Phi}^2$ through x_t coincide, this would contradict the assumption that o_x^2 has open end. This proves the claim. Items (i) and (ii) easily follow.

Let us suppose now that $\mathcal{U}_x^1 \neq \emptyset$. Then, either the second option of (4) does not hold for the orbit through x, and then

$$\mathcal{V}_x = \mathcal{U}_x^1,$$

or else there exists a point $x_+ \in \mathbb{R}^n$ such that

$$\mathcal{V}_x = \mathcal{U}_x^1 \cup \{u + t_+ \mid u \in \mathcal{U}_{x_+}^2\}.$$

In the former case, the same reasoning of the previous paragraph allows us to prove (i) and (ii). In the latter case, we have the three possible configurations (2.i–iii) for the times t_+, t and $t+s$ which are shown in Figure 1.

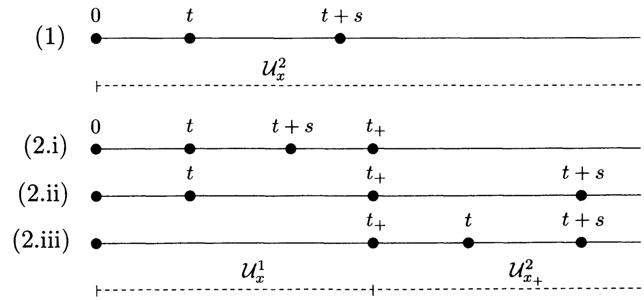

Figure 1. The four possible time scales for $\mathcal{V}_x$

Here again, it is easy to adapt the arguments used in the proof of the above claim to conclude the proof. □

When the hypothesis of the lemma hold, we shall call $\boldsymbol{\Psi}$ the *composed w-semiflow*, and note $\boldsymbol{\Psi} = [\boldsymbol{\Phi}^1, \boldsymbol{\Phi}^2]$.

LEMMA 3.4. — *If $\boldsymbol{\Psi} = (\mathcal{V}, \Psi)$ is the composition $[\boldsymbol{\Phi}^1, \boldsymbol{\Phi}^2]$ then $\mathcal{U}^1_x \subset \mathcal{V}_x$, for all $x \in \mathbb{R}^n$. Moreover,*

$$\operatorname{supp} \boldsymbol{\Psi} = \operatorname{supp} \boldsymbol{\Phi}^1 \cup \operatorname{supp} \boldsymbol{\Phi}^2.$$

Proof. — This is obvious from the definition of the composed flow. □

Notice that, if all orbits of a w-semiflow $\boldsymbol{\Phi}$ have open ends then $\boldsymbol{\Phi}$ is composable with itself. In this case, it is easy to verify that the composition $[\boldsymbol{\Phi}, \boldsymbol{\Phi}]$ is equal to $\boldsymbol{\Phi}$ (because the second option of (4) will never hold).

Remark 3.5. — Even if we assume that Φ^1 and Φ^2 are continuous maps, in general the Ψ will not be continuous, as the simple example in Figure 2 illustrates.

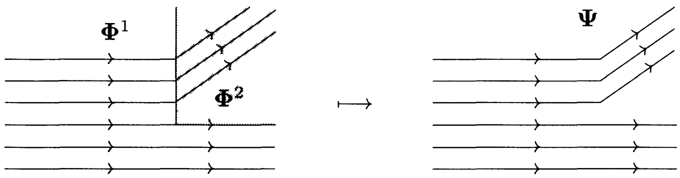

Figure 2. Example of composition of two continuous w-semiflows which gives a discontinuous w-semiflow

Remark 3.6. — It follows directly from (3) that, given two w-semiflows $\boldsymbol{\Phi}^1, \boldsymbol{\Phi}^2$, the condition $\operatorname{supp} \boldsymbol{\Phi}^1 \cap \operatorname{supp} \boldsymbol{\Phi}^2 = \emptyset$ is sufficient to guarantee that $\boldsymbol{\Phi}^1$ and $\boldsymbol{\Phi}^2$ are composable.

3.1.1. *Definable case.*

PROPOSITION 3.7. — *Suppose that two composable w-semiflows $\boldsymbol{\Phi}^1, \boldsymbol{\Phi}^2$ are definable in some o-minimal structure $\mathcal{S}$. Then, the w-semiflow $\boldsymbol{\Psi} = [\boldsymbol{\Phi}^1, \boldsymbol{\Phi}^2]$ is also a definable in such o-minimal structure.*

Proof. — We need to verify that the graph $\widetilde{\Psi}$ of the map given in (4) is definable. First of all, we define the the following subset $L^1 \subset \mathbb{R}^+ \times \mathbb{R}^n$:

$$(t,x) \in L^1 \iff \exists s\,(s \in \mathcal{U}^1_x) \wedge \forall s \in \mathbb{R}^+\,(s < t \Leftrightarrow s \in \mathcal{U}^1_x).$$

Thus, for each x, L^1_x is the length of the interval $\mathcal{U}^1_x$ (it is undefined if $\mathcal{U}^1_x$ is empty or unbounded). Now, we consider the set $C^1 \subset \mathbb{R}^n \times \mathbb{R}^n$ given by

$$(x,x_+) \in C^1 \Leftrightarrow \quad \exists s\;(s \in L^1_x) \;\wedge\; \big(\forall \varepsilon > 0,\; \exists \delta > 0,\; \forall t \in \mathcal{U}_x, \\ (|t - L^1_x| < \delta \Rightarrow \|\Phi^1(t,x) - x_+\| < \varepsilon)\big).$$

This relation associates to each point x the limit of its positive orbit (if it exists). Of course, if $L^1_x = \emptyset$ then $C^1_x = \emptyset$.

Let P^1 and P^2 be the graphs of Φ^i and Φ^2 on the space $(t,x,z) \in \mathbb{R}^+ \times \mathbb{R}^{2n}$. For simplicity, we introduce the following auxiliary subset of $\mathbb{R}^+ \times \mathbb{R}^{2n}$:

$$(t,x,z) \in P^3 \iff \exists x_+ \in \mathbb{R}^n\;\big((x,x_+) \in C^1 \wedge (t - L^1_x, x_+, z) \in P^2\big)$$

and denote by $\mathcal{U}^i$ the linear projection of P^i into the space $\{z = 0\}$, for $i = 1,2,3$.

Now, the definition of $\widetilde{\Psi}$ can be given as follows:

$$(t,x,z) \in \widetilde{\Psi} \iff (t,x,z) \in P^1 \vee \big(\neg((t,x) \in \mathcal{U}^1) \;\wedge\; (t,x,z) \in P^3\big) \\ \vee \big(\neg((t,x) \in \mathcal{U}^1 \cup \mathcal{U}^3) \wedge (t,x,z) \in P^2\big).$$

From these series of definitions, it is clear that if P^1 and P^2 belongs some o-minimal structure $\mathcal{S}$, then $\widetilde{\Psi}$ also belongs to such o-minimal structure. □

Remark 3.8. — In the definable case, the condition that the orbit $\Phi(\,.\,,x) : \mathcal{U}_x \to \mathbb{R}^n$ converges to a definite point $x_+ \in \mathbb{R}^n$ as $t \to t_+$ (where $t_+ \in \mathbb{R}^+ \cup \{\infty\}$ is the upper limit of the interval $\mathcal{U}_x$) is equivalent to require that $\Phi(\mathcal{U}_x, x)$ is a bounded set (see [D], Chap. 6, Section 4).

3.2. Restriction of a W-semiflow.

Given a subset $V \subset \mathbb{R}^n$ and a semiflow $\mathbf{\Phi} = (\mathcal{U}, \Phi)$ on $\mathbb{R}^n$, we consider the set $T_+(V) \subset \mathbb{R}^+ \times \mathbb{R}^n$ given as follows:

$$(5) \qquad T_+(V) := \big\{(t,x) \in \mathbb{R}^+_* \times \mathbb{R}^n \mid (t,x) \in \mathcal{U} \text{ and } \Phi((0,t],x) \subset V\big\}.$$

Intuitively, $T_+(V)_x$ is the interval of time during which the orbit through x remains in the set V.

LEMMA 3.9. — *Given a point $x \in \mathbb{R}^n$, following two situations can occur:*

(i) $x \in \overline{V}$: *then $T_+(V)_x$ is either empty, or has the form $(0, t_+)$, for some $t_+ \in \mathbb{R}^+ \cup \{+\infty\}$.*

(ii) $x \in \mathbb{R}^n \setminus \overline{V}$: *then $T_+(V)_x$ is empty.*

Here $\overline{V}$ denote the closure of V in $\mathbb{R}^n$.

Proof. — This follows immediately from continuity of the orbits of $\mathbf{\Phi}$. □

Let us consider now the subset $\mathcal{U}_V \subset \mathcal{U}$ given as follows:

$$\mathcal{U}_V := \big\{(t,x) \in \mathcal{U} \mid (t,x) \in T_+(V) \\ \text{or } \ (T_+(V)_x \text{ is nonempty and } t = 0\,)\big\} \tag{6}$$

and the restriction $\Phi_V := \Phi_{|\mathcal{U}_V}$ of Φ to such set. Clearly, the pair $\mathbf{\Phi}_V = (\mathcal{U}_V, \Phi_V)$ defines a new w-semiflow.

We will say that the w-semiflow $\mathbf{\Phi}_V = (\mathcal{U}_V, \Phi_V)$ is the *restriction* of $\mathbf{\Phi}$ to V. Given a point $x \in \operatorname{supp}\mathbf{\Phi}$ (the support of $\mathbf{\Phi}$ defined in (1)), we shall say that the orbit through x is

- *inward pointing to V* if $T_+(V)_x \neq \emptyset$,
- *outward pointing to V* if it is inward pointing to $\mathbb{R}^n \setminus V$.

Remark 3.10. — Of course, there can be orbits which are neither inward pointing nor outward pointing. For instance, take the one-dimensional flow $\Phi(t,x) = x + t$ and the set $V \subset \mathbb{R}$ given by all rational numbers. Then, no orbit is inward or outward pointing to V.

The following result is immediate:

LEMMA 3.11. — *The support of the restricted w-semiflow $\mathbf{\Phi}_V$ is the set of points $x \in \operatorname{supp}\mathbf{\Phi}$ such that the orbit of $\mathbf{\Phi}$ through x is inward pointing to V.*

Obviously, it follows also from Lemma 3.9 that $\operatorname{supp}\mathbf{\Phi}_V$ is a subset of $\overline{V}$.

Remark 3.12. — We can define exactly in the same way the restriction of a i-semiflow to a subset $V \subset \mathbb{R}^n$.

3.2.1. *Definable case.*

PROPOSITION 3.13. — *Suppose that $\mathbf{\Phi}$ and V are definable in some o-minimal structure $\mathcal{S}$. Then, $\mathbf{\Phi}_V$ is also definable in $\mathcal{S}$.*

Proof. — If $\mathbf{\Phi}$ is definable, the set $T_+(V)$ in (5) can be given by the formula

$$(7)\quad (t,x) \in T_+(V) \Leftrightarrow \big((t,x) \in \mathcal{U}\big) \wedge (t>0) \wedge \forall s \,\big(0 < s \leq t \Rightarrow \Phi(s,x) \in V\big)$$

which clearly shows that it is definable. The same is true for the set $\mathcal{U}_V$ given in (6), which is defined by the formula

$$(8)\qquad (t,x) \in \mathcal{U}_V \Leftrightarrow \big((t,x) \in T_+(V)\big) \vee \big((t=0) \wedge \exists s \mid (s,x) \in T_+(V)\big).$$

Thus, since the restriction of a definable function to a definable set is always definable, the proposition is proved. □

The following result is also an immediate consequence of the o-minimality.

COROLLARY 3.14. — *On the hypothesis of the above proposition, each orbit of $\mathbf{\Phi}$ is either inward or outward pointing to V.*

Remark 3.15. — Exactly the same results hold for the restriction of a definable i-semiflow to a definable subset $V \subset \mathbb{R}^n$.

3.3. Examples.

Example 3.16. — The map $\Phi^+(t,x,y) = (x+t,y)$ defines a w-semiflow $\mathbf{\Phi}^+ = (\mathcal{U},\Phi^+)$ on $\mathcal{U} = \mathbb{R}^+ \times \mathbb{R}^2$. If we consider the domains

$$V^+ = \big\{(x,y) \in \mathbb{R}^2 \mid x > 0\big\} \quad \text{and} \quad V^- = \big\{(x,y) \in \mathbb{R}^2 \mid x < 0\big\},$$

the domain $\mathcal{U}^+_{V^+}$ of the restricted w-semiflow $\mathbf{\Phi}^+_{V^+}$ is given by

$$\mathcal{U}^+_{V^+} = \big\{(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid x \geq 0\big\},$$

while the domain $\mathcal{U}^+_{V^-}$ of $\mathbf{\Phi}^+_{V^-}$ is given by

$$\mathcal{U}^+_{V^-} = \big\{(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid (x<0) \wedge (t < -x)\big\}.$$

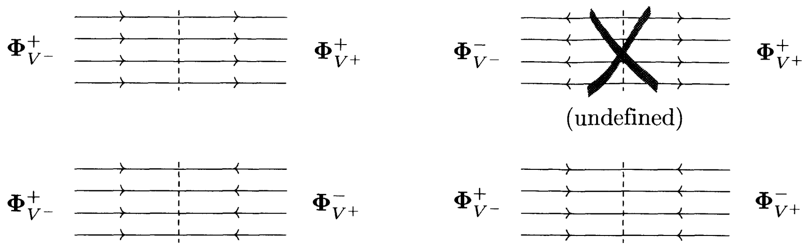

Figure 3. *Compositions of w-semiflows*

Similarly, if we consider the w-semiflow given by $\Phi^-(t,x,y) = (x-t,y)$ (on the same set $\mathcal{U}$),

$$\mathcal{U}^-_{V+} = \{(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid (x>0) \wedge (t<x)\},$$
$$\mathcal{U}^-_{V-} = \{(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid x \leq 0\}.$$

Let us consider the several possible compositions of such flows:

1) The composition $[\mathbf{\Phi}^+_{V-}, \mathbf{\Phi}^+_{V+}]$ gives the w-semiflow $\mathbf{\Phi}^+$ itself.

2) The composition $[\mathbf{\Phi}^+_{V+}, \mathbf{\Phi}^+_{V-}]$ gives the w-semiflow $\mathbf{\Psi} = (\mathcal{V}, \Psi)$, where

$$\mathcal{V} = \{(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid (x>0) \vee ((x<0) \wedge (t<-x))\}$$

and $\Psi(t,x,y) = (x+t,y)$.

3) The composition $[\mathbf{\Phi}^+_{V-}, \mathbf{\Phi}^-_{V+}]$ gives the w-semiflow $\mathbf{\Psi} = (\mathcal{V}, \Psi)$, where

$$\mathcal{V} = \{(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid ((x<0) \wedge (t<-x)) \vee ((x>0) \wedge (t<x))\}$$

and $\Psi(t,x,y) = (x+t,y)$ if $x<0$; $\Psi(t,x,y) = (x-t,y)$ if $x>0$.

4) Finally, the composition $[\mathbf{\Phi}^-_{V-}, \mathbf{\Phi}^+_{V+}]$ is undefined because these flows are not composable. Indeed, $\mathcal{U}^-_{V-} \cap \mathcal{U}^+_{V+}$ is the subset $S = \{(t,x,y) \mid x=0\}$ and the restriction of Φ^+_{V+} to S is different from the restriction of Φ^-_{V-} to S.

Example 3.17. — Keeping the notations of the previous example, it is also possible to introduce some *sliding* along axis $\{x=0\}$ by considering the flow $\mathbf{\Psi} = (\mathbb{R}^+ \times \{x=0\}, \Psi)$, with

$$\Psi(t,0,y) = (0, y-t).$$

The composition $\mathbf{\Psi}' = [\mathbf{\Phi}^+_{V-}, \mathbf{\Psi}]$ followed by the composition $\mathbf{\Psi}'' = [\mathbf{\Phi}^1_{V+}, \mathbf{\Psi}']$ gives the w-semiflow shown in the bottom of Figure 3.

Remark 3.18. — The above example illustrates the appearance of orbits with different initial points which collapse in finite time. Thus, in general it is not possible to *go back to the past* along the orbits of a w-semiflow.

We conclude this section by remarking that the notion of w-semiflows is analogous to the *local semi-dynamical systems*, which are treated extensively in [BH]. For instance, we can consider the following dynamical concepts:

• A *stationary point* is a point $x \in \mathbb{R}^n$ such that $|\mathcal{U}_x| > 0$ and $\Phi(t,x) = x$ for all $t \in \mathcal{U}_x$ (here $|\mathcal{U}_x|$ denotes the length of the interval $\mathcal{U}_x$).

• The orbit o_x through a point $x \in \mathbb{R}^n$ is *periodic* if there exists a strictly positive number $\tau \in \mathcal{U}_x$ such that

$$\Phi(\tau, x) = x$$

and $\Phi(t,x) \neq x$ for all $0 < t < \tau$. Such τ is called the *period* of the orbit.

• A subset $M \subset \mathbb{R}^n$ is *invariant* if for all $x \in M$, the orbit o_x is entirely contained in M.

• Given a point $x \in \mathbb{R}^n$ with a non-empty orbit o_x, we say that it is *attracted* to a subset $M \subset \mathbb{R}^n$ if for each neighborhood U of M there exists a $\tau \in \mathcal{U}_x$ such that $\Phi(t,x) \in U$ for all $\tau \leq t \in \mathcal{U}_x$.

Notice however that some properties proved in [BH] will not hold in our context because we do not require the continuity of the map $\Phi(t,x)$.

4. Piecewise linear vector fields.

From now on, we shall fix ourselves in the o-minimal structure

$$\mathbb{R}_{\mathrm{an,exp}} = \left(\mathbb{R}, 0, 1, +, \cdot, <, \{\widetilde{f}\}_{\tilde{f} \in \mathrm{an}}, \exp\right)$$

that is, the expansion of $(\mathbb{R}, 0, 1, +, \cdot, <)$ (the semi-algebraic sets) by adding the graphs of the exponential $\exp(x)$ and all *restricted analytic functions* $\widetilde{f} \in \mathrm{an}$.

We recall this last notion: Let $\mathbb{R}\{x_1, \ldots, x_m\}$ denote the ring of all real power series in $x_1, \ldots, x_m$ that converge in a neighborhood of I^m, with $I = [-1,1]$. For $f \in \mathbb{R}\{x_1,\ldots,x_m\}$, we define the *restricted analytic* function $\widetilde{f}:\mathbb{R}^m \to \mathbb{R}$ as follows:

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in I^m, \\ 0 & \text{if } x \notin I^m. \end{cases}$$

We refer to [DMM] and [LR] for general results on such structure.

For shortness, from now on we shall use the word *definable* as a synonym for the expression *definable in* $\mathbb{R}_{\text{an,exp}}$.

DEFINITION 4.1. — *Let $\mathcal{E}$ be a definable cell-decomposition of $\mathbb{R}^n$. A vector field $X(x)$ in $\mathbb{R}^n$ will be called* ***piecewise linear on $\mathcal{E}$*** *if for each cell $\mathfrak{C} \in \mathcal{E}$,*

$$X_{|\mathfrak{C}} = Ax + b$$

for some $(A,b) \in \mathbb{R}^{n^2} \times \mathbb{R}^n$ (we identify the space of real $n \times n$ matrices with $\mathbb{R}^{n^2}$).

Thus, fixing an enumeration $\mathcal{E} = \{\mathfrak{C}_i\}_{i=1}^s$ of the cells, the space $\mathrm{PL}^n(\mathcal{E})$ of all piecewise linear vector fields on $\mathcal{E}$ is isomorphic to $(\mathbb{R}^{n^2+n})^s$.

4.1. The exponential of a matrix.

Let us consider the analytic map

$$\begin{aligned} \mathrm{Exp} : \mathbb{R}^+ \times \mathbb{R}^{n^2} \times \mathbb{R}^n &\longrightarrow \mathbb{R}^{n^2} \times \mathbb{R}^n, \\ (t, A, x) &\longmapsto (A, \mathrm{e}^{tA}x), \end{aligned}$$

where $\mathrm{e}^{tA} = \sum_{j=0}^{\infty} (t^k/k!)\, A^k$. Then, the pair

$$\mathbb{E} = (\mathbb{R}^+ \times \mathbb{R}^{n^2} \times \mathbb{R}^n, \mathrm{Exp})$$

is a w-semiflow, according to Definition 3.1. We shall call that it the *exponential w-semiflow.*

It is easy to see that if $n \geq 2$, such w-semiflow cannot be defined in any o-minimal structure, as the following simple example shows:

Example 4.2. — Let $n = 2$. If we take the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

then $\mathrm{Exp}(t, A, x_1, x_2) = (A, x_1\cos(t) - x_2\sin(t), x_1\sin(t) + x_2\cos(t))$. Clearly, the graphs of $\sin(t)$ and $\cos(t)$ are needed to define $\mathbb{E}$.

Our next goal is to prove that we can obtain a definable semiflow by a suitable restriction of the domain of definition of $\mathbb{E}$.

LEMMA 4.3. — *There exists a semi-algebraic cell-decomposition $\mathcal{C} = \{\mathfrak{C}_i\}$ of the space $\mathbb{R}^{n^2}$ of $n \times n$ real matrices such that on each cell $\mathfrak{C} \in \mathcal{C}$, the number of distinct real and non-real eigenvalues, resp. $r = r(\mathfrak{C})$ and $c = c(\mathfrak{C})$, is constant. Moreover, on each cell the corresponding multiplicities $\nu_1, \ldots, \nu_r$ and $\mu_1, \ldots, \mu_c$ of such eigenvalues are constant and there is a semi-algebraic continuous map (the spectrum map)*

$$\begin{aligned} \mathrm{spec} : \mathfrak{C} &\longrightarrow \mathbb{R}^r \times (\mathbb{R}^{2c} \oplus i\,\mathbb{R}^{2c}_+), \\ A &\longmapsto \big((\lambda_1, \ldots, \lambda_r), (\alpha_1 \pm i\omega_1, \ldots, \alpha_c \pm i\omega_c)\big) \end{aligned}$$

which associates to each matrix its collection of distinct eigenvalues.

Proof. — This is an immediate consequence of the general theory of semi-algebraic sets (see e.g. [BR], [D]). □

The structure of the map Exp can now be described as follows:

PROPOSITION 4.4 (see also [A]). — *Suppose that the spectrum* spec(A) *of the matrix A is formed by real eigenvalues λ_j $(1 \le j \le r)$ of multiplicity ν_j and complex eigenvalues $\alpha_j \pm i\omega_j$ $(1 \le j \le c)$ of multiplicity μ_j. Then, each component $E_i(t, A, x)$ of the* Exp $= (E_1, \ldots, E_n)$ *is a sum*

$$E_i(t, A, x) = \sum_{k=1}^{n} e_{ik}(t, A) x_k$$

where the e_{ik} are functions of the form

$$\text{(9)} \quad \begin{aligned} e_{ik}(t, A) = &\sum_{j=1}^{r} \mathrm{e}^{\lambda_j t} p_{ikj}(t, A) \\ &+ \sum_{j=1}^{c} \mathrm{e}^{\alpha_j t} \big(q_{ikj}(t, A) \cos(\omega_j t) + r_{ikj}(t, A) \sin(\omega_j t)\big) \end{aligned}$$

where $p_{ikj}(t, A)$ is a polynomial of degree strictly less than ν_j in the t-variable and q_{ikj}, r_{ikj} are polynomials of degree strictly less than μ_j in the t-variable. The eigenvalues $\lambda_j, \alpha_j \pm i\omega_j$ and the coefficients of the polynomials $p_{ikj}, q_{ij\ell}$ and $r_{ij\ell}$ are real semi-algebraic functions of the entries of the matrix $A \in \mathbb{R}^{n^2}$.

Proof. — To prove such result, we will use the following beautiful characterization of the exponential of a matrix:

CLAIM. — *Each entry of the matrix* e^{tA} *satisfies the* n*-th order linear differential equation* $c(D)y = 0$*, where* $c(x) = \det(xI - A)$ *is the characteristic polynomial of* A *and* $D = d/dt$*.*

Indeed, the Hamilton-Cayley Theorem asserts that $c(A) = 0$. On the other hand, $D^k(e^{tA}) = A^k e^{tA}$ for each $k \in \mathbb{N}$. Therefore, $c(D)e^{tA} = e^{tA}c(A) = 0$. This proves the claim.

Let us follow the study made on [F]. From the claim, it follows that the matrix e^{tA} is the unique solution of the initial value problem

$$c(D)B(t) = 0, \quad B(0) = I, \quad B'(0) = A, \; \ldots, \; B^{(n-1)}(0) = A^{n-1}. \tag{10}$$

Suppose that A has n distinct real eigenvalues $\lambda_1, \ldots, \lambda_n$. Then, the general solution of the differential equation $c(D)B(t) = 0$ is

$$B(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \cdots + C_n e^{\lambda_n t}$$

where the initial conditions in (10) determine that the $n \times n$ matrices of constants C_k satisfy the following equations:

$$\begin{aligned}
I &= C_1 + C_2 + \cdots + C_n,\\
A &= \lambda_1 C_1 + \lambda_2 C_2 + \cdots + \lambda_n C_n,\\
A^2 &= \lambda_1^2 C_1 + \lambda_2^2 C_2 + \cdots + \lambda_n^2 C_n,\\
&\;\;\vdots\\
A^{n-1} &= \lambda_1^{n-1} C_1 + \lambda_2^{n-1} C_2 + \cdots + \lambda_n^{n-1} C_n.
\end{aligned}$$

Solving these equations, one obtains the C_k as polynomials of degree at most $n-1$ in the entries of the matrix A. The coefficients of these polynomials are the entries of the inverse of the Vandermonde matrix in $\lambda_1, \ldots, \lambda_n$ (the coefficient matrix of the above linear system of equations). Therefore, by Lemma 4.3, it is easy to see that each entry of C_k is a semi-algebraic function of the entries of A.

Suppose now that A has r distinct real eigenvalues $\lambda_1, \ldots, \lambda_r$ with multiplicities $\nu_1, \ldots, \nu_r$. Then, the general solution of $c(D)B(t) = 0$ is

$$(C_{11} + t\,C_{12} + \cdots + t^{\nu_1 - 1} C_{1\nu_1})\, e^{\lambda_1 t} + \cdots + (C_{r1} + t\,C_{r2} + \cdots + t^{\nu_r - 1} C_{r\nu_r})\, e^{\lambda_r t}.$$

The initial conditions in (10) yield again a linear system of equations for the matrices C_{ij}, but now with a coefficient matrix which is the following *confluent* Vandermonde matrix:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \lambda_1 & 1 & \cdots & 0 & \cdots & \lambda_r & 1 & \cdots & 0 \\ \lambda_1^2 & 2\lambda_1 & \cdots & 0 & \cdots & \lambda_r^2 & 2\lambda_r & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & (n-1)\lambda_1^{n-2} & \cdots & \frac{(n-1)!}{(n-\nu_1)!}\lambda_1^{n-\nu_1} & \cdots & \lambda_r^{n-1} & (n-1)\lambda_r^{n-2} & \cdots & \frac{(n-1)!}{(n-\nu_r)!}\lambda_r^{n-\nu_r} \end{bmatrix}.$$

The matrices C_j are again obtained as polynomials of degree at most $n-1$ in the entries of A and the coefficients are the entries of the inverse of the confluent Vandermonde matrix. By Lemma 4.3, it follows again that the entries of C_k are semi-algebraic functions of the entries of A.

The case where A contains complex eigenvalues is treated in a very similar way. □

Remark 4.5. — If we suppose that the characteristic polynomial $c(x)$ factors as

$$c(x) = \prod_{j=1}^{c} \left(x - 2\alpha_j x + (\alpha_j^2 + \omega_j^2)\right)^{\mu_j} \prod_{j=1}^{r} (x - \lambda_j)^{\nu_j}$$

each entry of e^{tA} is a linear combination of the elementary functions

$$\begin{aligned} &\{\mathrm{e}^{\lambda_j t}, t\,\mathrm{e}^{\lambda_j t}, \ldots, t^{\nu_j - 1}\mathrm{e}^{\lambda_j t}\}, \qquad \text{for } j = 1, \ldots, r; \\ &\{\mathrm{e}^{\alpha_j t}\cos(\omega_j t),\ \mathrm{e}^{\alpha_j t}\sin(\omega_j t), \\ &\quad \ldots, t^{\mu_j - 1}\mathrm{e}^{\alpha_j t}\cos(\omega_j t), t^{\mu_j - 1}\mathrm{e}^{\alpha_j t}\sin(\omega_j t)\}, \qquad \text{for } j = 1, \ldots, c. \end{aligned}$$

4.2. Exponential semiflow.

Let us consider the cell decomposition $\mathcal{C}$ of $\mathbb{R}^{n^2}$ which is described in Lemma 4.3. On each cell $\mathfrak{C} \in \mathcal{C}$, we can define the semi-algebraic function

$$(11) \qquad m(A) = \begin{cases} 0 & \text{if } c(\mathfrak{C}) = 0, \\ \max\{|\omega_1|, \ldots, |\omega_c|\} & \text{if } c(\mathfrak{C}) \geq 1, \end{cases}$$

where, we recall, $c(\mathfrak{C})$ is the function which describes the number of distinct eigenvalues with nonzero imaginary part.

Given a constant $k \in \mathbb{N}$, the *k-periodic region* (associated to the map Exp) is the semi-algebraic subset $U^k \subset \mathbb{R}^+ \times \mathbb{R}^{n^2+n}$ defined as

$$(12) \qquad U^k := \{(t, A, x) \in \mathbb{R}^+ \times \mathbb{R}^{n^2+n} \mid m(A)\,t < 2k\pi\}.$$

Remark 4.6. — Notice that if $\mathfrak{C} \in \mathcal{C}$ is a cell such that $c(\mathfrak{C}) = 0$ then

$$\{(t, A, x) \in \mathbb{R}^+ \times \mathfrak{C} \times \mathbb{R}^n\} \cap U^k = \{(t, A, x) \in \mathbb{R}^+ \times \mathfrak{C} \times \mathbb{R}^n\}$$

(i.e. *above* each cell in which all eigenvalues are real, the fiber of U_k contains the entire positive t-axis).

Let us consider the pair $\mathbb{E}^k = (U^k, E^k)$, where $E^k = \mathrm{Exp}_{|U^k}$ is the restriction of the exponential map Exp to U^k.

LEMMA 4.7. — *The pair $\mathbb{E}^k = (U^k, E^k)$ is an i-semiflow.*

Proof. — All properties in Definition 3.1 are trivially verified, except for $(b.i)$, which should be replaced by the condition $(b.i)'$ which is described in (2). □

We shall call $\mathbb{E}^k = (U^k, E^k)$ the ***k-periodic*** exponential i-semiflow.

THEOREM 4.8. — *For each constant $k \in \mathbb{N}$, the k-periodic exponential i-semiflow $\mathbb{E}^k = (U^k, E^k)$ is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$.*

Proof of the theorem. — It is convenient to extend the map E^k to the whole space $\mathbb{R}^+ \times \mathbb{R}^{n^2+n}$ by defining it as the null map on the complement of U^k. Let us call the resulting map $\widetilde{E}$. Thus, it suffices to prove that $\widetilde{E}$ is definable.

Consider the restricted analytic function $\widetilde{s}_k$ defined by

$$\widetilde{s}_k(t) = \begin{cases} \sin(2k\pi\, t), & \text{if } t \in [-1,1], \\ 0 & \text{if } t \notin [-1,1]. \end{cases}$$

Then, we have the equality

$$\sin(t) = \widetilde{s}_k\Big(\frac{t}{2k\pi}\Big), \quad \text{for } -1 \leq \frac{t}{2k\pi} \leq 1. \tag{13}$$

Similarly, we define the function

$$\widetilde{c}_k(t) = \begin{cases} \cos(2k\pi\, t) & \text{if } t \in [-1,1], \\ 0 & \text{if } t \notin [-1,1]. \end{cases}$$

From the Equation (9) in Proposition 4.4, it is clear that each column E_i of $\mathrm{Exp}(t, A, x)$ is a semi-algebraic function of the form

$$\begin{aligned} F_i(t, A, x, u, v, w, y) = \sum_{k=1}^{n} x_k \Big(\sum_{j=1}^{r} u_j p_{ikj}(t, A) \\ + \sum_{j=1}^{c} v_j \Big(q_{ikj}(t, A) w_j + r_{ikj}(t, A) y_j \Big)\Big) \end{aligned}$$

in which each variable u_j, v_j, w_j, y_j is replaced by the function $\exp(\lambda_j t)$, $\exp(\alpha_j t)$, $\cos(\omega_j t)$ and $\sin(\omega_j t)$, respectively.

Clearly, the corresponding column $\widetilde{E}_i(t, A, x)$ of the map $\widetilde{E}(t, A, x)$ can be written as

$$\widetilde{F}_i(t, A, x, u, v, w, y) = \phi_{U^k}(t, A, x) \cdot F_i(t, A, x, u, v, w, y)$$

where each variable u_j, v_j, w_j, y_j is replaced as above and ϕ_{U^k} is the (semi-algebraic) characteristic function of the domain U^k (i.e. $\phi_{U^k} \equiv 1$ on U^k and $\phi_{U^k} \equiv 0$ on $\mathbb{R}^+ \times \mathbb{R}^{n^2+n} \setminus U^k$). Using the expression of F_i, we obtain

$$\widetilde{F}_i = \sum_{k=1}^{n} x_k \Big(\sum_{j=1}^{r} (\phi\, u_j) p_{ikj}(t, A) \\ + \sum_{j=1}^{c} v_j \Big(q_{ikj}(t, A)(\phi\, w_j) + r_{ikj}(t, A)(\phi\, y_j) \Big) \Big)$$

where we have written $\phi = \phi_{U_k}$ for shortness. Therefore, $\widetilde{E}_i$ is simply obtained by replacing in F_i each u_j, v_j, w_j, y_j respectively by $\phi \exp(\lambda_j t)$, $\exp(\alpha_j t)$, $\phi \cos(\omega_j t)$ and $\phi \sin(\omega_j t)$.

Let us recall now the definition of the domain U^k. Since $m(A) \geq |\omega_j|$ for all $1 \leq j \leq c$,

$$\frac{2k\pi}{m(A)} \leq \frac{2k\pi}{|\omega_j|}.$$

Therefore, we know that the characteristic function $\phi(t, A, x)$ is identically zero for the values of t such that

$$t \notin \Big[0, \frac{2k\pi}{|\omega_j|}\Big).$$

On the other hand (13) implies that, for $t \in [0, 2k\pi/|\omega_j|)$, $\sin(\omega_j t)$ is identical to $\widetilde{s}_k\,(\omega_j t/2k\pi)$. Hence,

$$\phi\, \sin(\omega_j t) = \phi\, \widetilde{s}_k \Big(\frac{\omega_j t}{2k\pi} \Big)$$

and, similarly, $\phi\, \cos(\omega_j t) = \phi\, \widetilde{c}_k(\omega_j t/(2k\pi))$. Therefore, it is clear that each $\widetilde{E}_i$ is a function in $\mathbb{R}_{\text{an,exp}}$, and hence $\widetilde{E}$ is a map in $\mathbb{R}_{\text{an,exp}}$. $\square$

4.3. Non-homogeneous exponential semiflow.

The solution of the non-homogeneous linear differential equation $\dot{y} = Ay + b$, with $(A, b) \in \mathbb{R}^{n^2+n}$ and $y(0) = x$ is

$$L(t, A, b, x) = \mathrm{Exp}(t, A, x) + \int_0^t \mathrm{Exp}(t - s, A, b)\, \mathrm{d}s. \tag{14}$$

This motivates the introduction of another semiflow: the *non-homogeneous* exponential w-semiflow is the pair $\mathbb{L} = (\mathbb{R}^+ \times \mathbb{R}^{n^2} \times \mathbb{R}^n \times \mathbb{R}^n, \mathcal{L})$, where $\mathcal{L}$ is the analytic map

$$\begin{aligned} \mathcal{L} \colon \mathbb{R}^+ \times \mathbb{R}^{n^2} \times \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R}^{n^2} \times \mathbb{R}^n \times \mathbb{R}^n, \\ (t, A, b, x) &\longmapsto \big(A, b, L(t, A, b, x)\big). \end{aligned} \tag{15}$$

Now, in analogy with previous subsection, we introduce the following objects: given a $k \in \mathbb{N}$, the *k-periodic* non-homogeneous exponential i-semiflow is the pair $\mathbb{L}^k = (\mathcal{U}^k, \mathcal{L}^k)$, where

$$\mathcal{U}^k := \big\{(t, A, b, x) \in \mathbb{R}^+ \times \mathbb{R}^{n^2} \times \mathbb{R}^n \times \mathbb{R}^n \mid (t, A, x) \in U^k\big\}$$

(U^k being the k-periodic region in (12)), and $\mathcal{L}^k := \mathcal{L}_{|\mathcal{U}^k}$.

A simple integration yields the following corollary to Theorem 4.8:

Corollary 4.9. — *For each $k \in \mathbb{N}$, the i-semiflow $\mathbb{L}^k$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$.*

4.4. Semiflows associated to $\mathrm{PL}^n(\mathcal{E})$.

4.4.1. *Semiflow on a cell.* — Let $X(x) = Ax + b$ be a linear vector field on $\mathbb{R}^n$. The *w-semiflow associated to X* is the pair $\mathbf{\Phi}_X = (\mathbb{R}^+ \times \mathbb{R}^n, \Phi_X)$, where $\Phi_X(t, x) := L(t, A, b, x)$ is the map defined in (14).

Remark 4.10. — Equivalently, the map Φ_X can be defined as the composition

$$\Phi_X(t, x) := \pi_x \circ \mathcal{L}(t, A, b, x)$$

where $\mathcal{L}$ is the map defined in (15) and π_x is the linear projection $(A, b, x) \mapsto x$.

Let now $X \in \mathrm{PL}^n(\mathcal{E})$ be a piecewise linear vector field on a cell decomposition $\mathcal{E} = \{\mathfrak{E}_i\}_{i=1}^s$ of $\mathbb{R}^n$ and write $X_i(x) = A_i x + b_i$ for the restriction of the X to the cell $\mathfrak{C}_i$. We define the *ith-restricted w-semiflow* $\boldsymbol{\Phi}_{X,i}$ to be the restriction of the w-semiflow $\boldsymbol{\Phi}_{X_i}$ to $\mathfrak{C}_i$ (according to Subsection 3.2).

Writing such restricted w-semiflow as the pair $\boldsymbol{\Phi}_{X,i} = (\mathcal{U}, \boldsymbol{\Phi})$, we shall say that a piecewise linear vector field $X \in \mathrm{PL}^n(\mathcal{E})$ has *bounded spiraling* on the cell $\mathfrak{C}_i$ if there exists some constant $k \in \mathbb{N}$ such that for all $x \in \mathbb{R}^n$,

$$m(A_i)\,|\mathcal{U}_x| < 2k\pi \tag{16}$$

where $m(\cdot)$ is the function defined in (11) and $|\mathcal{U}_x|$ is the length of the time interval $\mathcal{U}_x$.

PROPOSITION 4.11. — *Suppose that X has bounded spiraling on a cell $\mathfrak{C}_i$. Then, the the ith-restricted w-semiflow $\boldsymbol{\Phi}_{X,i}$ is definable.*

Proof. — Write $\mathfrak{C} = \mathfrak{C}_i$ to simplify the notation. Let us denote by $\mathbb{L}_{\tilde{\mathfrak{C}}} = (\mathcal{U}_{\tilde{\mathfrak{C}}}, \mathcal{L}_{\tilde{\mathfrak{C}}})$ the restriction of the non-homogeneous exponential w-semiflow $\mathbb{L}$ to the subset

$$\widetilde{\mathfrak{C}} = \{A_i\} \times \{b_i\} \times \mathfrak{C} \subset \mathbb{R}^{n^2} \times \mathbb{R}^n \times \mathbb{R}^n.$$

Then, it is clear (see also Remark 4.10) that the w-semiflow $\boldsymbol{\Phi}_{X,i} = (\mathcal{U}, \Phi)$ can be obtained by setting

$$\mathcal{U} = \pi_x(\mathcal{U}_{\tilde{\mathfrak{C}}}) \quad \text{and} \quad \Phi(t,x) = \pi_x \circ \mathcal{L}_{\tilde{\mathfrak{C}}}(t, A_i, b_i, x), \tag{17}$$

where π_x is the linear projection $(A, b, x) \mapsto x$.

Let now $k \in \mathbb{N}$ be the constant of inequality (16). Then, it follows from the construction of restricted semiflows that the set $\mathcal{U}_{\tilde{\mathfrak{C}}}$ is entirely contained in the domain of definition $\mathcal{U}^k$ of the k-periodic non-homogeneous exponential i-semiflow $\mathbb{L}^k$ (defined in Subsection 4.3). Thus, the restriction of $\mathbb{L}$ to $\mathcal{U}_{\tilde{\mathfrak{C}}}$ is identical to the restriction of $\mathbb{L}^k$ to $\mathcal{U}_{\tilde{\mathfrak{C}}}$.

Since the i-semiflow $\mathbb{L}^k$ is definable, it follows from Proposition 3.13 that $\mathbb{L}_{\tilde{\mathfrak{C}}}$ is also definable. Now, the equation (17) immediately implies that $\boldsymbol{\Phi}_{X,i}$ is definable. This proves the result. □

Example 4.12. — The hypothesis of bounded spiraling is clearly necessary in the previous result. For instance, take the cell $\mathfrak{C} = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$, let X be the following linear vector field in $\mathbb{R}^3$:

$$\dot{x} = -x - y, \quad \dot{y} = x - y, \quad \dot{z} = z,$$

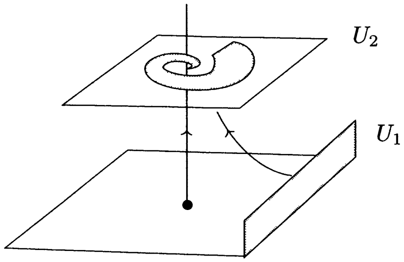

Figure 4. *A non-definable Poincaré map*

and consider the transversal sections $U_1 = \{(x, y, z) \mid y = 1,\ 0 < z < 1\}$ and $U_2 = \{(x, y, z) \mid z = 1\}$. It is clear that if the w-semiflow Φ_X is definable, the Poincaré map $P : U_1 \to U_2$ must also be definable.

Let us show that P is not definable. If we parameterize the points on U_1 by (x_1, z_1) and the points on U_2 by (x_2, y_2), the transition time between U_1 and U_2 will be given by the function

$$T(x_1, z_1) = -\ln(z_1).$$

Thus, the Poincaré map will be

$$P(x_1, z_1) = \begin{cases} x_2 = z_1\big(-\sin(-\ln(z_1)) + x_1 \cos(-\ln(z_1))\big), \\ y_2 = z_1\big(\cos(-\ln(z_1)) + x_1 \sin(-\ln(z_1))\big), \end{cases}$$

which is clearly not definable in any o-minimal structure.

Notice however that if we had considered the cell $\mathfrak{C} = \{(x, y, z) \in \mathbb{R}^3 \mid \mu < z < \lambda\}$ for some constants $\lambda > \mu > 0$, the restricted flow would be definable since X would have bounded spiraling on $\mathfrak{C}$ (just take any constant $k > \ln(\lambda/\mu)$ in inequality (16)).

4.4.2. *Semiflows on cell-lists.* — Let us generalize the previous discussion to define a semiflow on a sequence of cells.

An ordered list $\xi = (\xi_1, \ldots, \xi_\ell)$, where $\xi_i \in \{1, \ldots, s\}$, will be called a *cell-list.*

Let $\mathbf{\Phi}_{X,\xi_i}$ be the ξ_i-restricted w-semiflow corresponding to each one of such cells. We shall say that a cell-list $\xi = (\xi_1, \ldots, \xi_\ell)$ is *composable for X* when

$$\operatorname{supp} \mathbf{\Phi}_{X,\xi_i} \cap \operatorname{supp} \mathbf{\Phi}_{X,\xi_j} = \emptyset \quad \text{for each } 1 \le i \ne j \le s.$$

Thus, it follows directly from Lemma 3.4 and Remark 3.6 that the following definition makes sense:

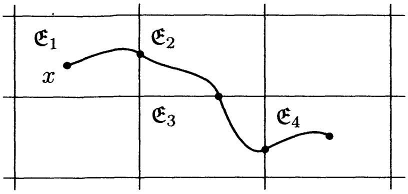

Figure 5. *The semiflow* $\mathbf{\Phi}_{X,\xi}$

DEFINITION 4.13. — *Given cell-list* ξ *which is composable for* X, *the* ξ***-restricted w-semiflow*** $\mathbf{\Phi}_{X,\xi}$ *is defined inductively as follows:*

(i) *If* $\ell = 1$ *then* $\mathbf{\Phi}_{X,\xi} := \mathbf{\Phi}_{X,\xi_1}$;

(ii) *If* $\ell \geq 2$, *we consider the sublist* $\widetilde{\xi} = (\xi_1, \dots, \xi_{\ell-1})$ *and define*

$$\mathbf{\Phi}_{X,\xi} := [\mathbf{\Phi}_{X,\widetilde{\xi}}, \mathbf{\Phi}_{X,\xi_\ell}],$$

where $[\,,]$ *is the composition operation defined in Subsection* 3.1.

We shall say that a piecewise linear vector field $X \in \mathrm{PL}^n(\mathcal{E})$ has *bounded spiraling* on the cell-list ξ if it has bounded spiraling on each cell $\mathfrak{C}_{\xi_i}$, for $1 \leq i \leq s$.

The next theorem is the main result of this paper:

THEOREM 4.14. — *Suppose that a piecewise linear vector field* $X \in \mathrm{PL}^n(\mathcal{E})$ *has bounded spiraling on a composable cell-list* ξ. *Then,* ξ*-restricted w-semiflow* $\mathbf{\Phi}_{X,\xi}$ *is definable.*

Proof. — It is a direct consequence of Proposition 4.11 and Proposition 3.7. □

5. Continuous piecewise linear vector fields.

In general, it is a subtle problem to associate a global flow to a piecewise linear vector field (see, for instance [Fi], [Ha]). In the previous section, we have bypassed such problem by considering semiflows defined only on *composable* cell-lists.

Let us see one special situation where such restriction can be dropped. We shall say that a piecewise linear vector field X is *continuous* if each one of its components $(X_1, \dots, X_n)$ is a continuous function on $\mathbb{R}^n$.

Remark 5.1. — It is easy to prove that the set of continuous vector fields is a definable subset of $\mathrm{PL}^n(\mathcal{E})$. We shall denote such subset by $\mathrm{PL}^n(\mathcal{E}, C^0)$.

LEMMA 5.2. — *Given a continuous vector field $X \in \mathrm{PL}^n(\mathcal{E}, C^0)$, there always exists a globally defined flow map*

$$\begin{aligned}\Phi_X : \mathbb{R} \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n,\\ (t,x) &\longmapsto \Phi_X(t,x)\end{aligned}$$

associated to it. Such flow map is continuous in (t,x) and C^1 in the t-variable.

Proof. — It is clear that X is a K-Lipschitz function on $\mathbb{R}^n$, for the Lipschitz constant $K := \max\{\|A_1\|, \ldots, \|A_s\|\}$. Thus, it suffices to apply the existence theorem for solutions of ordinary differential equations. □

If X is a continuous vector field, such global flow Φ_X induces the w-semiflow $\mathbf{\Phi}_X = (\mathbb{R}^+ \times \mathbb{R}^n, \Phi_X)$, obtained by considering only the positive orbits through each point.

PROPOSITION 5.3. — *Let $X \in \mathrm{PL}^n(\mathcal{E}, C^0)$. Then, given a cell $\mathfrak{C}_i \subset \mathcal{E}$, the i-restricted w-semiflow $\mathbf{\Phi}_{X,i}$ defined in Subsection* 4.4.1 *is equal to the restriction of the w-semiflow $\mathbf{\Phi}_X$ to $\mathfrak{C}_i$, as defined in Subsection* 3.2.

Proof. — This is a trivial consequence of the construction of the restricted semiflow. We omit the details for shortness. □

The following consequence is immediate.

COROLLARY 5.4. — *Let $X \in \mathrm{PL}^n(\mathcal{E}, C^0)$. Then, given a cell-list $\xi = (\xi_1, \ldots, \xi_s)$ which is composable for X, the ξ-restricted w-semiflow $\mathbf{\Phi}_{X,\xi}$ is equal to the semiflow obtained by the composing successively the restrictions of $\mathbf{\Phi}_X$ to $\mathfrak{C}_{\xi_1}, \ldots, \mathfrak{C}_{\xi_s}$.*

Another important consequence is that the condition of *composability* for cell-lists is immediately verified for continuous piecewise linear vector fields.

COROLLARY 5.5. — *Let $X \in \mathrm{PL}^n(\mathcal{E}, C^0)$. Then each cell-list $\xi = (\xi_1, \ldots, \xi_s)$ which is formed by distinct cells (i.e. $\xi_i \neq \xi_j$ for $i \neq j$) is composable for X.*

Proof. — We have to prove that $\operatorname{supp}\Phi_{X,\xi_i} \cap \operatorname{supp}\Phi_{X,\xi_j}$ is empty for $i \neq j$.

Suppose by absurd that there exists a point $x \in \operatorname{supp}\Phi_{X,\xi_i} \cap \operatorname{supp}\Phi_{X,\xi_j}$. Then, it follows from Lemma 3.11 and the previous proposition that the positive orbit $\Phi_X(\cdot,x): \mathbb{R}^+ \to \mathbb{R}^n$ through the point x is inward pointing to both $\mathfrak{C}_i$ and $\mathfrak{C}_j$. That is, there exists some $\tau > 0$ such that

$$\Phi_X(t,x) \in \mathfrak{C}_i \cap \mathfrak{C}_j, \quad \text{for all } 0 < t \leq \tau.$$

But this clearly contradicts the fact that the cells $\mathfrak{C}_i$ and $\mathfrak{C}_j$ are disjoint. □

For the rest of this section, we shall restrict our discussion to the set continuous piecewise linear vector fields. Thus, from now on the term *piecewise linear vector field* will always refer to an element of $\mathrm{PL}^n(\mathcal{E}, C^0)$.

5.1. Sufficient conditions for bounded spiraling.

Let us denote by $Z(X)$ the set of singularities of a vector field X. Clearly, if $X \in \mathrm{PL}^n(\mathcal{E}, C^0)$ then $Z(X)$ is a definable set.

Given a $n \times n$ real matrix A, we shall decompose its spectrum in the form

$$\operatorname{spec}(A) = \Lambda^s \cup \Lambda^c \cup \Lambda^u$$

where Λ^s (resp. Λ^c, Λ^u) is the collection of all eigenvalues with negative (resp. zero, positive) real part. Correspondingly, we have the direct sum decomposition of $\mathbb{R}^n$ into generalized eigenspaces,

$$\mathbb{R}^n = E^s(A) \oplus E^c(A) \oplus E^u(A).$$

The *subspace of centers* associated to A is the linear subspace $E^{\text{center}}(A) \subset E^c(A)$ generated by all pairs of vectors $x, y \in \mathbb{R}^n$ such that $z = x + iy \in \mathbb{C}^n$ is an eigenvector associated to some eigenvalue $\lambda \in \Lambda^c$.

Remark 5.6. — In particular, $E^{\text{center}}(A)$ contains $\operatorname{Ker}(A)$.

Given a non-homogeneous linear vector field $X = Ax + b$, we define the (*affine*) *subspace of centers* $E^{\text{center}}(X) \subset \mathbb{R}^n$ as follows:

- if $Z(X) = \emptyset$ then $E^{\text{center}}(X) = \emptyset$;
- otherwise, $E^{\text{center}}(X) = E^{\text{center}}(A) + c$, where $c \in Z(X)$ is an arbitrary singular point.

Clearly, $E^{\text{center}}(X)$ is independent of the choice of such c since for any other $c' \in Z(X)$, $c - c' \in E^{\text{center}}(A)$. We shall prove now the following result:

LEMMA 5.7. — *Let $K \subset \mathbb{R}^n$ be a compact subset and let $X = Ax + b$ be a non-homogeneous linear vector field in $\mathbb{R}^n$ such that*

$$K \cap E^{\text{center}}(X) = \emptyset.$$

Then, for each $x \in K$, there exists a positive time t such that $\Phi_X(t,x) \notin K$.

Proof. — We write $x(t) = \Phi(t,x)$ to simplify notation. Let $\operatorname{spec}(A) = \{\lambda_1, \dots, \lambda_k\} \subset \mathbb{C}$ be the collection of distinct eigenvalues of A. Up to a linear change of coordinates, we can suppose that A is in the real Jordan canonical form. Thus, it is a block-diagonal matrix

$$A = \operatorname{diag}(A_{\lambda_1}, \dots, A_{\lambda_k})$$

where A_{λ_i} is a $\mu_i \times \mu_i$ block diagonal matrix formed by Jordan elementary blocks with eigenvalue λ_i and $\bar{\lambda}_i$ for each $1 \leq i \leq k$. We consider also the decomposition of $\mathbb{R}^n$ as a direct sum of the corresponding generalized eigenspaces

$$\mathbb{R}^n = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}.$$

Let us suppose, first of all, that the vector field $Ax + b$ has no singular points in $\mathbb{R}^n$. Then the equation $Ax = -b$ has no solution. This implies that A is not an isomorphism, and so zero is an eigenvalue. Moreover, the projection of b in the corresponding eigenspace E_0 is nonzero.

Suppose that $(x_1, \dots, x_s)$ are the coordinates in the subspace E_0. We shall prove that there exists at least one $1 \leq i \leq s$ such that

$$\lim_{t \to \infty} \left| x_i(t) \right| = \infty.$$

Take the first Jordan elementary block in A_0 (of size, say, $m \times m$). Then, in the corresponding coordinates $(x_1, \dots, x_m)$, X projects into the vector field

$$X' \colon \dot{x}_1 = x_2 + b_1, \ \dot{x}_2 = x_3 + b_2, \ \dots, \ \dot{x}_{m-1} = x_m + b_{m-1}, \ \dot{x}_m = b_m.$$

Of course, if $b_m = 0$ then such vector field vanishes at the points $(x_1, \dots, x_m)$ such that $x_m = b_{m-1}, \dots, x_2 = b_1$. Using the same argument for each Jordan elementary block in A_0, we conclude (from the assumption that $Ax = -b$ has no solution) that there exists at least one of such blocks, say the first one, such that the last component b_m of the vector b in such block is non-vanishing. But this implies that the $|x_m(t)| = |x_m + b_m t|$ goes to infinity as $t \to \infty$. This concludes the proof in the case where $Ax + b$ has no singular points.

We assume now that $Ax + b$ has a singular point c. Then, up to the translation $x' = x - c$, it suffices to prove the claim for the homogeneous vector field $X = Ax$.

Let $E^s, E^u, E^c \subset \mathbb{R}^n$ denote the subspaces generated by the sum of the generalized eigenspaces corresponding to the eigenvalues with negative, positive and zero real part, respectively. Each vector $x \in \mathbb{R}^n$ can be uniquely written in the form $x = x_s + x_u + x_c$. The following two facts are obvious from the basic properties of e^{tA}:

- If $x_u \neq 0$, then $|x(t)| \to \infty$ as $t \to \infty$ (exponentially fast).
- If $x_u = 0$ and $x_s \neq 0$ then $x(t)_u = 0$ and $|x(t) - x(t)_c| \to 0$ as $t \to \infty$ (exponentially fast).

Therefore, it suffices to prove the lemma for $x \in E^c$. Recall that, by assumption, x does not belong to the subspace of centers $E^{\text{center}}(A) \subset E^c$.

For each purely imaginary eigenvalue $i\,\omega \in \Lambda^c$, the exponential of each elementary $2m \times 2m$ Jordan block B in $A_{i\,\omega}$ can be written, in appropriate complex coordinates $(z_1, \dots, z_m) = (x_1 + ix_2, \dots, x_{2m-1} + ix_{2m})$, as

$$\mathrm{e}^{tB} = \mathrm{e}^{i\,\omega} \begin{bmatrix} 1 & t & t^2/2 & \cdots & t^{m-1}/(n-1)! \\ 0 & 1 & t & \cdots & t^{m-2}/(n-2)! \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Since x is not contained in $E^{\text{center}}(A)$, there exists some set of coordinates as above such that $z_j \neq 0$ for some $2 \leq j \leq m$. Therefore, if we let j be the largest index for which this holds, the first component $z_1(t)$ of the solution $z(t)$ with $z(0) = z$ is given by

$$z_1(t) = \mathrm{e}^{i\omega}\Big(z_1 + z_2\,t + \cdots + z_j\,\frac{t^{j-1}}{(j-1)!}\Big)$$

which shows that $|z_1(t)| \to \infty$ as $t \to \infty$ ($t \in \mathbb{R}$). This finishes the proof of the lemma. □

PROPOSITION 5.8. — *Let $\mathfrak{C}_i \in \mathcal{E}$ be a relatively compact cell. Writing $X_{|\mathfrak{C}_i} = A_i x + b_i$, let us suppose that*

$$\overline{\mathfrak{C}_i} \cap E^{\text{center}}(A_i x + b_i) = \emptyset. \tag{18}$$

Then, if we consider the ith-restricted w-semiflow $\mathbf{\Phi}_{X,i} = (\mathcal{U}, \Phi)$, *there exists some* $T \in \mathbb{R}^+$ *such that, for all* $x \in \mathbb{R}^n$,

$$|\mathcal{U}_x| \leq T.$$

Remark 5.9. — Intuitively, this means that each orbit through a point $x \in \overline{\mathfrak{C}}_i$ leaves the cell in an (uniformly bounded) finite time.

Proof. — We have observed in Subsection 3.2 that $\mathcal{U}_x = \emptyset$ (and hence $|\mathcal{U}_x| = 0$) if $x \in \mathbb{R}^n \setminus \overline{\mathfrak{C}}_i$. Thus, we can suppose that x is in the compact set $K = \overline{\mathfrak{C}}_i$.

From the continuity of X, it follows that the restriction of X to K is also given by the vector field $A_i x + b_i$. By the continuity of the flow $\Phi_X(t,x)$ and the compactness of K, the proposition follows immediately from the previous lemma. □

Corollary 5.10. — *Keeping the notations of the proposition, let* $\mathfrak{C}_i$ *be a relatively compact cell such that either*

- $m(A_i) = 0$, *or*
- *Hypothesis* (18) *holds.*

Then, X *has bounded spiraling on* $\mathfrak{C}_i$.

Proof. — It suffices to look at the definition of bounded spiraling (see Condition (16) in Subsection 4.4). Nothing has to be proven if $m(A_i) = 0$. Otherwise, it follows from the above proposition that there exists some sufficiently large $k \in \mathbb{N}$ such that, for all $x \in \mathbb{R}^n$,

$$m(A_i)|\mathcal{U}_x| < 2k\pi. \qquad \square$$

5.2. Definable polycycles.

A *definable polycycle* for X is a simple closed curve Γ which is parameterized by a definable homeomorphism $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ and invariant by the flow of X.

Given an $\varepsilon > 0$, we define the ε-*neighborhood* $T_\varepsilon(\Gamma)$ of Γ as the image of the mapping

$$\begin{aligned} H_\varepsilon : \mathbb{S}^1 \times B_\varepsilon &\longrightarrow \mathbb{R}^n, \\ (\theta, y) &\longmapsto x = \gamma(t) + y \end{aligned}$$

where $B_\varepsilon = \{y \in \mathbb{R}^n \mid |y| < \varepsilon\}$. Of course, $T_\varepsilon(\Gamma)$ is a definable subset.

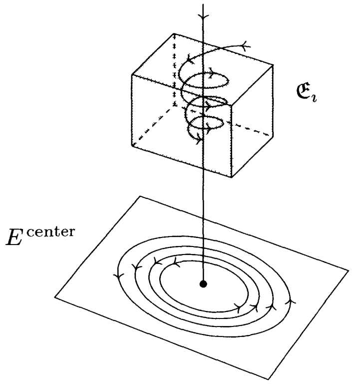

Figure 6. *Bounded spiraling*

A point $t \in \mathbb{S}^1$ will be called *smooth* for γ if there exists some open neighborhood of t in $\mathbb{S}^1$ where γ is C^1 and γ' is nonzero. Since $\gamma(t)$ is a definable map, there exists a finite number of points $t_1, \dots, t_m \in \mathbb{S}^1$ such that γ is C^1 in each interval (t_i, t_{i+1}) (with the identification $t_{m+1} = t_1$).

Remark 5.11. — This last statement follows from the C^1-cell decomposition and the fact that the derivative of a definable C^1 map is also definable (see [M]).

For a smooth point $t \in \mathbb{S}^1$, we can consider the affine orthogonal subspace

$$\Gamma^{\perp}_{\gamma(t)} = \{\gamma(t) + v \mid v \in \mathbb{R}^n \text{ and } \langle v, \gamma'(t)\rangle = 0\}.$$

For simplicity, we shall say that a point $x \in \Gamma$ is *smooth* if it is the image of a smooth point $t \in \mathbb{S}^1$ under γ.

Remark 5.12. — Notice that a smooth point $x \in \Gamma$ can be a singular point of the vector field X. In fact, we do not exclude the case where $\Gamma \subset Z(X)$.

LEMMA 5.13. — *Let $x = \gamma(t) \in \Gamma$ be a smooth point. Then, there exists an $\varepsilon_x > 0$ (depending on x) such that for all $0 < \varepsilon \leq \varepsilon_x$, the set*

$$\Sigma_\varepsilon(x) := B_\varepsilon(x) \cap \Gamma^{\perp}_x \tag{19}$$

is the connected component of $T_\varepsilon(\Gamma) \cap \Gamma^{\perp}_x$ which contains x (where $B_\varepsilon(x)$ is the open ball of radius ε centered at x).

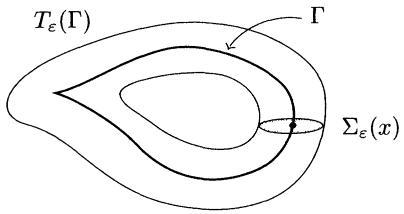

Figure 7. The ε-section

Remark 5.14. — If $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ is C^1 and γ' is nowhere vanishing, the result is an immediate consequence of the tubular neighborhood theorem (see e.g. [Hi]).

Proof. — Let $i_0 \in \{1, \ldots, m\}$ be the index such that $t \in (t_{i_0}, t_{i_0+1})$. We can choose constants $\eta_1, \eta_2, \rho > 0$ such that the closed interval $I := [t - \eta_1, t + \eta_2]$ is contained in (t_{i_0}, t_{i_0+1}) and $\gamma(I)$ is a *neat submanifold* with boundary of the closed ball $\overline{B}_\rho(x)$ (see definition in [Hi]). Let now

• $\varepsilon' > 0$ be the Hausdorff distance between the compact subsets $\gamma(\mathbb{S}^1 \setminus (t - \eta_1, t + \eta_2))$ and $\gamma([t - \frac{1}{2}\eta_1, t + \frac{1}{2}\eta_2])$;

• $\varepsilon'' > 0$ be such that the submanifold $\gamma(I)$ has a tubular neighborhood of radius ε'' in $\overline{B}_\rho(x)$ (see [Hi], Theorem 6.3).

Then, it suffices to take $\varepsilon_x < \min\{\frac{1}{2}\varepsilon', \varepsilon''\}$. □

Given a $0 < \varepsilon \leq \varepsilon_x$, the *$\varepsilon$-section* at the smooth point $x = \gamma(t)$ is the codimension-one disk $\Sigma_\varepsilon(x) = \Gamma_x^\perp \cap B_\varepsilon(x)$ given above.

Now, we would like to define the Poincaré first return map on such section. We start by defining the set of *returning points*

$$\widetilde{\Sigma}_\varepsilon(x) = \big\{ y \in \Sigma_\varepsilon(x) \mid \text{ the orbit through } y \text{ stays on } T_\varepsilon(\Gamma) \setminus \Sigma_\varepsilon(x) \\ \text{for a positive time interval } (0,t) \text{ and} \\ \text{intersects } \Sigma_\varepsilon(x) \text{ again at time } t \big\}.$$

Thus, for each $y \in \widetilde{\Sigma}_\varepsilon(x)$ there is associated a unique positive time, say $t_{\Sigma_\varepsilon(x)}(y)$, such that the positive orbit through y remains on the ε-neighborhood of Γ for all times $0 \leq s \leq t_{\Sigma_\varepsilon(x)}(y)$, and intersects $\Sigma_\varepsilon(x)$ again (for the first time) at $t_{\Sigma_\varepsilon(x)}(y)$. The Poincaré first return map is given by

$$P \colon \widetilde{\Sigma}_\varepsilon(x) \longrightarrow \Sigma_\varepsilon(x), \\ y \longmapsto \Phi_X\big(t_{\Sigma_\varepsilon(x)}(y), y\big).$$

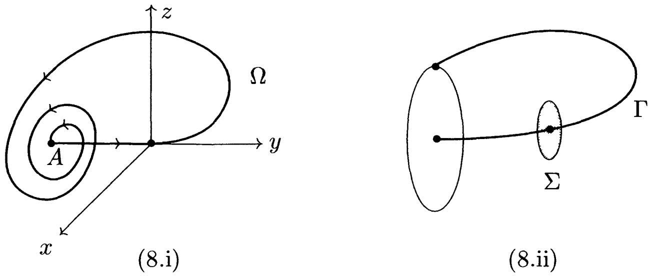

Figure 8. *Illustration for Example* 5.15

Even in the class of piecewise linear continuous vector field we can have Poincaré maps which are non-definable in any o-minimal structure.

Example 5.15. — In [ACT], the authors consider the following piecewise linear vector field X:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -y + \beta\, z + f(x), \end{cases}$$

where $\beta > 0$ and $f(x)$ is the piecewise linear function

$$f(x) = \begin{cases} 1 + ax & \text{if } x \leq 0, \\ 1 - \mu x & \text{if } x > 0. \end{cases}$$

If a and μ are positive, the system has two singularities at $A = (-1/a, 0, 0)$ and $B = (1/\mu, 0, 0)$. For a convenient choice of α, β, μ, it is possible to prove that A is a saddle-focus and there exists an invariant set Ω like the one illustrated in Figure (8.i), a *Shilnikov-type* homoclinic connection.

Although Ω is not a definable polycycle (the spiraling prevents this), we could (up to some technicalities) generalize the concept of definable polycycles to allow components of different dimensions. Using such generalized definition, we can show that the set Γ in Figure (8.ii) is a definable polycycle. Now, it follows directly from the results of [ACT] that the Poincaré map on the section Σ cannot be definable (for instance, because there is a countable set of horseshoes arbitrarily near Γ). A similar observation has been made in [Ka].

Example 5.16. — An even more *dramatic* example of non-definable Poincaré map is the so-called *Chua's circuit*

$$\begin{cases} \dot{x} = \alpha\big(y - x - h(x)\big), \\ \dot{y} = x - y + z, \\ \dot{z} = -\beta\, y, \end{cases}$$

where $h(x) = m_1 x + \frac{1}{2}(m_0 - m_1)\{|x+1| - |x-1|\}$. We refer to [S] for a detailed discussion.

We shall prove in the next section that the non-definability of the Poincaré map is exactly due to the spiraling behavior which is present in these examples.

5.3. Definable Poincaré maps.

Let $X \in \mathrm{PL}^n(\mathcal{E}, C^0)$. A definable polycycle Γ for X is said to be *δ-bounded away from spiraling* (for some $\delta > 0$) if for all cell $\mathfrak{E}_i \in \mathcal{E}$, writing $X_{|\mathfrak{E}_i} = A_i x + b_i$, we have

$$m(A_i) > 0 \implies d\big(\Gamma, E^{\mathrm{center}}(A_i x + b_i) \cap K\big) > \delta,$$

for all compact subset $K \subset \overline{\mathfrak{E}_i}$ (where d is the Hausdorff distance between compact sets in $\mathbb{R}^n$).

Theorem 5.17. — *Let Γ be a definable polycycle for X which is δ-bounded away from spiraling. Let $x \in \Gamma$ be a smooth point and let ε_x be the constant given by Lemma* 5.13. *Then, for all* $0 < \varepsilon \leq \min\{\varepsilon_x, \frac{1}{2}\delta\}$,

(i) *the set of returning points $\widetilde{\Sigma}_\varepsilon(x) \subset \Sigma_\varepsilon(x)$ is definable;*

(ii) *the Poincaré first return map $P : \widetilde{\Sigma}_\varepsilon(x) \to \Sigma_\varepsilon(x)$ is definable.*

Proof. — Let $\mathcal{F} = \{\mathfrak{F}_i\}$ be a definable cell decomposition which partitions simultaneously the sets $T_\varepsilon(\Gamma)$, $\Sigma_\varepsilon(x)$ and all the cells in $\mathcal{E}$. Clearly, the vector field X can be naturally seen as an element of $\mathrm{PL}^n(\mathcal{F}, C^0)$.

There exists a finite collection of distinct cells in $\mathcal{F}$, say $\mathfrak{F}_1, \ldots, \mathfrak{F}_m$, such that

$$T_\varepsilon(\Gamma) = \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \ldots \cup \mathfrak{F}_m.$$

Let σ_m denote the set of all lists obtained by permutation of the indices $(1, \ldots, m)$. Then, for each cell-list $\xi \in \sigma_m$, we can consider the associated ξ-restricted w-semiflow $\mathbf{\Phi}_{X,\xi} = (\mathcal{U}_{X,\xi}, \Phi_{X,\xi})$, defined as in Subsection 4.4.2. We now prove the following statement:

CLAIM. — *The w-semiflow $\Phi_{X,\xi}$ is definable.*

Indeed, notice that $\overline{\mathfrak{F}_i} \subset \overline{T_\varepsilon(\Gamma)}$. Thus, each $\mathfrak{F}_i$ is a relatively compact cell. Moreover, it follows from the choice of ε that, if we write $X_{|\mathfrak{F}_i} = A_i x + b_i$, either

- $m(A_i) = 0$, or
- $d(\overline{\mathfrak{F}_i}, E^{\text{center}}(A_i x + b_i)) \geq \frac{1}{2}\delta$.

From Corollary 5.10 it follows that X has bounded spiraling on $\mathfrak{F}_i$. Now, the proof of the claim is concluded by direct application of Theorem 4.14.

For each cell-list $\xi \in \sigma_m$, we introduce the set $\mathcal{P}_\xi \subset \Sigma_\varepsilon(x) \times \mathbb{R}^+_* \times \Sigma_\varepsilon(x)$ given as follows:

$$\begin{aligned}(y,t,z) \in \mathcal{P}_\xi \iff (t>0)\ &\wedge \big([0,t) \times \{y\} \subset \mathcal{U}_{X,\xi}\big)\\ &\wedge \big(\forall 0 < s < t, \Phi_{X,\xi}(s,y) \notin \Sigma_\varepsilon(x)\big)\\ &\wedge\ \big(z = \lim_{s\to t} \Phi_{X,\xi}(s,y)\big).\end{aligned}$$

Clearly, P_ξ is a definable subset (the relation $z = \lim_{s\to t} \Phi_{X,\xi}(s,y)$ is obviously definable). We define the *Poincaré domain*

$$\mathcal{P} \subset \Sigma_\varepsilon(x) \times \mathbb{R}^+_* \times \Sigma_\varepsilon(x)$$

as the finite union of all $\mathcal{P}_\xi$,

$$\mathcal{P} = \bigcup_{\xi \in \sigma_m} \mathcal{P}_\xi.$$

The set of returning points can now be defined as $\widetilde{\Sigma}_\varepsilon(x) = \pi_y(\mathcal{P})$, where π_y is the linear projection $(y,t,z) \mapsto y$. Clearly, $\mathcal{P}$ is the graph of a map over $\widetilde{\Sigma}_\varepsilon(x)$

$$y \longmapsto t = T(y), \quad z = P(y).$$

The first function $T(y)$ is the time of return and $P(y)$ is the Poincaré first return map. □

5.4. Accumulation of periodic orbits in polycycles.

Let γ be a periodic orbit of X. We shall say that γ is ε-*near* a definable polycycle Γ if $d(\gamma, \Gamma) < \varepsilon$ (for the Hausdorff metric d). This is equivalent to say that it is contained in the ε-neighborhood $T_\varepsilon(\Gamma)$ of Γ.

Let $x \in \Gamma$ be a smooth point and let $\Sigma_\varepsilon(x)$ be an ε-section as defined in Theorem 5.17. The ε-near periodic orbit γ will be said to be *k-intersecting* $\Sigma_\varepsilon(x)$ if $\gamma \cap \Sigma_\varepsilon(x)$ is composed of exactly k points.

Corollary 5.18 (to Theorem 5.17). — *Let Γ be a definable polycycle, δ-bounded away from spiraling and let $x \in \Gamma$ be a smooth point. For all $0 < \varepsilon \leq \min(\varepsilon_x, \frac{1}{2}\delta)$ and all $k \in \mathbb{N}$, let $\mathcal{O}_{\varepsilon,k}$ the set of periodic orbits which are ε-near Γ and k-intersecting $\Sigma_\varepsilon(x)$. Then $\mathcal{O}_{\varepsilon,k}$ is a definable subset of $T_\varepsilon(\Gamma)$.*

Proof. — First of all, for each $i \in \mathbb{N}$ we can consider the ith iterate of the Poincaré map

$$P^k(x) := \underbrace{P \circ \cdots \circ P}_{k \text{ times}}(x).$$

It is obvious that P^k is a definable map with domain on some definable subset $\widetilde{\Sigma}^i_\varepsilon(\Gamma) \subset \Sigma_\varepsilon(\Gamma)$ such that

$$\widetilde{\Sigma}_\varepsilon(\Gamma) = \widetilde{\Sigma}^1_\varepsilon(\Gamma) \supset \widetilde{\Sigma}^2_\varepsilon(\Gamma) \supset \widetilde{\Sigma}^3_\varepsilon(\Gamma) \supset \cdots.$$

Now, it suffices to consider the set of its fixed points of period i,

$$x \in \widetilde{F}^i \iff \big(x \in \widetilde{\Sigma}^i_\varepsilon(\Gamma)\big) \wedge \big(x = P^i(x)\big)$$

and define the set of periodic point of minimum period equal to k as

$$F^k = \widetilde{F}^k \setminus (\widetilde{F}^1 \cup \ldots \cup \widetilde{F}^{k-1}).$$

The set of ε-near, k-intersecting periodic orbits $\mathcal{O}_{\varepsilon,k} \subset T_\varepsilon(\Gamma)$ can now be defined by a straightforward procedure, very similar to the one used to define the set $\mathcal{P}$ in the proof of Theorem 5.17. □

5.5. Future work.

The corollary of the previous subsection has as an obvious consequence the non-accumulation of *semi-limit cycles* in definable polycycles of planar piecewise linear vector fields (a *semi-limit cycle* is a periodic orbit γ which is either ω-limit or α-limit of some point $x \in \mathbb{R}^2 \setminus \gamma$). In dimension 2, the hypothesis for a definable polycycle of being bounded away from spiraling can be dropped by using the Poincaré-Bendixson Theorem.

We intend to treat these matters in a forthcoming work, and prove the following uniform finiteness result:

Conjecture. — *Given a definable cell-decomposition $\mathcal{E}$ of $\mathbb{R}^2$, there exists a natural number $N(\mathcal{E})$ (depending only $\mathcal{E}$) such that each vector field in $\mathrm{PL}^2(\mathcal{E}, C^0)$ has at most $N(\mathcal{E})$ semi-limit cycles.*

In the spirit of the Hilbert's 16th Problem, we also treat the following question: given two finite collections of real numbers

$$\mu = \{\mu_1 < \cdots < \mu_r\} \quad \text{and} \quad \nu = \{\nu_1 < \cdots < \nu_s\},$$

we consider the (obviously defined) cell-decomposition $\mathcal{E}_{\mu,\nu}$ of $\mathbb{R}^2$ which partitions the family of lines $\{x = \mu_i\}_{i=1}^r$ and $\{y = \nu_j\}_{j=1}^s$.

Let $\mathrm{PL}^2(r, s, C^0)$ denote the set of continuous piecewise linear planar vector fields in $\mathrm{PL}^2(\mathcal{E}_{\mu,\nu}, C^0)$, for all possible collections $\mu \in \mathbb{R}^r$, $\nu \in \mathbb{R}^s$ as above.

The original question which was posed to me by Prof. Sotomayor can now be stated as follows:

CONJECTURE. — *There exists a natural number $N(r,s)$ (depending only on r, s) such that each vector field in $\mathrm{PL}^2(r, s, C^0)$ has at most $N(r,s)$ semi-limit cycles.*

In fact, using the uniform bounds for the number of roots of exponential polynomials which are given in [Kh], we believe that it is possible to obtain an explicit upper estimate for $N(r,s)$.

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