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SOME CLASSICAL FUNCTION THEORY THEOREMS AND THEIR MODERN VERSIONS

by J. L. DOOB

1. Introduction.

In 1940 Brelot defined a concept of thinness of a subset A of a Green space R at a point of the space. In 1957 Naïm extended the concept by defining thinness of $A \subset R$ at a point of the Martin boundary ∂R of R. Since a subset of a set thin at a point is also thin at the point and since the union of two sets thin at a point is thin at the point one can make the definition, for $A \subset R$, that ξ is a fine limit point of A if A is not thin at ξ. If f is a function on R, the concept of fine limit of f at ξ, and related limit concepts, are thereby well-defined, even without making R or $R \cup \partial R$ formally into topological spaces. For most applications, including those in this paper, this untopological approach to fine limit concepts is perfectly adequate. There is some interest however in going further and defining topologies as suggested by the thinness concept. The now classical fine topology on R can be defined by the convention that a point ξ of R is a fine limit point of a subset A of R if and only if A is not thin at ξ. A topology on $R \cup \partial R$ will be called compatible with the thinness concept if a point ξ of $R \cup \partial R$ is a limit point of a subset A of R if and only if A is not thin at ξ. Any such topology induces the fine topology on R, and all such topologies are equivalent in so far as limiting values at points of $R \cup \partial R$ of functions defined on R are concerned. Thus it is legitimate to discuss fine limits of such functions without specifying which of the topologies compatible with the thinness concept is involved. « Fine topology » on $R \cup \partial R$ will refer to any such topology. For example one simple topology compatible with the thinness concept (and maximal in a certain sense) is obtained by the definition that a subset A of $R \cup \partial R$ has a point of this space as a fine limit point if and only if $A \cap R$ is not thin at the point.

It is not yet obvious which topology compatible with the thinness concept is the most useful one. See Gowrisankaran [8] for a discussion of the possibilities. Cartan pointed out in 1940 that the topology on R which is the smallest topology (fewest open sets) making superharmonic functions continuous is precisely the fine topology on R, defined above in terms of thinness. In a natural extension of Cartan's idea Naïm [10] defined a (minimal in a certain sense) topology on $R \cup \partial R$ compatible with the thinness concept, the smallest topology making certain potentials on $R \cup \partial R$ (θ potentials) continuous.

In 1954 and 1957 Doob gave a probability interpretation of thinness, thereby giving a probability interpretation of the fine topology. The fine topology makes possible very elegant formulations of various results. For example, a boundary point of a Euclidean domain is regular for the Dirichlet problem if and only if the complement of the domain has the point as fine limit point; a conformal map from one hyperbolic Riemann surface to another has a fine limit at almost every Martin boundary point, and so on. These results were obtained in the natural course of various investigations, not inspired by the fine topology. By now it is clear, however, that the fine topology is intrinsic in potential theory and related subjects. It is natural therefore to investigate its possible applications to the cluster value theory of analytic functions. An obvious step is to find the relations between the angular and fine cluster sets (at boundary points) of functions defined on a ball or half-space. This step has already been carried out by Doob and by Constantinescu and Cornea, and more completely recently by Brelot and Doob [2] where detailed references will be found. The purpose of the first part of the present paper is to give some of the significant results of cluster value theory, in so far as they involve the fine topology, of superharmonic, harmonic, and meromorphic functions at the boundary of a half-space of definition and at an isolated singularity. The most interesting new results in this part are Theorems 4.1, 5.1, and 7.3. According to Theorem 4.1, if f is a function from a half-space to a compact metric space, and if Q is a boundary point of the half-space, the cluster set of f along the normal to Q is a subset of the fine topology cluster set at Q for almost all Q. This theorem makes it possible to derive the classical theorem on the almost everywhere existence of normal limits of a positive superharmonic function at boundary points of its (half-space) domain from the general theorem on the existence of almost everywhere fine limits at Martin boundary points of an arbitrary (Green space) domain. In fact Theorem 5.1 generalizes the classical limit theorem to apply to ratios of positive superharmonic functions. In Section 6, applying the results of this and previous papers, the classical Plessner theorem for meromorphic functions on a disk or half-plane is put into a new and more precise form (Theorem 6.2) involving normal and fine cluster values. Theorem 7.3 is an analogue involving fine cluster values of the Casorati–Weierstrass theorem for meromorphic functions in the neighborhood of a singularity.

In the second part of this paper a Hardy–Littlewood inequality (7.4) for positive subharmonic functions on a disk is extended to positive subharmonic functions defined on an arbitrary Green space of dimensionality $N \geqslant 2$. The generalization is given both in probabilistic language (Theorem 10.1) and non-probabilistic language (Theorem 11.1) but the probabilistic version is the more intuitive one. The Hardy–Littlewood inequality is put into a setting which makes it clear that the fundamental inequality underlying the work is a much simpler maximal inequality, an application of an elementary submartingale inequality. In Section 12 it is shown that the original Hardy–Littlewood inequality, in fact its generalization to $N \geqslant 2$ dimensions, is easily obtained from the general case.

I. APPLICATIONS OF THE FINE TOPOLOGY TO FUNCTION THEORY.

2. Cluster values at the boundary of a half-space.

If f is a function from an N-dimensional half-space into a compact metric space, it has a cluster set at the boundary point Q which depends on the admissible method of approach. Let A_Q be the cluster set along the line through Q normal to the boundary. Let B_Q be the cluster set for non-tangential approach, and let C_Q be the fine cluster set, that is, the cluster set for approach in the fine topology. These three sets are non-empty: A_Q and C_Q are compact; B_Q is a countable union of compact sets. In topological language, A_Q is the cluster set on approach to Q in the topology assigning as deleted neighborhood of Q the part of the half-space on an interval of the line through Q normal to the boundary, the interval to have Q as one endpoint. If (« non-tangential topology ») a deleted neighborhood of Q is defined as any subset of the half-space whose complement in the half-space is tangential, the non-tangential topology cluster set is the closure of B_Q. A function on the half-space has a non-tangential limit at Q, that is, a limit on every non-tangential sequence to Q, if and only if it has a limit in the non-tangential topology.

In the following, normal and fine limits will be denoted by $n \lim$ and $f \lim$ respectively.

Using the relations to be described between A_Q, B_Q, C_Q, Fatou's boundary limit theorem, involving non-tangential approach, for harmonic functions on a half-space, is equivalent to the same result for approach in the fine topology (see [2]). The fine topology approach is more natural however, for the following reasons. (*a*) In terms of the fine topology there is a natural extension of Fatou's theorem to the ratio u/h of two positive superharmonic functions [6]: u/h has a fine limit, but not necessarily a non-tangential limit or even a normal one, almost everywhere on the boundary, for the boundary measure determining the harmonic component of h in its Poisson–Stieltjes representation. (*b*) Even if $h = 1$ in (*a*), so that the boundary measure is a constant multiple of Lebesgue measure, the stated result does not become true for angular approach, although it is then true for normal approach. (*c*) The fine topology version of Fatou's theorem, even in the ratio form (*a*), remains true for functions on an arbitrary Green space.

It is clearly a present task for mathematicians to go through the extensive theory of cluster values of meromorphic functions and to see what if any contribution the fine topology has to offer.

3. A projection theorem.

In the following, if $\xi[A]$ is a point [set] in a given half-space R, $\xi^*[A^*]$ will denote its projection on the boundary ∂R. Lebesgue N-dimensional measure on ∂R will be denoted by ν. If $N > 2$ the area of the unit sphere in N-space multiplied by $N - 2$ will be denoted by $1/\alpha_N$; $\alpha_2 = 1/(2\pi)$. The distance from ξ to ∂R will be denoted by d_ξ. The Green function of R will be denoted by g and the harmonic measure of a set $A \subset \partial R$ relative to ξ by $\mu(\xi, A)$.

LEMMA 3.1.—*Let* $S \subset R$ *be a countable union of Borel sets, each on a hyperplane parallel to* ∂R, *with disjoint projections on* ∂R. *Let*

δ be the supremum of the distance from ∂R to a point of S. *Define*

$$(3.1) \qquad v(\xi) = \alpha_N \int_S \frac{g(\xi, \eta)}{d_\eta} \nu(d\eta^*).$$

Then there is a constant c_N independent of S, *δ, such that $v \leqslant c_N$. If* S *is contained in the ball of center ξ_0 and radius r, and if δ is sufficiently small, depending on ε, r, but not on ξ_0 or S,*

$$(3.2) \qquad v(\xi_0) \geqslant \mu(\xi_0, S^*) - \varepsilon.$$

When $\xi = \xi_0$ in (3.1) and all the points of S are within distance r of ξ_0, the integrand is uniformly within $\varepsilon/[\alpha_N \nu(S^*)]$ of the normal derivative of $g(\xi_0, .)$ at η^*, if δ is sufficiently small, depending on ε, N, r. The second assertion of the lemma is therefore true. In proving the first assertion we suppose that $N > 2$; the proof when $N = 2$ is similar. [The manipulations to follow are due to Mr. G. A. Brosamler.] Fix ξ in R and let M(s) be the supremum of $\alpha_N g(\xi, \eta)/d_\eta$ as η varies on the intersection with R of the sphere of center ξ and radius s. On this sphere

$$(3.3) \qquad \frac{\alpha_N g(\xi, \eta)}{d_\eta} = \frac{\alpha_N}{d_\eta}[s^{-(N-2)} - (s^2 + 4d_\xi d_\eta)^{-(N-2)/2}]$$

and the derivative of the right side with respect to d_η is negative. Applying this fact one finds that

$$(3.4) \qquad \begin{aligned} M(s) &= \alpha_N 2(N-2)\, d_\xi/s^N \qquad \text{if} \quad d_\xi \leqslant s \\ &= \alpha_N[s^{-(N-2)} - (2d_\xi - s)^{-(N-2)}]/(d_\xi - s) \quad \text{if} \quad d_\xi > s. \end{aligned}$$

An elementary calculus argument shows that M is a decreasing function. It follows that if $f(s)$ is the Lebesgue (N − 1)-dimensional measure of the part of S at distance $\leqslant s$ from ξ then $f(s) \leqslant as^{N-1}$ for some constant a depending only on N and

$$(3.5) \qquad \begin{aligned} v(\xi) &\leqslant \int_0^\infty M(s)\, df(s) = - \int_0^\infty f(s)\, dM(s) \\ &\leqslant - \int_0^\infty as^{N-1}\, dM(s) = a(N-1) \int_0^\infty M(s)s^{N-2}\, ds. \end{aligned}$$

Now from the above evaluation of M(s),

$$(3.6) \qquad \int_0^{d_\xi} M(s)s^{N-2}\, ds = \alpha_N \int_0^1 [1 - (2-t)^{-(N-2)}t^{N-2}]/(1-t)\, dt$$

and

$$(3.7) \qquad \int_{d_\xi}^{\infty} M(s)s^{N-2}\,ds = 2(N-2)\alpha_N.$$

Hence v is bounded as stated in the lemma.

Throughout this paper, if h is a positive superharmonic function on some domain and if A is a subset of the domain, $\hat{R}_h^A$ means the regularized (to be lower semi-continuous) reduced function of h on A, that is $\hat{R}_h^A$ is the positive superharmonic function which coincides off a set of capacity zero with the lower envelope of the positive superharmonic functions on the given domain which dominate h on A. We shall use in the proof of Theorem 3.1 the fact that if v is the Green potential of a positive measure on the domain, if the measure is carried by A, and if $v \leqslant h$ on A, then $v \leqslant \hat{R}_h^A$ on the whole domain.

In the following theorem we describe a point on the boundary of a half-space as a normal limit point of a subset of the half-space if the boundary point is a limit point of the part of the subset on the normal line through the boundary point.

THEOREM 3.1. — *Let* R *be a half-space and let* A *be a subset of* R. *Then almost every normal limit point of* A *on* ∂R *is a fine limit point of* A.

We can and shall suppose that A is a Borel set, and even a G_δ set. In fact let A′ be the set of fine limit points of A in R. Then it is known that A′ is a G_δ, and that $A - A \cap A'$ has capacity zero. Let A″ be the union of A′ and of a G_δ set of capacity zero covering $A - A \cap A'$. Then A″ is a G_δ set including A, with the same fine limit points as A on ∂R. Thus if A is not already a G_δ set we can replace it by A″. We can also suppose that A is bounded. Then the set B of normal limit points of A is also a bounded G_δ set. Let A_n be the part of A at distance $< 1/n$ from ∂R. Fix some point ξ_0 of R at which the regularized reduced function $\hat{R}_1^A$ is equal to the reduced function (the actual lower envelope involved) for $n \geqslant 1$. Given n, ε, there is an open set $G \supset A_n$ for which

$$\hat{R}_1^{A_n}(\xi_0) \geqslant \hat{R}_1^G(\xi_0) - \varepsilon$$

and for which each point of G is at distance $< 1/n$ from ∂R. Applying Vitali's theorem, there is a subset S of G, consisting of countably many (N − 1)-dimensional intervals, parallel to ∂R, whose projections on ∂R are disjoint and cover almost all of B. Now according to the lemma, n can be chosen to make, in the notation of the lemma,

$$v(\xi_0) \geqslant \mu(\xi_0, S^*) - \varepsilon \geqslant \mu(\xi_0, B) - \varepsilon, \tag{3.8}$$

and $v \leqslant c_N$ for some constant c_N independent of ξ_0 and n. But then, since v/c_N is a Green potential dominated by 1 on S, whose measure is carried by S,

$$\hat{R}_1^S(\xi_0) \geqslant v(\xi_0)/c_N \geqslant \mu(\xi_0, B)/c_N - \varepsilon/c_N. \tag{3.9}$$

Hence

$$\hat{R}_1^{A_n}(\xi_0) \geqslant \mu(\xi_0, B)/c_N - \varepsilon/c_N - \varepsilon \tag{3.10}$$

so that

$$\lim_{n\to\infty} \hat{R}_1^{A_n}(\xi_0) \geqslant \mu(\xi_0, B)/c_N. \tag{3.11}$$

Now according to a theorem of Naïm [10] the greatest harmonic minorant of $\hat{R}_1^{A_n}$ is $\mu(\,.\,, C)$, where C is the set of fine limit points of A_n, that is of A, on ∂R. Hence $\mu(\,.\,, C) \geqslant \mu(\,.\,, B)/c_N$. Since the harmonic measure of a boundary set has fine limit 1 almost everywhere on the set, 0 at almost all other boundary points, C must include almost all of B, as was to be proved.

4. Relations between A_Q, B_Q, C_Q.

THEOREM 4.1. — *Let f be a function from a half-space to a compact metric space. Then at almost every (Lebesgue measure) boundary point Q, $A_Q \subset C_Q$. In particular f has a normal limit at almost every boundary point where f has a fine limit, and the limits are the same.*

Note that no regularity hypotheses have been imposed on f. The sets A_Q and C_Q are compact, for each Q. Hence to prove the theorem it is sufficient to prove that there is an exceptional subset of the half-space boundary, of measure 0, such that, if Q is not in this set, C_Q meets (that is, has a non-empty intersection with) every closed ball which A_Q meets. It is even sufficient to consider only a properly chosen countable sequence of balls, and therefore even sufficient to consider a single ball S, and prove that if B is the set of boundary points Q for which A_Q meets S, C_Q meets S for almost all Q in B. Let S_n be the concentric ball of radius larger than that of S by $1/n$. Then $f^{-1}(S_n)$ is a set in the half-space which has every point of B as a normal cluster value. In view of Theorem 3.1 we conclude that $f^{-1}(S_n)$ has almost every point of B as a fine cluster value. That is, for almost every Q in B C_Q meets S_n. Hence for almost every Q in B C_Q meets S, as was to be proved.

The known relations, besides that given by Theorem 4.1, between A_Q, B_Q, C_Q for a function from a half-space to a compact metric space can be summarized as follows. (See [2] for proofs and references.) It is trivial that $A \subset B_Q$. For almost all (Lebesgue measure) Q, $C_Q \subset B_Q$. Moreover if the function *f* is positive and harmonic or the quotient of two such functions, $B_Q \subset C_Q$ for all Q, so that, under these special hypotheses on the function, $B_Q = C_Q$ for all Q, so that, under these special hypotheses on the function, $B_Q = C_Q$ for almost all Q.

5. A new generalization of Fatou's theorem.

A positive harmonic function on a half-space has a nontangential limit at almost every (Lebesgue measure) boundary point. A positive superharmonic function on a half-space need not have a non-tangential limit at almost every boundary point, but does have a normal limit at almost every boundary point. It is one of the beauties of the fine topology that using it as the approach topology no modification is needed in going from a Fatou-type boundary limit theorem for positive harmonic functions to one for positive superharmonic functions. In the following theorem the fine topology limit theorem is used to derive the normal one, and in fact a new normal one.

By a Stolz domain for a boundary point Q of a half-space we mean as usual the intersection with a ball of center Q of the interior of a right circular cone whose closure lies in the half-space except for its vertex Q.

THEOREM 5.1. — *Let h be superharmonic and positive on the half-space* R. *Suppose that u is defined and superharmonic on an open subset of* R *containing a variable Stolz domain whose vertex runs through a boundary set* B. *Suppose that on each of these Stolz domains u/h is bounded from below. Then u/h has a finite fine limit and equal normal limit at all points of* B *except for the union of a set of Lebesgue measure* 0 *and one of measure* 0 *for the measure associated with the harmonic component of h in its Poisson integral representation.*

Note that the half-angle of the Stolz domain and the orientation of its axis may vary with the vertex. In view of Theorem 4.1, it is sufficient to prove that *u/h* has a finite limit at all points of B except for a set of the type described, and this is precisely the first part of the proof of Theorem 10 in [2]. Furthermore, if u and h are harmonic the limit in Theorem 5.1 is even a non-tangential limit according to [2].

If h is a Green potential the theorem is vacuous. If $h = 1$ and if u is a Green potential on R, the theorem becomes the Littlewood–Privalov theorem that a Green potential on R has normal limit at almost every (Lebesgue measure) boundary point. Conversely the latter theorem can be used to derive the key Theorem 3.1.

Lebesgue measure is involved in Theorem 5.1 because of the use of Stolz domains. If instead it is supposed that u/h is bounded below in a deleted fine neighborhood of each point of B the conclusion becomes that u/h has a finite fine limit and equal normal limit at almost every boundary point for the measure associated with the Poisson integral representation of the harmonic component of h.

6. Cluster sets of superharmonic, harmonic, and meromorphic functions at the boundary of a half-space.

We consider functions f from a Green space into a compact metric space, specializing later. The Green space topology is the usual one unless « fine » is prefixed to a concept under discussion, when the fine topology is used. In every case the function f will be fine-continuous. In particular the results will be applicable to superharmonic functions, for which the range space is the extended compactified line, and to meromorphic functions, for which the range space is the extended compactified plane.

The continuity condition on f implies that the function is Borel measurable. In fact the continuity condition is equivalent to the combined condition that f be Borel measurable and that its restriction to a Brownian path from a point of the space be a continuous function of the path parameter for almost all paths from the point. The restriction of the function to a conditional Brownian path from the point to another point, either of the space or on the Martin boundary and minimal is then a continuous function of the path parameter for almost all these paths.

It follows from these facts, applied to subdomains of the Green space, that the image under an admissible function f of any open connected subset of the Green space is arcwise connected. In particular if the Green space is an N-dimensional half-space, the nontangential cluster set B_Q is connected. The fine cluster set C_Q

is also connected; in fact even if the domain is a general Green space the set of the fine cluster values of f at a minimal boundary point Q is connected. Indeed this cluster set is the cluster set at Q of the restriction of f to each conditional Brownian path from a point Q_0 of the Green space to Q, except for a set of paths of zero probability. If the Green space is an N-dimensional half-space and if f is extended real-valued, B_Q and C_Q must be sub-intervals of $[-\infty, \infty]$. The connectedness results can of course also be obtained without probability, although the probability approach shows better why they are true.

[If the function f is not necessarily fine continuous but at least if for every $\varepsilon > 0$ the restriction of f to some closed set whose complement has capacity $<\varepsilon$ is continuous, that is if f is fine-continuous except at the points of a set of zero capacity, f need no longer be Borel measurable. It remains true, however, that the restriction of f to a Brownian path from a point of the space is a continuous function of the path parameter, excluding the parameter value 0, for almost all paths from the point. Then C_Q (even in the general context in which Q is a minimal boundary point of a Green space) is still connected, but A_Q and B_Q need not be.]

The following theorem is a slight reorganization of the results we have obtained, in particular of Theorem 5.1 with $h = 1$, stated for comparison with Theorem 6.2 which refines Plessner's classical cluster value theorem for meromorphic functions.

THEOREM 6.1. — *If f is a superharmonic function on a half-space one of the following situations holds at almost every (Lebesgue measure) boundary point* Q.

(*a*) $-\infty \in C_Q \cap B_Q, C_Q \subset B_Q$.

(*b*) *f does not have a nontangential limit at* Q *but has a finite fine limit and an equal normal limit there.*

(*c*) *f has a finite nontangential limit and an equal fine limit at* Q.

According to Theorem 5.1, f has a finite fine limit at almost every (Lebesgue measure) point Q for which $-\infty$ is not in B_Q. Moreover as was noted in Section 4, $C_Q \subset B_Q$ for almost all Q. Thus Case (a) holds for almost all Q with $-\infty \in C_Q$. Moreover f has a finite fine limit, according to [6] at almost every Q for which $-\infty$ is not in C_Q. According to Theorem 4.1 f also has a normal limit, equal to its fine limit, at almost every such point. Thus (*b*) or (*c*) is true almost everywhere where (*a*) is false.

It is not known whether in (*a*) C_Q can be a proper subinterval of B_Q for a Q set of strictly positive measure. If $N = 2$ a fine neighborhood of a point Q necessarily contains circles of center Q and arbitrarily small radii, and linear segments with endpoint Q making angles arbitrarily near $\pi/2$ and $-\pi/2$ with the normal through Q. It then follows from the minimum principle for superharmonic functions, applied to the region bounded by two such segments and two such sufficiently small circles, that (*b*) can be strengthened to

$(b)_{N=2}$ *f does not have a nontangential limit at* Q *but has a finite fine limit and an equal normal limit there, namely the left endpoint of* B_Q.

It is not known whether or not this strengthening is possible when $N > 2$. Since the Green potential of a positive measure has normal and fine limit 0, but not necessarily a nontangential limit, at almost every boundary point Q, Case (*b*) actually can occur.

If *f* is harmonic, the three cases become:

$(a)_{\text{harm.}}$ $B_Q = C_Q = [-\infty, \infty]$.

$(b)_{\text{harm.}}$ $B_Q = [-\infty, \infty]$ *but there is a finite fine limit and an equal normal limit at* Q.

$(c)_{\text{harm.}}$ *There is a finite nontangential limit and an equal fine limit at* Q.

This specialization is proved by applying Theorem 6.1 to *f* and $-f$, taking into account the fact [2] that a harmonic function on a half-space has a finite fine limit at almost every point Q where B_Q does not contain both $+\infty$ and $-\infty$. Case $(b)_{\text{harm.}}$ can be omitted when $N = 2$ but it is not known whether it can arise when $N > 2$.

The version of Theorem 6.1 for meromorphic functions is an extension of Plessner's classical theorem. It was proved independently by Doob [7] and by Constantinescu and Cornea [4] aside from a further extension involving normal limits which follows from Theorem 4.1. With the extension the theorem becomes.

THEOREM 6.2. — *If f is meromorphic on a half-plane, one of the following situations holds at almost every point* Q *of the boundary.*

(*a*) $B_Q = C_Q =$ *extended plane.*

(*b*) $B_Q =$ *extended plane, but there is a finite fine limit and an equal normal limit at* Q.

(*c*) *There is a finite nontangential limit and equal fine limit at* Q.

Constantinescu and Cornea have shown by an example that (*b*) cannot be omitted, in general. If *f* omits a single value, however, (*b*) can be omitted. Constantinescu and Cornea have also treated the more general case in which f maps a half-plane into an arbitrary Riemann surface.

7. Cluster sets of superharmonic, harmonic, and meromorphic functions at a singularity.

We shall treat functions defined in a deleted neighborhood R of a point Q in N-space. Then Q is a minimal boundary point of R, and the deleted fine neighborhoods of Q relative to R are the intersections with R of the deleted fine neighborhoods of Q relative to N-space.

THEOREM 7.1. — *Let f be superharmonic in a deleted neighborhood* R *of a point* Q *of the plane. If $-\infty$ is not a fine cluster value of f at* Q, *f has a superharmonic extension to* $R \cup \{Q\}$. *Hence f has a fine limit at* Q. *If f is harmonic and if its fine cluster set at* Q *is bounded, f has a harmonic extension to* $R \cup \{Q\}$.

If f is superharmonic, its fine cluster set at Q is a compact subinterval of $[-\infty, \infty]$. If $-\infty$ is not a cluster value, f is bounded below in a deleted fine neighborhood of Q so that by a theorem of Brelot f is bounded below on a sequence of circles of center Q, having radii arbitrarily near 0. Applying the minimum principle it follows that f is bounded below in a deleted neighborhood of Q and hence has a superharmonic extension to $R \cup \{Q\}$. The rest of the theorem is now trivial.

In the following theorem we write r for the distance to Q.

THEOREM 7.2. — *Let f be superharmonic in a deleted neighborhood* R *of a point* Q *of the plane. Suppose that there is an open set which is a deleted fine neighborhood* R_0 *of* Q *on which f is harmonic and bounded from above. Then there is a constant $c \geqslant 0$ and a function v superharmonic in* $R \cup \{Q\}$ *such that* $f = c \log r + v$.

The point Q is an irregular boundary point of R_0; f is harmonic and bounded above in a deleted fine neighborhood of Q (relative to R_0). Hence according to a theorem of Brelot [1] $f/\log r$ has a finite fine limit at Q. But then, for a suitable constant c_1, $f - c_1 \log r$ is superharmonic in a deleted neighborhood of Q and bounded below in some deleted fine neighborhood of Q. According to Theorem 7.1 this difference has a superharmonic extension v_0 to $R \cup \{Q\}$. If the measure corresponding to v_0 has the value c_2 on the singleton $\{Q\}$, $v = v_0 + c_2 \log r$ can be defined at Q to be superharmonic in a neighborhood of Q, and $f = v + c \log r$, where $c = c_1 - c_2$. Since f is bounded above in a deleted fine neighborhood of Q, $c \geqslant 0$.

The following example, due to Brelot, shows that Theorem 7.1 is false when the dimensionality is greater than 2. Let R be the interior of an N-dimensional ball, $N \geqslant 3$, less the center Q. Let S be a smooth hypersurface of revolution with axis a ray with endpoint Q, at which point S has an exponential cusp. Then the part of S in R together with one of the parts of the boundary of R cut off by S bound an open subset R_0 of R which is thin at Q. Brelot [1] has shown how to construct a strictly positive harmonic function in R_0 with boundary limit 0 at every point of the boundary of R_0 except Q. The function is not bounded or a familiar extension of the maximum principle would imply that the function vanished identically. If the function is extended by 0 to R it becomes a subharmonic function there, which we denote by $-f$. The function f is superharmonic on R, with fine limit 0 at Q, and does not have a superharmonic extension to $R \cup \{Q\}$ because the value of the extension at Q would have to be $-\infty$, by lower semicontinuity. This function is a counterexample to Theorem 7.1 for $N > 2$. It is easily seen that f is also a counterexample to the obvious N-dimensional version of Theorem 7.2 for $N > 2$.

If R and Q are as in the preceding theorems and if f is now complex-valued and meromorphic in R, the fine cluster set of f at Q is a compact subset of the extended plane. According to the Casorati–Weierstrass theorem, if f does not have a meromorphic extension to a neighborhood of Q the cluster set of f at Q is the extended plane, and it is then even true, according to the Picard theorem, that f can omit at most two values near Q. The following theorem is the fine topology version of the Casorati–Weierstrass theorem.

THEOREM 7.3. — *If f is meromorphic in the deleted neighborhood* R *of* Q *and does not have a meromorphic extension to the full neighborhood, one of the following situations must hold.*

(*a*) *The fine cluster set of f at* Q *is the extended plane.*

(*b*) *f has a fine limit at* Q. *In this case each value in the extended plane is taken on arbitrarily near* Q.

Case (*b*) can actually arise. In fact if $\alpha_n \to \infty$ it is easy to choose c_n to make $\Sigma c_n(z - \alpha_n)^{-1}$ meromorphic on the finite plane with fine limit 0 at ∞. The function thus provides an example of (b) with Q the point at ∞. To prove the theorem we remark first that if the fine cluster set of f at Q is not the extended plane we can suppose, making a trivial transformation if necessary, that this set does not contain the point ∞. Then f is bounded in a deleted fine neighborhood of Q so by Theorem 7.1 its real and imaginary parts have fine limits at Q. Hence f has a fine limit at Q. To finish the proof we need only show that if f has a fine limit at Q it takes on each value in the extended plane arbitrarily near Q. If the assertion is false let α be the fine limit and we can suppose that the omitted value is ∞. But then f is regular in a deleted neighborhood of Q and has a fine limit α at Q. It follows at once from Theorem 7.1 applied to the real and imaginary parts of f that, if $\alpha \neq \infty$, Q is a removable singularity of f, contrary to hypothesis. On the other hand if $\alpha = \infty$, $-\log|f|$ is superharmonic in a deleted neighborhood of Q with fine limit $-\infty$ there. According to Theorem 7.2 $-\log|f|$ must have a certain form which makes $r^n|f|$ bounded near Q for some positive n. Hence f either has a pole at Q or a removable singularity. Since both possibilities have been excluded, the proof is complete.

II. THE HARDY-LITTLEWOOD MAXIMAL INEQUALITIES.

8. The basic inequalities.

Let f be a function from $[0, a]$ to the reals, Lebesgue measurable and integrable. Define

$$\hat{f}(t) = \sup_{0 \leqslant s < t} \frac{1}{t - s} \int_s^t f(\xi)\, d\xi. \tag{8.1}$$

The Hardy–Littlewood inequalities [9] in a modernized treatment depend on the inequality

$$\lambda \mathrm{L}\{t : \hat{f}(t) \geqslant \lambda\} \leqslant \int_{\{t:\, \hat{f}(t) \geqslant \lambda\}} f(t)\, dt \tag{8.2}$$

where L is Lebesgue measure and λ is an arbitrary real number. (See [13], vol. 1, p. 31 (13.11).) The positivity hypothesis imposed in [13] on f and λ is unnecessary.) The inequality is equivalent to the inequality obtained on replacing « $\geqslant$ » by « $>$ » in the definition of the set involved. The inequality (8.2) between two functions f and $\hat{f}$, disregarding the background which led to it, arises in many contexts in ergodic theory and martingale theory. If f and $\hat{f}$ are both positive and if $p > 1$ (8.2) implies

$$(8.3) \qquad \int_0^a \hat{f}(t)^p\,dt \leqslant q^p \int_0^a f(t)^p\,dt, \qquad (q^{-1} + p^{-1} = 1).$$

(See [5], p. 317, Theorem 3.4′, for this derivation, where in [5] y should have been replaced by $\min(y, n)$; after the inequality is proved for this bounded random variable, let $n \to \infty$. The probability context is irrelevant.) The analogous inequality for $p = 1$ involves $\int_0^a f \log^+ f\,dt$ on the right. To avoid pointless repetition we shall suppose $p > 1$ below.

The inequality (8.3) was applied by Hardy and Littlewood to derive an inequality for subharmonic and thereby for analytic functions. Let v be a positive subharmonic function on a ball in N-space, $N \geqslant 2$. Suppose that v belongs to the class H_p: the average of v^p over concentric spheres is bounded independently of the radius. Then v has a (radial limit) boundary function V and V^p is integrable. Let $\tilde{v}(\eta)$ be the supremum of v on the radius to the boundary point η. Hardy and Littlewood proved using (8.3) that if $N = 2$

$$(8.4) \qquad \int \tilde{v}(\eta)^p\,d\eta \leqslant A(p) \int V(\eta)^p\,d\eta, \qquad p > 1,$$

where the integration is over the ball boundary, $d\eta$ is Lebesgue measure, and $A(p)$ depends on p but not on v. There is a modified inequality for the case $p = 1$. We observe for later use that it is sufficient to prove (8.4) for v harmonic, since if v is subharmonic it can be replaced by the harmonic function $\geqslant v$ determined by the Poisson integral with boundary function V.

These inequalities recall strongly elementary martingale inequalities, as is only natural in view of the close relations between martingale theory and derivation, as well as between martingale theory and potential theory. In the following sections it will be shown that (8.4) can be derived from a function-theoretic version of (8.2), leading as usual to (8.4) just as (8.2) does to (8.3). The function-theoretic version of (8.2) is a consequence of an elementary martingale inequality, one so elementary that it is slightly ridiculous to refer to it as a « maximal inequality », although the name suits the role played by this inequality.

More precisely, it will be shown that the context of inequality (8.4): functions on a disk and suprema on radii, is an awkward specialization of a more general inequality, a specialization made possible by the metric properties of the disk and the Harnack inequalities. The general inequality is for functions on a Green space of dimension $\geqslant 2$ with the suprema along radii replaced by suprema along Brownian paths. For the benefit of readers wary of probability theory, the general inequality will also be stated in its (less intuitive) non-probabilistic form. The point is that in the Hardy–Littlewood approach the fundamental inequality (8.2) is natural but (8.4) is a somewhat artificial application of it. In the probabilistic approach the analogue of (8.2) is valid directly in the function-theoretic application. The specialization to functions on a disk carries with it no simplification in the general treatment. If in addition to this specialization suprema along radii are used instead of along Brownian paths, the Hardy–Littlewood inequality (8.4) can be retrieved from the general case, but it will be clear that the specialization is somewhat contrived: the more natural result involves suprema on Brownian paths.

9. Non-probabilistic context.

In this section the properties of the function classes involved are sketched. Let R be a Green space of $N \geqslant 2$ dimensions, let ∂R be its Martin boundary and let $\mu(\xi, .)$ be harmonic measure on ∂R relative to ξ. In the following, $p \geqslant 1$ and q is the conjugate index. If $R_1 \subset R_2 \subset \cdots$ is a sequence of relatively compact open subsets of R with union R, the sequence will be called a nested sequence of open sets. If u is a subharmonic function on R, the integral of its restriction to ∂R_n with respect to R_n harmonic measure does not decrease when n increases and its limit is either everywhere finite or identically infinite, independently of the nested sequence of open sets. If $u \geqslant 0$ and if the limit is finite u is said to be in class H_1. If $u \geqslant 0$, $p > 1$, and if $u^p \in H_1$, u is said to be in class H_p. If f is harmonic or regular analytic it is said to be in class H_p if $|f|$ is in this class. The harmonic functions in H_1 are the functions of the form $f_1 - f_2$, where f_1 and f_2 are positive harmonic functions. If u is positive and superharmonic on R, u has a fine limit at almost every (harmonic measure) point of ∂R. The limit defines the fine boundary function U, and

$$u(\xi) \geqslant \int_{\partial R} U(\eta)\mu(\xi, d\eta). \tag{9.1}$$

Hence harmonic functions of class H_p, $p \geqslant 1$ have fine boundary functions.

If u is a harmonic function defined as the harmonic average of some boundary function U,

$$u(\xi) = \int_{\partial R} U(\eta)\mu(\xi, d\eta), \tag{9.2}$$

u is called a Dirichlet solution. The function u is then in H_1 and U is its fine boundary function. A harmonic function u is a Dirichlet solution if and only if for some (equivalently *for every*) nested sequence $\{R_n\}$ of open sets the restriction of u to ∂R_n, for R_n harmonic measure at some (equivalently *each*) specified point is the nth term of a uniformly integrable sequence. It is sufficient if $u \in H_p$ for some $p > 1$.

Let v be a subharmonic function that is dominated from above by a Dirichlet solution v_1. Equivalently, the subharmonic function $\max[0, v]$ is to have the uniform integrability property of the preceding paragraph. The function v_1 is equal to the harmonic average of its fine boundary function. Since $v_1 - v$ is positive and superharmonic, $v_1 - v$ has a fine boundary function and is at least equal to its harmonic average. Thus v has a fine boundary function V and

$$v(\xi) \leqslant \int_{\partial R} V(\eta)\mu(\xi, d\eta). \tag{9.3}$$

The right side of (9.1) defines the Dirichlet solution for the boundary function V. This Dirichlet solution is the minimal choice of the dominating function v_1. In the following it will be convenient to suppose that v has been defined (as V) on ∂R.

10. Probabilistic inequalities.

Let $\{x(t), 0 \leqslant t \leqslant \infty\}$ be a separable Brownian motion process on R with initial point ξ, where $x(t)$ is defined as $x(\tau)$ for $t \geqslant \tau =$ time the path reaches ∂R. The point ξ is fixed throughout the following discussion. If η is a minimal point of ∂R, $\{x^\eta(t), 0 \leqslant t \leqslant \infty\}$ denotes the corresponding conditional process of paths from ξ to η. The notation P, E will be used for probabilities and expectations for the $x(t)$ process; P^η, E^η will be used for the $x^\eta(t)$ process. If K_η is the minimal harmonic function corresponding to the minimal boundary point η, with the normalization $K_\eta(\xi) = 1$, the $x^\eta(t)$ process is usually referred to as the K_η-path process, from ξ.

If v satisfies the conditions at the end of the preceding section, that is if v is subharmonic and dominated by a Dirichlet solution on R, the process $\{v[x(t)], 0 \leqslant t \leqslant \infty\}$ is a submartingale. In this context (9.3) becomes the submartingale inequality

$$v(\xi) = v[x(0)] \leqslant E\{v[x(\infty)]\}. \tag{10.1}$$

Define $\tilde{\tilde{v}}_p(\eta)(\geqslant 0)$ by

$$\tilde{\tilde{v}}_p(\eta)^p = E^\eta\{\sup_t v[x^\eta(t)]^p\} \tag{10.2}$$

if $v \geqslant 0$. This expectation will be evaluated probabilistically later.

THEOREM 10.1. — *If v is subharmonic and dominated from above by a Dirichlet solution,*

$$\begin{aligned} \lambda P\{\sup_t v[x(t)] \geqslant \lambda\} &= \lambda \int_{\partial R} P^\eta\{\sup_t v[x^\eta(t)] \geqslant \lambda\}\mu(\xi, d\eta) \\ &\leqslant \int_{\partial R} V(\eta) P^\eta\{\sup_t v[x^\eta(t)] \geqslant \lambda\}\mu(\xi, d\eta) \\ &= \int_{\{\sup v[x(t)] \geqslant \lambda\}} V[x(\infty)]\, dP \end{aligned} \tag{10.3}$$

for all real λ. If in addition $v \geqslant 0$, and if $p > 1$,

$$\begin{aligned} E\{\sup_t v[x(t)]^p\} &= \int_{\partial R} \tilde{\tilde{v}}_p(\eta)^p \mu(\xi, d\eta) \\ &\leqslant q^p \int_{\partial R} V(\eta)^p \mu(\xi, d\eta) = q^p E\{V[x(\infty)]^p\}. \end{aligned} \tag{10.4}$$

The inequality between the first term on the left in (10.3) and the last on the right is precisely the standard submartingale maximal inequality, which is much less deep than the Hardy–Littlewood maximal inequality (8.2), although related to it. The inequality between the first and second, and between the third and fourth terms in (10.3) is obtained by first conditioning the Brownian paths to terminate at η and then averaging over η. The equalities in (10.4) are derived by the same procedure. We have already remarked in Section 8 that (10.4) can be derived from (10.3) using standard manipulations; (10.3) is the fundamental inequality. Inequality (10.4) is significant only when the right side is finite, that is, as can be easily shown, only when $v \in H_p$.

We observe that (10.4) is a natural generalization of the Hardy–Littlewood inequality (8.4). There is no elegant version of (10.3) involving suprema on radii, although it is not difficult to derive a somewhat artificial version.

11. Non-probabilistic versions.

The fundamental tool in translating inequalities like those in Section 10 into non-probabilistic language is the following well-known fact. Let $\hat{R}_w^A$ be the regularized reduced function of the positive superharmonic function w on the set $A \subset R \cup \partial R$, that is the positive superharmonic function equal off a subset of $A \cap R$ of capacity 0 to the lower envelope of all positive superharmonic functions $\geqslant w$ either (*a*) on the intersection of R with a neighborhood of A relative to $R \cup \partial R$ or equivalently (*b*) on $A \cap R$ and, extending the functions to the Martin boundary by their fine boundary functions, also on $A \cap \partial R$. If w is the constant function 1, this reduced function at ξ is the probability that a Brownian path from ξ ever meets A. For example if A is a Borel subset of ∂R, $\hat{R}_w^A(\xi) = \mu(\xi, A)$ is the probability that a Brownian path from ξ meets ∂R in a point of A. More generally for any w $\hat{R}_w^A(\xi)/w(\xi)$ is the probability that a w-path from ξ ever meets A. We shall use the fact that if B is a Borel subset of the Martin boundary of strictly positive measure $\hat{R}_{\mu(.,B)}^A(\xi)$ is the probability that a Brownian path from ξ meets the Martin boundary in a point of B and also meets A.

The quantity $\tilde{\tilde{v}}_p(\xi)$ defined in Section 8 can now be evaluated in non-probabilistic terms. In fact if $v \geqslant 0$ and if we write $\{v \geqslant \lambda\}$ for the subset of $R \cup \partial R$ where the indicated inequality is satisfied, remembering that v has been extended to the Martin boundary by means of its fine boundary function,

$$(11.1) \qquad \tilde{\tilde{v}}_p(\xi)^p = \frac{1}{p}\int_0^\infty s^{p-1} P\{\sup_t v[x(t)] \geqslant s\}\, ds$$

$$= \frac{1}{p}\int_0^\infty s^{p-1} \hat{R}_1^{\{v \geqslant s\}}(\xi)\, ds.$$

Alternatively

$$(11.2)\quad \tilde{\tilde{v}}_p(\xi)^p = \frac{1}{p}\int_0^\infty s^{p-1}\,ds \int_{\partial R} P^\eta\{\sup_t v[x^\eta(t)] \geqslant s\}\mu(\xi, d\eta)$$
$$= \frac{1}{p}\int_0^\infty s^{p-1}\,ds \int_{\partial R} \hat{R}^{\{v \geqslant s\}}_{K_\xi}(\xi)\mu(\xi, d\eta).$$

Theorem 11.1 is less awkward in its non-probabilistic form when v is supposed positive, so this hypothesis is made in the following theorem.

THEOREM 11.1. — *If v is subharmonic and positive and dominated by a Dirichlet solution,* (10.3) *in non-probabilistic language takes the form*

$$(11.3)\quad \begin{aligned} \lambda R_1^{\{v \geqslant \lambda\}}(\xi) &= \lambda \int_{\partial R} \hat{R}^{\{v \geqslant \lambda\}}_{K\eta}(\xi)\mu(\xi, d\eta) \leqslant \int_{\partial R} V(\eta)\hat{R}^{\{v \geqslant \lambda\}}_{K\eta}(\xi)\mu(\xi, d\eta) \\ &= \int_{\{V \geqslant \lambda\}} [V(\eta) - \lambda]\mu(\xi, d\eta) + \int_0^\lambda \hat{R}^{\{v \geqslant \lambda\}}_{\mu(.,V \geqslant s)}(\xi)\,ds \end{aligned}$$

and (10.4) *takes the form*

$$(11.4)\quad \frac{1}{p}\int_0^\infty s^{p-1}\hat{R}_1^{\{v \geqslant s\}}(\xi)\,ds = \frac{1}{p}\int_{\partial R} \mu(\xi, d\eta)\int_0^\infty s^{p-1}\hat{R}^{\{v \geqslant s\}}_{K_\xi}(\xi)\,ds$$
$$\leqslant q^p \int_{\partial R} V(\eta)^p \mu(\xi, d\eta).$$

Since (11.4) can be derived from (11.3), we shall only discuss (11.3) which we shall first derive from (10.3) and then prove directly without using probability theory. According to the translation principle stated at the beginning of this section, taken together with the fact that almost all sample functions of the $v[x(t)]$ process are continuous, the first term of (10.3) is equal to the first term of (11.3). Furthermore

$$(11.5)\quad \begin{aligned} \int_{\{\sup_t v[x(t)] \geqslant \lambda\}} V[x(\infty)]\,dP &= \int_0^\infty P\{\sup_t v[x(t)] \geqslant \lambda, V[x(\infty)] \geqslant s\}\,ds \\ &= \int_\lambda^\infty P\{V[x(\infty)] \geqslant s\}\,ds + \int_0^\lambda P\{\sup_t v[x(t)] \geqslant \lambda, V[x(\infty)] \geqslant s\}\,ds \\ &= \int_{\{V \geqslant \lambda\}} [V(\xi) - \lambda]\mu(\xi, d\eta) + \int_0^\lambda \hat{R}^{\{v \geqslant \lambda\}}_{\mu(.,V \geqslant s)}(\xi)\,ds. \end{aligned}$$

Hence the last term of (10.3) is equal to the last term of (11.3). The equalities in (11.3) are established by a Fubini argument or can be derived from the corresponding equalities in (10.3).

To prove (11.3) directly we observe that the right side of the inequality defines a positive superharmonic function of ξ, so to prove that it majorizes the first term on the left we need only prove that the right side is at least equal to λ at any point ξ for which $v(\xi) \geqslant \lambda$ except for a set of zero capacity. We exclude ξ if $v(\xi) \geqslant \lambda$ but if ξ is not a fine limit point of $\{v \geqslant \lambda\}$. Since the set of all fine isolated points of a set which are in the set has zero capacity, this is a legitimate set to exclude. (Actually it can be shown that this excluded set is empty.) If $v(\xi) \geqslant \lambda$ and if ξ has not been excluded, the last integrand on the right in (11.3) is equal to $\mu(\xi, \mathrm{V} \geqslant s)$ so that the last integral becomes

$$\int_0^\lambda \mu(\xi, \mathrm{V} \geqslant s)\, ds = \int_{\{\mathrm{V} < \lambda\}} \mathrm{V}(\eta)\mu(\xi, d\eta) + \lambda\mu(\xi, \mathrm{V} \geqslant \lambda).$$

Thus the right side of (11.3) is

$$\int_{\partial \mathrm{R}} \mathrm{V}(\eta)\mu(\xi, d\eta) \geqslant v(\xi)/\lambda \geqslant 1$$

as was to be proved.

12. The ball case.

If R is a ball we shall show that the Hardy–Littlewood inequality (8.4) can be derived from (11.4). If R is a ball take $\xi = \mathrm{Q}$, the ball center. The minimal function K_η is the integrand in the Poisson integral corresponding to the boundary point η. We shall use the fact that if B_η is a ball with center η_1 on the radius to the boundary point η, of radius $\geqslant c|\eta - \eta_1|$ then the reduced function of K_η on B_η is $\geqslant \delta > 0$ at Q, where δ depends only on c. For the corresponding fact for a half-space see [2]. In deriving (8.4) we can assume, as remarked in Section 8, that v is harmonic. Suppose then that v is a positive harmonic function on the ball, in the class H_p, $p > 1$. Choose η_1 on the radius to the boundary point η to make $v(\eta_1) \geqslant \tilde{v}(\eta)/2$. Then let B_η be the ball of center η_1 of maximal radius to make $v \geqslant \tilde{v}(\eta)/3$ in this ball. According to Harnack's inequality, the radius of B_η is bounded below by $c|\eta - \eta_1|$ for some strictly positive constant c independently of η, η_1. Then for some strictly positive δ

$$\int_{\partial R} \hat{R}_{K_\eta}^{\{v \geqslant s\}}(Q)\mu(Q, d\eta) \geqslant \int_{\{\tilde{v} \geqslant 3s\}} \hat{R}_{K_\eta}^{B_\eta}(Q)\mu(Q, d\eta) \geqslant \delta\mu(Q, \tilde{v} \geqslant 3s). \tag{12.1}$$

Here $\mu(Q,.)$ is Lebesgue measure on the ball boundary, normalized to have value 1 for the whole boundary. Thus the left side of (11.4) is at least

$$\frac{\delta}{p}\int_0^\infty s^{p-1}\mu(Q, v \geqslant 3s)\, ds = 3^{-p}\delta \int_{\partial R} \tilde{v}(\eta)^p \mu(Q, d\eta)$$

and the inequality (11.4) thus yields the Hardy–Littlewood inequality (8.4), generalized to N dimensions, with $A(p) = (3q)^p/\delta$. The hypotheses are the same as those of Hardy and Littlewood except for their dimensionality restriction. Rauch [11] and Smith [12] have also generalized the Hardy–Littlewood inequality to N dimensions, and Smith went further, getting a version for a domain in N-space with a sufficiently smooth boundary.

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