ANNALES

DE

L'INSTITUT FOURIER

Carolyn S. GORDON & Juan Pablo ROSSETTI

Boundary volume and length spectra of Riemannian manifolds: what the middle degree Hodge spectrum doesn't reveal

BOUNDARY VOLUME AND LENGTH SPECTRA
OF RIEMANNIAN MANIFOLDS:
WHAT THE MIDDLE DEGREE HODGE SPECTRUM
DOESN'T REVEAL

by C.S. GORDON and J.P. ROSSETTI [1]

To what extent does spectral data associated with the Hodge Laplacian $dd^* + d^*d$ on a compact Riemannian manifold M determine the geometry and topology of M? Let $\mathrm{spec}_p(M)$ denote the spectrum of the Hodge Laplacian acting on the space of p-forms on M, with absolute boundary conditions in case the boundary of M is non-empty. (We will review the notion of absolute and relative boundary conditions in Section 1.) For each p, the spectrum $\mathrm{spec}_p(M)$ is known to contain considerable geometric information. For example, under genericity conditions, the p-spectrum of a closed Riemannian manifold M determines the geodesic length spectrum of M.

In this article, we will focus on even-dimensional manifolds and give a very simple method for obtaining manifolds with the same Hodge spectrum in the middle degree. Through examples, we will discover that this middle degree spectrum contains a perhaps surprising lack of topological and geometric information.

Among the examples of manifolds that we will construct with the same middle degree spectrum are:

- a cylinder, a Möbius strip, and a Klein bottle;

- pairs of cylinders with different boundary lengths;

• a hemisphere and a projective space;

• products $S(r) \times \mathbb{P}(s)$ and $\mathbb{P}(r) \times S(s)$, where $\mathbb{P}(t)$, resp. $S(t)$, denote a projective space, respectively sphere, of radius t (for generic r and s, the length spectra differ);

• non-orientable closed hyperbolic surfaces with different injectivity radius and length spectrum;

• pairs of surfaces, one with an arbitrarily large number of boundary components and the other closed. The metrics can be chosen to be hyperbolic.

We also obtain examples in all even dimensions exhibiting similar properties.

These examples prove:

THEOREM. — *The middle degree Hodge spectrum of an even-dimensional Riemannian manifold M does not determine* :

(i) *the volume of the boundary or even whether M has boundary;*

(ii) *the geodesic length spectrum or injectivity radius. In particular, it does not determine the length of the shortest closed geodesic.*

The result (i) is new. Concerning (ii), R. Miatello and the second author [MR3] recently constructed examples of flat manifolds, p-isospectral for various choices of p, such that the length of the shortest closed geodesic differed. (Here, we say two manifolds are p-isospectral if they have the same p-spectrum.)

Note that all the closed manifolds that we construct are non-orientable. For *orientable* surfaces, the 1-spectrum completely determines the 0-spectrum (which coincides with the 2-spectrum). Thus the behavior exhibited by our examples cannot occur for 1-isospectral orientable surfaces. For *non-orientable* surfaces, the 0-spectrum and 2-spectrum no longer coincide and the 1-spectrum is the join of the two (except for the 0-eigenspace). Our examples show that the join of the two spectra contains less geometric information than either one individually. In higher dimensions, it is perhaps surprising that all our examples are non-orientable. The Hodge $*$ operator intertwines the exact and co-exact middle degree forms and commutes with the Laplacian; thus the "exact" and "co-exact spectra" coincide. Hence the m-spectrum of a $2m$-dimensional *orientable* closed manifold, $m > 1$, contains half the information contained in the $(m-1)$-

spectrum and nothing more (except for the dimension of the space of harmonic forms). In contrast, for non-orientable manifolds, the "exact" and "co-exact spectra" no longer coincide, so the middle degree spectrum obstensibly contains the same amount of data as the other spectra.

One of the primary tools for recovering geometric and topological information from the spectra is through the small time asymptotics of the heat equation. See, for example, [Gi] or [BBG1], [BBG2]. For closed manifolds, the trace of the heat kernel associated with the Hodge Laplacian on p-forms has an asymptotic expansion of the form

$$\zeta(t) = (4\pi t)^{-n/2}\big(a_0(p) + a_1(p)t + a_2(p)t^2 + \cdots\big).$$

The coefficients $a_i(p)$ are spectral invariants and are given by integrals, with respect to the Riemannian measure, of expressions involving the curvature and its covariant derivatives. The first three heat invariants are given as follows:

- $a_0 = C_0(p)\operatorname{vol}(M)$;

- $a_1(p) = C_1(p)\int_M \tau$, where τ is the scalar curvature and $C_i(p)$, $i = 0, 1$, is a constant depending only on p; and

- $a_2(p)$ is a linear combination, with coefficients depending only on p, of $\int_M \tau^2$, $\int_M \|\operatorname{Ric}\|^2$ and $\int_M \|R\|^2$.

For manifolds with boundary, the expansion is instead in powers of $t^{\frac{1}{2}}$. The coefficient $a_{\frac{1}{2}}(p)$ is given by $c(p)\operatorname{vol}(\partial M)$, where $c(p) = \binom{n-1}{p} - \binom{n-1}{p-1}$.

A very difficult open question is whether the heat invariants are the only integral invariants of the spectrum, i.e., the only invariants which are integrals of curvature expressions either over the manifold or over the boundary. This article and work of Dorothee Schueth lend support (albeit in a small way) towards an affirmative answer since:

- The coefficient $a_{\frac{1}{2}}(p) = c(p)\operatorname{vol}(\partial M)$ vanishes precisely when $\dim(M)$ is even and $p = \frac{1}{2}\dim(M)$. Our examples show in this case that the p-spectrum indeed does not determine the volume of the boundary.

- Dorothee Schueth [S1] showed that the three individual integral terms in the expression for $a_2(0)$ are not spectral invariants.

A secondary theme of this paper is the spectrum of Riemannian orbifolds. Riemannian orbifolds are analogs of Riemannian manifolds but with singularities. They are locally modelled on quotients of Riemannian

manifolds by finite effective isometric group actions. The singularities in the orbifold correspond to the singular orbits. One can define the notion of Laplacian on p-forms on orbifolds. A natural question is whether the spectrum contains information about the singularities. We will see:

THEOREM. — *The middle degree Hodge spectrum cannot distinguish Riemannian manifolds from Riemannian orbifolds with singularities.*

For example, we will see that the mutually 1-isospectral cylinder, Klein bottle and Möbius strip are also 1-isospectral to a "pillow", a 2-dimensional orbifold with four singular points, so named because of its shape. In dimension $2m$, $m > 1$, we will also construct orbifolds that are m-isospectral to a projective space and to a hemisphere. To our knowledge, the examples given here are the first examples of orbifolds p-isospectral to manifolds for some p. We do not know whether the spectrum of the Laplacian on functions can distinguish orbifolds from manifolds, although we will give some positive results in §3 and we will verify that none of the middle degree isospectral orbifolds and manifolds we construct are 0-isospectral.

We will also construct orbifolds, isospectral in the middle degree, having singular sets of arbitrarily different dimensions.

In this paper we have concentrated on m-isospectrality of $2m$-manifolds and orbifolds. However, for various choices of p, one can also obtain p-isospectral orbifolds with singular sets of different dimensions. This phenomenon may be discussed in a later paper.

We conclude these introductory remarks with a partial history of the isospectral problem for the Hodge Laplacian. For a general survey of isospectral manifolds, see [Go3]. Most known examples of isospectral manifolds, e.g., those constructed by the method of Sunada [Sun], are strongly isospectral; in particular, the manifolds are p-isospectral for all p. Thus they do not reveal possible differences in the geometric information contained in the various p-spectra. The article [Go1] gave the first example of 0-isospectral manifolds which are not 1-isospectral; further examples were given in [Gt1], [Gt2]. (The 0-spectrum, i.e., the spectrum of the Laplacian acting on smooth functions, is frequently referred to simply as the spectrum of the manifold.) Ikeda [Ik] constructed, for each positive integer k, spherical space forms which are p-isospectral for all p less than k but not for $p = k$. In the past decade, many examples have been constructed of 0-isospectral manifolds with different local as well as global

geometry, e.g., [Sz1], [Sz2], [Go2], [Go4], [GW], [GGSWW], [GSz], [S1], [S2], [S3], and [Gt1]; at least in those cases in which comparisons of the higher p-spectra have been carried out, these manifolds are not p-isospectral for $p \geq 1$. The first examples of manifolds which are p-isospectral for some values of p but not for $p = 0$ were flat manifolds constructed by R. Miatello and the second author in [MR1], [MR2], [MR3]. R. Gornet and J. McGowan (private communication) recently constructed examples of spherical space forms which are simultaneously p-isospectral for $p = 0$ and for various other, but not all, p.

The second author wishes to thank the mathematics department at Dartmouth College for its great hospitality during the time this paper was written.

1. Method for constructing manifolds isospectral in the middle degree.

1.1. DEFINITION. — *Suppose M is a compact Riemannian manifold with boundary. Let ω be a smooth p-form on M. For $x \in \partial M$, write $\omega_x = \omega_x^T + \omega_x^N$, where $\omega_x^T \in \bigwedge^p (\partial M)_x^*$ and $\omega_x^N = \sigma \wedge \mu$ with $\sigma \in \bigwedge^{p-1} (\partial M)_x^*$ and μ a vector in $T_x^*(M)$ normal to the cotangent space of ∂M at x. The form ω is said to satisfy* absolute (resp. relative) boundary conditions *if*

$$\omega^N = 0 = (\mathrm{d}\omega)^N, \quad \left(resp. \ \omega^T = 0 = (\delta\omega)^T \right),$$

everywhere on ∂M. If M is orientable, observe that ω satisfies the relative boundary conditions if and only if $*\omega$ satisfies the absolute boundary conditions, where $*$ is the Hodge duality operator.

1.2. *Notation and remarks.* — Let M be a compact Riemannian manifold of dimension $n = 2m$.

(i) If M is closed let $\mathrm{spec}_m(M)$ denote the spectrum of the Hodge Laplacian of M acting on smooth m-forms.

(ii) If M has boundary, we denote by $\mathrm{spec}_m(M)$ the absolute spectrum of M. If M is orientable, then since the Hodge $*$ operator carries $\mathcal{A}^m(M)$ to itself and commutes with the Hodge Laplacian, we see from 1.1 that the absolute m-spectrum coincides with the relative m-spectrum. However, this coincidence of the spectra does not in general occur when M is non-orientable.

(iii) Suppose that M is closed and τ is an orientation reversing isometric involution. If τ does not act freely, then $\langle\tau\rangle\backslash M$ is a Riemannian orbifold. Its m-spectrum is defined to be the spectrum of the Hodge Laplacian of M acting on the space of τ-invariant m-forms. We denote this spectrum by $\operatorname{spec}_m(\langle\tau\rangle\backslash M)$. (In this article we will be concerned only with orbifolds of this form. For a more general discussion of orbifolds, see for example [Sc] or [T].)

If the fixed point set of τ is a submanifold of codimension one, then the underlying space of the orbifold $\langle\tau\rangle\backslash M$ is a Riemannian manifold with boundary. The boundary corresponds to the singular set of the orbifold. (E.g., if M is a sphere and τ is reflection across the equator, then $\langle\tau\rangle\backslash M$ is an orbifold whose underlying space is a hemisphere.) Since the orbifold m-spectrum $\operatorname{spec}_m(\langle\tau\rangle\backslash M)$ just defined agrees with the m-spectrum of the underlying manifold with absolute boundary conditions, we will ignore the orbifold structure and view $\langle\tau\rangle\backslash M$ as a manifold with boundary in what follows. This point of view allows us to obtain manifolds with boundary as quotients of closed manifolds.

1.3. THEOREM. — *Let M be a $2m$-dimensional orientable closed Riemannian manifold. Suppose that τ is an orientation reversing involutive isometry of M. Then $\operatorname{spec}_m(\langle\tau\rangle\backslash M)$ consists precisely of the eigenvalues of $\operatorname{spec}_m(M)$ but with all multiplicities multiplied by $\frac{1}{2}$.*

Proof. — Let λ be an eigenvalue in $\operatorname{spec}_m(M)$, say of multiplicity m_λ. Both τ and the Hodge $*$ operator commute with the Hodge Laplacian and thus leave the λ-eigenspace $H_\lambda \subset \mathcal{A}^m(M)$ invariant. Letting H_λ^+, respectively H_λ^-, denote the space of τ-invariant, respectively τ-anti-invariant, forms in H_λ, then $H_\lambda = H_\lambda^+ \oplus H_\lambda^-$. The λ-eigenspace in $\mathcal{A}^m(\langle\tau\rangle\backslash M)$ corresponds to H_λ^+.

The Hodge $*$ operator interchanges H_λ^+ and H_λ^-. To see this, let Ω denote the Riemannian volume form of M. For $\alpha \in \mathcal{A}^m(M)$, we have

$$\alpha \wedge *\alpha = \|\alpha\|^2\Omega \quad \text{and} \quad \tau^*\alpha \wedge \tau^*(*\alpha) = \|\alpha\|^2\tau^*\Omega = -\|\alpha\|^2\Omega,$$

since τ is an orientation reversing isometry. Thus $*$ interchanges H_λ^+ and H_λ^-.

Consequently, H_λ^+ and H_λ^- both have dimension $\frac{1}{2}m_\lambda$, and the theorem follows. $\square$

1.4. COROLLARY. — *Let M_1 and M_2 be $2m$-dimensional orientable closed Riemannian manifolds with $\operatorname{spec}_m(M_1) = \operatorname{spec}_m(M_2)$. Suppose that*

τ_1 and τ_2 are orientation reversing involutive isometries of M_1 and M_2, respectively. Then

$$\mathrm{spec}_m\big(\langle\tau_1\rangle\backslash M_1\big) = \mathrm{spec}_m\big(\langle\tau_2\rangle\backslash M_2\big).$$

Moreover, if N is a $2k$-dimensional closed Riemannian manifold, then

$$\mathrm{spec}_{m+k}\big((\langle\tau_1\rangle\backslash M_1)\times N\big) = \mathrm{spec}_{m+k}\big((\langle\tau_2\rangle\backslash M_2)\times N\big).$$

For the second statement, observe that τ_i extends to an involutive orientation reversing isometry of $M_i \times N$.

Remark. — The conclusion fails to hold if we drop the hypothesis that the involutions be orientation reversing. For example, let $M_1 = M_2 = (\mathbb{Z}\times 2\mathbb{Z})\backslash\mathbb{R}^2$, and let τ_1 and τ_2 be the translations $(x,y)\mapsto(x+\frac{1}{2},y)$ and $(x,y)\mapsto(x,y+1)$, respectively. Then the quotient tori are not 1-isospectral.

In our applications of Corollary 1.4 below, we will take $M_1 = M_2$.

A useful special case of Corollary 1.4 is the following:

1.5. COROLLARY. — *Let M and M' be manifolds of dimension k and k', respectively, with $k + k'$ even, say $k + k' = 2m$. Suppose ϕ and ϕ' are orientation reversing involutive isometries of M and M' respectively. Then $(\langle\phi\rangle\backslash M)\times M'$ is m-isospectral to $M\times(\langle\phi'\rangle\backslash M')$.*

Given a Riemannian manifold M and a non-negative integer $k \leq \dim(M)$, let $\{\lambda_j\}_{j=1}^\infty$ be the spectrum of the Laplacian acting on k-forms on M. The complexified heat trace is given by $Z(z) = \Sigma_{n=0}^\infty \exp(-\lambda_n/z)$ for $z \in \mathbb{C}$. Recall that Y. Colin de Verdière [C] obtained an expansion of the complexifed heat trace as a sum of oscillating terms whose periods are related to the lengths of the closed geodesics in M. The authors are grateful to the referee for pointing out the following additional corollary of Theorem 1.3.

1.6. COROLLARY. — *Suppose that N is a nonorientable closed Riemannian manifold of dimension $2m$. Let γ be an isolated, nondegenerate simple closed geodesic such that the holonomy about γ is orientation reversing. Then γ does not contribute to the complexified heat trace associated with the Laplacian acting on m-forms.*

Proof. — Let M be the orientation covering of N with the lifted Riemannian metric. By Theorem 1.3, the complexified heat traces of M and N associated with the Laplacian on m-forms coincide except for a factor of $\frac{1}{2}$. Since the lift of γ to M is not closed, it does not contribute to

the complexified heat trace of M and hence γ does not contribute to that of N.   □

1.7. Remark. — In the setting of Corollary 1.6 with $m = 1$, it was already known that the contribution of γ to the complexified heat trace on 1-forms must be zero up to principal order since the principal order coefficient is the trace of the holonomy map about the geodesic.

2. Examples of manifolds isospectral in the middle degree.

2.1. Remark. — Most, though not all, of the examples we construct below will be surfaces. However, by taking products of the manifolds in these examples with a closed manifold N and applying the second statement of Corollary 1.4, one obtains examples in arbitrary even dimensions.

We begin with manifolds of positive curvature.

2.2. Hemispheres and projective spaces. — Let M be the $2m$-sphere, $m \geq 1$, let τ denote the antipodal map and let σ denote reflection about an equatorial sphere. Then Corollary 1.4 implies that the projective space $\langle\tau\rangle\backslash M$ is m-isospectral to the hemisphere $\langle\sigma\rangle\backslash M$. (I.e., the m-spectrum of the projective space coincides with the absolute m-spectrum of the hemisphere.)

2.3. Half ellipsoids with different boundary volume. — Let M be a $2m$-dimensional ellipsoid (not round) and let σ and τ denote the reflections across two different hyperplanes of symmetry. Then $\langle\tau\rangle\backslash M$ and $\langle\sigma\rangle\backslash M$ are m-isospectral manifolds with boundary, but their boundaries have different volume.

2.4. Products of spheres and projective spaces with different length spectra and injectivity radius. — Let $S^k(r)$ and $\mathbb{P}^k(r)$ denote the sphere and projective space, respectively, of radius r and dimension k. Then by Corollary 1.5, for $m, n \geq 1$ and for any $r, s > 0$, the manifolds $S^{2m}(r) \times \mathbb{P}^{2n}(s)$ and $\mathbb{P}^{2m}(r) \times S^{2n}(s)$ have the same $(m + n)$-spectrum. However, for generic choices of r and s, the lengths of the shortest closed geodesics in the two manifolds differ.

These closed manifolds are also isospectral to the product of a hemisphere and a sphere, as can be seen by taking an involution of $S^{2m}(r) \times S^{2n}(s)$ given by the equatorial reflection in one factor.

We next consider flat manifolds.

2.5. *Cylinders, Klein bottles, and Möbius strips.* — The following, with appropriately chosen sizes, are mutually 1-isospectral:

  (i) a Klein bottle;

 (ii) a cylinder, say of perimeter p;

(iii) a Möbius strip of perimeter $p/\sqrt{2}$.

Indeed each of these is a quotient of $T = \mathbb{Z}^2\backslash\mathbb{R}^2$ by an involution τ arising from an involution $\tilde{\tau}$ of $\mathbb{R}^2$ given as follows: For the Klein bottle, $\tilde{\tau}$ is the composition of translation by $(\frac{1}{2},0)$ with reflection R about the x-axis. For the cylinder, $\tilde{\tau}$ is the reflection R just defined. For the Möbius strip, $\tilde{\tau}$ is reflection across the line $x = y$.

Thus by Corollary 1.4, the Klein bottle, the cylinder and the Möbius strip all have the same 1-form spectrum.

2.6. *Cylinders of different shapes.* — Let $C_{(h,w)}$ denote the flat cylinder of height h and circumference w, hence of perimeter $2w$. Then for all $a, b > 0$, we have

$$\mathrm{spec}_1\left(C_{(a,\frac{b}{2})}\right) = \mathrm{spec}_1\left(C_{(b,\frac{a}{2})}\right)$$

since both cylinders are quotients by a reflection of a rectangular torus of height a and width b. Thus cylinders of arbitrarily different perimeters can have the same 1-form spectrum.

2.7. *Remarks.* — (i) One similarly obtains pairs of non-isometric Klein bottles with the same 1-spectra and with different lengths of closed geodesics.

 (ii) By taking M to be a rhomboidal torus and letting τ_1 and τ_2 be reflections across the "diagonals" one similarly obtains pairs of 1-isospectral Möbius strips of arbitrarily different perimeters.

(iii) By taking products of the various 1-isospectral flat manifolds in 2.5, 2.6 and in these remarks with a torus of dimension $2m - 2$ and recalling Remark 2.1, we obtain pairs of m-isospectral $2m$-dimensional flat manifolds with the same boundary behavior as the surfaces in these examples.

Finally we consider hyperbolic surfaces.

2.8. THEOREM. — *In each genus $g > 1$, there exist non-orientable hyperbolic closed surfaces S_1, S_2, S_3, and S_4 of genus g such that*

(a) *All four surfaces have the same Hodge spectrum on 1-forms.*

(b) *They have different length spectra (different in the strong sense that different lengths occur, not just different multiplicities), and S_4 has a smaller injectivity radius than S_1, S_2, and S_3.*

(c) *The Laplacians acting on functions on these surfaces are not isospectral.*

These surfaces are also 1-isospectral to each of four hyperbolic surfaces with boundary having 2, 4, $2g - 2$, and $2g$ boundary components, respectively.

We remark that (b) implies (c), since the 0-spectrum determines the weak length spectrum (i.e., the spectrum of lengths of closed geodesics, ignoring multiplicities) in the case of hyperbolic manifolds. (Aside: For Riemann surfaces, it is a classical result of Huber that the 0-spectrum and the strong length spectrum, i.e., the spectrum of lengths, counting multiplicities, determine each other. This result has been extended to non-orientable hyperbolic surfaces by Peter Doyle and the second author in an article in preparation.)
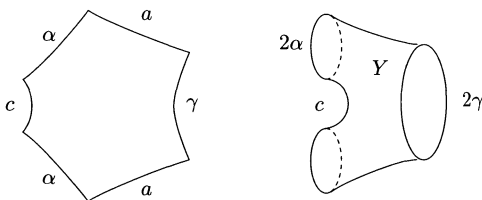

Figure 1. *Right-angled hexagon and the associated pair of pants.*

Proof. — We construct an orientable surface S as follows: Let Y be a pair of hyperbolic pants, as shown in Figure 1; the boundary geodesics of the pant legs have the same length, while the waist may have a different length. The pants are formed by gluing together two identical right-angled geodesic hexagons along three sides. The hexagon is also pictured in Figure 1. Choose a positive integer t and glue together $4t$ isometric copies $Y_1, \ldots, Y_{4t}$ of Y to obtain a Riemann surface S of genus $2t + 1$. To describe the symmetries, we will visualize S (as shown in Figure 2 in case $t = 1$) as obtained from a surface N with boundary in $\mathbb{R}^3$ with appropriate identifications of boundary edges. The three symmetries τ_H, τ_{V1}, and τ_P of N given by reflection across

the xy-plane, the xz-plane and the yz-plane, respectively, define commuting involutive isometries of S. (We are using the indices H, V, and P here to indicate reflection across planes that are horizontal, vertical, or the plane of the paper.) There are additional isometric involutions of S (not of N) given by reflections across vertical planes passing through the waists of two of the pants. Choose such a symmetry and denote it by τ_{V2}. E.g., in case $t = 1$, a choice of τ_{V2} interchanges pants Y_1 with Y_2 and Y_3 with Y_4 in Figure 2. Finally, let ρ be the orientation preserving isometric involution (rotation) ρ sending the pair of pants Y_i to the pants $Y_{i+2t} \bmod 4t$, for each i. Note that τ_H and τ_P commute with ρ. Thus the isometries $\tau_1 : = \tau_H \circ \tau_{V1} \circ \tau_P$, $\tau_2 : = \tau_P \circ \rho$, $\tau_3 : = \tau_H \circ \rho$, and $\tau_4 : = \tau_H \circ \tau_{V2} \circ \tau_P$ are involutive, orientation-reversing, fixed-point-free isometries of S. We set $S_i = \langle \tau_i \rangle \backslash S$, for $i = 1, 2, 3, 4$. The surfaces S_i are depicted in Figure 3 in case $t = 1$. These non-orientable surfaces have genus $g = t + 1$, and, topologically, they are spheres with $t - 1$ handles and two cross handles, or equivalently, spheres with $t + 1$ cross handles.
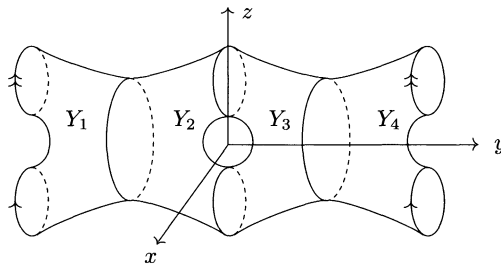

Figure 2. Four congruent pants glued to form S.

For generic choices of the hyperbolic pants Y used to build the surfaces above, the four surfaces will have different geodesic length spectra. For concreteness, we will make a specific choice of Y to guarantee that one of the surfaces has a strictly smaller injectivity radius than the others. There exists a unique right-angled hexagon as in Figure 1 for each given choice of α and γ. (See Buser [B], Theorem 2.4.2.) We choose α and γ to be less than $\mathrm{arcsinh}(1)$ and to satisfy $\gamma < 2\alpha$. Since the waist of the resulting pants has length 2γ and the boundary geodesics of the legs have length 2α, the resulting surface S has $2t$ closed geodesics of length 2γ and $4t$ of length 2α. By Theorem 4.1.6 of [B], a surface of genus g can have at most $3g - 3$ simple closed geodesics of length less than or equal to $2\mathrm{arcsinh}(1)$. Thus the geodesics in S corresponding to the boundary geodesics of the pants are the only simple closed geodesics satisfying this bound on their lengths.

Now for $i = 1, 2, 3, 4$, consider the shortest "new" closed geodesic in the surface S_i, i.e., the shortest closed geodesic that does not lift to a closed geodesic of the same length in S. These geodesics are depicted in Figure 3 in case $t = 1$. (The relative lengths of the geodesics in the figures are distorted.) The surface S_4 contains a geodesic of length γ. This geodesic is shorter than any closed geodesic in the orther surfaces. Thus S_4 has smaller injectivity radius than the others. Thus statement (b) of the Theorem is satisfied.
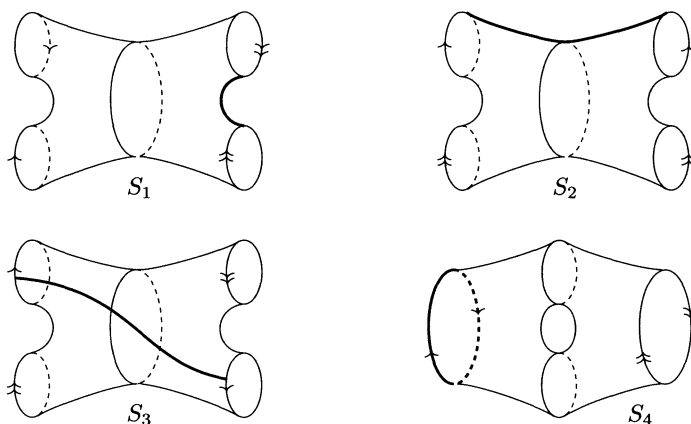

Figure 3. The non-orientable surfaces S_i; the shortest "new" geodesics are thicker.

For the final statement of the theorem, we use Corollary 1.4 to observe that the surfaces S_i are also 1-isospectral to the four surfaces $\langle \tau_P \rangle \backslash S$, $\langle \tau_H \rangle \backslash S$, $\langle \tau_{V1} \rangle \backslash S$, and $\langle \tau_{V2} \rangle \backslash S$ with boundary. These surfaces have $2g$, $2g - 2$, 4 and 2 boundary components, respectively.   □

2.9. Remark. — Letting S be the surface of genus $2n$, $n \geq 1$, pictured in Figure 4, one may take the quotient of S by each of the three visual symmetries (reflection across a vertical plane, horizontal plane, and plane of the paper) to obtain 1-isospectral surfaces with boundary. The first has only one boundary component, while the others have $2n + 1$.

2.10. Surfaces with boundaries of arbitrarily different shapes. — This final example involves manifolds of mixed curvature. Let M be a 2-sphere with a non-standard metric, symmetric with respect to reflection across two orthogonal planes. The quotients of M by these isometric involutions are 1-isospectral manifolds with boundaries. By choosing the metrics on M
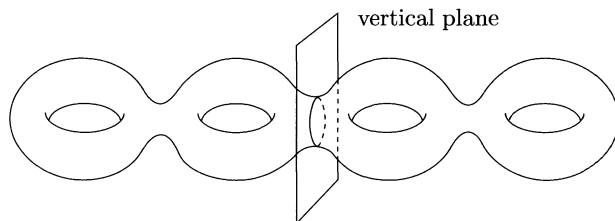
Figure 4. A Riemann surface and a reflection plane.

appropriately, one can more or less arbitrarily prescribe independently the geodesic curvatures of the boundary curves.

3. Examples of orbifolds and manifolds isospectral in the middle degree.

Does the spectrum distinguish orbifolds with singularities from smooth manifolds? We will see below that the middle degree Hodge spectrum cannot detect the presence of singularities. We do not know whether the 0-spectrum always detects singularities. However, we will show in Propositions 3.4 and 3.5 that it does in many situations, including all the examples we will give of orbifolds and manifolds which are isospectral in the middle degree.

3.1. THEOREM. — *The cylinder, Klein bottle and Möbius strip of Example 2.5 are also 1-isospectral to a four pillow $\mathcal{O}$.*

Proof. — The four pillow is the quotient of $T = \mathbb{Z}^2 \backslash \mathbb{R}^2$ by the involution ρ induced from the map $-\mathrm{Id} : \mathbb{R}^2 \to \mathbb{R}^2$. It is a pillow-shaped orbifold with isolated singularities at the four corners. Since ρ is not orientation reversing, we cannot apply Corollary 1.4. Instead we will compare the spectra directly.

Every 1-form on T may be written in the form $f \, \mathrm{d}x + g \, \mathrm{d}y$, where $\mathrm{d}x$ and $\mathrm{d}y$ are the forms induced on T by the standard forms on $\mathbb{R}^2$. Since

$$\Delta(f \, \mathrm{d}x + g \, \mathrm{d}y) = \Delta(f) \, \mathrm{d}x + \Delta(g) \, \mathrm{d}y,$$

$\mathrm{spec}_1(T)$ consists of two copies of $\mathrm{spec}_0(T)$. Hence each of the manifolds in Example 2.5 has 1-spectrum equal to the 0-spectrum of T.

Since both $\mathrm{d}x$ and $\mathrm{d}y$ are ρ-anti-invariant, the space of ρ-invariant one-forms on T is given by

$$\{f\,\mathrm{d}x + g\,\mathrm{d}y : \tau^* f = -f,\ \tau^* g = -g\}.$$

Thus $\mathrm{spec}_1(\mathcal{O})$ is formed by two copies of the spectrum of the Laplacian of T acting on the space of ρ-anti-invariant smooth functions. By Fourier decomposition on the torus, one sees that the Laplacian of T restricted to the ρ-anti-invariant smooth functions has the same spectrum as the Laplacian restricted to the ρ-invariant smooth functions. Consequently $\mathrm{spec}_1(\mathcal{O}) = \mathrm{spec}_0(T)$ and the theorem follows.   □

3.2. Theorem. — (i) *For each positive integer $m > 1$, there exists a collection of $m - 1$ distinct mutually m-isospectral $2m$-dimensional flat orbifolds having singular sets of dimension $1, 3, 5, \ldots, 2m - 3$, respectively. Moreover, these orbifolds are also m-isospectral to the direct product of a cubical $(2m - 2)$-torus with each of the manifolds of Example 2.5.*

(ii) *For each positive integer $m > 1$, there exists a collection of $m - 1$ distinct mutually m-isospectral $2m$-dimensional spherical orbifolds having singular sets of dimension $1, 3, \ldots, 2m - 3$, respectively. Moreover, these orbifolds are also m-isospectral to a $2m$-dimensional projective space and to a $2m$-dimensional hemisphere.*

Remark. — One can also incorporate into the family in Theorem 3.2 (i) of mutually m-isospectral orbifolds and manifolds a $2m$-dimensional analogue of the pillow, obtained as the quotient of T^{2m} under the action of minus the identity. This orbifold has only isolated singularities; i.e., the dimension of its singular set is zero. The proof of isospectrality is similar to that of Theorem 3.1.

Proof. — (i) For k odd with $1 \leq k \leq 2m - 3$, let τ_k be the orthogonal involution of $\mathbb{R}^{2m}$ given by the $2m \times 2m$ diagonal matrix

$$\mathrm{diag}\big(\underbrace{-1, \ldots, -1}_{2m-k},\ \underbrace{1, 1, \ldots, 1}_{k}\big).$$

Then τ_k induces an orientation reversing involution, which we also denote by τ_k, of the $2m$-dimensional cubical flat torus T. Let $\mathcal{O}_k$ be the quotient of T by τ_k. By Corollary 1.4, these orbifolds are mutually m-isospectral. Since the direct product of a cubical $(2m-2)$-torus with any of the manifolds or orbifolds in Theorem 3.1 may also be viewed as a quotient of T by an

involution, the final statement of the theorem follows. (Aside: One could also take $k = 2m - 1$ to obtain an orbifold with singular set of dimension $2m - 1$. However, under the identification described in Remark 1.2 (iii), this orbifold is identified with the product of the cyliner in Theorem 3.1 with the cubical torus and thus is redundant.)

(ii) We now let τ_k, for k even with $2 \le k \le 2m - 2$, be the orthogonal involution of $\mathbb{R}^{2m+1}$ given by the $(2m + 1) \times (2m + 1)$ diagonal matrix

$$\mathrm{diag}\big(\underbrace{-1, \ldots, -1}_{2m+1-k}, \underbrace{1, 1, \ldots, 1}_{k}\big).$$

The quotient of the $2m$-sphere by τ_k is an orbifold with singular set of dimension $k - 1$. Again by Corollary 1.4, these orbifolds are mutually m-isospectral and are also m-isospectral to the projective space and hemisphere of Example 2.2. (Aside: If we take $k = 2m$, we obtain the hemisphere under the identification in 1.2.) $\qquad\square$

In [MR1], [MR2], R. Miatello and the second author computed all the various p-spectra of flat manifolds. Their method extends to flat orbifolds and can be used to give an alternative proof of Theorem 3.2 (i) as well as of the earlier examples of flat manifolds and/or orbifolds.

We next show that the orbifolds and manifolds in Theorems 3.1 and 3.2 are not 0-isospectral. The orbifolds in these examples belong to the class of so-called "good" Riemannian orbifolds; that is, they are quotients of Riemannian manifolds by discrete, effective, properly discontinuous isometric group actions.

3.3. LEMMA (see [D1], [D2]). — *Let $\mathcal{O}$ be a good Riemannian orbifold, and let $\lambda_1, \lambda_2, \ldots$ be the 0-spectrum of the Laplacian. Then there is an asymptotic expansion as $t \downarrow 0$ of the form*

$$\sum_{i=1}^{\infty} e^{-\lambda_i t} \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} a_k t^k + \sum_{S} B_S(t),$$

where S varies over the strata of the singular set and where $B_S(t)$ is of the form

$$B_S(t) = (4\pi t)^{-\frac{1}{2} \dim(S)} \sum_{k=0}^{\infty} b_{k,S}\, t^k$$

with $b_{0,S} \ne 0$. The coefficients a_k in the first part of the expansion are given by the same curvature integrals as in the heat expansion for manifolds; in particular, $a_0 = \mathrm{vol}(\mathcal{O})$.

Remarks. (i) There is an apparent typographical error in the statement of this result as Theorem 4.8 in the article [D2] in that the power $(4\pi t)^{-\frac{1}{2}\dim(S)}$ is missing. However, the proof makes the expression clear.

(ii) S. Greenwald, D. Webb, S. Zhu and the first author recently generalized Lemma 3.3 to arbitrary Riemannian orbifolds. An article is in preparation. With this generalization, the hypothesis that $\mathcal{O}$ be good in the first statement of Proposition 3.4 below can be dropped.

3.4. PROPOSITION. — *Let $\mathcal{O}$ be a good Riemannian orbifold with singularities. Then*

(i) *If $\mathcal{O}$ is even-dimensional (resp., odd-dimensional) and some strata of the singular set is odd-dimensional (resp., even-dimensional), then $\mathcal{O}$ cannot be 0-isospectral to a Riemannian manifold.*

(ii) *If N is a manifold such that $\mathcal{O}$ and N have a common Riemannian cover, then M and $\mathcal{O}$ are not 0-isospectral.*

Proof. — (i) In the two cases, the fact that $\mathcal{O}$ is an orbifold can be gleaned from the presence of half-integer powers, respectively integer powers, of t in the asymptotic expansion in Lemma 3.3.

(ii) Suppose N and $\mathcal{O}$ have a common covering. By Lemma 3.3, the spectrum of $\mathcal{O}$ determines its volume. The first part of the heat expansion in Lemma 3.3 (involving the a_k) depends only on the volume of the orbifold and the curvature of the covering manifold; thus it must be identical for N and $\mathcal{O}$. The second part of the expansion vanishes for N but not for $\mathcal{O}$. Hence the trace of the heat kernels of N and $\mathcal{O}$ have different asymptotic expansions, so N and $\mathcal{O}$ can't be 0-isospectral.   □

3.5. PROPOSITION. — *None of the manifolds and orbifolds in Theorems 3.1 and 3.2 are 0-isospectral.*

Proof. — By Proposition 3.4 (ii), the closed manifolds in these theorems are not 0-isospectral to the orbifolds. As discussed in Remark 1.2, the manifolds with boundary in these theorems are the underlying spaces of (closed) orbifolds; the boundaries of the manifolds form the singular sets of the orbifolds. Moreover the Neumann spectrum of each such manifold coincides with the 0-spectrum of the associated orbifold. Thus we are reduced to comparing the 0-spectra of various orbifolds, each of which has a singular set consisting of a single strata. Moreover, the orbifolds in each collection have the same constant curvature and have a common covering

as orbifolds. In particular, the first part of the heat expansion in Lemma 3.3 is identical for the various orbifolds. However, since their singular strata have different dimensions, the second part of the expansion differs. Thus they cannot be 0-isospectral. □

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Carolyn S. GORDON,
Dartmouth College
Department of Mathematics
Hanover, N.H. 03755 (USA).
carolyn.s.gordon@dartmouth.edu

Juan Pablo ROSSETTI,
FAMAF (CIEM)
Universidad Nacional de Córdoba
Ciudad Universitaria
5000 Córdoba (Argentina).
rossetti@mate.uncor.edu