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AN OBSTRUCTION TO HOMOGENEOUS MANIFOLDS BEING KÄHLER

by Bruce GILLIGAN

1. Introduction.

Throughout this paper we consider a complex homogeneous manifold $X = G/H$, where G is a connected complex Lie group and H is a closed complex subgroup. From the point of view of complex analysis one would like to understand the structure of such a complex manifold X with respect to complex analytic objects on it, like nonconstant holomorphic functions, and there has been much progress on problems of this type. One particular problem in this area that is central, because of its connections with the existence of plurisubharmonic functions and analytic hypersurfaces, is to understand under what conditions such an X can be Kähler. We note that the structure of compact homogeneous Kähler manifolds has been known for some time [5] and the structure in the case of a G-invariant metric was handled by Dorfmeister and Nakajima in [7]. Thus we consider a noncompact complex homogeneous manifold $X = G/H$ with a Kähler metric that is not necessarily G-invariant. The purpose of this paper is to present an obstruction to the existence of such a Kähler metric on X under certain conditions.

Results are known if the group G is either semisimple or solvable. For G semisimple and H a closed complex subgroup of G, if the coset space G/H is Kähler, then H is an algebraic subgroup of G [4]. If G is solvable

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and H is a closed complex subgroup of G such that G/H is Kähler and has no nonconstant holomorphic functions, then G/H is a Cousin group; as a consequence, G is abelian, see [16]. Also there is a result in the case that $G = R \times S$ is a group theoretic direct product of its radical R with a maximal semisimple subgroup S of G, see [17]. The motivation for the present paper came from our attempts to construct examples where the group $G = R \rtimes S$ is a nontrivial semidirect product. The full affine group of $\mathbb{C}^n$ for $n > 1$ is such an example. The affine group is an algebraic group whose radical R is a vector group and a maximal semisimple subgroup S of this group acts by an irreducible linear representation that has no nonzero invariant vector. One consequence of our main result is that it is not possible to have an "interesting" example involving a group of this type. After introducing some terminology, we give a precise statement of this result below.

Suppose $X = G/H$ is a noncompact Kähler homogeneous manifold. If $\mathcal{O}(G/H) \neq \mathbb{C}$, then there exists a closed complex subgroup L of G containing H such that the homogeneous fibration

$$p : G/H \longrightarrow G/L$$

is the *holomorphic reduction* of G/H, see [9]. This means that G/L is holomorphically separable and $p^*(\mathcal{O}(G/L)) = \mathcal{O}(G/H)$. Now a holomorphically separable complex homogeneous manifold is known to be Kähler [12]. Also if G/H is Kähler, then L/H, as a complex submanifold, will inherit this condition. Thus, as a kind of irreducibility assumption, we may suppose that $\mathcal{O}(G/H) = \mathbb{C}$. We also note that if H is contained in a proper parabolic subgroup P of G, then one has a homogeneous fibration

$$G/H \longrightarrow G/P.$$

Since G/P is a projective variety, it is Kähler. It is thus a further irreducibility assumption to suppose that H is not contained in any proper parabolic subgroup of G. Under the assumptions that $\mathcal{O}(G/H) = \mathbb{C}$ and H is not contained in any proper parabolic subgroup of G it is then easy to see that H° is normal in G [17]. If one assumes G is acting effectively on X, this means that $H = \Gamma$ is discrete in G.

Throughout this paper we consider the setting where $H = \Gamma$ is discrete in G and there is a homogeneous fibration

$$X = G/\Gamma \longrightarrow G/I =: Y,$$

where I° is an abelian normal subgroup of G and the fiber $I^\circ/(I^\circ \cap \Gamma)$ of this fibration is an abelian complex Lie group that has no nonconstant holomorphic functions. We call such complex Lie groups *Cousin groups*, see [6], although in some circles they are called *toroidal groups*, *e.g.*, see the work of Abe and Kopfermann [1]. Since I° is normal in G, we set $\widehat{G} := G/I^\circ$ and $\widehat{\Gamma} := I/I^\circ$. Then $Y = \widehat{G}/\widehat{\Gamma}$ with $\widehat{G}$ acting almost effectively.

If X is Kähler, one would like to determine if Y is Kähler too. But we do not know of any method of proving such a result in full generality in the present setting. However, one can write $\widehat{G} = \widehat{S} \ltimes \widehat{R}$, where $\widehat{R}$ is the radical of $\widehat{G}$ and $\widehat{S}$ is a maximal complex semisimple subgroup of $\widehat{G}$. If $Y = \widehat{G}/\widehat{\Gamma}$ were Kähler, then $\widehat{S} \cap \widehat{\Gamma}$ would be finite [4]. Our approach will be to prove by other means that if X is Kähler, then $\widehat{S} \cap \widehat{\Gamma}$ is finite. This means that if the intersection $\widehat{S} \cap \widehat{\Gamma}$ is infinite, then this is an obstruction to the existence of a Kähler metric on G/Γ. What we actually show in the third section of this paper (the Main Lemma) is that no element of infinite order in $\widehat{\Gamma}$ is contained in a semisimple subgroup of $\widehat{G}$. Since every complex semisimple Lie group has the structure of a linear algebraic group, this is equivalent to the statement that $\widehat{S} \cap \widehat{\Gamma}$ is finite for every semisimple subgroup $\widehat{S}$ of $\widehat{G}$. This is surprising because this result is basically an algebraic conclusion. For, it tells us that for every complex semisimple subgroup $\widehat{S}$ of $\widehat{G}$ **the $\widehat{S}$-orbit in $\widehat{G}/\widehat{\Gamma}$ is algebraic**, even though there are no additional assumptions on the group $\widehat{G}$ beyond its being a complex Lie group. Of course, the assumption that X is Kähler is essential.

In the fourth and final section we place some restrictions on the group $\widehat{G}$. Our main result is the following. Assume that $\widehat{G}$ is a linear algebraic group whose radical $\widehat{R}$ is a vector group with a (positive dimensional) maximal semisimple subgroup of $\widehat{G}$ acting on $\widehat{R}$ by a linear representation that has no nonzero invariant vector. Then G/Γ does not admit a Kähler structure. A particular example of such a linear algebraic group $\widehat{G}$ is the affine group of $\mathbb{C}^n$ for $n > 1$. Therefore, if $X = G/\Gamma$ is Kähler and $Y = \widehat{G}/\widehat{\Gamma}$ is the base of a Cousin group bundle as we have described above, then $\widehat{G}$ cannot be the affine group.

2. Existence of the Cousin bundle.

Under the assumptions that $\mathcal{O}(G/\Gamma) = \mathbb{C}$ and Γ is not contained in any proper parabolic subgroup of G, the orbits of the radical R of G are closed in G/Γ, *e.g.,* see [17]. Thus one has a holomorphic fibration

$$G/\Gamma \longrightarrow G/R \cdot \Gamma$$

where $R \cdot \Gamma$ is a closed subgroup of G. The fiber of this fibration has the complex solvmanifold $R \cdot \Gamma/\Gamma$ as its fiber. There is the holomorphic reduction of the homogeneous manifold $R \cdot \Gamma/\Gamma$, *i.e.,* there exists a closed complex subgroup I of $R \cdot \Gamma$ that contains Γ such that one has the holomorphic fibration

$$\pi : R \cdot \Gamma/\Gamma \longrightarrow R \cdot \Gamma/I$$

where for $x_1, x_2 \in R \cdot \Gamma/\Gamma$ one has $\pi(x_1) = \pi(x_2)$ iff $f(x_1) = f(x_2)$ for all $f \in \mathcal{O}(R \cdot \Gamma/\Gamma)$, see [9]. In words, we can mod out by the level sets of the holomorphic function algebra $\mathcal{O}(R{\cdot}\Gamma/\Gamma)$ using a holomorphic homogeneous fibration, the *holomorphic reduction* of the space $R \cdot \Gamma/\Gamma$. The base of this holomorphic reduction is a holomorphically separable solvmanifold that is Stein [12] and this implies the fiber of the holomorphic reduction is connected by Proposition 1 in [9]. Further, because the solvmanifold $R{\cdot}\Gamma/\Gamma$ is Kähler, the fiber I/Γ of its holomorphic reduction is a Cousin group [16]. (*Note:* The group $R{\cdot}\Gamma$ need not be solvable, in general. However, the radical R acts transitively on the connected manifold $R \cdot \Gamma/\Gamma$ and this, together with the fact that the holomorphic reduction is equivariant with respect to any Lie group acting holomorphically on the space, justifies the various claims that we have made concerning solvmanifolds.)

3. The Main Lemma.

We remind the reader of the situation. Suppose G is a connected, simply connected complex Lie group and Γ is a discrete subgroup of G. We noted above that the irreducibility assumptions $\mathcal{O}(G/\Gamma) = \mathbb{C}$ and Γ not contained in a proper parabolic subgroup of G, imply that there is a homogeneous fibration

$$X := G/\Gamma \xrightarrow{I/\Gamma} G/I =: Y, \tag{1}$$

where I° is an abelian, normal subgroup of G and the fiber $I^\circ/(I^\circ \cap \Gamma)$ is a Cousin group. From now on we will simply make the assumption that the fibration given in (1) exists for the given space G/Γ. Set $\widehat{G} := G/I^\circ$ and $\widehat{\Gamma} := I/I^\circ$. Thus $Y = \widehat{G}/\widehat{\Gamma}$ with $\widehat{G}$ acting almost effectively. Since G is simply connected and I° is connected and normal, $\widehat{G}$ is simply connected. Let $\widehat{G} = \widehat{S} \ltimes \widehat{R}$ be a Levi decomposition of $\widehat{G}$, where $\widehat{S}$ is a maximal semisimple subgroup of G and $\widehat{R}$ is the radical of $\widehat{G}$. Our approach will be to assume that $\dim \widehat{S} > 0$ and $\dim \widehat{R} > 0$. In the Main Lemma we prove that if X is Kähler, then no element $\hat{g} \in \widehat{\Gamma}$ that is of infinite order lies in a semisimple subgroup of $\widehat{G}$.

We first prove a technical result that generalizes Proposition 1 in [10]. In [10] we considered compact, complex torus bundles over $\mathbb{C}^*$ that are Kähler. In the present setting we need to consider bundles whose fiber is a Cousin group and whose base is some positive power of $\mathbb{C}^*$.

We use the notation from [16] which we now recall. Suppose G is a solvable complex Lie group and Γ is a discrete subgroup of G. Assume G_0 is a connected, closed subgroup of G containing Γ with G_0/Γ compact and generic as a Cauchy-Riemann submanifold of G/Γ. This means that the complexification of the tangent space of G_0/Γ is not contained in any proper complex vector subspace of the tangent space of G/Γ. In terms of Lie algebras this is equivalent to

$$\mathfrak{g} = \mathfrak{g}_0 + i \cdot \mathfrak{g}_0.$$

Further we assume that G/Γ is Kähler and following [16] we call such a triple (G, G_0, Γ) a KCRS. Set

$$\mathfrak{m} = \mathfrak{g}_0 \cap i \cdot \mathfrak{g}_0.$$

Then $\mathfrak{m}$ is an ideal in $\mathfrak{g}_0$ and is the maximal complex Lie subalgebra in $\mathfrak{g}_0$.

Lemma 1. — *Suppose (G, G_0, Γ) is a KCRS with G acting effectively on G/Γ. Let H be a closed complex subgroup of G that contains Γ such that H/Γ is a Cousin group and G/H is biholomorphic to $(\mathbb{C}^*)^n$ for some $n > 0$. Then the kernel Γ_{ker} of the adjoint action of Γ on $\mathfrak{m}$ is a subgroup of finite index in Γ.*

Proof. — Let $H_0 := G_0 \cap H \supset \Gamma$. Then one has the fibration

$$G_0/\Gamma \xrightarrow{H_0/\Gamma} G_0/H_0 = (S^1)^n,$$

where H_0/Γ is the maximal compact subgroup of the Cousin group H/Γ. Since the maximal compact subgroup $G_0/H_0 = (S^1)^n \subset (\mathbb{C}^*)^n = G/H$ is totally real, it follows that $\mathfrak{m} \subset \mathfrak{h}_0$. Since H/Γ is an abelian group, $H' \subset \Gamma^\circ = \{e\}$. Thus $\mathfrak{h}$ is abelian and so $\mathfrak{h}_0$ is also abelian. This implies $\mathfrak{m}$ is an abelian ideal, a fact we use later.

Let ω be a Kähler form on G/Γ. In paragraphs 1 and 2 on p. 166 in the proof of Proposition 1 in [10], we showed that in the setting where (G, G_0, Γ) is a KCRS that fibers as a compact torus bundle over $\mathbb{C}^*$, one can "average" the form $\omega' = p^*\omega$, where $p : G \to G/\Gamma$ is the natural projection, by integration over the compact manifold G_0/Γ to get a closed $(1,1)$-form $\tilde{\omega}$ that is right G_0-invariant and positive definite near G_0.

We now adapt that proof to the present setting. Assume ω is a positive semidefinite (1,1)-form on G/Γ that is positive definite on a neighbourhood of G_0/Γ and let $\omega' = p^*\omega$, where $p : G \to G/\Gamma$ is the projection. Here one can "average" the form ω' to obtain a closed $(1,1)$-form $\tilde{\omega}$ that is right G_0-invariant and positive definite near G_0.

Since G_0 is solvable, G_0/Γ has a G_0-invariant finite measure μ. Define

$$\tilde{\omega}(v,w) := \int_{G_0/\Gamma} f d\mu,$$

where

$$G_0 \ni g \mapsto r_g^*\omega_p'(v,w) = \omega_{p\cdot g}'(dr_g v, dr_g w) =: f(g)$$

for $v, w \in T_p(G)$. Then $\tilde{\omega}$ is a closed (1,1)-form on G that is right G_0-invariant. Note that the form $\tilde{\omega}$ is positive definite on some neighbourhood of G_0.

From now on we suppress the tilde for notational convenience. There exists a closed form ω that is right G_0-invariant. So applying these facts to a standard formula for the exterior derivative of ω one gets

$$\omega(X,[Y,Z]) + \omega(Z,[X,Y]) + \omega(Y,[Z,X]) = 0 \tag{2}$$

for all $X, Y, Z \in \mathfrak{g}_0$, where $\mathfrak{g}_0$ is the Lie algebra of right invariant vector fields on G_0. Following what is done in [16] we will use formula (2) in an essential way.

Given $\mathfrak{m}$ and ω, we set

$$\mathfrak{a} := \{X \in \mathfrak{g}_0 |\ \omega(X,\mathfrak{m}) = 0\}.$$

Since ω is bilinear, it is clear that $\mathfrak{a}$ is a vector subspace of $\mathfrak{g}_0$. In fact, $\mathfrak{a}$ is a subalgebra and one has

$$\mathfrak{g}_0 = \mathfrak{a} \ltimes \mathfrak{m}.$$

This is proved on p. 409 in [16] by means of a straightforward calculation using (2).

For $X \in \mathfrak{a}$, let w_1 be an eigenvector of ad_X, and set

$$w_1^{\perp} := \{v \in \mathfrak{m} |\ \omega(v, \langle w_1\rangle_{\mathbb{C}}) = 0\}.$$

For $v \in w_1^{\perp}$ one can plug into (2) to get

$$\omega(X,[v,w_1]) + \omega(w_1,[X,v]) + \omega(v,[w_1,X]) = 0.$$

Since $\mathfrak{m}$ is abelian, one has $[v, w_1] = 0$ and the first term vanishes. But $[w_1, X] = \lambda_1 w_1$, for some $\lambda_1 \in \mathbb{C}$, so the third term vanishes too. Thus $[X, v] \in w_1^\perp$ for every $v \in w_1^\perp$. It now follows by recursion that the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}_0$ restricted to $\mathfrak{m}$ is diagonalizable. As well, the eigenvalues of the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}_0$ restricted to $\mathfrak{m}$ are purely imaginary; see Theorem 2 in [16]. As a consequence, $\mathrm{Ad}_{G_0} \subset GL_{\mathbb{C}}(\mathfrak{m})$ is contained in a compact real torus of $GL_{\mathbb{C}}(\mathfrak{m})$. Also Ad_Γ stabilizes the lattice $M \cap \Gamma$ and thus $\mathrm{Ad}_\Gamma \subset GL_{\mathbb{C}}(\mathfrak{m})$ is discrete. This implies Ad_Γ is finite. Thus the kernel Γ_{ker} of the adjoint representation of Γ on $\mathfrak{m}$ is a subgroup of finite index in Γ. □

We now prove the Main Lemma.

Lemma 2 (Main Lemma). — *Assume the notation is as above with G/Γ Kähler. Then no element $\hat{g}$ of infinite order in $\widehat{\Gamma}$ is contained in a semisimple subgroup $\widehat{S}$ of $\widehat{G}$.*

Proof. — We will assume that G/Γ is Kähler and that $\hat{g} \in \widehat{S} \cap \widehat{\Gamma}$ is of infinite order where $\widehat{S}$ is a semisimple subgroup of $\widehat{G}$. From this we will derive a contradiction. Let $\hat{g} = \hat{s} \cdot \hat{v}$ be its Jordan decomposition, where $\hat{s}$ is diagonalizable, $\hat{v}$ is unipotent, and $\hat{s}$ and $\hat{v}$ commute. The algebraic closure $A_{\hat{s}}$ of the infinite cyclic group $\langle \hat{s} \rangle_{\mathbb{Z}}$ is biholomorphic to $(\mathbb{C}^*)^q$ for some positive integer q with $A_{\hat{s}}/\langle \hat{s} \rangle_{\mathbb{Z}}$ a Cousin group. The algebraic closure $A_{\hat{v}}$ of the infinite cyclic group $\langle \hat{v} \rangle_{\mathbb{Z}}$ is biholomorphic to $\mathbb{C}$ with $A_{\hat{v}}/\langle \hat{v} \rangle_{\mathbb{Z}} = \mathbb{C}^*$. Further $A_{\hat{g}} = A_{\hat{s}} \times A_{\hat{v}}$.

Note that if $\hat{v} = e$, then $\hat{g}$ is a semisimple element in $\widehat{S}$. This case is handled by Theorem 1 in [8]. So throughout the rest of the proof, we assume that $\hat{g}$ is not a semisimple element. If $\hat{s} = e$, then

$$A_{\hat{g}}/\langle \hat{g} \rangle_{\mathbb{Z}} = A_{\hat{v}}/\langle \hat{v} \rangle_{\mathbb{Z}} = \mathbb{C}^*.$$

This case is handled by the proof given below, where the base of the Cousin bundle fibration is simply $\mathbb{C}^*$, instead of $\mathbb{C}^*$ to some higher power. In other words, the following proof works for any *positive* power of $\mathbb{C}^*$ and so handles this special case. We now turn to the general case.

The abelian group $A_{\hat{g}}/\langle \hat{g} \rangle_{\mathbb{Z}}$ admits a holomorphic fibration

$$\psi : A_{\hat{g}}/\langle \hat{g} \rangle_{\mathbb{Z}} \xrightarrow{A_{\hat{s}}} A_{\hat{v}}/\langle \hat{v} \rangle_{\mathbb{Z}} = \mathbb{C}^*.$$

Since the base of this equivariant fibration is holomorphically separable, the map ψ factors through the holomorphic reduction $A_{\hat{g}}/\langle \hat{g} \rangle_{\mathbb{Z}} \to A_{\hat{g}}/B$, where B is some closed complex subgroup of $A_{\hat{g}}$ that contains the lattice

$\langle \hat{g} \rangle_{\mathbb{Z}}$. But the fiber $B/\langle \hat{g} \rangle_{\mathbb{Z}}$ of the holomorphic reduction (the Steinizer in the terminology of [14]) of a (connected) complex Lie group is always a Cousin group. Thus we have a Cousin group contained in $A_{\hat{s}} = (\mathbb{C}^*)^q$. This is impossible, unless the dimension of B is zero. But since the fiber of the holomorphic reduction of a complex Lie group is connected, it follows that $B = \langle \hat{g} \rangle_{\mathbb{Z}}$. (These facts are proved in [14].) We therefore conclude that $A_{\hat{g}}/\langle \hat{g} \rangle_{\mathbb{Z}}$ is a holomorphically separable abelian complex Lie group and thus is isomorphic to $(\mathbb{C}^*)^{q+1}$ with $q \geqslant 0$.

For the time being we restrict to a special case; namely, we assume that $\widehat{S} \cap \widehat{\Gamma} = \langle \hat{g} \rangle_{\mathbb{Z}}$. We will return to the general case later in the proof. Let $\pi : G \to \widehat{G} := G/I^\circ$ denote the projection. Since $I^\circ \lhd R$, where R denotes the radical of G, we can find a complex semisimple subgroup $S \subset G$ such that S is isomorphic to $\widehat{S}$ under the map π. Pick $g \in \pi^{-1}(\hat{g}) \cap S$ and let A_g denote the algebraic closure of $\langle g \rangle_{\mathbb{Z}}$ in S. Then $\pi : A_g \to A_{\hat{g}}$ is an isomorphism. Set $G^* = A_g \ltimes I^\circ$ and let $\tilde{G^*}$ be the universal covering of G^*, where $p : \tilde{G^*} \to G^*$ is the covering map. Since I° is a connected normal subgroup of the simply connected Lie group G it follows that I° is simply connected. Hence $\tilde{G^*} = p^{-1}(A_g) \ltimes I^\circ$. Let Γ^* denote the preimage in $\tilde{G^*}$ of $\Gamma \cap G^* = \langle g \rangle_{\mathbb{Z}} \ltimes (\Gamma \cap I^\circ)$.

The simply connected, solvable, complex Lie group $\tilde{G^*}$ admits a Hochschild-Mostow hull, *i.e.*, there exists a solvable linear algebraic group, which we will denote by G_a^*, with

$$G_a^* = (\mathbb{C}^*)^k \ltimes \tilde{G^*}$$

such that G_a^* contains $\tilde{G^*}$ as a Zariski dense, topologically closed, normal subgroup, see [11]. Also G_a^*/Γ^* is biholomorphic to $(\mathbb{C}^*)^k \times \tilde{G^*}/\Gamma^*$. In particular, because $\tilde{G^*}/\Gamma^*$ is Kähler, G_a^*/Γ^* is Kähler too. Let A_{Γ^*} denote the algebraic closure of Γ^* in G_a^*. Then A_{Γ^*}/Γ^* is Kähler, and Γ^* is Zariski dense in A_{Γ^*}. As a consequence, A_{Γ^*} is nilpotent; see Lemma 3 in [16].

Note that

$$G_a^*/\Gamma^* = (\mathbb{C}^*)^k \times \tilde{G^*}/\Gamma^* \overset{I^\circ/I^\circ\cap\Gamma^*}{\longrightarrow} (\mathbb{C}^*)^k \times \tilde{G^*}/\Gamma^* \cdot I^\circ = G_a^*/\Gamma^* \cdot I^\circ$$

is the holomorphic reduction of G_a^*/Γ^*, because its fiber $I^\circ/I^\circ \cap \Gamma^*$ is a Cousin group and its base $(\mathbb{C}^*)^k \times \tilde{G^*}/\Gamma^* \cdot I^\circ$ is biholomorphic to $(\mathbb{C}^*)^n$ with $n = k+q+1$ and so is holomorphically separable. Now G_a^*/A_{Γ^*} is affine and thus is holomorphically separable. So the universality of the holomorphic reduction implies that there is a G_a^*-equivariant holomorphic map

$$G_a^*/\Gamma^* \cdot I^\circ \longrightarrow G_a^*/A_{\Gamma^*}.$$

One particular consequence is the fact that I° is a subgroup of A_{Γ^*}. Also $\langle g\rangle_{\mathbb{Z}} \subset \Gamma^*$ and thus $A_g \subset A_{\Gamma^*}$. As a consequence, $A_g \ltimes I^\circ \subset A_{\Gamma^*}$. Thus $A_g \ltimes I^\circ$ is nilpotent. But this implies that its homomorphic image $G^* = p(A_g \ltimes I^\circ)$ is also nilpotent. Hence there exists a connected, closed (real) subgroup G_0^* of G^* that contains Γ^* cocompactly, see [13]. Thus (G^*, G_0^*, Γ^*) is a KCRS.

Now let $\mathfrak{m}^* := \mathfrak{g}_0^* \cap i \cdot \mathfrak{g}_0^*$. One can now apply Lemma 1 to the triple (G^*, G_0^*, Γ^*) to see that Γ^* contains a subgroup Γ^*_{ker} of finite index which is the kernel of the adjoint representation $\rho : \Gamma^* \to GL(\mathfrak{m}^*)$. Note that $\pi(\Gamma^*_{\mathrm{ker}}) = \langle \hat{g}^k\rangle_{\mathbb{Z}}$ for some positive integer k. In particular, this group is infinite. If $\gamma \in \langle \hat{g}^k\rangle_{\mathbb{Z}}$, then $\mathrm{Ad}_\gamma = \{e\} \in GL(\mathfrak{m}^*)$. Thus Ad_γ acts trivially on $M^* := \exp(\mathfrak{m}^*)$ in $C := I^\circ / I^\circ \cap \Gamma$. Since M^* is dense in the maximal compact subgroup K of C, this implies Ad_γ acts trivially on K in C. But since the action is holomorphic, this implies Ad_γ acts trivially on C itself.

The proof now relates to the following diagram

$$\begin{array}{ccccc} \tilde{Y} & \longrightarrow & Y & \hookrightarrow & X \\ {\scriptstyle I/\Gamma}\big\downarrow & & \big\downarrow{\scriptstyle I/\Gamma} & & \big\downarrow{\scriptstyle I/\Gamma} \\ \widehat{S}/\langle \hat{g}^k\rangle_{\mathbb{Z}} & \xrightarrow{\alpha} & \widehat{S}/\widehat{S}\cap\widehat{\Gamma} & \hookrightarrow & \widehat{G}/\widehat{\Gamma} \end{array} \tag{3}$$

Recall that we are assuming that $\widehat{S} \cap \widehat{\Gamma} = \langle \hat{g}\rangle_{\mathbb{Z}}$ and thus the map α is a finite covering map. We point out that the $\widehat{S}$-orbit $\widehat{S}/\widehat{S}\cap\widehat{\Gamma}$ need not be closed in $\widehat{G}/\widehat{\Gamma}$. However, the space Y, which is the restriction of the bundle space X over the orbit $\widehat{S}/\widehat{S}\cap\widehat{\Gamma}$, is Kähler in any case. The space $\tilde{Y}$, as a covering of Y, is clearly Kähler. It now follows that $\tilde{Y}$ is biholomorphic to the product

$$\widehat{S}/\langle \hat{g}^k\rangle_{\mathbb{Z}} \times I/\Gamma$$

because $\langle \hat{g}^k\rangle_{\mathbb{Z}}$ lies in the kernel of ρ, where $\rho : \Gamma^* \to \mathrm{Aut}(I/\Gamma)$ is the adjoint representation. This means $\widehat{S}/\langle \hat{g}^k\rangle_{\mathbb{Z}}$ is Kähler. But then the discrete isotropy subgroup $\langle \hat{g}^k\rangle_{\mathbb{Z}}$ is algebraic and thus finite by [4]. This is the contradiction that we seek, since the group $\langle \hat{g}^k\rangle_{\mathbb{Z}}$ is infinite.

In the general case, we consider the following diagram

$$\begin{array}{ccccc} \tilde{Y} & \longrightarrow & Y & \hookrightarrow & X \\ {\scriptstyle I/\Gamma}\big\downarrow & & \big\downarrow{\scriptstyle I/\Gamma} & & \big\downarrow{\scriptstyle I/\Gamma} \\ \widehat{S}/\langle \hat{g}^k\rangle_{\mathbb{Z}} & \xrightarrow{\alpha} & \widehat{S}/\widehat{S}\cap\widehat{\Gamma} & \hookrightarrow & \widehat{G}/\widehat{\Gamma} \end{array} \tag{4}$$

Since $\langle \hat{g}^k\rangle_{\mathbb{Z}}$ is a subgroup of $\widehat{S}\cap\widehat{\Gamma}$, the map α is a covering map. The space $\tilde{Y}$, as a covering of Y, is again Kähler. The argument given in the special case using (3) now applies in this setting using (4). One obtains the same contradiction as before and this proves the Main Lemma. $\square$

4. Vector group as radical.

Throughout this section we assume Γ is a discrete subgroup of a complex Lie group G such that $\mathcal{O}(G/\Gamma) = \mathbb{C}$ and Γ is not contained in a proper parabolic subgroup of G. The next result uses the Main Lemma to show that if such a G/Γ is Kähler and the group $\widehat{G}$ is linear algebraic, then the group $\widehat{G}$ cannot be a semidirect product of a vector group with a (positive dimensional) semisimple group acting by means of a linear representation that has no nonzero invariant vector.

THEOREM 1. — *Suppose Γ is a discrete subgroup of the complex Lie group G such that $\mathcal{O}(G/\Gamma) = \mathbb{C}$ and Γ is not contained in any proper parabolic subgroup of G. Assume $\widehat{G}$ is a simply connected linear algebraic group such that the radical $\widehat{R}$ of $\widehat{G}$ is a vector group and the representation of $\widehat{S}$ on $\widehat{R}$ is linear with no nonzero invariant vector, where $\widehat{G} = \widehat{S} \ltimes \widehat{R}$ is a Levi decomposition of $\widehat{G}$. Then G/Γ cannot be Kähler.*

Proof. — First we note that in the setting of this theorem, Lemma 7 in [3] implies that the union of all conjugates to $\widehat{S}$ in $\widehat{G}$ contains a Zariski open set of the form $\pi_1^{-1}(\widehat{S} - Z)$, where $\pi_1 : \widehat{G} \to \widehat{S}$ is the projection mapping and Z is a proper Zariski closed subset in $\widehat{S}$.

Now, if G/Γ were Kähler, then $\widehat{S} \cap \widehat{\Gamma}$ would be finite, by the Main Lemma, for every complex semisimple subgroup $\widehat{S}$ of $\widehat{G}$. In particular, if we fix a maximal semisimple subgroup $\widehat{S}$ of $\widehat{G}$, then this statement holds for every conjugate of $\widehat{S}$ in $\widehat{G}$. In the proof of Lemma 8 on p. 906 in [3] we showed how to construct a subgroup $\widehat{\Gamma}_2$ of $\widehat{\Gamma}$ with the property that, if $\widehat{\Gamma}_2 \cap \hat{g}\widehat{S}\hat{g}^{-1}$ is finite for any $\hat{g} \in \widehat{G}$, then

$$\widehat{\Gamma}_2 \cap \hat{g}\widehat{S}\hat{g}^{-1} = \{e\}.$$

(By the Tits' Alternative, if $\pi_1(\widehat{\Gamma})$ is Zariski dense in $\widehat{S}$, then $\pi_1(\widehat{\Gamma})$ contains a subgroup that is a free group on two generators. The preimages in $\widehat{G}$ of those two generators yield a subgroup $\widehat{\Gamma}_1$ of $\widehat{\Gamma}$ and by a theorem of Selberg, *e.g.*, see [18], Corollary 6.13, the group $\widehat{\Gamma}_1$ contains a subgroup $\widehat{\Gamma}_2$ of finite index that is without torsion.) But then

$$\widehat{\Gamma}_2 - \{e\} \subset \widehat{G} - \bigcup_{\hat{g} \in \widehat{G}} \hat{g}\widehat{S}\hat{g}^{-1} \subset \pi_1^{-1}(Z),$$

where Z is the proper Zariski closed subset in $\widehat{S}$ alluded to in the previous paragraph. Thus $\pi_1(\widehat{\Gamma}_2) \subset Z$. But this would be a contradiction, unless $\widehat{\Gamma}$ were not Zariski dense in $\widehat{G}$, *i.e.*, unless $\widehat{\Gamma}$ would be contained in a proper algebraic subgroup A of $\widehat{G}$. But then A would be either reductive or would be contained in a proper parabolic subgroup of $\widehat{G}$. The first would contradict the assumption that $\mathcal{O}(G/\Gamma) = \mathbb{C}$ and the second the assumption that Γ is not contained in any proper parabolic subgroup of G. As a consequence, G/Γ cannot be Kähler. □

Corollary 1. — *Suppose G/Γ as above and $\dim \widehat{S} > 0$. Then $\dim_{\mathbb{C}} \widehat{R} \neq 2$.*

Proof. — If $\dim_{\mathbb{C}} \widehat{R} = 2$, the proof of Lemma 8 in [3] handles both the case when the adjoint action of $\widehat{S}$ on $\widehat{R}$ is trivial and the case when it is not trivial. In both cases it is shown that $\widehat{G}$ has the structure of a linear algebraic group and that $\widehat{\Gamma}$ lies in a proper algebraic subgroup of $\widehat{G}$. In particular, this implies that either $\mathcal{O}(\widehat{G}/\widehat{\Gamma}) \neq \mathbb{C}$ or else $\widehat{\Gamma}$ is contained in a proper parabolic subgroup of $\widehat{G}$. But then one can draw the same conclusion about G and Γ; either $\mathcal{O}(G/\Gamma) \neq \mathbb{C}$ or else Γ is contained in a proper parabolic subgroup of G. This is a contradiction. □

Remark. — As a consequence, this result affirmatively answers the following question of Akhiezer [2] in this setting. Suppose G is a connected complex Lie group and H is a closed complex subgroup of G such that G/H has no nonconstant holomorphic functions and H is not contained in any proper parabolic subgroup of G. Let $G/H \to G/J$ be the hypersurface reduction of G/H, *e.g.*, see [15]. Is $G' \subset J$, where G' denotes the commutator subgroup of G? This follows from what we have proved, because it is known that a hypersurface separable homogeneous complex manifold is Kähler, *e.g.*, see the Appendix in [10].

In Theorem 4 in [8] we also answered Ahkiezer's question affirmatively when the group $\widehat{G} = \widehat{S} \times \widehat{R}$ is a group theoretic direct product. Using the Main Lemma we get a simpler proof of this fact.

Theorem 2. — *With the assumptions as above and G/Γ Kähler, suppose $\widehat{G} = \widehat{S} \times \widehat{R}$ is a group theoretic direct product. Then $S = \{e\}$, and G/Γ is a Cousin group.*

Proof. — Since $\widehat{S} \cap \widehat{\Gamma}$ is finite, it follows from Lemma 3 in [3] that $\widehat{\Gamma}$

is contained in a group of the form $A \times \widehat{R}$, where A is an algebraic subgroup of $\widehat{S}$ with A° solvable. Consequently, A is a proper subgroup of $\widehat{S}$ and one has the homogeneous fibration

$$\widehat{G}/\widehat{\Gamma} \longrightarrow \widehat{G}/(A \times \widehat{R}) = \widehat{S}/A.$$

As in the proof of the previous theorem, A is either reductive or contained in a proper parabolic subgroup of $\widehat{S}$. This yields a contradiction, unless $\widehat{S} = \{e\}$. But then $S = \{e\}$ too. □

Remark. — Our considerations do not handle $\widehat{G} = \widehat{S} \ltimes_{\hat{\rho}} \widehat{R}$, where the representation $\hat{\rho}$ has zero weights. An example can be constructed using the adjoint representation of $SL(2, \mathbb{C})$ on its Lie algebra. Other methods will be needed for this.

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