ANNALES

DE

L'INSTITUT FOURIER

Francesco VACCARINO

The ring of multisymmetric functions

Tome 55, nº 3 (2005), p. 717-731.

<http://aif.cedram.org/item?id=AIF_2005__55_3_717_0>

Ann. Inst. Fourier, Grenoble
55, 3 (2005), 717–731

THE RING OF MULTISYMMETRIC FUNCTIONS

by Francesco VACCARINO

Introduction.

Let R be a commutative ring and let n, m be two positive integers. Let $A_R(n,m)$ be the polynomial ring in the commuting independent variables $x_i(j)$ with $i = 1, \ldots, m\,; j = 1, \ldots, n$ and coefficients in R. The symmetric group on n letters S_n acts on $A_R(n,m)$ by means of $\sigma(x_i(j)) = x_i(\sigma(j))$ for all $\sigma \in S_n$ and $i = 1, \ldots, m\,; j = 1, \ldots, n$. Let us denote by $A_R(n,m)^{S_n}$ the ring of invariants for this action: its elements are usually called multisymmetric functions and they are the usual symmetric functions when $m = 1$. In this case, $A_R(n,1) \cong R[x_1, x_2, \ldots, x_n]$, and $R[x_1, x_2, \ldots, x_n]^{S_n}$ is freely generated by the elementary symmetric functions $e_1, \ldots, e_n$ given by the equality

$$\sum_{k=0}^{n} t^k e_k := \prod_{i=1}^{n} (1 + t x_i). \tag{0.1}$$

Here $e_0 = 1$ and t is a commuting independent variable (see [M]). Furthermore one has

$$e_k(x_1, \ldots, x_n) = \sum_{i_1 < i_2 < \ldots < i_k \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_k} \tag{0.2}$$

Unless otherwise stated, we now assume that $m > 1$. We first obtain generators of the ring $A_R(n,m)^{S_n}$.

Keywords: Characteristic-free invariant theory, symmetric functions, representations of symmetric groups.
Math. classification: 05E05, 13A50, 20C30.

Let $A_R(m) := R[y_1, \dots, y_m]$, where $y_1, \dots, y_m$ are commuting independent variables, let $f = f(y_1, \dots, y_m) \in A_R(m)$ and define

$$f(j) := f(x_1(j), \dots, x_m(j)) \text{ for } 1 \leqslant j \leqslant n. \tag{0.3}$$

Notice that $f(j) \in A_R(n,m)$ for all $1 \leqslant j \leqslant n$ and that $\sigma(f(j)) = f(\sigma(j))$, for all $\sigma \in S_n$ and $j = 1, \dots, n$.

Define $e_k(f) := e_k(f(1), f(2), \dots, f(n))$ i.e.

$$\sum_{k=0}^{n} t^k e_k(f) := \prod_{i=1}^{n} (1 + tf(i)), \tag{0.4}$$

where t is a commuting independent variable. Then $e_k(f) \in A_R(n,m)^{S_n}$.

One may think about the y_i as diagonal matrices in the following sense: let $M_n(A_R(n,m))$ be the full ring of $n \times n$ matrices with coefficients in $A_R(n,m)$. Then there is an embedding

$$\rho_n : A_R(m) \hookrightarrow M_n(A_R(n,m)) \tag{0.5}$$

given by

$$\rho_n(y_i) := \begin{pmatrix} x_i(1) & 0 & \dots & 0 \\ 0 & x_i(2) & \dots & 0 \\ 0 & 0 & \dots & x_i(n) \end{pmatrix} \quad \text{for } i = 1, \dots, m. \tag{0.6}$$

Now (0.4) gives

$$\sum_{k=0}^{n} t^k e_k(f) = \prod_{j=1}^{n} (1 + t\rho_n(f)_{jj}) = \det(1 + t\rho_n(f)), \tag{0.7}$$

where $\det(-)$ is the usual determinant of $n \times n$ matrices.

Let $\mathcal{M}_m$ be the set of monomials in $A_R(m)$. For $\mu \in \mathcal{M}_m$ let $\partial_i(\mu)$ denote the degree of μ in y_i, for all $i = 1, \dots, m$. We set

$$\partial(\mu) := (\partial_1(\mu), \dots, \partial_m(\mu)) \tag{0.8}$$

for its multidegree. The total degree of μ is $\sum_i \partial_i(\mu)$. Let $\mathcal{M}_m^+$ be the set of monomials of positive degree. A monomial $\mu \in \mathcal{M}_m^+$ is called *primitive* it is not a power of another one. We denote by $\mathfrak{M}_m^+$ the set of primitive monomials. We define an S_n invariant multidegree on $A_R(n,m)$ by setting $\partial(x_i(j)) = \partial(y_i) \in \mathbb{N}^m$ for all $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant m$. If $f \in A_R(m)$ is homogeneous of total degree l, then $e_k(f)$ has total degree kl (for all k and n).

We are now in a position to state the first part of our result (recall that $m > 1$).

THEOREM 1 (generators). — *The ring of multisymmetric functions $A_R(n,m)^{S_n}$ is generated by the $e_k(\mu)$, where $\mu \in \mathfrak{M}_m^+$, $k = 1, \dots n$ and the total degree of $e_k(\mu)$ is less or equal than $m(n-1)$. If $n = p^s$ is a power of a prime and $R = \mathbb{Z}$ or $p \cdot 1_R = 0$, then at least one generator has degree equal to $m(n-1)$.*

If $R \supset \mathbb{Q}$ then $A_R(n,m)^{S_n}$ is generated by the $e_1(\mu)$, where $\mu \in \mathcal{M}_m^+$ and the degree of μ is less or equal than n.

To obtain the relations between these generators, we need more notation on (multi)symmetric functions.

The action of S_n on $A_R(n,1) \cong R[x_1, x_2, \dots, x_n]$ preserves the usual degree. We denote by $\Lambda_{R,n}^k$ the R-submodule of invariants of degree k.

Let $q_n : R[x_1, x_2, \dots, x_n] \longrightarrow R[x_1, x_2, \dots, x_{n-1}]$ be given by $x_n \mapsto 0$ and $x_i \mapsto x_i$, for $i = 1, \dots, n-1$. This map sends $\Lambda_{n,R}^k$ to $\Lambda_{n-1,R}^k$ and it is easy to see that $\Lambda_{n,R}^k \cong \Lambda_{k,R}^k$ for all $n \geqslant k$. Denote by Λ_R^k the limit of the inverse system obtained in this way.

The ring $\Lambda_R := \bigoplus_{k \geqslant 0} \Lambda_R^k$ is called the ring of *symmetric functions* (over R).

It can be shown [M] that Λ_R is a polynomial ring, freely generated by the (limits of the) e_k, that are given by

$$\sum_{k=0}^{\infty} t^k e_k := \prod_{i=1}^{\infty} (1 + t x_i). \tag{0.9}$$

Furthermore the kernel of the natural projection $\pi_n : \Lambda_R \longrightarrow \Lambda_{n,R}$ is generated by the e_{n+k}, where $k \geqslant 1$.

In a similar way we build a limit of multisymmetric functions. For any $a \in \mathbb{N}^m$ we set $A_R(n,m,a)$ for the linear span of the monomials of multidegree a. One has

$$A_R(n,m) = \bigoplus_{a \in \mathbb{N}^m} A_R(n,m,a). \tag{0.10}$$

Let $\pi_n : A_R(n,m) \longrightarrow A_R(n-1,m)$ be given by

$$\pi_n(x_i(j)) = \begin{cases} 0 & \text{if } j = n \\ x_i(j) & \text{if } j \leqslant n-1 \end{cases} \qquad \text{for all } i. \tag{0.11}$$

Then (see (3.5)) we prove that, for all $a \in \mathbb{N}^m$

$$\pi_n(A_R(n,m,a)^{S_n}) = A_R(n-1,m,a)^{S_{n-1}}. \tag{0.12}$$

For any $a \in \mathbb{N}^m$ set

$$A_R(\infty, m, a) := \lim_{\leftarrow} A_R(n, m, a)^{S_n}, \tag{0.13}$$

where the projective limit is taken with respect to n over the projective system $(A_R(n, m, a)^{S_n}, \pi_n)$.

Set

$$A_R(\infty, m) := \bigoplus_{a \in \mathbb{N}^m} A_R(\infty, m, a). \tag{0.14}$$

We set, by abuse of notation,

$$e_k(f) := \lim_{\leftarrow} e_k(f) \in A_R(\infty, m) \tag{0.15}$$

with $k \in \mathbb{N}$ and $f \in A(m)^+$, the augmentation ideal, i.e.

$$\sum_{k=0}^{\infty} t^k e_k(f) := \prod_{j=1}^{\infty} (1 + tf(j)). \tag{0.16}$$

Then e_k is a homogeneous polynomial of degree k. Now, if $f = \sum_{\mu \in \mathcal{M}_m^+} \lambda_\mu \mu$, we set

$$e_k(f) := \sum_{\alpha} \lambda^\alpha e_\alpha \tag{0.16}$$

where $\alpha := (\alpha_\mu)_{\mu \in \mathcal{M}_m^+}$ is such that $\alpha_\mu \in \mathbb{N}$, $\sum_{\mu \in \mathcal{M}_m^+} \alpha_\mu \leqslant k$ and $\lambda^\alpha := \prod_{\mu \in \mathcal{M}_m^+} \lambda^{\alpha_\mu}$.

We can now state the second part of our main result.

Theorem 2 (relations). — (1) *The ring $A_R(\infty, m)$ is a polynomial ring, freely generated by the (limits of) the $e_k(\mu)$, where $\mu \in \mathfrak{M}_m^+$ and $k \in \mathbb{N}$.*

The kernel of the natural projection

$$A_R(\infty, m) \longrightarrow A_R(n, m)^{S_n}$$

is generated as R-module by the coefficients e_α of the elements

$$e_{n+k}(f), \text{ where } k \geqslant 1 \text{ and } f \in A_R(m)^+.$$

(2) *If $R \supset \mathbb{Q}$ then $A_R(\infty, m)$ is freely generated by the $e_1(\mu)$, where $\mu \in \mathcal{M}_m^+$.*

The kernel of the natural projection is generated as an ideal by the $e_{n+1}(f)$, where $f \in A_R(m)^+$.

In Dalbec's paper [D] generators and relations are found in the case where $R \supset \mathbb{Q}$. The relations found there are actually the same we find: indeed what Dalbec calls *monomial multisymmetric functions* are exactly those e_α we introduced in (0.17), so that his Proposition 1.9 is a special case of our Proposition 3.1(1) when $R \supset \mathbb{Q}$. Another paper on this theme, giving a minimal presentation when the base ring is a characteristic 2 field, is [A]. Again, its main results on multisymmetric functions are a corollary of ours when R is a characteristic 2 field.

The results of this paper were presented in 1997 at a congress on algebraic groups representations in Ascona (CH) organized by H.P. Kraft. They are published only now for personal reasons.

1. Notations and basic facts.

The monomials of $A_R(n,m)$ form a R-basis, permuted by the action of S_n. Thus, the sums of monomials over the orbits form a R-basis of the ring of multisymmetric functions. We now introduce some notation and preliminary results concerning these functions and orbit sums.

Let $k \in \mathbb{N}$, we denote by $\mathbf{f}$ the sequence $(f_1 \dots, f_k)$ in $A_R(m)$ and by α the element $(\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$, where $\sum \alpha_j \leqslant n$. Let $t_1, \dots, t_k$ be commuting independent variables, we set as usual $t^\alpha := \prod_i t_i^{\alpha_i}$. We define elements $e_\alpha(\mathbf{f}) \in A_R(n,m)^{S_n}$ by

$$\sum_\alpha t^\alpha e_\alpha(\mathbf{f}) := \det\Big(1 + \sum_h t_h \rho_n(f_h)\Big) = \prod_{i=1}^n \Big(1 + \sum_h t_h f_h(i)\Big). \tag{1.1}$$

Example 1.1. — Let $n = 3$ and $f, g \in A_R(m)$ then

$$e_{(2,1)}(f,g) = f(1)f(2)g(3) + f(1)g(2)f(3) + g(1)f(2)f(3).$$

If $n = 4$ then

$$\begin{aligned} e_{(2,1)}(f,g) = {} & f(1)f(2)g(3) + f(1)g(2)f(3) + g(1)f(2)f(3) \\ & + f(1)f(2)g(4) + f(1)g(2)f(4) + g(1)f(2)f(4) \\ & + f(1)f(3)g(4) + f(1)g(3)f(4) + g(1)f(3)f(4) \\ & + f(2)f(3)g(4) + f(2)g(3)f(4) + g(2)f(3)f(4) \end{aligned}$$

Let $k = m$ and $f_j = y_j$ for $j = 1, \dots, m$, then the $e_\alpha(\mathbf{y}) = e_{(\alpha_1,\dots,\alpha_m)}(y_1, \dots, y_m)$ where $\sum \alpha_j \leqslant n$ are the well–known *elementary* *multisymmetric functions*. These generate $A_R(n,m)^{S_n}$ when $R \supset \mathbb{Q}$ (see [G] or [W]), and satisfy

$$\sum_\alpha t^\alpha e_\alpha(\mathbf{y}) = \det\Big(1 + \sum_j t_j \rho_n(y_j)\Big) = \prod_{i=1}^{n}\Big(1 + \sum_{j=1}^{m} t_j x_j(i)\Big). \tag{1.2}$$

Lemma 1.2. — *The multisymmetric function $e_{(\alpha_1,\dots,\alpha_k)}(f_1,\dots,f_k)$ is the orbit sum (under the considered action of S_n) of*

$$f_1(1)f_1(2)\cdots f_1(\alpha_1)f_2(\alpha_1+1)\cdots f_2(\alpha_1+\alpha_2)\cdots f_k\big(\sum_h \alpha_h\big).$$

Proof. — Let E be the set of mappings $\phi\colon\{1,\dots,n\}\to\{1,\dots,k+1\}$. We define a mapping $\phi\mapsto\phi^*$ of E into $\mathbb{N}^{k+1}$ by putting $\phi^*(i)$ equal to the cardinality of $\phi^{-1}(i)$. For two elements ϕ_1,ϕ_2 of E, to satisfy $\phi_1^* = \phi_2^*$ it is necessary and sufficient that there should exist $\sigma\in S_n$ such that $\phi_2 = \phi_1\circ\sigma$. Set $f_{k+1} := 1_R$ and $E(\alpha) := \{\phi\in E \mid \phi^* = (\alpha_1,\dots,\alpha_k, n-\sum_i \alpha_i)\}$, then we have

$$e_\alpha(\mathbf{f}) = \sum_{\phi\in E(\alpha)} f_{\phi(1)}(1)f_{\phi(2)}(2)\cdots f_{\phi(n)}(n) \tag{1.3}$$

and the lemma is proved. □

It is clear that $e_{(\alpha_1,\dots,\alpha_k)}(f_1,\dots,f_k) = e_{(\alpha_{\tau(1)},\dots,\alpha_{\tau(k)})}(f_{\tau(1)},\dots,f_{\tau(k)})$ for all $\tau\in S_k$. If two entries are equal, say $f_1 = f_2$, then, by (1.1)

$$e_{(\alpha_1,\dots,\alpha_k)}(f_1,\dots,f_k) = \frac{(\alpha_1+\alpha_2)!}{\alpha_1!\alpha_2!}\, e_{(\alpha_1+\alpha_2,\dots,\alpha_k)}(f_1,f_3\dots,f_k). \tag{1.4}$$

Let $\mathbb{N}^{(\mathcal{M}_m^+)}$ be the set of functions $\mathcal{M}_m^+ \longrightarrow \mathbb{N}$ with finite support. We set

$$\mid\alpha\mid := \sum_{\mu\in\mathcal{M}_m^+} \alpha(\mu). \tag{1.5}$$

Let $\alpha\in\mathbb{N}^{(\mathcal{M}_m^+)}$, then there exist $k\in\mathbb{N}$ and $\mu_1,\dots,\mu_k\in\mathcal{M}_m^+$ such that $\alpha(\mu_i) = \alpha_i \neq 0$ for $i = 1,\dots,k$ and $\alpha(\mu) = 0$ when $\mu\neq\mu_1,\dots,\mu_k$. We set

$$e_\alpha := e_{(\alpha_1,\dots,\alpha_k)}(\mu_1,\dots,\mu_k), \tag{1.6}$$

i.e. we substitute $(\mu_1,\dots,\mu_k)$ to variables in the elementary multisymmetric function $e_{(\alpha_1,\dots,\alpha_k)}(y_1,\dots,y_k)$.

Then

$$\sum_{|\alpha|\leqslant n} t^{\alpha} e_{\alpha} = \prod_{i=1}^{n}\Big(1 + \sum_{\mu\in\mathcal{M}_m^+} t_{\mu}\mu(i)\Big), \tag{1.7}$$

where t_μ are commuting independent variables indexed by monomials and

$$t^{\alpha} := \prod_{\mu\in\mathcal{M}_m^+} t_{\mu}^{\alpha(\mu)} \tag{1.8}$$

for all $\alpha \in \mathbb{N}^{(\mathcal{M}_m^+)}$.

If $\alpha \in \mathbb{N}^{(\mathcal{M}_m^+)}$ is such that $\alpha(\mu) = k$ for some $\mu \in \mathcal{M}_m^+$ and $\alpha(\nu) = 0$ for all $\nu \in \mathcal{M}_m^+$ with $\nu \neq \mu$, we see that $e_\alpha = e_k(\mu)$, the k-th elementary symmetric function evaluated at $(\mu(1), \mu(2), \dots, \mu(n))$.

Lemma 1.3. — *Given a monomial $\mu \in A_R(n, m)$, there exist $\mu_1, \dots, \mu_n \in A_R(m)$ such that $\mu = \mu_1(1)\cdots\mu_n(n)$.*

Proof. — Let $\mu = \prod_{ij} x_i(j)^{a_{ij}}$ then $\mu_j = \prod_i y_i^{a_{ij}}$ for $j = 1, \dots, n$. □

Proposition 1.4. — *The set*

$$\mathcal{B}_{n,m,R} := \{e_\alpha : |\ \alpha\ | \leqslant n\}$$

is a R-basis of $A_R(n, m)^{S_n}$.

The set

$$\mathcal{B}_{n,m,a,R} := \{e_\alpha\ :|\ \alpha\ |\leqslant n \text{ and } \partial(e_\alpha) = a\}$$

is a R-basis of $A_R(n, m, a)^{S_n}$, for all $a \in \mathbb{N}^m$.

Proof. — By Lemma 1.2 and (1.6), the e_α are a complete system of representatives (for the action of S_n) of the orbit sums of the products

$$\{\mu_1(1)\mu_2(2)\cdots\mu_n(n)\ :\ \mu_i \in \mathcal{M}_m\,,\ i = 1, \dots, n\}.$$

So the first statement follows by Lemma 1.3.

Notice that $\partial(e_\alpha) = \sum_{\mu\in\mathcal{M}_m^+} \alpha_\mu \partial(\mu)$ to prove the second statement. □

2. Generators.

Let us calculate the product between two elements $e_\alpha, e_\beta \in \mathcal{B}_{n,m,R}$ of the basis $\mathcal{B}_{n,m,R}$.

THEOREM 2.1 (Product Formula). — *Let $k, h \in \mathbb{N}$, $f_1 \dots, f_k$, $g_1, \dots, g_h \in A_R(m)$ and $t_1, \dots, t_k, s_1, \dots, s_h$ be commuting independent variables. Set as in (1.1)*

$$e_\alpha(\mathbf{f}) := e_{(\alpha_1,\dots,\alpha_k)}(f_1,\dots,f_k) \text{ and } e_\beta(\mathbf{g}) := e_{(\beta_1,\dots,\beta_h)}(g_1,\dots,g_h).$$

Then

$$e_\alpha(\mathbf{f}) e_\beta(\mathbf{g}) = \sum_\gamma e_\gamma(\mathbf{f}, \mathbf{g}, \mathbf{fg}),$$

where $\mathbf{fg} := (f_1g_1, f_1g_2, \dots, f_1g_h, f_2g_1, \dots, f_2g_h, \dots, f_kg_h)$ and $\gamma := (\gamma_{10}, \dots, \gamma_{k0}, \gamma_{01}, \dots, \gamma_{0h}, \gamma_{11}, \gamma_{12}, \dots, \gamma_{kh})$ are such that

$$\begin{cases} \gamma_{ij} \in \mathbb{N} \\ \mid \gamma \mid \leqslant n \\ \sum_{j=0}^h \gamma_{ij} = \alpha_i \ \text{ for } \ i = 1, \dots, k \\ \sum_{i=0}^k \gamma_{ij} = \beta_j \ \text{ for } \ j = 1, \dots, h. \end{cases}$$

Proof. — The result follows from

$$\begin{aligned}
&\Bigg(\sum_{\sum \alpha_j \leqslant n} \prod_{j=1}^k t_j^{\alpha_j} e_\alpha(\mathbf{f}) \Bigg) \Bigg(\sum_{\sum \beta_l \leqslant n} \prod_{l=1}^h s_l^{\beta_l} e_\beta(\mathbf{g}) \Bigg) \\
&= \Bigg(\sum_\alpha t^\alpha e_\alpha(\mathbf{f}) \Bigg) \Bigg(\sum_\beta s^\beta e_\beta(\mathbf{g}) \Bigg) \\
&= \prod_{i=1}^n \Bigg(1 + \sum_{j=1}^k t_j f_j(i) \Bigg) \prod_{i=1}^n \Bigg(1 + \sum_{l=1}^h s_l g_l(i) \Bigg) \\
&= \prod_{i=1}^n \Bigg(1 + \sum_{j=1}^k t_j f_j(i) + \sum_{l=1}^h s_l g_l(i) + \sum_{j,l} t_j s_l f_j(i) g_l(i) \Bigg).
\end{aligned}$$

Introduce the new variables u_{jl} with $j = 1, \dots, k$ and $l = 1, \dots, h$, then

$$\begin{aligned}
&\prod_{i=1}^n \Bigg(1 + \sum_{j=1}^k t_j f_j(i) + \sum_{l=1}^h s_l g_l(i) + \sum_{j,l} t_j s_l f_j(i) g_l(i) \Bigg) \\
&= \prod_{i=1}^n \Bigg(1 + \sum_{j=1}^k t_j f_j(i) + \sum_{l=1}^h s_l g_l(i) + \sum_{j,l} u_{jl}(i) g_l(i) \Bigg) \\
&= \sum_\gamma v^\gamma e_\gamma(\mathbf{f}, \mathbf{g}, \mathbf{fg})
\end{aligned}$$

where v is the cumulative variable t, s, u. Then substitute $u_{jl} = t_j s_l$ to obtain

$$\begin{aligned}\sum_\gamma & v^\gamma e_\gamma(\mathbf{f}, \mathbf{g}, \mathbf{fg}) \\ &= \sum_\gamma \left(\prod_{a=1}^{k} t_a^{\gamma_{a0}} \prod_{b=1}^{h} s_b^{\gamma_{0b}} \prod_{a=1}^{k} \prod_{b=1}^{h} (t_a s_b)^{\gamma_{ab}} e_\gamma(\mathbf{f}, \mathbf{g}, \mathbf{fg}) \right),\end{aligned}$$

where $\mathbf{fg} = (f_1 g_1, f_1 g_2, \ldots, f_k g_1, \ldots, f_k g_h)$ and γ satisfy the condition of the theorem.

Example 2.2. — Let us calculate in $A_R(2,3)^{S_2}$

$$e_{(1,1)}(a,b) e_2(c) = \sum_{0 \leqslant k,h \leqslant 1} e_{(1-k,1-h,2-k-h,h,k)}(a,b,c,ac,bc) = e_{(1,1)}(ac,bc),$$

since $1-k+1-h+2-k-h+h+k = 4-k-h \leqslant 2$.

Corollary 2.3. — *Let $k \in \mathbb{N}$, $a_1, \ldots, a_k \in A_R(m)$, $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ with $\sum \alpha_j \leqslant n$. Then $e_{(\alpha_1,\ldots,\alpha_k)}(a_1, \ldots, a_k)$ belongs to the subring of $A_R(n,m)^{S_n}$ generated by the $e_i(\mu)$, where $i = 1, \ldots, n$ and μ is a monomial in the $a_1, \ldots, a_k$.*

Proof. — We prove the claim by induction on $\sum_j \alpha_j$ (notice that $1 \leqslant k \leqslant \sum_j \alpha_j$) assuming that $\alpha_i > 0$ for all i. If $\sum_j \alpha_j = 1$ then $k = 1$ and $e_{(\alpha_1,\ldots,\alpha_k)}(a_1, \ldots, a_k) = e_1(a_1)$. Suppose the claim true for all $e_{(\beta_1,\ldots,\beta_h)}(b_1, \ldots, b_h)$ with $b_1, \ldots, b_h \in A_R(m)$ and $\sum_i \beta_i < \sum_j \alpha_j$. Let $k, a_1, \ldots, a_k, \alpha$ be as in the statement, then we have by Theorem 2.1

$$\begin{aligned} e_{\alpha_1}(a_1) e_{(\alpha_2,\ldots,\alpha_k)}(a_2, \ldots, a_k) &= e_{(\alpha_1,\ldots,\alpha_k)}(a_1, \ldots, a_k) \\ &\quad + \sum e_\gamma(a_1, \ldots, a_k, a_1 a_2, \ldots, a_1 a_k),\end{aligned}$$

where

$$\gamma = (\gamma_{10}, \gamma_{01}, \ldots, \gamma_{0h}, \gamma_{11}, \gamma_{12}, \ldots, \gamma_{1h})$$

with $h = k-1$, $\sum_{j=0}^h \gamma_{1j} = \alpha_1$ with $\sum_{j=1}^h \gamma_{1j} > 0$, and $\gamma_{0j} + \gamma_{1j} = \alpha_j$ for $j = 1, \ldots, h$. Thus

$$\gamma_{10} + \gamma_{01} + \ldots + \gamma_{0h} + \gamma_{11} + \ldots + \gamma_{1h} = \sum_j \alpha_j - \sum_{j=1}^h \gamma_{1j} < \sum_j \alpha_j.$$

Hence

$$\begin{aligned} e_{(\alpha_1,\ldots,\alpha_k)}(a_1, \ldots, a_k) &= e_{\alpha_1}(a_1) e_{(\alpha_2,\ldots,\alpha_k)}(a_2, \ldots, a_k) \\ &\quad - \sum e_\gamma(a_1, \ldots, a_k, a_1 a_2, a_1 a_3, \ldots, a_1 a_k),\end{aligned}$$

where $\sum_{r,s}\gamma_{rs} < \sum_j \alpha_j$. So the claim follows by induction hypothesis.

Example 2.4. — Consider $e_{(2,1)}(a,b)$ in $A_R(3,m)$ as in Example 1.2, then

$$e_{(2,1)}(a,b) = e_2(a)e_1(b) - e_{(1,1)}(a,ab) = e_2(a)e_1(b) - e_1(a)e_1(ab) + e_1(a^2b).$$

We now recall some basic facts about classical symmetric functions, for further reading on this topic see [M].

We have another distinguished kind of functions in Λ_R beside the elementary symmetric ones: the *power sums.*

For any $r \in \mathbb{N}$ the r-th power sum is

$$p_r := \sum_{i\geqslant 1} x_i^r.$$

Let $g \in \Lambda_R$, set $g\cdot p_r = g(x_1^r, x_2^r, \ldots, x_k^r, \ldots)$, this is again a symmetric function. Since the e_i generate Λ_R we have that $g \cdot p_r$ can be expressed as a polynomial in the e_i. In particular,

$$P_{h,k} := e_h \cdot p_k$$

is a polynomial in the e_i.

Proposition 2.5. — *For all $f \in A_R(m)$, and $k, h \in \mathbb{N}$, $e_h(f^k)$ belongs to the subring of $A_R(n,m)^{S_n}$ generated by the $e_j(f)$.*

Proof. — Let $f \in A_R(m)$ and consider $e_h(f^k) \in A_R(n,m)^{S_n}$, we have (see Introduction)

$$e_h(f^k) = e_h(f(1)^k, \ldots, f(n)^k) = P_{h,k}(e_1(f(1), \ldots, f(n)), \ldots, e_n(f(1), \ldots, f(n)))$$

and the result is proved.

We are now ready to prove Theorem 1 stated in the introduction.

Proof of Theorem 1. — Recall that a monomial $\mu \in \mathcal{M}_m^+$ is called *primitive* if it is not a power of another one and we denote by $\mathfrak{M}_m^+$ the set of primitive monomials. The elements $e_\alpha \in \mathcal{B}_{n,m,R}$, that form a R-basis by Proposition 1.4, can be expressed as polynomials in $e_i(\mu)$ with $i = 1, \ldots, n$ and $\mu \in \mathcal{M}_m^+$, by Corollary 2.3. If $\mu = \nu^k$ with $\nu \in \mathfrak{M}_m^+$, then $e_i(\mu)$ can be expressed as a polynomial in the $e_j(\nu)$, by Proposition 2.5. Since for all $\mu \in \mathcal{M}_m^+$ there exist $k \in \mathbb{N}$ and $\nu \in \mathfrak{M}_m^+$ such that $\mu = \nu^k$, we have that

$A(n,m)^{S_n}$ is generated as a commutative ring by the $e_j(\nu)$, where $\nu \in \mathfrak{M}_m^+$ and $j = 1, \ldots, n$.

The theorem then follows by the following result due to Fleischmann [F]: the ring $A_R(n,m)^{S_n}$ is generated by elements of total degree $\ell \leqslant m(n-1)$, for any commutative ring R, with sharp bound if $n = p^s$ a power of a prime and $R = \mathbb{Z}$ or $p \cdot 1_R = 0$. If $R \supset \mathbb{Q}$ then the result follows from Newton's Formulas and a well–known result of H.Weyl (see [G], [W]). $\square$

3. Relations.

We write a generating series for the orbits of monomials

$$G_n(t) := \prod_{i=1}^{n} \left(1 + \sum_{\mathcal{M}_m^+} t_\mu \mu(i) \right) = \sum_{\alpha, |\alpha| \leqslant n} t^\alpha e_\alpha(n), \tag{3.1}$$

where $\alpha \in \mathbb{N}^{(\mathcal{M}_m^+)}$ and $t^\alpha e_\alpha(n) = 0$ when $\alpha = 0$.

Recall the map $\pi_n : A_R(n,m) \longrightarrow A_R(n-1,m)$ defined by

$$\pi_n(x_i(j)) = \begin{cases} 0 & \text{if } \ j = n \\ x_i(j) & \text{if } \ j \leqslant n-1 \end{cases} \qquad \text{for all } i. \tag{3.2}$$

Then we have of course that $\pi_n(G_n(t)) = G_{n-1}(t)$, so that

$$\pi_n((e_\alpha)) = \begin{cases} e_\alpha & \text{if } \ \mid \alpha \mid < n \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Thus, by Proposition 1.4, for all $a \in \mathbb{N}^m$ the restriction

$$\pi_{n,a} : A_R(n,m,a) \longrightarrow A_R(n-1,m,a) \tag{3.4}$$

is such that

$$\pi_{n,a}(A_R(n,m,a)^{S_n}) = A_R(n-1,m,a)^{S_{n-1}} \tag{3.5}$$

and then $(A_R(n,m,a)^{S_n}, \pi_{n,a})$ is a projective sytem.

For any $a \in \mathbb{N}^m$ set

$$A_R(\infty,m,a) := \varprojlim A_R(n,m,a)^{S_n}, \tag{3.6}$$

where the projective limit is taken with respect to n over the above projective system and set

$$\tilde{\pi}_{n,a} : A_R(\infty,m,a) \longrightarrow A_R(n,m,a)^{S_n} \tag{3.7}$$

for the natural projection.

Set

$$A_R(\infty, m) := \bigoplus_{a\in\mathbb{N}^m} A_R(\infty, m, a) \tag{3.8}$$

and

$$\tilde{\pi}_n := \bigoplus_{a\in\mathbb{N}^m} \tilde{\pi}_{n,a}. \tag{3.9}$$

Similarly to the classical case ($m = 1$) and recalling (3.1), (3.3) we make an abuse of notation and set

$$e_\alpha := \lim_{\leftarrow} \; e_\alpha(n),$$

for any $\alpha \in \mathbb{N}^{(\mathcal{M}_m^+)}$. In the same way we set $e_j(f) := \lim\limits_{\leftarrow} \; e_j(f)$ with $j \in \mathbb{N}$, where $f \in A_R(m)^+$ is homogeneous of positive multidegree, so that $j\, \partial(f) = a$.

Proposition 3.1. — *Let $a \in \mathbb{N}^m$.*

(1) *The R-module $\ker \tilde{\pi}_{n,a}$ is the linear span of*

$$\{e_\alpha \in A_R(\infty, m, a) \;: \mid \alpha \mid > n\}.$$

(2) *The R-module homomorphisms $\tilde{\pi}_{n,a} : A_R(\infty, m, a) \to A_R(n, m, a)^{S_n}$ are onto for all $n \in \mathbb{N}$ and $A_R(\infty, m, a) \cong A_R(n, m, a)^{S_n}$ for all $n \geqslant \mid a \mid$.*

(3) *The R-module $A_R(\infty, m, a)$ is free with basis*

$$\{e_\alpha \;:\; \partial(e_\alpha) = a\},$$

(4) *The R-module $A_R(\infty, m)$ is free with basis*

$$\{e_\alpha \;:\; \alpha \in \mathbb{N}^{(\mathcal{M}_m^+)}\}.$$

Proof. — (1) By (3.3) and (3.5), for all $a \in \mathbb{N}^m$, the following is a split exact sequence of R-modules

$$0 \longrightarrow \ker \pi_{n,a} \longrightarrow A(n, m, a)^{S_n} \xrightarrow{\pi_{n,a}} A(n-1, m, a)^{S_{n-1}} \longrightarrow 0,$$

and the claim follows.

(2) If $\sum_{j=1}^m a_j < n$, then $\ker \tilde{\pi}_{n,a} = 0$, indeed

$$\partial(e_\alpha) = \sum_{\mu\in\mathcal{M}_m^+} \alpha_\mu \; \partial(\mu) = a \Longrightarrow \mid \alpha \mid \leqslant \sum_{j=1}^m a_j < n.$$

Hence $A(h,m,a)^{S_h} \cong A(b,m,a)^{S_b}$ where $b := \sum_{j=1}^m a_j$, for all $h \geqslant \sum_{j=1}^m a_j$ and the claim follows by (3.5).

(3) follows from (1) and (2).

(4) follows from (3) and (3.8) □

Remark 3.2. — Notice that $A_R(m)^{\otimes n} \cong A_R(n,m)$ as multigraded S_n-algebras by means of

$$f_1 \otimes \cdots \otimes f_n \leftrightarrow f_1(1) f_2(2) \cdots f_n(n) \tag{3.10}$$

for all $f_1, \ldots, f_n \in A_R(m)$. Hence $A_R(n,m)^{S_n} \cong TS^n(A_R(m))$, where $TS^n(\ -\ )$ denotes the symmetric tensors functor. Since $TS^n(A_R(m)) \cong R \bigotimes TS^n(A_{\mathbb{Z}}(m))$ (see [B]), we have

$$A_R(n,m)^{S_n} \cong R \otimes A_{\mathbb{Z}}(n,m)^{S_n} \tag{3.11}$$

for any commutative ring R.

We then work with $R = \mathbb{Z}$ and we suppress the $\mathbb{Z}$ subscript for the sake of simplicity.

Remark 3.3. — The $\mathbb{Z}$-module $A(\infty, m)$ can be endowed with a structure of $\mathbb{N}^m$-graded ring such that the π_n are $\mathbb{N}^m$-graded ring homomorphisms: the product $e_\alpha e_\beta$, where $\alpha, \beta \in \mathbb{N}^{(\mathcal{M}_m^+)}$, is defined by using the product formula of Theorem 2.1 with no upper bound on $\mid \gamma \mid$, where γ appears in the summation.

Proposition 3.4. — *Consider the free polynomial ring*

$$C(m) := \bigoplus_{a \in \mathbb{N}^m} C(m,a) := \mathbb{Z}[e_{i,\mu}]_{i \in \mathbb{N}, \mu \in \mathfrak{M}_m^+}$$

with multidegree given by $\partial(e_{i,\mu}) = \partial(\mu) i$.

Then the multigraded ring homomorphism

$$\sigma_m : \mathbb{Z}[e_{i,\mu}]_{i \in \mathbb{N}, \mu \in \mathfrak{M}_m^+} \longrightarrow A(\infty, m)$$

given by

$$\sigma_m : e_{i,\mu} \mapsto e_i(\mu), \ \textit{for all } i \in \mathbb{N}, \ \mu \in \mathfrak{M}_m^+$$

is an isomorphism, i.e. $A(\infty, m)$ *is freely generated as a commutative ring by the* $e_i(\mu)$*, where* $i \in \mathbb{N}$ *and* $\mu \in \mathfrak{M}_m^+$.

Proof. — Since we defined the product in $A(\infty, m)$ as in Theorem 2.1, it is easy to verify, repeating the reasoning of the previous section, that $A(\infty, m)$ is generated as a commutative ring by the $e_i(\mu)$, where $i \in \mathbb{N}$, $\mu \in \mathfrak{M}_m^+$. Hence σ_m is onto for all $m \in \mathbb{N}$.

Let $a \in \mathbb{N}^m$ and consider the restriction $\sigma_{m,a} : C(m,a) \longrightarrow A(\infty, m, a)$. It is onto as we have just seen. A $\mathbb{Z}$-basis of $C(m,a)$ is

$$\Big\{ \prod_{i\in\mathbb{N},k\in\mathbb{N},\mu\in\mathfrak{M}_m^+} e_{i,\mu} \ : \sum_{i\in\mathbb{N},k\in\mathbb{N},\mu\in\mathfrak{M}_m^+} i\ k\ \partial(\mu) \ = \ a \Big\}.$$

On the other hand, a $\mathbb{Z}$-basis of $A(\infty, m, a)$ is

$$\Big\{ e_\alpha \ : \sum_{\alpha_\mu\in\mathbb{N},\mu\in\mathcal{M}_m^+} \alpha_\mu\ \partial(\mu) \ = \ a \Big\}.$$

Let $\mu \in \mathcal{M}_m^+$, then there are an unique $k \in \mathbb{N}$ and an unique $\nu \in \mathfrak{M}_m^+$ such that $\mu = \nu^k$. Hence

$$\sum_{\alpha_\mu\in\mathbb{N},\mu\in\mathcal{M}_m^+} \alpha_\mu\ \partial(\mu) = \sum_{k\in\mathbb{N},\alpha_\mu\in\mathbb{N},\nu\in\mathfrak{M}_m^+} \alpha_\mu\ k\ \partial(\nu),$$

so that $C(m,a)$ and $A(\infty, m, a)$ have the same (finite) $\mathbb{Z}$-rank and thus are isomorphic via $\sigma_{m,a}$. □

Corollary 3.5. — *Let $R \supset \mathbb{Q}$ then $A_R(\infty, m)$ is a polynomial ring freely generated by the $e_1(\mu)$, where $\mu \in \mathcal{M}_m^+$.*

Proof. — By Proposition 3.4 and Theorem 1. □

Proof of Theorem 2. — (1) As before we set $R = \mathbb{Z}$ and the result follows by Remark 3.2, Proposition 3.4. and Proposition 3.1.

(2) By Proposition 3.1 the kernel of

$$A(\infty, m) \xrightarrow{\tilde{\pi}_n} A(n,m)^{S_n}$$

has basis $\{e_\alpha :\mid \alpha \mid > n\}$. Let V_k be the submodule of $A(\infty, m)$ with basis $\{e_\alpha :\mid \alpha \mid = k\}$. Let A_k be the sub-$\mathbb{Z}$-module of $\mathbb{Q} \otimes V_k$ generated by the $e_k(f)$ with $f \in A(m)^+$. Let $g : \mathbb{Q} \otimes V_k \longrightarrow \mathbb{Q}$ be a linear form identically zero on A_k. Then

$$0 = g(e_k(f)) = g\Bigg(e_k\Big(\sum_{\mu\in\mathcal{M}_m^+} \lambda_\mu\ \mu\Big)\Bigg) = \Bigg(\sum_{|\alpha|=k} \Big(\prod_{\mu\in\mathcal{M}_m^+} \lambda_\mu^{\alpha_\mu}\Big)\ g(e_\alpha)\Bigg),$$

for all $\sum_{\mu\in\mathcal{M}_m^+} \lambda_\mu\ \mu \in A(m)^+$. Hence $g(e_\alpha) = 0$ for all e_α with $\mid \alpha \mid = k$; thus $g = 0$. If $R \supset \mathbb{Q}$ the result then follows from Newton's formulas and Corollary 3.5. □

Aknowledgement. — I would like to thank M. Brion, C. De Concini and C. Procesi, in alphabetical order, for useful discussions. I would also like to thank the referee for its valuable suggestions.

BIBLIOGRAPHY

[A] M. Feschbach, The mod 2 cohomology rings of the symmetric groups and invariants, Topology, (2002), 57–84.

[B] N. Bourbaki, Elements of mathematics - Algebra II Chapters 4-7, Springer-Verlag, Berlin, (1988).

[D] J. Dalbec, Multisymmetric functions, Beiträge Algebra Geom., 40(1) (1999), 27–51.

[F] P. Fleischmann, A new degree bound for vector invariants of symmetric groups, Trans. Am. Math. Soc., 350 (1998), 1703–1712.

[G] I. Gelfand, M. Kapranov, A. Zelevinsky, Discriminants, resultants and multidimensional determinants, Birkahuser, Boston, (1994).

[J1] F. Junker, Die Relationen, welche zwischen den elementaren symmetrischen Functionen bestehen, Math. Ann., 38 (1891), 91–114.

[J2] F. Junker, Über symmetrische Functionen von mehreren Reihen von Veränderlichen, Math. Ann., 43 (1893), 225–270.

[J3] F. Junker, Die symmetrische Functionen und die Relationen zwischen den Elementarfunctionen derselben, Math. Ann., 45 (1894), 1–84.

[M] I.G. Macdonald, Symmetric Functions and Hall Polynomials - second edition, Oxford mathematical monograph, (1995).

[W] H. Weyl, The classical groups, Princeton University Press, Princeton N.J., (1946).

Manuscrit reçu le 1er juillet 2004,
Accepté le 12 septembre 2004.

Francesco VACCARINO,
Politecnico di Torino
Dipartimento di Matematica
Corso Duca degli Abruzzi 24
10129 Torino (Italy)
vaccarino@syzygie.it