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SIGN FUNCTIONS OF IMAGINARY QUADRATIC FIELDS AND APPLICATIONS

by Hassan OUKHABA

1. Introduction.

In this paper we introduce the concept of a sign function of a imaginary quadratic field. As we will prove below this concept is very helpful in the study of some arithmetical problems. Classically by sign we mean the extension to $\mathbb{R}^\times$ of the continuous homomorphism $s : \mathbb{Q}^\times \rightarrow \{-1, 1\}$ satisfying $s(-1) = -1$. Here $A^\times$ is the multiplicative group of the ring A. In 1985 David R. Hayes introduced the concept of a sign function of a global function field and used this notion to normalize Drinfel'd modules of rank one, *cf.* [7]. The torsion points of these modules have many important arithmetical properties. They are essential in the construction of Stickelberger elements, Stark units, Euler systems, groups of cyclotomic units in characteristic p, etc. To recall this definition we let K be a global function field. We denote by ∞ a fixed place of K, and by $\widehat{K}$ the completion of K at ∞. Let us also denote by $\mathbb{F}_q$ both the finite field of q elements and the constant field of $\widehat{K}$. Then a sign function, with respect to (K, ∞), is a continuous homomorphism $s : \widehat{K}^\times \rightarrow \mathbb{F}_q^\times$ satisfying $s(a) = a$ for all $a \in \mathbb{F}_q^\times$. See [7] and [8] for more details. Our definition of a sign function in the case of a imaginary quadratic field $k \subset \mathbb{C}$ is as follows. Let $H \subset \mathbb{C}$ be the Hilbert class field of k. Then a sign function of k is a surjective

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group homomorphism $s : \Lambda_6 \to \mu_H$ satisfying $s(\mathcal{U}_{12}) = 1$ and such that $s(\xi) = \xi$ for all $\xi \in \mu_k$ (see the notation below). As one may check the homomorphism $\widetilde{\kappa} : \Lambda_6 \to \mu_H$ induced by κ^{-1}, where κ is the character defined by Hajir-Villegas in [6], satisfies all these properties. Hence $\widetilde{\kappa}$ is a sign function of k.

In section 2 below we associate to each couple $(s, \mathfrak{m})$, where s is a sign function and $\mathfrak{m}$ is a non zero integral ideal of k prime to 6, a finite abelian extension $k_{\mathfrak{m},s} \subset \mathbb{C}$ of k. The field $k_{\mathfrak{m},s}$ is well described by class field theory. In particular $k_{\mathfrak{m},s}$ contains the ray class field modulo $\mathfrak{m}$, which we denote by $k_{\mathfrak{m}}$. The extension $k_{\mathfrak{m},s}/k_{\mathfrak{m}}$ is cyclic of degree w_H (resp. w_H/w_k) if $\mathfrak{m} \neq (1)$ (resp $\mathfrak{m} = (1)$). As explained below the properties of the ramification in the extension $k_{\mathfrak{m},s}/H_s$, where $H_s = k_{(1),s}$ lead us to consider $k_{\mathfrak{m},s}$ as the analog of a cyclotomic number field or a cyclotomic function field as well.

In section 3 we associate to each integral ideal $\mathfrak{c}$ of k prime to $6N(\mathfrak{m})$ an algebraic integer $\Gamma_{\mathfrak{m}}(\mathfrak{c})$, which is a root of the Ramachandra invariant, see definition 3.1. The construction of $\Gamma_{\mathfrak{m}}(\mathfrak{c})$ involves the Klein function and the eta function of Hajir-Villegas. In Theorem 3.1 we describe the Galois action on $\Gamma_{\mathfrak{m}}(\mathfrak{c})$. This is essentially done by using the Shimura reciprocity law. In particular we prove that $\Gamma_{\mathfrak{m}}(\mathfrak{c}) \in k_{m,s}$, where $m = N(\mathfrak{m})$ and s is the sign defined by the formula (3.2). In Theorem 3.2 and Corollary 3.1 below we describe the behavior under the norm map of a certain power of $\Gamma_{\mathfrak{m}}(1)$. In this we use the distribution law of the Siegel function stated in [12] §2. The result we get is a refinement of the well known Theorem 2 of [17] that gives the norm formulas satisfied by the Ramachandra invariants.

In section 4 we define the level $\mathfrak{m}$ universal ordinary s-distribution $U_s(\mathfrak{m})$, in spirit of those considered in [11], [1] or [24]. We give the structure of $U_s(\mathfrak{m})$ as an abelian group and compute the Tate cohomology groups $\widehat{H}^n(J, U_s(\mathfrak{m}))$, where $J = \mathrm{Gal}(k_{\mathfrak{m},s}/k_{\mathfrak{m}})$. In this we follow the method of Ouyang, *cf.* [16], which essentially uses Anderson's resolution and related spectral sequences. These cohomology groups naturally appear in many settings. See for instance Anderson's theory of epsilon extensions and it's analog for function fields in [2] and [3]. Let us remark that $U_s(\mathfrak{m})$ is naturally a $\mathrm{Gal}(k_{\mathfrak{m},s}/k)$-module. It's Galois module structure is closely related to a certain group of elliptic units. This connexion will be made clear in a forthcoming paper in which we extend some results of Ouyang's paper [16] to our case and use them to improve Theorem B of [14].

1.1. Notations.

In this subsection we give some of the notation we will use in this paper.

k^{ab}:= the maximal abelian extension of k in $\mathbb{C}$.

μ_∞:= The group of roots of unity in $\mathbb{C}$ and $\mu_k = \mu_\infty \cap k$.

$\mathcal{O}_k$:= the ring of integers of k and $D_k < 0$ the discriminant of k.

T_0:= the monoïde of non-zero integral ideals of $\mathcal{O}_k$ prime to 6.

$\overline{T}_0$:= the group of fractional ideals of k prime to 6.

Λ_6:= the group of elements $x \in k^*$ that are prime to 6

$\mathcal{U}_{12}$:=$\{x \in k^*$ such that $x \equiv 1 \mod^\times 12\}$

Further if $\mathfrak{n} \in T_0$ then we call $\mathfrak{n}$ primitive if it is not of the form $t\mathfrak{u}$, with $\mathfrak{u} \in T_0$ and $2 \leqslant t \in \mathbb{N}$. It is always possible to write $\mathfrak{n} = n_1\mathfrak{n}'$, where $n_1 \in \mathbb{N}^*$ and $\mathfrak{n}'$ primitive. This decomposition is unique. We will denote the primitive part $\mathfrak{n}'$ by $pr(\mathfrak{n})$. Let $L \subset k^{ab}$ be a finite abelian extension of k. Then we let $\mu_L = \mu_\infty \cap L$. If $\mathfrak{a} \in T_0$ is prime to the conductor of L/k then we denote by $(\mathfrak{a}, L/k)$ the automorphism of L/k associated to $\mathfrak{a}$ by the Artin map.

$I_\mathfrak{n}$:= the group of fractional ideals of k prime to $6\mathfrak{n}$.

$\mathcal{U}_\mathfrak{n}$:=$\{x \in \Lambda_6$ such that $x \equiv 1 \mod^\times \mathfrak{n}\}$

$\mathcal{R}_\mathfrak{n}$:= the sub-group of $I_\mathfrak{n}$ formed of those principal ideals $x\mathcal{O}_k$ with $x \in \mathcal{U}_\mathfrak{n}$

$k_\mathfrak{n}$:=the ray class field modulo $\mathfrak{n}$

$e_\mathfrak{n}$:=the positive generator of $\mathfrak{n} \cap \mathbb{Z}$ and $N(\mathfrak{n})$:= the cardinal of $\mathcal{O}_k/\mathfrak{n}$.

$\overline{\mathfrak{n}}$:= the image of $\mathfrak{n}$ by the complex conjugation.

$\deg(\mathfrak{n})$:= the number of non zero prime ideals dividing $\mathfrak{n}$

2. The narrow ray class fields $k_{\mathfrak{m},s}$.

Let s be a sign function. Then for all $\mathfrak{m} \in T_0$ we define $\mathcal{R}_{\mathfrak{m},s}$ to be the group of fractional principal ideals $x\mathcal{O}_k$ such that $x \in \Lambda_6 \cap \mathcal{U}_\mathfrak{m}$ and

$s(x)=1$. By class field theory there exists a unique finite abelian extension $k_{\mathfrak{m},s}\subset\mathbb{C}$ of k such that the Artin map gives the isomorphism

$$\mathrm{Gal}(k_{\mathfrak{m},s}/k)\simeq I_{\mathfrak{m}}/\mathcal{R}_{\mathfrak{m},s}.$$

If $\mathfrak{m}\neq(1)$ then the map $x\mathcal{O}_k\mapsto s(x)$, where $x\in\Lambda_6\cap\mathcal{U}_{\mathfrak{m}}$ induces an isomorphism from $\mathcal{R}_{\mathfrak{m}}/\mathcal{R}_{\mathfrak{m},s}$ into μ_H. In particular the tautologic exact sequence

$$1\longrightarrow\mathcal{R}_{\mathfrak{m}}/\mathcal{R}_{\mathfrak{m},s}\longrightarrow I_{\mathfrak{m}}/\mathcal{R}_{\mathfrak{m},s}\longrightarrow I_{\mathfrak{m}}/\mathcal{R}_{\mathfrak{m}}\longrightarrow 1$$

clearly shows that $k_{\mathfrak{m},s}$ is a cyclic extension of $k_{\mathfrak{m}}$ of degree w_H. Let us put $H_s=k_{(1),s}$. Since $\mathcal{R}_{(1)}/\mathcal{R}_{(1),s}$ is isomorphic to μ_H/μ_k via the map $x\mathcal{O}_k\mapsto s(x)\mu_k$ for $x\in\Lambda_6$ we see that H_s/H is a cyclic extension of degree w_H/w_k. Moreover for $\mathfrak{m}\in T_0$ and $\mathfrak{m}\neq(1)$ we have

$$\mathrm{Gal}(k_{\mathfrak{m},s}/H_s)\simeq I_{\mathfrak{m}}\cap\mathcal{R}_{(1),s}/\mathcal{R}_{\mathfrak{m},s}\simeq(\mathcal{O}_k/\mathfrak{m})^\times.$$

The inertia group of a prime ideal $\mathfrak{q}|\mathfrak{m}$ in $k_{\mathfrak{m},s}/k$ is isomorphic to $(\mathcal{O}_k/\mathfrak{q}^e)^\times$, where $\mathfrak{q}^e$ is the exact power of $\mathfrak{q}$ that divide $\mathfrak{m}$. In particular $\mathrm{Gal}(k_{\mathfrak{m},s}/H_s)$ is the direct product of the inertia groups of the prime ideals $\mathfrak{q}|\mathfrak{m}$. We call $k_{\mathfrak{m},s}$ the narrow ray class field of k modulo $\mathfrak{m}$ relative to s. One may consider $k_{\mathfrak{m},s}$ as the analog of a cyclotomic number field or the analog of the narrow ray class field of Hayes. See for instance [8] page 27.

3. A fundamental example.

Let us recall the definition of the character $\kappa:(\mathcal{O}_k/12\mathcal{O}_k)^\times\rightarrow\mu_H$ constructed in [6] definition 11. If $\lambda\in\mathcal{O}_k\cap\Lambda_6$ is such that $\lambda\mathcal{O}_k$ is primitive then

$$\kappa(\lambda)=(-1)^{\frac{N(\lambda)-1}{2}}\frac{1}{\lambda}\frac{\eta^2(\lambda\mathcal{O}_k)}{\eta^2(\mathcal{O}_k)} \tag{3.1}$$

where $N(\lambda)=N(\lambda\mathcal{O}_k)$ and $\mathfrak{a}\mapsto\eta(\mathfrak{a})$ is the eta function on primitive ideals $\mathfrak{a}\in T_0$, *cf.* [6] definition 8.

If $x\in\mathcal{O}_k\cap\Lambda_6$ then we denote by $q(x)$ the class of x in $\mathcal{O}_k/12\mathcal{O}_k$. We extend multiplicatively this definition to obtain a group homomorphism $q:\Lambda_6\rightarrow(\mathcal{O}_k/12\mathcal{O}_k)^\times$. In this section we investigate some aspects of the abelian extensions $k_{\mathfrak{m},s}$ where s is the sign function satisfying

$$s(x)=(-1)^{\frac{N(x)-1}{2}}\kappa(q(x))^{-1},\ \text{for all } x\in\mathcal{O}_k\cap\Lambda_6, \tag{3.2}$$

Using the properies of κ one may easily prove that $s(t)=(-1)^{\frac{t-1}{2}}$ for all positive integer t prime to 6.

Let $\mathfrak{a} \in T_0$ be a primitive ideal of $\mathcal{O}_k$. Let $\lambda \in \Lambda_6$ and denote by σ_λ the automorphism of k^{ab}/k associated to $\lambda\mathcal{O}_k$ by the Artin map. Then by Proposition 10 (i) of [6] we have

$$\frac{\eta^2(\mathcal{O}_k)}{\eta^2(\mathfrak{a})} \in k^{ab} \quad \text{and} \quad \Big(\frac{\eta^2(\mathcal{O}_k)}{\eta^2(\mathfrak{a})}\Big)^{\sigma_\lambda - 1} = s(\lambda)^{N(\mathfrak{a})-1}. \tag{3.3}$$

Thus $\eta^2(\mathcal{O}_k)/\eta^2(\mathfrak{a}) \in H_s$ and H_s is the extension of H generated by all these quotients. Let $\mathfrak{a}, \mathfrak{b} \in T_0$ be primitive ideals of $\mathcal{O}_k$, prime to D_k and such that $N(\mathfrak{a})$ and $N(\mathfrak{b})$ are coprime. Then $\mathfrak{ab}$ is primitive. Let us put

$$\eta(\mathfrak{a}, \mathfrak{b}) = \frac{\eta(\mathfrak{a})\eta(\mathfrak{b})}{\eta(\mathcal{O}_k)\eta(\mathfrak{ab})}. \tag{3.4}$$

Then we have

$$\eta(\mathfrak{a}, \mathfrak{b}) \in H_s \quad \text{and} \quad \eta(\mathfrak{a}, \mathfrak{b})^{\sigma_\lambda - 1} = s(\lambda)^{\frac{1}{2}(N(\mathfrak{a})-1)(N(\mathfrak{b})-1)}, \tag{3.5}$$

thanks to Theorem 19 (i) of [6].

Let L be a lattice of $\mathbb{C}$ and let (ω_1, ω_2) be a positive $\mathbb{Z}$-basis of L, which means that $Im(\omega_1/\omega_2) > 0$, then we denote by $\mathfrak{f}(z, L)$ and $g(z, \omega_1, \omega_2)$ respectively the Klein function and Siegel function as defined in [12], formulas (2.8) and (2.12).

Definition 3.1. — *For all $\mathfrak{m}, \mathfrak{c} \in T_0$ such that $\mathfrak{m} \neq (1)$ and $\mathfrak{c}$ is prime to $6N(\mathfrak{m})$ we put*

$$\Gamma_{\mathfrak{m}}(\mathfrak{c}) = \frac{-2\pi\mathfrak{f}\big(N(\mathfrak{c}), \mathfrak{m}\overline{\mathfrak{c}}\big)\eta^2\big(pr(\overline{\mathfrak{m}}\,\mathfrak{c})\big)}{e_{\mathfrak{mc}}} s(c_1),$$

where $c_1 \in \mathbb{N}^*$ is defined by $\mathfrak{c} = c_1 pr(\mathfrak{c})$

Proposition 3.1. — *Let us set $m = N(\mathfrak{m})$. If $(\mathfrak{c}, k_{(m),s}/k) = (\mathfrak{c}', k_{(m),s}/k)$ then $\Gamma_{\mathfrak{m}}(\mathfrak{c}) = \Gamma_{\mathfrak{m}}(\mathfrak{c}')$.*

Proof. — Let $\lambda, \mu \in \Lambda_6 \cap \mathcal{O}_k$ be such that $\lambda \equiv \mu$ modulo $N(\mathfrak{m})$, $s(\lambda) = s(\mu)$ and $\lambda\mathfrak{c} = \mu\mathfrak{c}'$. Then we have

$$\lambda\overline{\lambda}\mathfrak{f}(N(\mathfrak{c}), \mathfrak{m}\overline{\mathfrak{c}}\big) = \mathfrak{f}(\lambda\overline{\lambda}N(\mathfrak{c}), \lambda\overline{\lambda}\mathfrak{m}\overline{\mathfrak{c}}\big) = \mathfrak{f}(\mu\overline{\mu}N(\mathfrak{c}'), \lambda\overline{\mu}\mathfrak{m}\overline{\mathfrak{c}'}\big) = \overline{\mu}\,\mathfrak{f}(\mu N(\mathfrak{c}'), \lambda\mathfrak{m}\overline{\mathfrak{c}'}\big)$$

But since λ, μ are prime to 2 and $\lambda \equiv \mu$ modulo $N(\mathfrak{m})$ we also have the equality $\mathfrak{f}(\mu N(\mathfrak{c}'), \lambda\mathfrak{m}\overline{\mathfrak{c}'}\big) = \mathfrak{f}(\lambda N(\mathfrak{c}'), \lambda\mathfrak{m}\overline{\mathfrak{c}'}\big)$, thanks to the transformation law $(K3)$ of the Klein function in [12] page 232. Thus we have

$$\overline{\lambda}\mathfrak{f}(N(\mathfrak{c}), \mathfrak{m}\overline{\mathfrak{c}}\big) = \overline{\mu}\,\mathfrak{f}(N(\mathfrak{c}'), \mathfrak{m}\overline{\mathfrak{c}'}\big). \tag{3.6}$$

let us set $\mathfrak{c}_2 = pr(\mathfrak{c})$, $\mathfrak{c}_2' = pr(\mathfrak{c}')$, $\lambda = \lambda_1\lambda_2$ and $\mu = \mu_1\mu_2$ where λ_1 and μ_1 are positive integers, moreover, $\lambda_2\mathcal{O}_k$ and $\mu_2\mathcal{O}_k$ are primitive integral ideals. Let us denote by $\mathfrak{d}$ the ideal $pr(\lambda_2\mathfrak{c}_2) = pr(\mu_2\mathfrak{c}_2')$. In particular we have

$$\frac{\eta^2\big(pr(\overline{\mathfrak{m}})\mathfrak{c}_2\big)}{\eta^2\big(pr(\overline{\mathfrak{m}})\mathfrak{c}_2'\big)} = \Big[\frac{\eta^2\big(\mathfrak{c}_2\big)}{\eta^2\big(\mathfrak{d}\big)}\frac{\eta^2\big(\mathfrak{d}\big)}{\eta^2\big(\mathfrak{c}_2'\big)}\Big]^{\tau}, \tag{3.7}$$

where $\tau = (pr(\overline{\mathfrak{m}}), H_s/k)$. Let $\mathfrak{a}$ be a primitive integral ideal of $\mathcal{O}_k$ prime to $6D_kN(\lambda_2\mathfrak{c}_2)$ and such that $\mathfrak{a}\mathfrak{c}_2 = \alpha\mathcal{O}_k$ for some $\alpha \in \mathcal{O}_k$. Then Proposition 10 (i) of [6] give

$$\Big[\frac{\eta^2\big(\mathfrak{c}_2\big)}{\eta^2\big(\mathfrak{d}\big)}\Big]^{(\mathfrak{a},H_s/k)} = \frac{\eta^2\big(\alpha\mathcal{O}_k\big)}{\eta^2\big((\lambda_2\alpha/t)\mathcal{O}_k\big)} = s(\lambda_2/t)t/\lambda_2, \tag{3.8}$$

where t is the positive integer defined by $\lambda_2\mathfrak{c}_2 = t\mathfrak{d}$. In the same manner we have

$$\Big[\frac{\eta^2\big(\mathfrak{c}_2'\big)}{\eta^2\big(\mathfrak{d}\big)}\Big]^{(\mathfrak{c}_2',H/k)^{-1}} = s(\mu_2/t')t'/\mu_2, \tag{3.9}$$

with $\mu_2\mathfrak{c}_2' = t'\mathfrak{d}$. Now to complete the proof we have just to use the definition 3.1 and the equations (3.6), (3.7), (3.8) and (3.9). □

THEOREM 3.1. — *Let $\mathfrak{m}, \mathfrak{c} \in T_0$ be such that $\mathfrak{m} \neq (1)$ and $\mathfrak{c}$ is prime to $6N(\mathfrak{m})$. Put $m = N(\mathfrak{m})$ and $\sigma_{\mathfrak{c}} = (\mathfrak{c}, k_{(m),s}/k)$ then*

$$\Gamma_{\mathfrak{m}}(\mathfrak{c}) \in k_{(m),s} \quad \textit{and} \quad \Gamma_{\mathfrak{m}}(\mathfrak{c}) = \Gamma_{\mathfrak{m}}(1)^{\sigma_{\mathfrak{c}}}.$$

If $\mathfrak{c} = \lambda\mathcal{O}_k$, with $\lambda \in \mathcal{U}_{(m)}$ then $\Gamma_{\mathfrak{m}}(1)^{\sigma_{\lambda\mathcal{O}_k}} = s(\lambda)^{-m}\Gamma_{\mathfrak{m}}(1)$.

Proof. — By the above proposition 3.1 and Chebotarev theorem we may assume without loss of generality that $\mathfrak{c}$ is a prime ideal $\mathfrak{p}$ of residual degree 1 in $k/\mathbb{Q}$ and such that $\mathfrak{p} \nmid 6D_kN(\mathfrak{m})$. Let us write $\mathfrak{m} = m_1 pr(\mathfrak{m})$, where m_1 is a positive integer. Let $u \in \mathbb{Z}$ be such that $u \equiv D_k$ modulo 4 and $u \equiv -\sqrt{D}_k$ modulo $pr(\mathfrak{m})\overline{\mathfrak{p}}$. Put $\alpha = (u + \sqrt{D}_k)/2$, $m_2 = N(pr(\mathfrak{m}))$ and $p = N(\mathfrak{p})$ then $(\alpha, 1)$ is a positive $\mathbb{Z}$-basis of $\mathcal{O}_k$. Moreover we have

$$pr(\mathfrak{m}) = \mathbb{Z}\alpha + \mathbb{Z}m_2, \quad \overline{\mathfrak{p}} = \mathbb{Z}\alpha + \mathbb{Z}p \quad \text{and} \quad pr(\mathfrak{m})\overline{\mathfrak{p}} = \mathbb{Z}\alpha + \mathbb{Z}pm_2.$$

On the other hand recall that

$$2\pi i\eta^2(pr(\overline{\mathfrak{m}})\mathfrak{p}) = e_{24}(pm_2(-u + 3\widetilde{\omega}))\eta^2(\frac{\alpha}{pm_2}),$$

where $e_n(z) = e^{\frac{2\pi i z}{n}}$ and $\widetilde{\omega} = \gcd(\frac{w_H}{2}, 2)$, *cf.* [6] definition 8. The Dedekind η^2 function is given in [12] formula (2.11). Thus we have the identity

$$\Gamma_{\mathfrak{m}}(\mathfrak{p}) = ig(p, m_1\alpha, e_{\mathfrak{m}\mathfrak{p}})e_{24}(pm_2(-u + 3\widetilde{\omega})). \tag{3.10}$$

Let us also remark that $\Gamma_{\mathfrak{m}}(1) = ig(1, m_1\alpha, e_{\mathfrak{m}})e_{24}(m_2(-u+3\widetilde{\omega}))$. Now we may use the Shimura reciprocity law as stated in [21] Theorem 3. Indeed, it is well known that $h_1 : z \mapsto ig(\frac{1}{e_{\mathfrak{m}}}, z, 1)$ is in $\mathcal{F}_{12e_{\mathfrak{m}}^2}$, that is the set of modular functions of level $12e_{\mathfrak{m}}^2$ with Fourier coefficients in $\mathbb{Q}(\zeta_{12e_{\mathfrak{m}}^2})$ at every cusp. Hence by Theorem 3 of loc.cit. we have

$$ig(1, m_1\alpha, e_{\mathfrak{m}}) = h_1(\theta) \in k_{12e_{\mathfrak{m}}^2} \text{ and } \big(ig(1, m_1\alpha, e_{\mathfrak{m}})\big)^{\widetilde{\sigma}_{\mathfrak{p}}} = [h_1 \circ (pB^{-1})](B\theta)$$

where $\theta = \alpha/m_2$, $\widetilde{\sigma}_{\mathfrak{p}} = (\mathfrak{p}, k_{12e_{\mathfrak{m}}^2}/k)$ and $B = \left(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right)$. The operation $f \circ A$ for $f \in \mathcal{F}_{12e_{\mathfrak{m}}^2}$ and A an integral matrix of determinant prime to $12e_{\mathfrak{m}}^2$ is defined in loc.cit. pages 210 and 211. Now we have $pB^{-1} = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right)\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. Moreover $h_1 \circ \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right) = h_2 : z \mapsto g(\frac{z}{e_{\mathfrak{m}}}, z, 1)$. On the other hand the function $z \mapsto g(u_1z + u_2, z, 1)$ has the q-expansion

$$g(u_1z+u_2, z, 1) = -q_z^{B_2(u_1)/2} e_1(u_2(u_1-1)/2)(1-q_u)\prod_{n=1}^{\infty}(1-q_z^n q_u)(1-q_z^n/q_u),$$

where $u = u_1z + u_2$, $u_1, u_2 \in \mathbb{R}$, $q_z = e_1(z)$, $q_u = e_1(u)$ and $B_2(x) = x^2 - x + 1/6$, *cf.* [13] page 29. In particular the q-expansion of h_2 has rational coefficients. Thus we have $h_2 \circ \left(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right) = h_2$ and $h_1 \circ (pB^{-1}) = h_1$. Now since $e_{24}(m_2(-u+3\widetilde{\omega})) \in k_{12e_{\mathfrak{m}}^2}$ our first conclusion is

$$\Gamma_{\mathfrak{m}}(1) \in k_{12e_{\mathfrak{m}}^2} \quad \text{and} \quad \Gamma_{\mathfrak{m}}(\mathfrak{c}) = \Gamma_{\mathfrak{m}}(1)^{\widetilde{\sigma}_{\mathfrak{c}}},$$

for all integral ideal $\mathfrak{c}$ prime to $6N(\mathfrak{m})$. In order to complete the proof of the theorem we must compute $\Gamma_{\mathfrak{m}}(1)^{\widetilde{\sigma}_{\lambda\mathcal{O}_k}}$ for $\lambda \in \mathcal{O}_k \cap \mathcal{U}_{(m)}$. Let $\mathfrak{c} = \lambda\mathcal{O}_k = c_1 pr(\mathfrak{c})$. The equation $(K2)$ and the transformation law $(K3)$ of the Klein function in [12] give

$$\begin{aligned}
\Gamma_{\mathfrak{m}}(1)^{\widetilde{\sigma}_{\lambda\mathcal{O}_k}} = \Gamma_{\mathfrak{m}}(\lambda\mathcal{O}_k) &= \frac{-2\pi\mathfrak{f}\big(\lambda\overline{\lambda}, \overline{\lambda}\mathfrak{m}\big)\eta^2\big(pr(\lambda\overline{\mathfrak{m}})\big)}{e_{\mathfrak{c}}e_{\mathfrak{m}}}s(c_1)\\
&= \frac{-2\pi c_1\mathfrak{f}\big(\lambda, \mathfrak{m}\big)\eta^2\big(pr(\lambda\overline{\mathfrak{m}})\big)}{\lambda e_{\mathfrak{m}}}s(c_1)\\
&= \frac{-2\pi c_1\mathfrak{f}\big(1, \mathfrak{m}\big)\eta^2\big(pr(\lambda\overline{\mathfrak{m}})\big)}{\lambda e_{\mathfrak{m}}}s(c_1)\\
&= \frac{c_1}{\lambda}\frac{\eta^2(pr(\lambda\overline{\mathfrak{m}}))}{\eta^2(pr(\overline{\mathfrak{m}}))}\Gamma_{\mathfrak{m}}(1)s(c_1)\\
&= \left(\frac{1}{\mu}\frac{\eta^2(\mu\mathcal{O}_k)}{\eta^2(\mathcal{O}_k)}\right)^{\tau} s(c_1)\Gamma_{\mathfrak{m}}(1) = s(\lambda)^{-m}\Gamma_{\mathfrak{m}}(1)
\end{aligned}$$

where $\tau = (pr(\overline{\mathfrak{m}}), H_s/k)$ and $\mu = \lambda/c_1$. The sixth equality is an application of Proposition 10 (i) in [6]. The proof of the theorem is now complete. □

Remark 3.1. — Here we draw the attention of the reader that the computation of the congugates of $g(1, \omega_1, \omega_2)$, where (ω_1, ω_2) is any $\mathbb{Z}$-basis of the ideal $\mathfrak{m}$ with $\Im(\omega_1/\omega_2) > 0$, is already made by several authors even in the case $\mathfrak{m}$ is not prime to 6. See for instance Satz(1.1) in [19] or the Corollaire in page 228 of[18]. Perhaps the most recent such computations are those made by A. Hayward in his thesis, *cf.* [9] Proposition 5.2 and Corollary 5.3. What is really new here is the fact that we succeded in defining invariants by using the algebraic numbers $g(1,\omega_1,\omega_2)$ which depend on the $\mathbb{Z}$-basis of $\mathfrak{m}$.

Let us put

$$\widehat{\Gamma}_{\mathfrak{m}}(\mathfrak{c}) = \big(\Gamma_{\mathfrak{m}}(\mathfrak{c})\big)^{e_{\mathfrak{m}}}, \tag{3.11}$$

where $\mathfrak{m}$ and $\mathfrak{c}$ are as in the above theorem 3.1.

One may prove that

$$\widehat{\Gamma}_{\mathfrak{m}}(1) \in k_{\mathfrak{m},s} \quad \text{and} \quad \widehat{\Gamma}_{\mathfrak{m}}(1)^{\sigma_{\lambda\mathcal{O}_k}} = s(\lambda)^{-m_1}\widehat{\Gamma}_{\mathfrak{m}}(1), \tag{3.12}$$

for all $\lambda \in \mathcal{O}_k \cap \mathcal{U}_{\mathfrak{m}}$ prime to $N(\mathfrak{m})$. The proof is similar to that of theorem 3.1. One has just to remark the following. If $\lambda \in \mathcal{O}_k \cap \mathcal{U}_{\mathfrak{m}}$ then $\mathfrak{f}(\lambda,\mathfrak{m})^{e_{\mathfrak{m}}} = \mathfrak{f}(1,\mathfrak{m})^{e_{\mathfrak{m}}}$. In particular we have

$$k_{(N(\mathfrak{m})),s} = k_{(N(\mathfrak{m}))}\big(\Gamma_{\mathfrak{m}}(1)\big) \quad \text{and} \quad k_{\mathfrak{m},s} = k_{\mathfrak{m}}\big(\widehat{\Gamma}_{\mathfrak{m}}(1)\big). \tag{3.13}$$

Let us remark that (3.12) is the analog of formula (16.4) of [8].

THEOREM 3.2. — *Let $\mathfrak{n}$ and $\mathfrak{q}$ be ideals in T_0 such that $\mathfrak{q}$ is a prime ideal. Let us put $\mathfrak{m} = \mathfrak{n}\mathfrak{q}$, $E = k_{\mathfrak{m},s}$ and $F = k_{\mathfrak{n},s}$. Then we have*

$$N_{E/F}\big(\widehat{\Gamma}_{\mathfrak{m}}(1)\big) = \begin{cases} s(e_{\mathfrak{q}})\big(\widehat{\Gamma}_{\mathfrak{n}}(1)\big)^{\frac{e_{\mathfrak{m}}}{e_{\mathfrak{n}}}} & \text{if } \mathfrak{q}|\mathfrak{n}, \\ \big(\widehat{\Gamma}_{\mathfrak{n}}(1)\big)^{(1-(\mathfrak{q},F/k)^{-1})\frac{e_{\mathfrak{m}}}{e_{\mathfrak{n}}}} & \text{if } \mathfrak{q}\nmid\mathfrak{n} \text{ and } \mathfrak{n}\neq(1) \\ s(e_{\mathfrak{q}})\big(e_{\mathfrak{q}}\frac{\eta^2(\mathcal{O}_k)}{\eta^2(pr(\overline{\mathfrak{q}}))}\big)^{e_{\mathfrak{q}}} & \text{if } \mathfrak{n}=(1) \end{cases} \tag{3.14}$$

Proof. — Let us choose $\alpha = (u+\sqrt{D_k})/2 \in \mathcal{O}_k$, as we did in the proof of theorem 3.1, such that

$$\mathcal{O}_k = \mathbb{Z}\alpha + \mathbb{Z}, \quad pr(\mathfrak{n}) = \mathbb{Z}\alpha + \mathbb{Z}n_2 \quad \text{and} \quad pr(\mathfrak{m}) = \mathbb{Z}\alpha + \mathbb{Z}m_2.$$

Also let us fix X a set of elements $\lambda \in \mathcal{O}_k$ prime to $6N(\mathfrak{m})$ such that $\lambda \equiv 1$ modulo $12\mathfrak{n}$, and $\lambda\mathcal{O}_k$ is primitive. Moreover the map $\lambda \mapsto (\lambda\mathcal{O}_k, E/k)$ is a bijection from X to $\mathrm{Gal}(E/F)$. If $\lambda \in X$ then we have

$$\begin{aligned} \Gamma_{\mathfrak{m}}(1)^{\sigma_{\lambda\mathcal{O}_k}} = \Gamma_{\mathfrak{m}}(\lambda\mathcal{O}_k) &= \frac{-2\pi\mathfrak{f}\big(\lambda,\mathfrak{m}\big)\eta^2\big(\lambda pr(\overline{\mathfrak{m}})\big)}{\lambda e_{\mathfrak{m}}} \\ &= \frac{1}{\lambda}\Big(\frac{\eta^2(\lambda\mathcal{O}_k)}{\eta^2(\mathcal{O}_k)}\Big)^{\tau}\frac{-2\pi\mathfrak{f}\big(\lambda,\mathfrak{m}\big)\eta^2\big(pr(\overline{\mathfrak{m}})\big)}{e_{\mathfrak{m}}} \\ &= s(\lambda)^{-m}\, ig(\lambda, m_1\alpha, e_{\mathfrak{m}})e_{24}(m_2(-u+3\widetilde{\omega})) \\ &= ig(\lambda, m_1\alpha, e_{\mathfrak{m}})e_{24}(m_2(-u+3\widetilde{\omega})) \end{aligned}$$

where $\tau = (pr(\overline{\mathfrak{m}}), H/k)$. Now, for each $\lambda \in X$ there is unique non negative integers a_λ and b_λ satisfying $0 \leqslant a_\lambda < m_1/n_1$, $0 \leqslant b_\lambda < e_{\mathfrak{m}}/e_{\mathfrak{n}}$ and such that $\lambda - (1 + a_\lambda n_1\alpha + b_\lambda e_{\mathfrak{n}}) \in \mathfrak{m}$. Let us put $\nu(a_\lambda, b_\lambda) = -1$ if $2|a_\lambda$ and $2 \nmid b_\lambda$, and $\nu(a_\lambda, b_\lambda) = 1$ otherwise. Then we have $\mathfrak{f}(\lambda, \mathfrak{m})^{e_{\mathfrak{m}}} = \mathfrak{f}(1 + a_\lambda n_1\alpha + b_\lambda e_{\mathfrak{n}}, \mathfrak{m})^{e_{\mathfrak{m}}}\nu(a_\lambda, b_\lambda)$ by the transformation law (K_3) in [12]. This implies

$$(3.15)\quad N_{E/F}(\widehat{\Gamma}_{\mathfrak{m}}(1)) = (i)^{e_{\mathfrak{m}}[E:F]}\Upsilon\Theta \prod_{\lambda\in X} g(1 + a_\lambda n_1\alpha + b_\lambda e_{\mathfrak{n}}, m_1\alpha, e_{\mathfrak{m}})^{e_{\mathfrak{m}}},$$

where $\Upsilon = \prod_{\lambda\in X} \nu(a_\lambda, b_\lambda)$ and $\Theta = \big(e_{24}(m_2(-u + 3\widetilde{\omega}))\big)^{e_{\mathfrak{m}}[E:F]}$. We have

$$\Theta = \begin{cases} \big(e_{24}(n_2(-u + 3\widetilde{\omega}))\big)^{e_{\mathfrak{m}}} & \text{if } \mathfrak{q}|\mathfrak{n}, \\ \big(e_{24}(n_2(-u + 3\widetilde{\omega}))\big)^{e_{\mathfrak{m}}(1-(\mathfrak{q},F/k)^{-1})} & \text{if } \mathfrak{q}\nmid\mathfrak{n}. \end{cases}$$

Moreover, if $\mathfrak{q}|\mathfrak{n}$ then the map $\lambda \mapsto (a_\lambda, b_\lambda)$ is a bijection from X to the set Σ of couples (x, y) with $x \in \{0, \dots, \frac{m_1}{n_1} - 1\}$ and $y \in \{0, \dots, \frac{e_{\mathfrak{m}}}{e_{\mathfrak{n}}} - 1\}$. In particular we have

$$\prod_{\lambda\in X} g(1 + a_\lambda n_1\alpha + b_\lambda e_{\mathfrak{n}}, m_1\alpha, e_{\mathfrak{m}}) = \varepsilon(1)g(1, n_1\alpha, e_{\mathfrak{n}})$$

thanks to Theorem 2.2 (b) of [12], where $\varepsilon(1)$ is the root of unity defined in Theorem 2.2 (a) of loc. cit. We have

$$(3.16)\qquad (\varepsilon(1))^{e_{\mathfrak{m}}} = e_8\Big(e_{\mathfrak{m}}\big(3N(\mathfrak{q}) + \frac{m_1}{n_1} - \frac{e_{\mathfrak{m}}}{e_{\mathfrak{n}}} - 3\big)\Big)e_4\Big(\frac{e_{\mathfrak{m}}}{e_{\mathfrak{n}}} - N(\mathfrak{q})\Big).$$

On the other hand the integers $U = \frac{m_1}{n_1}$ and $V = \frac{e_{\mathfrak{m}}}{e_{\mathfrak{n}}}$ are such that

$$(3.17)\qquad \Upsilon = \prod_{(x,y)\in\Sigma} \nu(x, y) = (-1)^{\left(\frac{U+1}{2}\right)\left(\frac{V-1}{2}\right)}$$
$$= e_8\Big(e_{\mathfrak{m}}\big(N(\mathfrak{q}) - \frac{m_1}{n_1} + \frac{e_{\mathfrak{m}}}{e_{\mathfrak{n}}} - 1\big)\Big).$$

This completes the proof of the theorem in case $\mathfrak{q}|\mathfrak{n}$. If $\mathfrak{q} \nmid \mathfrak{n}$ there exists a unique $(x_0, y_0) \in \Sigma$ such that $\mu := 1 + x_0n_1\alpha + y_0e_{\mathfrak{n}} \in \mathfrak{q}$. The map $\lambda \mapsto (a_\lambda, b_\lambda)$ is a bijection from X to $\Sigma - \{((x_0, y_0)\}$. Let us put

$$A = \Upsilon \prod_{\lambda\in X} g(1 + a_\lambda n_1\alpha + b_\lambda e_{\mathfrak{n}}, m_1\alpha, e_{\mathfrak{m}}).$$

Then we have

$$A = \begin{cases} \Upsilon\nu(x_0, y_0)\varepsilon(1)\frac{g(1,n_1\alpha,e_{\mathfrak{n}})}{\nu(x_0,y_0)g(\mu,m_1\alpha,e_{\mathfrak{m}})} & \text{if } \mathfrak{n} \neq (1) \\ \Upsilon\varepsilon(1) \lim_{z\to1} \frac{\mathfrak{f}(z,\mathcal{O}_k)}{\mathfrak{f}(z+\mu-1,\mathfrak{q})} \frac{\eta^2(\alpha)}{\eta^2(\frac{m_1\alpha}{e_{\mathfrak{m}}})} \frac{1}{e_{\mathfrak{m}}} & \text{if } \mathfrak{n} = (1) \end{cases}$$

by Theorem2.2 (b) of [12]. Now suppose $\mathfrak{n} \neq (1)$ and fix an ideal $\mathfrak{c}$ of $\mathcal{O}_k$ prime to $6N(\mathfrak{n})$ such that $\mathfrak{q}\mathfrak{c}$ is principal generated by $\lambda \in \mathcal{O}_k$ satisfying $\lambda \equiv 1$ modulo $12\mathfrak{n}$. Moreover, if $\overline{\mathfrak{q}} \nmid \mathfrak{n}$ we suppose that $\lambda \in e_{\mathfrak{q}}\mathcal{O}_k$ and $\lambda/e_{\mathfrak{q}}$ is prime to $6N(\mathfrak{q})$. Let us also choose our α such that $u \equiv -\sqrt{D_k}$ modulo $pr(\mathfrak{n}\overline{\mathfrak{c}})$ and put $\rho = (\mathfrak{q}, F/k)$. The two identities

$$\mathfrak{f}(\mu, \mathfrak{m})^{e_\mathfrak{m}} = \mathfrak{f}(\lambda, \mathfrak{m})^{e_\mathfrak{m}}\nu(x_0, y_0) \quad \text{and} \quad \overline{\lambda}\mathfrak{f}(\lambda, \mathfrak{m}) = N(\mathfrak{q})\mathfrak{f}(N(\mathfrak{c}), \mathfrak{n}\overline{\mathfrak{c}}),$$

which are easy to check then imply

$$\Big(ig(\mu, m_1\alpha, e_\mathfrak{m})\nu(x_0, y_0)\big(e_{24}(n_2(-u + 3\widetilde{\omega}))\big)^{\rho^{-1}}\Big)^{e_\mathfrak{m}} = \widehat{\Gamma}_\mathfrak{n}(\mathfrak{c})^{\frac{e_\mathfrak{m}}{e_\mathfrak{n}}} Q^{e_\mathfrak{m}}$$

where

$$Q = s(c_1)\frac{N(\mathfrak{q})e_{\mathfrak{n}\mathfrak{c}}}{\overline{\lambda}e_\mathfrak{m}}\frac{\eta^2(pr(\overline{\mathfrak{m}}))}{\eta^2(pr(\overline{\mathfrak{n}}\,\mathfrak{c}))} = \begin{cases} s(e_\mathfrak{q}) & \text{if } \ \overline{\mathfrak{q}} \nmid \mathfrak{n} \\ 1 & \text{if } \ \overline{\mathfrak{q}} | \mathfrak{n}. \end{cases}$$

Since $\Upsilon\nu(x_0, y_0)$ is equal to the term on the right of (3.17) the theorem is proved in case $\mathfrak{n} \neq (1)$ and $\mathfrak{q} \nmid \mathfrak{n}$. If $\mathfrak{n} = (1)$ then we deduce from (K_3) and Proposition 2.5 of [12] the limit $\lim_{z\to 1} \frac{\mathfrak{f}(z, \mathcal{O}_k)}{\mathfrak{f}(z+\mu-1, \mathfrak{q})} = \nu(x_0, y_0)$, and this concludes the proof of the theorem in all cases. □

Let us associate to a prime ideal $\mathfrak{q}$ in T_0 the number $\chi(\mathfrak{q}) = s(e_\mathfrak{q})$. Then extend χ by multiplicativity to all ideals $\mathfrak{m} \in T_0$ and set

$$\ddot{\Gamma}_\mathfrak{m}(\mathfrak{c}) = \chi(\mathfrak{m})\widehat{\Gamma}_\mathfrak{m}(\mathfrak{c}). \tag{3.18}$$

Then we have

Corollary 3.1. — *Let $\mathfrak{n}$ and $\mathfrak{q}$ be ideals in T_0 such that $\mathfrak{q}$ is a prime ideal. Let us put $\mathfrak{m} = \mathfrak{n}\mathfrak{q}$, $E = k_{\mathfrak{m},s}$ and $F = k_{\mathfrak{n},s}$. Then we have*

$$N_{E/F}\big(\ddot{\Gamma}_\mathfrak{m}(1)\big) = \begin{cases} \big(\ddot{\Gamma}_\mathfrak{n}(1)\big)^{\frac{e_\mathfrak{m}}{e_\mathfrak{n}}} & \text{if } \ \mathfrak{q}|\mathfrak{n}, \\ \big(\ddot{\Gamma}_\mathfrak{n}(1)\big)^{(1-(\mathfrak{q},F/k)^{-1})\frac{e_\mathfrak{m}}{e_\mathfrak{n}}} & \text{if } \ \mathfrak{q} \nmid \mathfrak{n} \text{ and } \mathfrak{n} \neq (1) \\ s(e_\mathfrak{q})\Big(e_\mathfrak{q}\frac{\eta^2(\mathcal{O}_k)}{\eta^2(pr(\overline{\mathfrak{q}}))}\Big)^{e_\mathfrak{q}} & \text{if } \ \mathfrak{n} = (1). \end{cases}$$

4. The universal ordinary s-distribution U_s.

Let s be a sign function. Then the map $\xi x \mapsto \xi s(x)$ for $\xi \in \mu_H$ and $x \in \Lambda_6$, is a well defined homomorphism from $\mu_H\Lambda_6$ to the group μ_H. It coincides with s (resp. the identity map) when restricted to Λ_6 (resp. μ_H). This homomorphism will also be noted s. In this § we define both the universal ordinary s-distribution U_s and the level $\mathfrak{m}$ universal ordinary s-distribution denoted by $U_s(\mathfrak{m})$, where $\mathfrak{m} \in T_0$. We give some of their properties. The first definition we propose below for U_s and $U_s(\mathfrak{m})$ follows the construction of G. Anderson in [1], section 3. Recall that the main goal of [1] is to study the Galois-module structure of the sign-cohomology of the modules appeared in the course of Yin's calculation of the unit index, *cf.* [22]and [23]. The approach of G. Anderson is based on the use of Farrell-Tate theory which extends the theory of Tate cohomology of finite groups to groups of finite virtual cohomological dimension. He also introduces a certain double complex to compute this cohomology. The Anderson's double complex turns out to be a powerful means for other cohomological computations like those made in [15].

Of course we have to make some adaptation of the definitions used in [1] subsections 3.2 and 3.3. But we will keep almost all the notations of Anderson.

By a lattice we mean a set $W = \xi\mathfrak{c}$, where $\xi \in \mu_H$ and $\mathfrak{c} \in \overline{T}_0$. Two lattices W_1 and W_2 are homothetic if $W_2 = \lambda W_1$, with $\lambda \in \mu_H\Lambda_6$ and is positive, *i.e.* such that $s(\lambda) = 1$. A lattice translate is a set $x + W$, where $x \in \mu_H\Lambda_6$ and W is a lattice. Two lattice translate x_1+W_1 and x_2+W_2 are homothetic if there exists a positive $\alpha \in \mu_H\Lambda_6$ such that $W_2 = \alpha W_1$ and $\alpha x_1 - x_2 \in W_2$. We say that $x + W$ is torsion if the ideal $\mathfrak{m} = x^{-1}W \cap \mathcal{O}_k$ is nonzero. In this case $x + W$ is said to be torsion of order $\mathfrak{m}$. Note that $\mathfrak{m}$ is necessarily prime to 6. We let Ξ be the set of homothety classes of torsion lattice translates. We denote the homothety class of a torsion lattice translate $x + W$ by [x+W].

For each ideal $\mathfrak{m} \in T_0$ we put $G_{\mathfrak{m},s} = \mathrm{Gal}(k_{\mathfrak{m},s}/k)$ and $J_{\mathfrak{m},s} = \mathrm{Gal}(k_{\mathfrak{m},s}/k_{\mathfrak{m}})$. Let us consider the inverse limits

$$G_s = \varprojlim G_{\mathfrak{m},s} \quad \text{and} \quad G_\infty = \varprojlim D_{\mathfrak{m},s}.$$

Also in our case we have a left action of G_s on Ξ and an isomorphism $G_\infty \to \mu_H$, *cf.* [1] Proposition 3.4.3 and Proposition 3.5.3. Let $\mathcal{A}_s$ be the free abelian group generated by Ξ. We may view $\mathcal{A}_s$ as a $\mathbb{Z}[G_s]$-module by extending linearly the action of G_s on Ξ. By definition U_s is the quotient of $\mathcal{A}_s$ by the $\mathbb{Z}$-module generated by the sums

$$\xi - \sum_{\eta \in Y_{\mathfrak{m}}^{-1}(\xi)} \eta$$

where $\mathfrak{m} \in T_0$, $\xi \in \Xi$ and $Y_{\mathfrak{m}} : \Xi \to \Xi$ is the G_s-equivariant map defined in [1] Proposition 3.3.1. Let us recall that $Y_{\mathfrak{m}}([x + W]) = [x + \mathfrak{m}^{-1}W]$. In fact U_s is also a $\mathbb{Z}[G_s]$-module. In the same manner if $\mathfrak{m} \in T_0$ we define

$\mathcal{A}_s(\mathfrak{m})$ to be the free abelian group generated by $\Xi(\mathfrak{m})$, which is the set of homothety classes of torsion lattice translates of order dividing $\mathfrak{m}$. Its quotient by the $\mathbb{Z}$-module generated by the sums

$$\xi - \sum_{\eta \in Y_{\mathfrak{n}}^{-1}(\xi)} \eta,$$

where $\mathfrak{n}$ divides $\mathfrak{m}$ and $\xi \in \Xi(\mathfrak{m}/\mathfrak{n})$, is denoted $U_s(\mathfrak{m})$. The reader may also check that $U_s(\mathfrak{m})$ is a $\mathbb{Z}[G_s]$-module and even a $\mathbb{Z}[G_{\mathfrak{m},s}]$-module.

There is an other way to define U_s and $U_s(\mathfrak{m})$ (here we take our inspiration from both [4] and [16]). Indeed, let Ω_s (resp. $\Omega_s(\mathfrak{m})$) be the free abelian group generated by the disjoint union $\coprod_{\mathfrak{f} \in T_0} G_{\mathfrak{f},s}$ (resp. $\coprod_{\mathfrak{f}|\mathfrak{m}} G_{\mathfrak{f},s}$). Let U'_s be the quotient of Ω_s by its $\mathbb{Z}$-submodule generated by

$$S(\mathfrak{f},\mathfrak{g},\sigma) = \sigma - \sum_{\substack{\mathfrak{n}|\mathfrak{g} \\ (\mathfrak{n},\mathfrak{f})=1}} \sum_{\tau \in Z_{\mathfrak{g},\mathfrak{n}}(\sigma)} \tau$$

for all $\mathfrak{f},\mathfrak{g} \in T_0$ and all $\sigma \in G_{\mathfrak{f},s}$. Here $Z_{\mathfrak{g},\mathfrak{n}}(\sigma)$ is the set of the automorphisms $\tau \in G_{\mathfrak{f}\mathfrak{g}/\mathfrak{n},s}$ such that τ coincides with $\sigma(\mathfrak{n}, k_{\mathfrak{f},s}/k)^{-1}$ on $k_{\mathfrak{f},s}$. Then U'_s is G_s-isomorphic to U_s. Let $U'_s(\mathfrak{m})$ be the quotient of $\Omega_s(\mathfrak{m})$ by its $\mathbb{Z}$-submodule generated by $S(\mathfrak{f},\mathfrak{g},\sigma)$, for all $\mathfrak{f},\mathfrak{g} \in T_0$ such that $\mathfrak{f}\mathfrak{g}|\mathfrak{m}$ and all $\sigma \in G_{\mathfrak{f},s}$. Then $U'_s(\mathfrak{m})$ is $G_{\mathfrak{m},s}$-isomorphic to $U_s(\mathfrak{m})$.

Let $\Sigma_{\mathfrak{m}}$ be the set of the ideals $\mathfrak{n}$ dividing $\mathfrak{m}$ such that the $\gcd(\mathfrak{n},\mathfrak{m}/\mathfrak{n}) = 1$. Let $\Omega_s^0(\mathfrak{m})$ be the free abelian group generated by $\Upsilon_{\mathfrak{m}} = \coprod_{\mathfrak{f} \in \Sigma_{\mathfrak{m}}} G_{\mathfrak{f},s}$. Let $D^0_{\mathfrak{m}}$ be the $G_{\mathfrak{m},s}$-submodule of $\Omega_s^0(\mathfrak{m})$ generated by

$$S^0(\mathfrak{f},\mathfrak{p}^e,\sigma) = \sigma - \sigma(\mathfrak{p}, k_{\mathfrak{f},s}/k)^{-1} - \sum_{\tau \in Z_{\mathfrak{p}^e,(1)}(\sigma)} \tau$$

for all $\mathfrak{f},\mathfrak{p}^e \in \Sigma_{\mathfrak{m}}$ such that $\mathfrak{p}$ is a prime ideal not dividing $\mathfrak{f}$. It is easy to prove that $\Omega_s^0(\mathfrak{m})/D^0_{\mathfrak{m}}$ and $U'_s(\mathfrak{m})$ are isomorphic as $G_{\mathfrak{m},s}$-modules. Let us put $U_s^0(\mathfrak{m}) = \Omega_s^0(\mathfrak{m})/D^0_{\mathfrak{m}}$. Our next step now is to prove that $U_s^0(\mathfrak{m})$ is $\mathbb{Z}$-free and give an explicit basis. But let us first fix some notation. If $\mathfrak{p}$ is a prime ideal dividing $\mathfrak{m}$ then we denote by $\widehat{\mathfrak{p}}$ the $\mathfrak{p}$-power satisfying $\widehat{\mathfrak{p}} \in \Sigma_{\mathfrak{m}}$. We associate to $\mathfrak{p}$ a $G_{\mathfrak{m},s}$-operator

$$X_{\mathfrak{p}} : \Omega_s^0(\mathfrak{m}/\widehat{\mathfrak{p}}) \longrightarrow \Omega_s^0(\mathfrak{m})$$

such that $X_{\mathfrak{p}}(\sigma) = S^0(\mathfrak{f},\widehat{\mathfrak{p}},\sigma)$ for all $\mathfrak{f} \in \Sigma_{\mathfrak{m}}$ and all $\sigma \in G_{\mathfrak{f},s}$. Let us put $X_{(1)} = 1$ and $X_{\mathfrak{g}} = \prod_{\mathfrak{p}|\mathfrak{g}} X_{\mathfrak{p}}$ for $\mathfrak{g} \in \Sigma_{\mathfrak{m}}$. This is a $G_{\mathfrak{m},s}$-operator from $\Omega_s^0(\mathfrak{m}/\mathfrak{g})$ to $\Omega_s^0(\mathfrak{m})$.

For a ideal $\mathfrak{g} \in T_0$ we fix $S(\mathfrak{g}) \subset G_{\mathfrak{g},s}$ such that $1 \in S(\mathfrak{g})$ and the restriction map $S(\mathfrak{g}) \to G_{(1),s}$ is a bijection. Let $\sigma \in G_{\mathfrak{g},s}$. Then there exists a unique $\tau \in S(\mathfrak{g})$ such that $\sigma\tau^{-1}$ is the identity on $H_s = k_{(1),s}$. Moreover $\sigma\tau^{-1}$ can be uniquely written as

$$\sigma\tau^{-1} = \prod_{\mathfrak{p}|\mathfrak{g}} \sigma_{\mathfrak{p}}$$

where for each prime ideal $\mathfrak{p}$ dividing $\mathfrak{g}$ the automorphism $\sigma_{\mathfrak{p}} \in G_{\mathfrak{g},s}$ and is the identity on $k_{\mathfrak{g}/\mathfrak{p}^t,s}$, with $\mathfrak{p}^t$ being the exact power of $\mathfrak{p}$ dividing $\mathfrak{g}$. Indeed, $\mathrm{Gal}(k_{\mathfrak{g},s}/H_s)$ is equal to the direct product

$$\mathrm{Gal}(k_{\mathfrak{g},s}/H_s) = \prod_{\mathfrak{p}|\mathfrak{g}} \mathrm{Gal}(k_{\mathfrak{g},s}/k_{\mathfrak{g}/\mathfrak{p}^t,s}).$$

We say that $\sigma \in B_n$ if there exist exactly n prime ideals $\mathfrak{p}$ such that $\sigma_{\mathfrak{p}} = 1$. By definition $G_{(1),s} \subset B_0$. Thus we have

$$\coprod_{\mathfrak{f}\in T_0} G_{\mathfrak{f},s} = \coprod_{n\geqslant 0} B_n.$$

THEOREM 4.1. — *Let $\mathfrak{m} \in T_0$. Then the set*

$$\mathfrak{X}_{\mathfrak{m}} = \left\{X_{\mathfrak{g}}(\sigma) \text{ such that } \mathfrak{g} \in \Sigma_{\mathfrak{m}} \text{ and } \sigma \in B_0 \cap \Upsilon_{\mathfrak{m}/\mathfrak{g}}\right\}$$

is a $\mathbb{Z}$-basis of $\Omega_s^0(\mathfrak{m})$. Moreover $U_s^0(\mathfrak{m})$ is $\mathbb{Z}$-free of rank $\#G_{\mathfrak{m},s}$ with basis the set $B_0 \cap \Upsilon_{\mathfrak{m}}$.

Proof. — This result is nothing both Theorem 4.1.1. assertion 1 and Theorem 4.3.1 assertion 3 of [1]. See also Proposition 3.1 (i) and (ii) of [16]. For the convenience of the reader we repeat the proof here. Let $\sigma \in G_{\mathfrak{g},s}$ for a certain $\mathfrak{g} \in \Sigma_{\mathfrak{m}}$. Suppose that $\sigma \in B_n$, with $n > 0$. Let σ' be the restriction of σ to $k_{\mathfrak{f},s}$, where $\mathfrak{f} = \mathfrak{g}/\widehat{\mathfrak{p}}$. Then, by considering $X_{\mathfrak{p}}(\sigma')$ we see that

$$\sigma \in X_{\mathfrak{p}}\big(\Omega_s^0(\mathfrak{f})\big) + \Omega_s^0(\mathfrak{f}) + B_{n-1}(\mathfrak{g}).$$

Here $B_{n-1}(\mathfrak{g})$ is the $\mathbb{Z}$-module generated by the automorphisms $\tau \in G_{\mathfrak{g},s} \cap B_{n-1}$. This allows us to conclude by induction that $\mathfrak{X}_{\mathfrak{m}}$ generates $\Omega_s^0(\mathfrak{m})$. On the other hand we have

$$\#(B_0 \cap G_{\mathfrak{g},s}) = \#G_{(1),s} \prod_{\mathfrak{p}|\mathfrak{g}} \Big(\frac{\#G_{\mathfrak{g},s}}{\#G_{\mathfrak{g}/\widehat{\mathfrak{p}},s}} - 1\Big) = \sum_{\mathfrak{f}\in\Sigma_{\mathfrak{g}}} (-1)^{\deg(\mathfrak{g}/\mathfrak{f})} \#G_{\mathfrak{f},s}.$$

In particular $\#(B_0 \cap \Upsilon_{\mathfrak{g}}) = \#G_{\mathfrak{g}}$ and $\#\mathfrak{X}_{\mathfrak{m}} = \#\Upsilon_{\mathfrak{m}}$. Thus $\mathfrak{X}_{\mathfrak{m}}$ is indeed a $\mathbb{Z}$-basis of $\Omega_s^0(\mathfrak{m})$. Now the second claim of the theorem is a straightforward consequence of the first one. □

Our goal now is to compute the Tate cohomology groups $\widehat{H}^i(G_\infty, U_s^0(\mathfrak{m}))$ for $\mathfrak{m} \neq (1)$. Let us point out that these groups are useful in many purposes. See for instance the index calculation of Sinnott in [20]. They are naturally equipped with a $G_{\mathfrak{m},s}$ action. Let us identify G_∞ with the group $J = \mathrm{gal}(k_{\mathfrak{m},s}/k_{\mathfrak{m}})$. As mentioned in the introduction we will use Anderson's method introduced in [1] and improved by Ouyang in [16] and also in [15]. The first step is to define the Anderson's resolution of $U_s^0(\mathfrak{m})$. Let $\mathrm{supp}(\mathfrak{m})$ be the set of prime ideals dividing $\mathfrak{m}$ and let $\prec$ be a total order of $\mathrm{supp}(\mathfrak{m})$. If $\mathfrak{g} \in \Sigma_{\mathfrak{m}}$ and $\mathfrak{p} \in \mathrm{supp}(\mathfrak{g})$ then we let $\omega(\mathfrak{p}, \mathfrak{g}) = \#\{\mathfrak{q} \in \mathrm{supp}(\mathfrak{g}) \text{ such that } \mathfrak{q} \prec \mathfrak{p} \text{ and } \mathfrak{q} \neq \mathfrak{p}\}$. Now consider the free abelian group $L_{\mathfrak{m}}$ generated by the symbols

$$[\sigma, \mathfrak{g}], \quad \sigma \in \Upsilon_{\mathfrak{m}/\mathfrak{g}} \quad \text{and} \quad \mathfrak{g} \in \Sigma_{\mathfrak{m}}.$$

If $x = \sum n_\sigma(\sigma) \in \Omega_s^0(\mathfrak{m}/\mathfrak{g})$ then some times we will write $[x, \mathfrak{g}]$ instead of $\sum n_\sigma[\sigma, \mathfrak{g}]$. It is easy to see that $L_{\mathfrak{m}}$ is a graded $G_{\mathfrak{m},s}$-module with the definition of degree given by $\deg[x, \mathfrak{g}] = -\deg(\mathfrak{g})$. We define a differential d on $L_{\mathfrak{m}}$ as follows

$$d([\sigma, \mathfrak{g}]) = \begin{cases} 0 & \text{if } \mathfrak{g} = (1) \\ \sum_{\mathfrak{p}|\mathfrak{g}} (-1)^{\omega(\mathfrak{p},\mathfrak{g})} [S^0(\mathfrak{f}, \widehat{\mathfrak{p}}, \sigma), \mathfrak{g}/\widehat{\mathfrak{p}}] & \text{if } \mathfrak{g} \neq (1) \text{ and } \sigma \in G_{\mathfrak{f},s}. \end{cases}$$

It is easy to check that $d^2 = 0$. Moreover the 0-cohomology group $H^0(L_{\mathfrak{m}}, d) \simeq U_s^0(\mathfrak{m})$. The isomorphism is induced by $[\sigma, (1)] \mapsto \sigma$.

Proposition 4.1. — *The complex $(L_{\mathfrak{m}}, d)$ is acyclic in nonzero degree.*

Proof. — We refer the reader to the appendix by Greg W. Anderson in [15]. □

As in [16] Anderson's resolution $(L_{\mathfrak{m}}, d)$ can be used to compute the cohomology groups $\widehat{H}^i(J, U_s^0(\mathfrak{m}))$ for $\mathfrak{m} \neq (1)$. Indeed let

$$(P, \partial): \quad \cdots \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$$

be a complete resolution for J, *cf.* [5] page 132, and consider the double complex C and its total cochain complex Tot C defined by

$$C_{p,q} = \mathrm{Hom}_J(P_q, L_{\mathfrak{m}}^p) \quad \text{and} \quad (\mathrm{Tot}\ C)_n = \bigoplus_{u+v=n} C_{u,v}$$

Here we consider the differential D on Tot C whose restriction to $C_{p,q}$ is the sum $d + (-1)^p \partial$. Also recall the classical filtrations of Tot C

$$'F^p(\mathrm{Tot}\ C)_n = \bigoplus_{\substack{u+v=n \\ u \geqslant p}} C_{u,v} \quad \text{and} \quad ''F^p(\mathrm{Tot}\ C)_n = \bigoplus_{\substack{u+v=n \\ v \geqslant p}} C_{u,v}.$$

Since we have $C_{u,v} = 0$ for $u \gg 0$ or $u \ll 0$ each one of these two filtrations is finite, *cf.* [10] page 267, and hence gives rise to a spectral sequence which converges finitely to the cohomology of Tot C, see Theorem 3.5. of loc. cit. The E_1-term and the E_2-term of the *first filtration* are

$$'E_1^{p,q} = \widehat{H}^q(J, L_{\mathfrak{m}}^p) \quad \text{and} \quad 'E_2^{p,q} = H_d^p\big(\widehat{H}^q(J, L_{\mathfrak{m}}^\bullet)\big).$$

Moreover the E_1-term of the *second filtration* is $''E_1^{p,q} = H_d^p\big(\mathrm{Hom}_J(P_q, L_{\mathfrak{m}}^\bullet)\big)$. But since P_q is projectif we have $''E_1^{p,q} = \mathrm{Hom}_J(P_q, H_d^p(L_{\mathfrak{m}}^\bullet))$. On the other hand we have

$$''E_2^{p,q} = H_\partial^q(''E_1^{p,q}) = \begin{cases} 0 & \text{if } \ p \neq 0 \\ \widehat{H}^q(J, U_s^0(\mathfrak{m})) & \text{if } \ p = 0. \end{cases}$$

It is easy to check that the spectral sequence $''E$ degenerates at $''E_2$. In particular the convergence $''E_2^{p,q} \to H^{p+q}(Tot\, C)$ gives the identity $H^{p+q}(Tot\, C) = \widehat{H}^{p+q}(J, U_s^0(\mathfrak{m}))$. Let us compute $'E_1^{p,q}$ and $'E_2^{p,q}$. If $\mathfrak{f} \in \Sigma_{\mathfrak{m}}$ is such that $\mathfrak{f} \neq (1)$ then $\mathbb{Z}[G_{\mathfrak{f},s}]$ is $\mathbb{Z}[J]$-free. Moreover if q is odd then $\widehat{H}^q(J, \mathbb{Z}[\mathrm{Gal}(H_s/k)]) = 0$. We deduce from these two remarks that $'E_2^{p,q} =' E_1^{p,q} = \widehat{H}^q(J, L_{\mathfrak{m}}^p) = 0$ if q is odd. If q is even then

$$'E_1^{p,q} = \widehat{H}^q(J, L_{\mathfrak{m}}^p) = \bigoplus_{\substack{\mathfrak{g} \in \Sigma_{\mathfrak{m}} \\ \deg(\mathfrak{g}) = -p}} \big[A, \mathfrak{g}\big].$$

where $A = \mathbb{Z}/w_k\mathbb{Z}[\mathrm{Gal}(H/k)]$ and $\big[A, \mathfrak{g}\big] = \{[x, \mathfrak{g}],\ x \in A\}$. Let us denote by $\bar{d}$ the differential on $'E_1$ induced by d. We have

$$\bar{d}([\sigma, \mathfrak{g}]) = \begin{cases} 0 & \text{if } \ \mathfrak{g} = (1) \\ \displaystyle\sum_{\mathfrak{p}|\mathfrak{g}} (-1)^{\omega(\mathfrak{p},\mathfrak{g})}[\sigma - \sigma(\mathfrak{p}, H/k)^{-1}, \mathfrak{g}/\widehat{\mathfrak{p}}] & \text{if } \ \mathfrak{g} \neq (1). \end{cases}$$

THEOREM 4.2. — *Let* $(1) \neq \mathfrak{m} \in T_0$. *Let* D *be the subgroup of* $\mathrm{Gal}(H/k)$ *generated by the Frobenius automorphims* $(\mathfrak{p}, H/k) = \tau_{\mathfrak{p}}$ *at the prime ideals* $\mathfrak{p} \in \mathrm{supp}(\mathfrak{m})$. *Then for* q *even we have an isomorphism*

$$'E_2^{p,q} \simeq \bigoplus_{\substack{\mathfrak{g} \in \Sigma_{\mathfrak{m}} \\ \deg(\mathfrak{g}) = -p}} \big[\bar{A}, \mathfrak{g}\big].$$

where $\bar{A} = \mathbb{Z}/w_k\mathbb{Z}[\mathrm{Gal}(H/k)/D]$.

Proof. — The Theorem is obvious if $w_k = 4$ or $w_k = 6$ because in these two cases we have $\bar{d} = 0$. Let us suppose $w_k = 2$. If $S \subset \mathrm{supp}(\mathfrak{m})$ is not empty then we denote by D_S the subgroup of D generated by the Frobenius automorphims $\tau_{\mathfrak{p}}$, $\mathfrak{p} \in S$. Also we set $D_\emptyset = 1$. In the following we will use the group algebra $A_S = \mathbb{Z}/2\mathbb{Z}[\mathrm{Gal}(H/k)/D_S]$ and the graded A_S-module

$$M_S = \bigoplus_{\substack{\mathfrak{g}\in\Sigma_{\mathfrak{m}}\\ \mathrm{supp}(\mathfrak{g})\cap S=\emptyset}} [A_S, \mathfrak{g}], \qquad \deg[x,\mathfrak{g}] = -\deg(\mathfrak{g})$$

on which we define the differential $\bar{d}_S$ naturally induced by $\bar{d}$. Let Z_S^p and B_S^p be the kernel and the image of $\bar{d}_S$ in degree p. Thus we have $H^p(M_S, \bar{d}_S) = Z_S^p/B_S^p$ and $\#S - \deg(\mathfrak{m}) \leqslant p \leqslant 0$. These groups satisfy some interesting properties that we give below

1) $H^p(M_S, \bar{d}_S)$ is naturally an $\bar{A}$-module.

2) If $S \subset S'$ then the natural map $\lambda_{S,S'} : A_S \to A_{S'}$ induces a morphism $(M_S, \bar{d}_S) \to (M_{S'}, \bar{d}_{S'})$ (which sends $[x,\mathfrak{g}]$ to 0 if $\mathrm{supp}(\mathfrak{g}) \cap S' \neq \emptyset$). This gives us an $\bar{A}$-homomorphism

$$\lambda_{S,S'}^p : H^p(M_S, \bar{d}_S) \longrightarrow H^p(M_{S'}, \bar{d}_{S'}).$$

3) The map $\lambda_{S,S'}^0$ is an isomorphism of $\bar{A}$-modules.

4) If $S \subset S'$ and $\#S' = \deg(\mathfrak{m}) + p$ then the homomorphism $\lambda_{S,S'}^p$ is onto. To prove this claim we may take $S = \emptyset$ without loss of generality. On the other hand the sum $\sum \sigma$, where $\sigma \in < \tau_{\mathfrak{q},S'} \mid \mathfrak{q}|\mathfrak{m}$ and $\mathfrak{q} \notin S' >$, is an $\bar{A}$-basis of $H^p(M_{S'}, \bar{d}_{S'})$. Here $\tau_{\mathfrak{q},S'}$ is the image of $\tau_{\mathfrak{q}}$ in $\mathrm{Gal}(H/k)/D_{S'}$. One may easily prove that $\sum \sigma$ may be lifted to a element of $H^p(M_\emptyset, \bar{d}_\emptyset)$.

5) Let $S \subset S'$ and let $\mathfrak{u}$ be the unique ideal in $\Sigma_{\mathfrak{m}}$ such that $\mathrm{supp}(\mathfrak{u}) = S' - S$. Then we have a well defined morphism $\mu_{S,S'} : (M_{S'}, \bar{d}_{S'}) \to (M_S, \bar{d}_S)$ satisfying

$$\mu_{S,S'}([x,\mathfrak{g}]) = [t(S,S')x', \mathfrak{g}\mathfrak{u}].$$

where $t(S,S')$ is the sum in A_S of the elements of the group generated by $\tau_{\mathfrak{p},S}$, $\mathfrak{p} \in S' - S$ and x' is any element of A_S whose image is x via the natural map $A_S \to A_{S'}$. Let us remark that $x \mapsto t(S,S')x'$ is a homomorphism of the $\mathbb{Z}/2\mathbb{Z}$-vector spaces $A_{S'}$ and A_S. We also denote it by $\mu_{S,S'}$ and we let $\mu_{S,S'}^p$ be the $\bar{A}$-homomorphism

$$\mu_{S,S'}^p : H^{p+\deg(\mathfrak{u})}(M_{S'}, \bar{d}_{S'}) \longrightarrow H^p(M_S, \bar{d}_S)$$

induced by $\mu_{S,S'}$.

6) Suppose we have $S' = S \cup \{\mathfrak{p}\}$. Then it is not hard to prove that the sequence

$$H^{p+1}(M_{S'}, \bar{d}_{S'}) \xrightarrow{\mu_{S,S'}^p} H^p(M_S, \bar{d}_S) \xrightarrow{\lambda_{S,S'}^p} H^p(M_{S'}, \bar{d}_{S'})$$

is exact whenever these groups exist.

7) Let us prove by induction that $\lambda^p_{S,S'}$ is onto for all $S\subset S'$. The induction is initialized by remarks 3) and 4). Suppose we have proved that $\lambda^i_{X,X'}$ is onto for all $i>p$ and all X,X' such that $\lambda^i_{X,X'}$ is defined. Suppose also we have proved that $\lambda^p_{S,S'\cup\{\mathfrak{p}\}}$ is onto. Let us put $U=S\cup\{\mathfrak{p}\}$ and $U'=S'\cup\{\mathfrak{p}\}$. We deduce from 8) and our hypotheses that the following commutative diagram

$$\begin{array}{ccccccc}
H^{p+1}(M_U,\bar d_U) & \xrightarrow{\mu^p_{S,U}} & H^p(M_S,\bar d_S) & \xrightarrow{\lambda^p_{S,U}} & \mathrm{Im}(\lambda^p_{S,U}) & \longrightarrow & 0\\
\downarrow{\scriptstyle \lambda^{p+1}_{U,U'}} & & \downarrow{\scriptstyle \lambda^p_{S,S'}} & & \downarrow{\scriptstyle \lambda^p_{U,U'}} & & \downarrow\\
H^{p+1}(M_{U'},\bar d_{U'}) & \xrightarrow{\mu^p_{S',U'}} & H^p(M_{S'},\bar d_{S'}) & \xrightarrow{\lambda^p_{S',U'}} & H^p(M_{U'},\bar d_{U'}) & \longrightarrow & 0
\end{array}$$

has exact rows. Moreover $\lambda^p_{U,U'}$ and $\lambda^{p+1}_{U,U'}$ are epimorphisms. Hence by the four lemma $\lambda^p_{S,S'}$ is also an epimorphism.

8) Let us prove that $\mu^p_{S,S'}$ is injective whenever it is defined. Actually since $\mu^p_{S,S"}=\mu^p_{S,S'}\circ\mu^{p+i}_{S',S"}$ for all $S\subset S'\subset S"$ and $i=\#(S'-S)$ we may take $S'=S\cup\{\mathfrak{p}\}$. Let $X\in Z^{p+1}_{S'}$ and $Y\in M_S$ be such that $\mu^p_{S,S'}(X)=\bar d_S(Y)$. We have

$$X=\sum_{\mathfrak{g}\in\mathcal{G}}[X_\mathfrak{g},\mathfrak{g}]\quad\text{and}\quad Y=\sum_{\mathfrak{f}\in\mathcal{F}}[Y_\mathfrak{f},\mathfrak{f}].$$

where $\mathcal{F}$ and $\mathcal{G}$ are defined as follows

$$\mathcal{F}=\{\mathfrak{f}\in\Sigma_\mathfrak{m}\mid \mathrm{supp}(\mathfrak{f})\cap S=\emptyset\quad\text{and}\quad \deg(\mathfrak{f})=-p+1\}$$
$$\mathcal{G}=\{\mathfrak{g}\in\Sigma_\mathfrak{m}\mid \mathrm{supp}(\mathfrak{g})\cap S'=\emptyset\quad\text{and}\quad \deg(\mathfrak{g})=-p-1\}.$$

Further, for a fixed $\mathfrak{u}\in\mathcal{U}_{S'}=\{\mathfrak{u}\in\Sigma_\mathfrak{m}|\ \mathrm{supp}(\mathfrak{u})\cap S'=\emptyset\,\text{and}\ \deg(\mathfrak{u})=-p\}$ we must have

$$(1-\tau^{-1}_{\mathfrak{p},S})Y_{\widehat{\mathfrak{u}\mathfrak{p}}}=\sum_{\mathfrak{r}\notin S'\cup\,\mathrm{supp}(\mathfrak{u})}(1-\tau^{-1}_{\mathfrak{r},S})Y_{\widehat{\mathfrak{u}\mathfrak{r}}}\tag{4.1}$$

Let us set $\mathcal{H}=\{\mathfrak{h}\in\Sigma_\mathfrak{m}|\ \mathrm{supp}(\mathfrak{h})\cap S'=\emptyset\,\text{and}\ \deg(\mathfrak{h})=-p+1\}$. Then the image of $\sum_{\mathfrak{h}\in\mathcal{H}}[Y_\mathfrak{h},\mathfrak{h}]$ in $M_{S'}$ gives us a element of $H^{p-1}(M_{S'},\bar d_{S'})$. But we have proved in 9) that $\lambda^{p-1}_{S,S'}$ is an epimorphism. Thus there exist $Z=\sum_{\mathfrak{f}\in\mathcal{F}}[Z_\mathfrak{f},\mathfrak{f}]\in Z^{p-1}_S$ and $Z'=\sum_{\mathfrak{h}\in\mathcal{H}}[Z'_\mathfrak{h},\mathfrak{h}]\in M_S$ such that

$$Y_\mathfrak{h}=Z_\mathfrak{h}+(1-\tau^{-1}_{\mathfrak{p},S})Z'_\mathfrak{h},$$

for all $\mathfrak{h}\in\mathcal{H}$. Moreover we deduce from (4.1) the identity

$$Y_{\widehat{\mathfrak{u}\mathfrak{p}}}=Z_{\widehat{\mathfrak{u}\mathfrak{p}}}+\sum_{\mathfrak{r}\notin S'\cup\,\mathrm{supp}(\mathfrak{u})}(1-\tau^{-1}_{\mathfrak{r},S})Z'_{\widehat{\mathfrak{u}\mathfrak{r}}}+t(S,S')Z''_\mathfrak{u},\tag{4.2}$$

where $Z''_{\mathfrak{u}} \in A_S$. Now (4.2) and the equation $\mu^p_{S,S'}(X) = \bar{d}_S(Y)$ clearly show that

$$X_{\mathfrak{g}} = \sum_{\mathfrak{q} \notin S' \cup \operatorname{supp}(\mathfrak{g})} (1 - \tau^{-1}_{\mathfrak{q},S'}) \lambda_{S,S'}(Z''_{\widehat{\mathfrak{g}\mathfrak{q}}}).$$

In particular $X \in B^{p+1}_{S'}$.

9) Now we are ready to prove by induction that $H^p(M_S, \bar{d}_S)$ is isomorphic (as an $\bar{A}$-module) to a direct sum $\bigoplus_{\mathfrak{u} \in \mathcal{U}_S} [\bar{A}, \mathfrak{u}]$. This is easy to check if $p = 0$ or $-p = \deg(\mathfrak{m}) - \#S$. On the other hand if this property is satisfied for all $H^i(M_S, \bar{d}_S)$ with $i > p$ and for all $H^j(M_{S'}, \bar{d}_{S'})$ with $S' = S \cup \{\mathfrak{p}\}$ and $j \geqslant p$ then the exact sequence

$$0 \longrightarrow H^{p+1}(M_{S'}, \bar{d}_{S'}) \xrightarrow{\mu^p_{S,S'}} H^p(M_S, \bar{d}_S) \xrightarrow{\lambda^p_{S,S'}} H^p(M_{S'}, \bar{d}_{S'}) \longrightarrow 0$$

of $\bar{A}$-modules splits and this gives us the desired property for $H^p(M_S, \bar{d}_S)$. The proof of the theorem is now complete since we have $'E_2^{p,q} = H^p(M_\emptyset, \bar{d}_\emptyset)$. □

Lemma 4.1. — *The spectral sequence $'E_2^{p,q} \to \widehat{H}^{p+q}(J, U^0_s(\mathfrak{m}))$ degenerates at $'E_2$. In other words we have $'E_r^{p,q} = {}'E_2^{p,q}$ for all $r \geqslant 2$.*

Proof. — Since J is cyclic we can use the following complete resolution of J

$$\cdots \xrightarrow{N} \mathbb{Z}[J] = P_1 \xrightarrow{j-1} \mathbb{Z}[J] = P_0 \xrightarrow{N} \mathbb{Z}[J] = P_{-1} \xrightarrow{j-1} \cdots$$

where j is a fixed generator of J and $N = 1 + j + j^2 + \cdots + j^{w_H - 1}$. On the other hand we consider the double cochain complex $\widehat{C}$, with

$$\widehat{C}_{p,q} = \bigoplus_{\substack{\mathfrak{g} \in \Sigma_{\mathfrak{m}} \\ \deg(\mathfrak{g}) = -p}} [B, \mathfrak{g}],$$

where $B = \nu\mathbb{Z}[\mathrm{Gal}(H/k)/D]$ and $\nu = w_H/w_k$, equipped with the unique differential $\widehat{D}$ whose restriction to $\widehat{C}_{p,q}$ is given by

$$\widehat{D}(x) = \begin{cases} 0 & \text{if } q \text{ is even} \\ (-1)^p w_k x & \text{if } q \text{ is odd.} \end{cases}$$

If $\varphi \in \mathrm{Hom}_J(P_q = \mathbb{Z}[J], L^p_{\mathfrak{m}})$ then $\nu\varphi(1)$ has a component in

$$\bigoplus_{\substack{\mathfrak{g} \in \Sigma_{\mathfrak{m}} \\ \deg(\mathfrak{g}) = -p}} [\nu\mathbb{Z}[\mathrm{Gal}(H_s/k)], \mathfrak{g}].$$

Let us denote by $\mathcal{F}(\varphi)$ the projection of this component in $\widehat{C}_{p,q}$. This gives us a filtration-preserving cochain map $\mathcal{F} : \mathrm{Tot}\, C \to \mathrm{Tot}\, \widehat{C}$. The corresponding family of maps $\mathcal{F}_r : {}'E_r \to {}'\widehat{E}_r$ from the spectral sequence $({}'E_r)$ into $({}'\widehat{E}_r)$ is such that $\mathcal{F}_2$ is an isomorphism. Therefore $\mathcal{F}_r$ is an isomorphism for all $r \geqslant 2$. Now it is clear that ${}'\widehat{E}_r \simeq {}'\widehat{E}_1$, for all $r \geqslant 1$. □

THEOREM 4.3. — *Suppose* $\mathfrak{m} \neq (1)$ *then* $\widehat{H}^i(J, U_s^0(\mathfrak{m}))$ *is isomorphic as a* $G_{\mathfrak{m},s}$*-module to*

$$\big(\mathbb{Z}/w_k\mathbb{Z}[\mathrm{Gal}(H/k)/D]\big)^{2^{\deg(\mathfrak{m})-1}}$$

Proof. — Since the two filtrations are finite in each dimension the theorem follows from Theorem 4.2 and Lemma 4.1. □

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