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## SUMS OF COMMUTATORS IN IDEALS AND MODULES OF TYPE II FACTORS

by Kenneth J. DYKEMA & Nigel J. KALTON (\*)

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### 1. Introduction and description of results.

Let  $\mathcal{M}$  be a von Neumann algebra of type  $\text{II}_\infty$  having separable predual. We will study the commutator structure of ideals of  $\mathcal{M}$  and, more generally, of modules of operators affiliated to  $\mathcal{M}$ .

Fix a faithful semifinite trace  $\tau$  on  $\mathcal{M}$ , and let  $\mathcal{M}$  be represented on a Hilbert space  $\mathcal{H}$ . Segal [26] introduced measurability for unbounded operators on  $\mathcal{H}$  affiliated to  $\mathcal{M}$ . Later Nelson [23], in a slightly different approach, defined the completion  $\overline{\mathcal{M}}$  of  $\mathcal{M}$  with respect to a notion of convergence in measure, and showed that the operations on  $\mathcal{M}$  extend to make  $\overline{\mathcal{M}}$  a topological  $*$ -algebra. He also showed that  $\overline{\mathcal{M}}$  is the set of all  $\tau$ -measurable operators, i.e. the closed, densely defined, possibly unbounded operators  $T$  on  $\mathcal{H}$ , affiliated with  $\mathcal{M}$ , such that for every  $\epsilon > 0$  there is a projection  $E \in \mathcal{M}$  with  $\tau(1 - E) < \epsilon$  and with  $TE$  bounded. Note that  $\overline{\mathcal{M}}$  is defined independently of the Hilbert space  $\mathcal{H}$  on which  $\mathcal{M}$  acts, but is then characterized in terms of operators on  $\mathcal{H}$ . Nelson's work was done in the more general context of a von Neumann algebra  $\mathcal{M}$  equipped with a fixed finite or semifinite faithful normal trace. (See [4] for a proof that Segal's and Nelson's definitions are equivalent in  $\text{II}_\infty$  factors.)

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We consider subspaces  $\mathcal{I} \subseteq \overline{\mathcal{M}}$  that are globally invariant under left and right multiplication by elements from  $\mathcal{M}$ . These are thus sub- $(\mathcal{M}, \mathcal{M})$ -bimodules of  $\overline{\mathcal{M}}$ ; for brevity we will call them *submodules* of  $\overline{\mathcal{M}}$ . Note that if such a submodule  $\mathcal{I}$  is actually contained in  $\mathcal{M}$ , then it is a two-sided ideal of  $\mathcal{M}$ . Submodules of  $\overline{\mathcal{M}}$  are analogues in the type  $\text{II}_\infty$  context of ideals of  $B(\mathcal{H})$  in the type I context. The submodules of  $\overline{\mathcal{M}}$  can be classified in terms of the singular numbers of their elements, analogously to Calkin’s classification [3] of the ideals of  $B(\mathcal{H})$ . If  $T \in \overline{\mathcal{M}}$  and  $t > 0$ , the  $t$ -th singular number of  $T$  is

$$(1.1) \quad \mu_t(T) = \inf \{ \|T(1 - E)\| : E \in \mathcal{M} \text{ a projection with } \tau(E) \leq t \},$$

and we denote by  $\mu(T)$  the function  $t \mapsto \mu_t(T)$ . If  $\mathcal{I} \subseteq \overline{\mathcal{M}}$  is a submodule, we set

$$\mu(\mathcal{I}) = \{ \mu(T) \mid T \in \mathcal{I} \}$$

and we call  $\mu(\mathcal{I})$  the *characteristic set* of  $\mathcal{I}$ . The aforementioned classification is the bijection  $\mathcal{I} \mapsto \mu(\mathcal{I})$  from the set of all submodules of  $\overline{\mathcal{M}}$  to the set of all characteristic sets, where, abstractly, a characteristic set is a set of decreasing functions on  $(0, \infty)$  satisfying certain properties. Several authors have used singular numbers to characterize ideals of  $\mathcal{M}$  and modules of  $\overline{\mathcal{M}}$  (see [5], [6], [9], [27] and [29]), and the full classification result was derived by Guido and Isola in [16].

One interesting facet of submodules of  $\overline{\mathcal{M}}$  is that their classification involves both asymptotics at infinity (the rate of decay of  $\mu_t(T)$  as  $t \rightarrow \infty$ ) and asymptotics at zero (the rate of increase of  $\mu_t(T)$  as  $t \rightarrow 0$ ).

We consider additive commutators  $[A, B] = AB - BA$  of elements of  $\overline{\mathcal{M}}$  and study the commutator spaces

$$[\mathcal{I}, \mathcal{J}] = \left\{ \sum_{k=1}^n [A_k, B_k] \mid n \in \mathbb{N}, A_k \in \mathcal{I}, B_k \in \mathcal{J} \right\}$$

of submodules  $\mathcal{I}$  and  $\mathcal{J}$  of  $\overline{\mathcal{M}}$ . Note  $[\mathcal{I}, \mathcal{J}] \subseteq \mathcal{I}\mathcal{J}$ , where  $\mathcal{I}\mathcal{J}$  is the submodule of  $\overline{\mathcal{M}}$  spanned by all products  $AB$  with  $A \in \mathcal{I}$  and  $B \in \mathcal{J}$ . Using properties of singular numbers (which are reviewed in § 2), one easily shows that  $\mu(\mathcal{I}\mathcal{J})$  is the set of all decreasing functions  $f : (0, \infty) \rightarrow [0, \infty)$  bounded above by products  $gh$  with  $g \in \mu(\mathcal{I})$  and  $h \in \mu(\mathcal{J})$ . Since an element of  $\mathcal{I}\mathcal{J}$  belongs to  $[\mathcal{I}, \mathcal{J}]$  if and only if its real and imaginary parts belong to  $[\mathcal{I}, \mathcal{J}]$ , to characterize  $[\mathcal{I}, \mathcal{J}]$  it will suffice to describe the normal elements of it. This we do as follows: given a normal element  $T \in \mathcal{I}\mathcal{J}$ , let  $E_{|T|}$  denote the spectral measure of the positive part  $|T|$  of  $T$ . Then  $T \in [\mathcal{I}, \mathcal{J}]$  if and only if there is  $h \in \mu(\mathcal{I}\mathcal{J})$  such that

$$(1.2) \quad |\tau(TE_{|T|}(\mu_s(T), \mu_r(T)))| \leq rh(r) + sh(s)$$

for all  $0 < r < s < \infty$ . This is analogous, though for asymptotics in both directions, to the characterization of commutator spaces for ideals of  $B(\mathcal{H})$  found in [7] (see also [19] for the earlier result in the case of the trace-class operators). Our proof relies on a result of Fack and de la Harpe [11], expressing any trace-zero element of a  $\text{II}_1$ -factor as a sum of a fixed number of commutators of elements whose norms are controlled. A corollary of our characterization is

$$[\mathcal{I}, \mathcal{J}] = [\mathcal{I}\mathcal{J}, \mathcal{M}]$$

for any submodules  $\mathcal{I}$  and  $\mathcal{J}$  of  $\overline{\mathcal{M}}$ . We also give a characterization of  $T \in [\mathcal{I}, \mathcal{J}]$  for  $T$  normal that considers separately the asymptotics at 0 and at  $\infty$ .

As an alternative to using the characteristic set  $\mu(\mathcal{I})$  of a submodule  $\mathcal{I} \subseteq \overline{\mathcal{M}}$  for the classification of submodules, one can use the corresponding rearrangement invariant function space  $S(\mathcal{I})$ , which is the set of all measurable functions  $f : (0, \infty) \rightarrow \mathbb{C}$  such that the decreasing rearrangement of the absolute value of  $f$  lies in  $\mu(\mathcal{I})$ . Then every normal element  $T \in \mathcal{I}$  gives rise to a unique (up to rearrangement) function  $f_T \in S(\mathcal{I})$  defined as follows: Fix any measure preserving transformation from  $(0, \infty)$  with Lebesgue measure to the disjoint union of four copies of  $(0, \infty)$  with Lebesgue measure, in order to define the measurable function  $g_1 \oplus g_2 \oplus g_3 \oplus g_4 : (0, \infty) \rightarrow \mathbb{C}$ , given measurable functions  $g_j : (0, \infty) \rightarrow \mathbb{C}$ . Now let  $f_T = f_1 \oplus (-f_2) \oplus (if_3) \oplus (-if_4) \in S(\mathcal{I})$ , where

$$\begin{aligned} f_1(t) &= \mu_t((\text{Re } T)_+) & f_2(t) &= \mu_t((\text{Re } T)_-) \\ f_3(t) &= \mu_t((\text{Im } T)_+) & f_4(t) &= \mu_t((\text{Im } T)_-), \end{aligned}$$

with  $\text{Re } T = (T + T^*)/2 = (\text{Re } T)_+ - (\text{Re } T)_-$ , where  $(\text{Re } T)_+$  and  $(\text{Re } T)_-$  are commuting positive operators whose product is zero, and similarly for  $\text{Im } T = (T - T^*)/2i = (\text{Im } T)_+ - (\text{Im } T)_-$ . Then in the case when  $\lim_{t \rightarrow \infty} \mu_t(T) = 0$  for all elements  $T \in \mathcal{I}\mathcal{J}$ , the condition (1.2) above for  $T \in [\mathcal{I}, \mathcal{J}]$  with  $T$  normal can be rephrased in terms of  $f_T$  and is seen to be equivalent to the condition found in [13] for  $f_T$  to belong to the kernel of every symmetric functional on  $S(\mathcal{I}\mathcal{J})$ . Thus, our main result can be seen as a noncommutative analogue of this result from [13]. See also [6] for related results on Banach symmetric functions spaces and the corresponding submodules of  $\overline{\mathcal{M}}$ .

In the case of a  $\text{II}_1$ -factor  $\mathcal{M}$ , we give an analogous characterization of the commutator spaces  $[\mathcal{I}, \mathcal{J}]$  for submodules  $\mathcal{I}$  and  $\mathcal{J}$  of  $\overline{\mathcal{M}}$ .

In the case of ideals in  $\mathcal{B}(\mathcal{H})$  it was shown in [20] that for quasi-Banach ideals  $\mathcal{I}$  the subspace  $[\mathcal{I}, \mathcal{B}(\mathcal{H})]$  can be characterized purely in

spectral terms (see also [19] for an earlier result in this direction). More generally this result was established for the class of geometrically stable ideals. This means that for such ideals if two operators  $S, T$  in  $\mathcal{I}$  have the same spectrum (counting algebraic multiplicities) and  $S \in [\mathcal{I}, \mathcal{B}(\mathcal{H})]$  then  $T \in [\mathcal{I}, \mathcal{B}(\mathcal{H})]$ . This was known for hermitian operators (and hence normal operators) from the results in [7], but is generally false (see [8]). We study the same phenomenon in type  $\text{II}_\infty$ -factors. In this case, since we need a notion corresponding to multiplicity we employ the Brown measure [2] as a substitute for the notion of spectrum. The Brown measure of an operator is a measure with support contained in its spectrum. It is, however, only defined for certain special types of operators. Nevertheless we obtain a quite satisfactory analogue of the result of [20]. If  $\mathcal{I}$  is a geometrically stable submodule of  $\overline{\mathcal{M}}$  and  $T \in \mathcal{I}$  admits a Brown measure  $\nu_T$  then  $T \in [\mathcal{I}, \mathcal{M}]$  if and only if there is a positive operator  $V \in \mathcal{I}$  so that

$$\left| \int_{r < |z| \leq s} z \, d\nu_T(z) \right| \leq r\tau(E_V(r, \infty)) + s\tau(E_V(s, \infty)), \quad 0 < r < s < \infty.$$

This condition depends only on the Brown measure associated to  $T$ .

The paper is organized as follows: In § 2, we recall some facts about singular numbers of elements of  $\overline{\mathcal{M}}$ . In § 3, we describe the classification of submodules of  $\overline{\mathcal{M}}$  when  $\mathcal{M}$  is a type  $\text{II}_\infty$  or  $\text{II}_1$  factor with separable predual. In § 4, we prove the main results characterizing  $[\mathcal{I}, \mathcal{J}]$ . In § 5, we give a characterization of  $[\mathcal{I}, \mathcal{J}]$  in the  $\text{II}_\infty$  case, separating the asymptotics at 0 and  $\infty$ . Results on the Brown measure are discussed in § 6.

After a first version of this paper was distributed, we were made aware of the recent results of T. Fack [10], which are independent of and have some overlap with our results. In particular, he proves the characterization of  $T \in [\mathcal{I}_b, \mathcal{M}]$  for  $T$  self-adjoint, when  $\mathcal{I}_b = \mathcal{I} \cap \mathcal{M}$  for a Banach submodule  $\mathcal{I}$  of  $\overline{\mathcal{M}}$ . He also proves the characterization of  $T \in [\mathcal{I}_b, \mathcal{M}]$  when  $\mathcal{I}$  is a Banach module contained in  $L^p$  for some  $0 < p < \infty$ . Both of these results are generalized in this paper to (possibly) unbounded operators and to submodules that are not necessarily Banach. Some of our methods are quite parallel to Fack's and in the text we will remark on these similarities. Fack also gives some nice applications of his commutator results to Dixmier traces in the  $\text{II}_\infty$  setting.

## 2. Preliminaries on singular numbers.

If  $\mathcal{M}$  is a von Neumann algebra with a fixed finite or semifinite normal trace  $\tau$ , then the singular numbers (sometimes called generalized singular

numbers) of elements of  $\mathcal{M}$  and more generally of  $\tau$ -measurable operators affiliated to  $\mathcal{M}$  have been understood for many years; see, for example, [22], [15], [24], [9] and [12]. In this section, we review these concepts and some results, introduce the notation we will use throughout the paper and prove a technical result that will be of use later.

Recall that the  $t$ -th singular number of  $T \in \overline{\mathcal{M}}$  is defined for  $t > 0$  by (1.1). Since  $T$  is  $\tau$ -measurable, we have  $0 \leq \mu_t(T) < +\infty$ . We will also use the convention  $\mu_0(T) = \|T\|$ , where  $\|T\| = \infty$  if  $T \notin \mathcal{M}$ . Note that  $t \mapsto \mu_t(T)$  is a nonincreasing function from  $[0, \infty)$  into  $[0, \infty]$ . If  $\tau$  is a finite trace, then by our convention that  $\tau(1) = 1$  we have  $\mu_t(T) = 0$  whenever  $t \geq 1$ . We will use the following properties of singular numbers; see [9] or [12] for proofs.

PROPOSITION 2.1. — *Let  $\mathcal{M}$  be a von Neumann algebra with a distinguished finite or semifinite normal faithful trace, let  $S, T \in \overline{\mathcal{M}}$  and  $s, t \geq 0$ . Then*

- (i)  $\mu_t(T) = \mu_t(T^*) = \mu_t(|T|)$ ,
- (ii)  $\mu_{s+t}(S + T) \leq \mu_s(S) + \mu_t(T)$ ,
- (iii)  $\mu_{s+t}(ST) \leq \mu_s(S)\mu_t(T)$ ,
- (iv) if  $A, B \in \mathcal{M}$ , then  $\mu_t(ATB) \leq \|A\| \|B\| \mu_t(T)$ .

Moreover,

- (v) *the function  $[0, \infty) \ni t \mapsto \mu_t(T) \in [0, \infty]$  is continuous from the right.*

Given  $T \in \overline{\mathcal{M}}$ , let  $A \mapsto E_{|T|}(A)$  be the projection-valued spectral measure of the positive part  $|T|$  of  $T$ . (To avoid clutter, when  $A$  is an interval we will frequently omit to write parenthesis, writing just  $E_{|T|}A$ .)

PROPOSITION 2.2 ([12], 2.2). — *For  $t \geq 0$  we have*

$$(2.1) \quad \mu_t(T) = \inf (\{s \geq 0 \mid \tau(E_{|T|}(s, \infty)) \leq t\} \cup \{\infty\})$$

*and the infimum is attained, giving*

$$(2.2) \quad \tau(E_{|T|}(\mu_t(T), \infty)) \leq t$$

*whenever  $\mu_t(T) < \infty$ .*

LEMMA 2.3. — *Let  $\mathcal{M}$  be a nonatomic von Neumann algebra with a normal faithful semifinite trace  $\tau$ , let  $T \in \overline{\mathcal{M}}$  and let  $x \in \mathbb{R}, x \geq 0$ . Then*

$$(2.3) \quad \tau(E_{|T|}(x, \infty)) = \inf (\{s \geq 0 \mid \mu_s(T) \leq x\} \cup \{\infty\}),$$

$$(2.4) \quad \tau(E_{|T|}[x, \infty)) = \sup (\{s \geq 0 \mid \mu_s(T) \geq x\} \cup \{0\}),$$

and the infimum in (2.3) is attained.

*Proof.* — The infimum in (2.3) is attained because  $s \mapsto \mu_s(T)$  is continuous from the right. If  $a = \tau(E_{|T|}(x, \infty)) < \infty$ , then, since  $\|T(1 - E_{|T|}(x, \infty))\| \leq x$ , we have  $\mu_a(T) \leq x$ , proving  $\geq$  in (2.3). On the other hand, if  $\mu_s(T) \leq x < \infty$ , then using (2.2) we have

$$\tau(E_{|T|}(x, \infty)) \leq \tau(E_{|T|}(\mu_s(T), \infty)) \leq s,$$

proving  $\leq$  in (2.3).

If  $s < \tau(E_{|T|}[x, \infty))$ , then for any projection  $P \in \mathcal{M}$  with  $\tau(P) = s$ , we have  $(1 - P) \wedge E_{|T|}[x, \infty) \neq 0$ . Hence  $\|T(1 - P)\| \geq x$ . Therefore  $\mu_s(T) \geq x$ , which proves  $\leq$  in (2.4). If  $\tau(E_{|T|}[x, \infty)) < s' < \infty$ , then since  $[x, \infty) = \bigcap_{0 < r < x} (r, \infty)$ , there is  $r < x$  such that  $\tau(E_{|T|}(r, \infty)) \leq s'$ . But then  $\mu_{s'}(T) \leq r < x$ , which implies  $s' \geq \sup(\{s \geq 0 \mid \mu_s(T) \geq x\} \cup \{0\})$ . This proves  $\geq$  in (2.4).  $\square$

DEFINITION 2.4. — Let  $\mathcal{M}$  be a  $\text{II}_\infty$ -factor and let us introduce the natural notation  $\oplus$ . Since  $\overline{\mathcal{M}}$  consists of (in general unbounded) operators on a Hilbert space  $\mathcal{H}$ , by choosing an isomorphism  $\mathcal{H} \cong \mathcal{H} \oplus \mathcal{H}$ , we may realize  $\overline{\mathcal{M}} \oplus \overline{\mathcal{M}}$  as a subalgebra of  $\overline{\mathcal{M}}$  in such a way that  $\tau(S \oplus T) = \tau(S) + \tau(T)$  whenever  $S$  and  $T$  are in  $L^1(\mathcal{M}, \tau) \subseteq \overline{\mathcal{M}}$ . Thus for  $S, T \in \overline{\mathcal{M}}$ ,  $S \oplus T$  defines an element of  $\overline{\mathcal{M}}$  uniquely up to conjugation by a unitary in  $\mathcal{M}$ . Since  $U^*AU = A + [AU, U^*]$  whenever  $U$  is unitary and  $A \in \overline{\mathcal{M}}$ , if  $S, T \in \mathcal{I}$  for any submodule  $\mathcal{I} \subseteq \overline{\mathcal{M}}$ , the direct sum  $S \oplus T$  is defined uniquely up to addition of a commutator from  $[\mathcal{I}, \mathcal{M}]$ . Moreover, we have  $\mathcal{I} \oplus \mathcal{I} \subseteq \mathcal{I}$  and for every  $T \in \mathcal{I}$  we get

$$T \oplus 0 \in T + [\mathcal{I}, \mathcal{M}]$$

by using an appropriate nonunitary isometry in  $\mathcal{M}$ .

PROPOSITION 2.5. — Let  $S, T \in \overline{\mathcal{M}}$  and let  $a \geq 0$ . Then

$$\mu_a(S \oplus T) = \inf \{ \max(\mu_b(S), \mu_c(T)) \mid b, c \geq 0, b + c = a \}.$$

*Proof.* — The case  $a = 0$  is straightforward, so we may assume  $a > 0$ . It is clearly equivalent to show

$$(2.5) \quad \mu_a(S \oplus T) = \inf \{ \max(\mu_b(S), \mu_c(T)) \mid b, c \geq 0, b + c \leq a \}.$$

Given  $b, c \geq 0$  such that  $b + c \leq a$ , by (2.2) we have

$$\tau(E_{|S|}(\mu_b(S), \infty) \oplus E_{|T|}(\mu_c(T), \infty)) \leq b + c \leq a,$$

so using the definition (1.1) of singular numbers, we get

$$\mu_a(S \oplus T) \leq \|SE_{|S|}[0, \mu_b(S)] \oplus TE_{|T|}[0, \mu_c(T)]\| \leq \max(\mu_b(S), \mu_c(T)).$$

This shows  $\leq$  in (2.5). For the reverse inclusion, by (2.1) we have

$$\mu_a(S \oplus T) = \inf \{r \geq 0 \mid \tau(E_{|S \oplus T|}(r, \infty)) \leq a\}.$$

But

$$E_{|S \oplus T|}(r, \infty) = E_{|S| \oplus |T|}(r, \infty) = E_{|S|}(r, \infty) \oplus E_{|T|}(r, \infty),$$

so

$$\mu_a(S \oplus T) = \inf \{r \geq 0 \mid \tau(E_{|S|}(r, \infty)) + \tau(E_{|T|}(r, \infty)) \leq a\}.$$

By Lemma 2.3, if  $b = \tau(E_{|S|}(r, \infty))$  and  $c = \tau(E_{|T|}(r, \infty))$ , then  $\mu_b(S) \leq r$  and  $\mu_c(T) \leq r$ . This implies  $\geq$  in (2.5). □

The next lemma can be described as mashing the atoms of  $E_{|T|}$ . It is both straightforward and similar to [9, Lemme 1.13] and [16, Lemma 1.8]. However, for completeness we include a proof.

LEMMA 2.6. — *Let  $\mathcal{M}$  be a von Neumann algebra without minimal projections and with a distinguished semifinite normal faithful trace  $\tau$ . Let  $T \in \overline{\mathcal{M}}$ . Then there is a family  $(P_t)_{t \geq 0}$  of projections in  $\mathcal{M}$  such that for all  $s$  and  $t$ ,*

- (i)  $s \leq t$  implies  $P_s \leq P_t$ ,
- (ii)  $\tau(P_t) = t$ ,
- (iii)  $P_t$  and  $|T|$  commute, and if  $T$  is normal then  $P_t$  and  $T$  commute,
- (iv)  $E_{|T|}(\mu_t(T), \infty) \leq P_t \leq E_{|T|}[\mu_t(T), \infty)$ ,
- (v) if  $x > 0$ , then  $E_{|T|}(x, \infty) = P_y$ , where  $y = \inf \{t > 0 \mid \mu_t(T) \leq x\}$ .

Furthermore, suppose  $\lim_{t \rightarrow \infty} \mu_t(T) = 0$ . Then, letting  $F$  be the projection-valued Borel measure in  $\mathcal{M}$  supported on  $(0, \infty)$  and satisfying  $F((a, b)) = P_b - P_a$ , we have

$$|T| = \int_{(0, \infty)} \mu_t(T) dF(t).$$

*Proof.* — If  $T$  is not normal, then we may replace  $T$  by  $|T|$ , so assume  $T$  is normal. Set  $P_t = E_{|T|}(\mu_t(T), \infty)$  whenever  $E_{|T|}(\{\mu_t(T)\}) = 0$ . For these values of  $t$ , it follows from Lemma 2.3 that  $\tau(P_t) = t$ . The set

$$\mathcal{E} = \{E_{|T|}(\{\mu_t(T)\}) \mid t > 0\}$$



is finite or countable. We index  $\mathcal{E}$  by letting  $I$  be a set and  $I \ni i \mapsto t(i) \in [0, \infty)$  be an injective map such that

$$\mathcal{E} = \{0\} \cup \{E_{|T|}(\{\mu_{t(i)}(T)\}) \mid i \in I\},$$

$E_{|T|}(\{\mu_{t(i)}(T)\}) \neq 0$  and

$$t(i) = \inf \{s \mid \mu_s(T) = \mu_{t(i)}(T)\}.$$

Let  $a_i = \tau(E_{|T|}(\{\mu_{t(i)}(T)\}))$ .

Fix  $i \in I$ . Applying the spectral theorem to the normal operator  $TE_{|T|}(\{\mu_{t(i)}(T)\})$  and putting an atomless resolution of the identity under any of its atoms, we find a family  $(Q_r)_{0 \leq r < a_i}$  of projections in  $\mathcal{M}$  such that

- (1)  $r_1 \leq r_2$  implies  $Q_{r_1} \leq Q_{r_2}$ ,
- (2)  $\tau(Q_r) = r$ ,
- (3)  $Q_r T = T Q_r$ ,
- (4)  $Q_r \leq E_{|T|}(\{\mu_{t(i)}(T)\})$ .

Let

$$P_{t(i)+r} = E_{|T|}(\mu_{t(i)}(T), \infty) + Q_r$$

for all  $r \in [0, a_i)$ . If  $a_i \neq \infty$  then set  $P_{t(i)+a_i} = E_{|T|}[\mu_{t(i)}(T), \infty)$ . Now it is easily seen that the family  $(P_t)_{t \geq 0}$  satisfies (i)–(v).

Suppose  $\lim_{t \rightarrow \infty} \mu_t(T) = 0$  and let

$$S = \int_{(0, \infty)} \mu_t(T) dF(t).$$

Clearly  $S \geq 0$ . In order to show  $S = |T|$ , it will suffice to show  $E_S(x, \infty) = E_{|T|}(x, \infty)$  for all  $x > 0$ . We have

$$E_S(x, \infty) = F(\{t > 0 \mid \mu_t(T) > x\}).$$

But  $\{t > 0 \mid \mu_t(T) > x\} = (0, y)$  where

$$y = \sup \{t > 0 \mid \mu_t(T) > x\} = \inf \{t > 0 \mid \mu_t(T) \leq x\}.$$

From Lemma 2.3,  $y = \tau(E_{|T|}(x, \infty))$  and, furthermore,  $\mu_y(T) \leq x$ . By construction,

$$F((0, y)) = P_y = E_{|T|}(x, \infty). \quad \square$$

### 3. Classification of modules of a type II factor.

Let  $D^+(0, \infty)$ , respectively  $D^+(0, 1)$ , denote the cone of all decreasing (i.e. nonincreasing) functions  $f$  from the interval  $(0, \infty)$ , respectively  $(0, 1)$ , into  $[0, \infty)$  that are continuous from the right.

DEFINITION 3.1. — *Let  $\mathcal{D}$  be either  $D^+(0, \infty)$  or  $D^+(0, 1)$ . A subset  $\Lambda$  of  $\mathcal{D}$  is called a hereditary subcone of  $\mathcal{D}$  if*

- (i)  $f, g \in \Lambda$  implies  $f + g \in \Lambda$ ,
- (ii)  $f \in \Lambda, g \in D^+(0, \infty)$  and  $g \leq f$  imply  $g \in \Lambda$ .

The subset  $\Lambda \subseteq \mathcal{D}$  is called a characteristic set in  $\mathcal{D}$  if it is a hereditary subcone and if

- (iii)  $f \in \Lambda$  implies  $D_2f \in \Lambda$ ,

where  $D_2f(t) = f(t/2)$ .

Let  $\mathcal{M}$  be either a type  $\text{II}_\infty$  factor with a fixed semifinite normal trace  $\tau$  or a type  $\text{II}_1$  factor with tracial state  $\tau$ . Let  $\mathcal{D}$  be  $D^+(0, \infty)$  if  $\mathcal{M}$  is type  $\text{II}_\infty$  and  $D^+(0, 1)$  if  $\mathcal{M}$  is type  $\text{II}_1$ . We will recall from [16] the classification of submodules of the algebra  $\overline{\mathcal{M}}$  of  $\tau$ -measurable operators in terms of characteristic sets in  $\mathcal{D}$ .

For  $T \in \overline{\mathcal{M}}$ , let  $\mu(T) \in \mathcal{D}$  be the function which at  $t$  takes the value  $\mu_t(T)$  of the  $t$ -th singular number of  $T$ . Given a submodule  $\mathcal{I} \subseteq \overline{\mathcal{M}}$ , let

$$\mu(\mathcal{I}) = \{\mu(T) : T \in \mathcal{I}\} \subseteq \mathcal{D}.$$

PROPOSITION 3.2 ([16]). — *Let  $\mathcal{M}$  be a factor of type  $\text{II}_\infty$  or  $\text{II}_1$ . Then the map  $\mathcal{I} \mapsto \mu(\mathcal{I})$  is a bijection from the set of all submodules of  $\overline{\mathcal{M}}$  onto the set of all characteristic sets in  $\mathcal{D}$ .*

Remark 3.3. — A few well known observations are perhaps in order. If  $\mathcal{M}$  is type  $\text{II}_\infty$ , then  $\mu(\mathcal{M})$  is the set of all bounded functions in  $D^+(0, \infty)$ . Thus the smallest nonzero ideal of  $\mathcal{M}$  is the set  $\mathcal{F}$  of all  $\tau$ -finite rank operators in  $\mathcal{M}$ , where an operator  $T$  has  $\tau$ -finite rank if  $T = ET$  for some projection  $E \subseteq \mathcal{M}$  with  $\tau(E) < \infty$ ; the largest proper ideal of  $\mathcal{M}$  is the set  $\mathcal{K}$  of all  $\tau$ -compact operators in  $\mathcal{M}$ , where (see [9]), an operator  $T$  is  $\tau$ -compact if  $\lim_{t \rightarrow \infty} \mu_t(T) = 0$ .

On the other hand, if  $\mathcal{M}$  is type  $\text{II}_1$ , then  $\mu(\mathcal{M})$  is the set of all bounded functions in  $D^+(0, 1)$ , and  $\mathcal{M}$  itself has no proper nonzero ideals.

### 4. Sums of commutators.

The proof of the following lemma is similar to the proof of (i) $\implies$ (iii) of [10].

LEMMA 4.1. — *Let  $\mathcal{M}$  be a von Neumann algebra without minimal projections and with a normal faithful semifinite trace  $\tau$ . Let  $T \in \overline{\mathcal{M}}$ . If  $T = \sum_{i=1}^N [A_i, B_i]$  with  $A_i, B_i \in \overline{\mathcal{M}}$ , then*

$$(4.1) \quad |\tau(T E_{|T|}(\mu_s(T), \mu_r(T)))| \leq rh(r) + sh(s)$$

whenever  $0 < r < s < \infty$ , where

$$h(t) = (8N + 2)\mu_t(T) + (16N + 4) \sum_{i=1}^N \mu_t(A_i)\mu_t(B_i).$$

*Proof.* — We have  $E_{|T|}(\mu_s(T), \mu_r(T)) = F_s - F_r$  where  $F_t = E_{|T|}(\mu_t(T), \infty)$ . Note

$$\begin{aligned} \|T(1 - F_s)\| &\leq \mu_s(T), & \tau(F_s) &\leq s, \\ \|T(1 - F_r)\| &\leq \mu_r(T), & \tau(F_r) &\leq r. \end{aligned}$$

We can find a projection  $P \geq F_r$  in  $\mathcal{M}$  such that  $\tau(P) \leq (4N + 1)r$  and

$$\begin{aligned} \|A_i(1 - P)\| &\leq \mu_r(A_i), & \|(1 - P)A_i\| &\leq \mu_r(A_i), \\ \|B_i(1 - P)\| &\leq \mu_r(B_i), & \|(1 - P)B_i\| &\leq \mu_r(B_i), \end{aligned} \quad (1 \leq i \leq N).$$

Then we can find a projection  $Q \geq F_s \vee P$  such that  $\tau(Q) \leq (4N + 1)(r + s) \leq (8N + 2)s$  and such that for all  $i \in \{1, \dots, N\}$ ,

$$\begin{aligned} \|A_i(1 - Q)\| &\leq \mu_s(A_i), & \|(1 - Q)A_i\| &\leq \mu_s(A_i), \\ \|B_i(1 - Q)\| &\leq \mu_s(B_i), & \|(1 - Q)B_i\| &\leq \mu_s(B_i), \end{aligned} \quad (1 \leq i \leq N).$$

Hence

$$\begin{aligned} |\tau(T(F_s - F_r))| &\leq |\tau(T(Q - P))| + |\tau(T(Q - F_s))| + |\tau(T(P - F_r))| \\ &\leq |\tau(T(Q - P))| + (8N + 2)s\mu_s(T) + (4N + 1)r\mu_r(T). \end{aligned}$$

Since  $Q - P$  is a finite projection and  $T(Q - P)$  is bounded,

$$|\tau(T(Q - P))| = |\tau((Q - P)T(Q - P))| \leq \sum_{i=1}^N |\tau((Q - P)[A_i, B_i](Q - P))|.$$

Since also  $A_i(Q - P)$ ,  $(Q - P)A_i$ ,  $B_i(Q - P)$  and  $(Q - P)B_i$  are bounded, we have

$$\begin{aligned} \tau((Q - P)[A_i, B_i](Q - P)) &= \tau((Q - P)A_i(1 - Q + P)B_i(Q - P)) \\ &\quad - \tau(((Q - P)B_i(1 - Q + P)A_i(Q - P))). \end{aligned}$$

But

$$\begin{aligned}
 & |\tau((Q - P)A_i(1 - Q + P)B_i(Q - P))| \\
 & \leq |\tau((Q - P)A_i(1 - Q)B_i)| + |\tau(B_i(Q - P)A_iP)| \\
 & \leq \tau(Q - P)\|A_i(1 - Q)\| \|(1 - Q)B_i\| + \tau(P)\|B_i(Q - P)\| \|(Q - P)A_i\| \\
 & \leq (8N + 2)s\mu_s(A_i)\mu_s(B_i) + (4N + 1)r\mu_r(B_i)\mu_r(A_i),
 \end{aligned}$$

and also with  $A_i$  and  $B_i$  interchanged. Adding these several upper bounds gives (4.1). □

The proof of the following lemma is similar to the proof of Lemma 3 of [10].

LEMMA 4.2. — *Let  $\mathcal{M}$  be a  $\text{II}_\infty$  factor with a specified normal faithful semifinite trace  $\tau$ . Let  $h \in D^+(0, \infty)$  and suppose  $T \in \overline{\mathcal{M}}$  is a normal operator satisfying*

$$(4.2) \quad \lim_{t \rightarrow \infty} \mu_t(T) = 0$$

and

$$(4.3) \quad |\tau(TE|_T(\mu_s(T), \mu_r(T)))| \leq rh(r) + sh(s), \quad (0 < r < s < \infty).$$

Let  $\Lambda$  be the characteristic subset in  $D^+(0, \infty)$  generated by  $h$  and  $\mu(T)$ . Then there are  $X_1, \dots, X_{14} \in \overline{\mathcal{M}}$  and  $Y_1, \dots, Y_{14} \in \mathcal{M}$  such that

$$(4.4) \quad T = \sum_{i=1}^{14} [X_i, Y_i]$$

and  $\mu(X_i) \in \Lambda$  for all  $i$ .

*Proof.* — Let  $(P_t)_{t \geq 0}$  be a family of projections obtained from Lemma 2.6. Assumption (4.2) implies  $P_\infty T = TP_\infty = T$ , where  $P_\infty = \bigvee_{t \geq 0} P_t$ . Let  $P[s, t] = P_t - P_s$  when  $s < t$ , let

$$\alpha_n = 2^{-n} \tau(TP[2^n, 2^{n+1}]), \quad (n \in \mathbb{Z})$$

and let

$$A = \sum_{n \in \mathbb{Z}} \alpha_n P[2^n, 2^{n+1}].$$

Note that  $T - A = \sum_{n \in \mathbb{Z}} S_n$  where  $S_n = (T - A)P[2^n, 2^{n+1}]$  is an element of the  $\text{II}_1$ -factor  $P[2^n, 2^{n+1}]\mathcal{M}P[2^n, 2^{n+1}]$  having zero trace. By [11, Thm. 2.3], there are  $X_1^{(n)}, \dots, X_{10}^{(n)}, Y_1^{(n)}, \dots, Y_{10}^{(n)} \in P[2^n, 2^{n+1}]\mathcal{M}P[2^n, 2^{n+1}]$  such that

$$S_n = \sum_{i=1}^{10} [X_i^{(n)}, Y_i^{(n)}]$$

and for all  $i$ ,  $\|X_i^{(n)}\| \leq 12\|S_n\|$  and  $\|Y_i^{(n)}\| \leq 2$ . We therefore have

$$T - A = \sum_{i=1}^{10} [X_i, Y_i],$$

where

$$X_i = \sum_{n \in \mathbb{Z}} X_i^{(n)}, \quad Y_i = \sum_{n \in \mathbb{Z}} Y_i^{(n)}.$$

Clearly  $Y_i \in \mathcal{M}$ . Since  $\|X_i^{(n)}\| \leq 12\|T_n\| \leq 12\mu_{2^n}(T)$ , it follows that  $\mu_{2^{n+1}}(X_i) \leq 12\mu_{2^n}(T)$ , and therefore that  $\mu(X_i) \in \Lambda$ .

It remains to show that  $A$  is a sum of four commutators. For  $t > 0$  let  $F_t = E_{|T|}(\mu_t(T), \infty)$ . For  $k, \ell \in \mathbb{Z}$ ,  $k < \ell$ , using the hypothesis (4.3) we get

$$\begin{aligned} \left| \sum_{j=k}^{\ell-1} 2^j \alpha_j \right| &= |\tau(T(P_{2^\ell} - P_{2^k}))| \\ &\leq |\tau(T(F_{2^\ell} - F_{2^k}))| + |\tau(T(P_{2^k} - F_{2^k}))| + |\tau(T(P_{2^\ell} - F_{2^\ell}))| \\ &\leq 2^k h(2^k) + 2^\ell h(2^\ell) + 2^k \mu_{2^k}(T) + 2^\ell \mu_{2^\ell}(T). \end{aligned}$$

Letting  $\phi(t) = h(t) + \mu_t(T)$ , we have  $\phi \in \Lambda$  and

$$(4.5) \quad \left| \sum_{j=k}^{\ell-1} 2^j \alpha_j \right| \leq 2^k \phi(2^k) + 2^\ell \phi(2^\ell).$$

We will now write  $\text{Re } A$  as a sum of two commutators. Note that inequality (4.5) continues to hold when each  $\alpha_j$  is replaced by  $\text{Re } \alpha_j$ . We will find real numbers  $\beta_n$  satisfying

$$(4.6) \quad \text{Re } \alpha_n = \beta_{n-1} - 2\beta_n, \quad |\beta_n| \leq \phi(2^n), \quad (n \in \mathbb{Z}).$$

Treating  $\beta_0$  as the independent variable, solving the equality in (4.6) recursively yields

$$\begin{aligned} \beta_{-m} &= 2^m \beta_0 + 2^{m-1} \sum_{j=-m+1}^0 2^j \text{Re } \alpha_j \\ \beta_m &= 2^{-m} \beta_0 - 2^{-m-1} \sum_{j=1}^m 2^j \text{Re } \alpha_j \end{aligned} \quad (m \geq 1).$$

The condition  $|\beta_n| \leq \phi(2^n)$  for all  $n \in \mathbb{Z}$  is thus equivalent to the inequalities

$$\begin{aligned} -2^{-m} \phi(2^{-m}) - \frac{1}{2} \sum_{j=-m+1}^0 2^j \text{Re } \alpha_j &\leq \beta_0 \leq 2^{-m} \phi(2^{-m}) - \frac{1}{2} \sum_{j=-m+1}^0 2^j \text{Re } \alpha_j \\ -2^m \phi(2^m) + \frac{1}{2} \sum_{j=1}^m 2^j \text{Re } \alpha_j &\leq \beta_0 \leq 2^m \phi(2^m) + \frac{1}{2} \sum_{j=1}^m 2^j \text{Re } \alpha_j \end{aligned}$$

for all  $m \in \mathbb{N}$ . The existence of a real number  $\beta_0$  satisfying all of these relations is equivalent to the following four inequalities holding for all integers  $k, \ell \geq 1$ :

$$(4.7) \quad -2^{-k}\phi(2^{-k}) - \frac{1}{2} \sum_{j=-k+1}^0 2^j \operatorname{Re} \alpha_j \leq 2^{-\ell}\phi(2^{-\ell}) - \frac{1}{2} \sum_{j=-\ell+1}^0 2^j \operatorname{Re} \alpha_j$$

$$(4.8) \quad -2^{-k}\phi(2^{-k}) - \frac{1}{2} \sum_{j=-k+1}^0 2^j \operatorname{Re} \alpha_j \leq 2^\ell\phi(2^\ell) + \frac{1}{2} \sum_{j=1}^\ell 2^j \operatorname{Re} \alpha_j$$

$$(4.9) \quad -2^k\phi(2^k) + \frac{1}{2} \sum_{j=1}^k 2^j \operatorname{Re} \alpha_j \leq 2^{-\ell}\phi(2^{-\ell}) - \frac{1}{2} \sum_{j=-\ell+1}^0 2^j \operatorname{Re} \alpha_j$$

$$(4.10) \quad -2^k\phi(2^k) + \frac{1}{2} \sum_{j=1}^k 2^j \operatorname{Re} \alpha_j \leq 2^\ell\phi(2^\ell) + \frac{1}{2} \sum_{j=1}^\ell 2^j \operatorname{Re} \alpha_j.$$

But these inequalities are easily verified. For example, (4.7) is equivalent to

$$(4.11) \quad \frac{1}{2} \sum_{j=-\ell+1}^{-k} 2^j \operatorname{Re} \alpha_j \leq 2^{-\ell}\phi(2^{-\ell}) + 2^{-k}\phi(2^{-k}) \quad \text{if } k < \ell$$

$$(4.12) \quad -2^{-k}\phi(2^{-k}) - 2^{-\ell}\phi(2^{-\ell}) \leq \frac{1}{2} \sum_{j=-k+1}^{-\ell} 2^j \operatorname{Re} \alpha_j \quad \text{if } k > \ell,$$

while (4.8) is equivalent to

$$(4.13) \quad -2^{-k}\phi(2^{-k}) - 2^\ell\phi(2^\ell) \leq \frac{1}{2} \sum_{j=-k+1}^\ell 2^j \operatorname{Re} \alpha_j;$$

keeping in mind that  $\phi$  is nonnegative and nonincreasing, inequalities (4.11)–(4.13) follow directly from (4.5). Inequalities (4.9) and (4.10) are verified for all  $k$  and  $\ell$  similarly. We have succeeded in proving the existence of  $\beta_n$  satisfying (4.6).

Now let  $V_n, W_n \in \mathcal{M}$ , ( $n \in \mathbb{Z}$ ), be such that

$$\begin{aligned} V_n^* V_n &= P[2^{n-1}, 2^n], & V_n V_n^* &= P[2^n, 2^n + 2^{n-1}], \\ W_n^* W_n &= P[2^{n-1}, 2^n], & W_n W_n^* &= P[2^n + 2^{n-1}, 2^{n+1}] \end{aligned}$$

and let

$$\begin{aligned} X_{11} &= \sum_{n \in \mathbb{Z}} \beta_{n-1} V_n & Y_{11} &= \sum_{n \in \mathbb{Z}} V_n^* \\ X_{12} &= \sum_{n \in \mathbb{Z}} \beta_{n-1} W_n & Y_{12} &= \sum_{n \in \mathbb{Z}} W_n^*. \end{aligned}$$

Then  $X_i \in \overline{\mathcal{M}}$ ,  $\mu(X_i) \in \Lambda$  and  $Y_i \in \mathcal{M}$  ( $i = 11, 12$ ), and

$$[X_{11}, Y_{11}] + [X_{12}, Y_{12}] = \operatorname{Re} A.$$

We may do the same for  $\operatorname{Im} A$ . □

We now prove an analogous result in a  $\text{II}_1$ -factor.

LEMMA 4.3. — *Let  $\mathcal{M}$  be a  $\text{II}_1$ -factor with tracial state  $\tau$  and let  $T \in \overline{\mathcal{M}}$  be a normal operator. Suppose there is  $h \in D^+(0, 1)$  such that*

$$(4.14) \quad |\tau(TE_{|T|}[0, \mu_r(T)])| \leq rh(r), \quad (0 < r \leq 1).$$

*Let  $\Lambda$  be the characteristic set in  $D^+(0, 1)$  generated by  $h$  and  $\mu(T)$ . Then there are  $X_1, \dots, X_{12} \in \overline{\mathcal{M}}$  and  $Y_1, \dots, Y_{12} \in \mathcal{M}$  such that*

$$(4.15) \quad T = \sum_{i=1}^{12} [X_i, Y_i]$$

*and  $\mu(X_i) \in \Lambda$  for all  $i$ .*

*Proof.* — Lemma 2.6 (formally applied in  $\mathcal{M} \otimes B(\mathcal{H})$ , if we like) gives a family of projections  $(P_t)_{0 \leq t \leq 1}$  satisfying (i)–(v) of that proposition. Let  $P[s, t] = P_t - P_s$  ( $s < t$ ), let

$$\alpha_n = \tau(TP[2^n, 2^{n+1}]), \quad (n \in \mathbb{Z}, n < 0)$$

and let

$$A = \sum_{n=-\infty}^{-1} \alpha_n P[2^n, 2^{n+1}].$$

Applying the result of Fack and de la Harpe [11] as in the proof of Lemma 4.2, we can show

$$T - A = \sum_{i=1}^{10} [X_i, Y_i]$$

with  $Y_i \in \mathcal{M}$ ,  $X_i \in \overline{\mathcal{M}}$  and  $\mu(X_i) \in \Lambda$ . Letting  $F_t = E_{|T|}(\mu_t(T), \infty)$  and using the hypothesis (4.14), for  $n \in \mathbb{Z}$ ,  $n \leq -1$  we have

$$\begin{aligned} \left| \sum_{j=n}^{-1} 2^j \alpha_j \right| &= |\tau(T(1 - P_{2^n}))| \\ &\leq |\tau(T(1 - F_{2^n}))| + |\tau(T(P_{2^n} - F_{2^n}))| \\ &\leq 2^n h(2^n) + 2^n \mu_{2^n}(T). \end{aligned}$$

Let  $\beta_{-1} = 0$  and

$$\beta_n = 2^{-n-1} \sum_{j=n+1}^{-1} 2^j \alpha_j, \quad (n \in \mathbb{Z}, n \leq -2).$$

Then we have

$$|\beta_n| \leq \frac{1}{2} (h(2^{n+1}) + \mu_{2^{n+1}}(T))$$

and

$$\beta_{n-1} - 2\beta_n = \alpha_n, \quad (n \leq -1).$$

Let  $V_n, W_n \in \mathcal{M}$  ( $n \leq -1$ ) be as in the proof of Lemma 4.2 and let

$$\begin{aligned} X_{11} &= \sum_{n=-\infty}^{-1} \beta_{n-1} V_n & Y_{11} &= \sum_{n=-\infty}^{-1} V_n \\ X_{12} &= \sum_{n=-\infty}^{-1} \beta_{n-1} W_n & Y_{12} &= \sum_{n=-\infty}^{-1} W_n. \end{aligned}$$

Then  $X_i \in \overline{\mathcal{M}}$ ,  $\mu(X_i) \in \Lambda$  and  $Y_i \in \mathcal{M}$  ( $i = 11, 12$ ), and

$$[X_{11}, Y_{11}] + [X_{12}, Y_{12}] = A. \quad \square$$

The following result is well known and indeed follows from the stronger result of Halpern [17].

PROPOSITION 4.4. — *If  $\mathcal{M}$  is a  $\text{II}_\infty$ -factor, then  $[\mathcal{M}, \mathcal{M}] = \mathcal{M}$ .*

LEMMA 4.5. — *Let  $\mathcal{M}$  be a  $\text{II}_1$ - or a  $\text{II}_\infty$ -factor, and let  $\mathcal{I} \subseteq \overline{\mathcal{M}}$  and  $\mathcal{J} \subseteq \overline{\mathcal{M}}$  be submodules. Then  $[\mathcal{I}\mathcal{J}, \mathcal{M}] \subseteq [\mathcal{I}, \mathcal{J}]$ .*

*Proof.* — If  $X \in \mathcal{I}\mathcal{J}$  then  $X = AB$  for  $A \in \mathcal{I}$  and  $B \in \mathcal{J}$ . This can be seen by writing  $X = V|X|$  for a partial isometry  $V$ .

If also  $Y \in \mathcal{M}$ , then we have

$$[X, Y] = ABY - YAB = [A, BY] + [B, YA] \in [\mathcal{I}, \mathcal{J}]. \quad \square$$

THEOREM 4.6. — *Let  $\mathcal{M}$  be a type  $\text{II}_1$  factor and let  $\mathcal{I} \subseteq \overline{\mathcal{M}}$  and  $\mathcal{J} \subseteq \overline{\mathcal{M}}$  be submodules. Let  $T \in \mathcal{I}\mathcal{J}$  be normal. Then  $T \in [\mathcal{I}, \mathcal{J}]$  if and only if there is  $h \in \mu(\mathcal{I}\mathcal{J})$  such that*

$$(4.16) \quad |\tau(TE_{|T|}[0, \mu_r(T)])| \leq rh(r), \quad (0 < r < 1).$$



*Proof.* — We may embed  $\mathcal{M}$  in the  $\text{II}_\infty$  factor  $\mathcal{M} \otimes B(\mathcal{H})$  in a trace-preserving manner. If  $T \in [\mathcal{I}, \mathcal{J}]$  then letting  $h$  be as in Lemma 4.1, we have  $h \in \mu(\mathcal{I}\mathcal{J})$  with  $h(1) = 0$ . So taking  $s = 1$  in equation (4.16), we get that  $h$  satisfies (4.16).

Now suppose  $h \in \mu(\mathcal{I}\mathcal{J})$  is such that (4.16) holds. By Lemma 4.3,  $T \in [\mathcal{I}\mathcal{J}, \mathcal{M}]$ . Now Lemma 4.5 gives  $T \in [\mathcal{I}, \mathcal{J}]$ . □

Compare the following to Theorem 1 of [10].

**THEOREM 4.7.** — *Let  $\mathcal{M}$  be a type  $\text{II}_\infty$  factor and let  $\mathcal{I} \subseteq \overline{\mathcal{M}}$  and  $\mathcal{J} \subseteq \overline{\mathcal{M}}$  be submodules. Let  $T \in \mathcal{I}\mathcal{J}$  be normal. Then  $T \in [\mathcal{I}, \mathcal{J}]$  if and only if there is  $h \in \mu(\mathcal{I}\mathcal{J})$  such that*

$$(4.17) \quad |\tau(TE_{|T|}(\mu_s(T), \mu_r(T)))| \leq rh(r) + sh(s), \quad (0 < r < s < \infty).$$

*Proof.* — If  $T \in [\mathcal{I}, \mathcal{J}]$  then by Lemma 4.1 there is  $h \in \mu(\mathcal{I}\mathcal{J})$  satisfying (4.17).

Now suppose  $h \in \mu(\mathcal{I}\mathcal{J})$  is such that (4.17) holds, and let us show  $T \in [\mathcal{I}\mathcal{J}, \mathcal{M}]$ . If  $\lim_{t \rightarrow \infty} \mu_t(T) = 0$ , then by Lemma 4.2 we have  $T \in [\mathcal{I}\mathcal{J}, \mathcal{M}]$ . Suppose  $d := \lim_{t \rightarrow \infty} \mu_t(T) > 0$ . If  $T$  is bounded, then by Halpern’s result, Proposition 4.4, we have  $T \in [\mathcal{M}, \mathcal{M}] \subseteq [\mathcal{I}\mathcal{J}, \mathcal{M}]$ . Suppose  $T$  is unbounded, let  $a > 0$  be such that  $\mu_a(T) > d$  and let  $Q = E_{|T|}(\mu_a(T), \infty)$ . Then  $0 < \tau(Q) \leq a$ ,  $QT = TQ$  and  $\|(1 - Q)T\| \leq \mu_a(T)$ . By Proposition 4.4,  $(1 - Q)T \in [\mathcal{I}\mathcal{J}, \mathcal{M}]$ . We have

$$\mu_t(QT) = \begin{cases} \mu_t(T) & \text{if } t < \tau(Q) \\ 0 & \text{if } t \geq \tau(Q). \end{cases}$$

Let  $0 < r < s < \infty$ . Then

$$(QT)E_{|QT|}(\mu_s(QT), \mu_r(QT)) = \begin{cases} TE_{|T|}(\mu_s(T), \mu_r(T)) & \text{if } s < \tau(Q) \\ TE_{|T|}(\mu_a(T), \mu_r(T)) & \text{if } r < \tau(Q) \leq s \\ 0 & \text{if } \tau(Q) \leq r. \end{cases}$$

If  $r < \tau(Q) \leq s$  then we have

$$\begin{aligned} |\tau(TE_{|T|}(\mu_a(T), \mu_r(T)))| &\leq rh(r) + ah(a) \leq rh(r) + ah(\tau(Q)) \\ &\leq rh(r) + s \frac{ah(\tau(Q))}{\tau(Q)}. \end{aligned}$$

Let  $\tilde{h}(t) = \max(h(t), \frac{ah(\tau(Q))}{\tau(Q)})$ . Then  $\tilde{h} \in \mu(\mathcal{I}\mathcal{J})$  and we have

$$|\tau((QT)E_{|QT|}(\mu_s(QT), \mu_r(QT)))| \leq r\tilde{h}(r) + s\tilde{h}(s), \quad (0 < r < s < \infty).$$

Now Lemma 4.2 implies  $QT \in [\mathcal{I}\mathcal{J}, \mathcal{M}]$ .

We have shown  $T \in [\mathcal{I}\mathcal{J}, \mathcal{M}]$ . From Lemma 4.5, it follows that  $T$  belongs to  $[\mathcal{I}, \mathcal{J}]$ . □

COROLLARY 4.8. — *Let  $\mathcal{M}$  be a type  $\text{II}_\infty$  factor or a type  $\text{II}_1$  factor, and let  $\mathcal{I} \subseteq \overline{\mathcal{M}}$  and  $\mathcal{J} \subseteq \overline{\mathcal{M}}$  be submodules. Then*

$$[\mathcal{I}, \mathcal{J}] = [\mathcal{I}\mathcal{J}, \mathcal{M}].$$

### 5. Separated asymptotic behaviour.

Throughout this section,  $\mathcal{M}$  will be a type  $\text{II}_\infty$  factor with semifinite trace  $\tau$  and  $\mathcal{I} \subseteq \overline{\mathcal{M}}$  will be a nonzero submodule. Theorem 4.7 gives a necessary and sufficient condition for a normal operator  $T$  to belong to the commutator space  $[\mathcal{I}, \mathcal{M}]$ , but this condition considers simultaneous asymptotics at 0 and  $\infty$ . In this section, we give an equivalent characterization which separates the behaviour at 0 and  $\infty$ .

We have

$$(5.1) \quad \mathcal{I} = \mathcal{I}_{f_s} + \mathcal{I}_b$$

where

$$\begin{aligned} \mathcal{I}_{f_s} &= \{T \in \mathcal{I} \mid \mu_s(T) = 0 \text{ for some } s > 0\} \\ \mathcal{I}_b &= \{T \in \mathcal{I} \mid \mu(T) \text{ bounded}\}. \end{aligned}$$

Thus  $\mathcal{I}_{f_s}$  is the set of  $T \in \mathcal{I}$  that are supported on finite projections and  $\mathcal{I}_b = \mathcal{I} \cap \mathcal{M}$ . From (5.1), we have

$$(5.2) \quad [\mathcal{I}, \mathcal{M}] = [\mathcal{I}_{f_s}, \mathcal{M}] + [\mathcal{I}_b, \mathcal{M}].$$

Given a normal element  $T \in \mathcal{I}$ , using a spectral projection of  $|T|$  we can easily write  $T = T_{f_s} + T_b$  for some normal elements  $T_{f_s} \in \mathcal{I}_{f_s}$  and  $T_b \in \mathcal{I}_b$ . It is our purpose to use Theorem 4.7 to give necessary and sufficient conditions for  $T \in [\mathcal{I}, \mathcal{M}]$  in terms of  $T_{f_s}$  and  $T_b$ .

LEMMA 5.1. — *Let  $\mathcal{I} \subseteq \overline{\mathcal{M}}$  be a submodule.*

(i) *Let  $T \in \mathcal{I}_{f_s}$  be normal. Then  $T \in [\mathcal{I}_{f_s}, \mathcal{M}]$  if and only if there is  $h \in \mu(\mathcal{I}_{f_s})$  such that*

$$(5.3) \quad |\tau(TE_{|T|}[0, \mu_r(T)])| \leq rh(r), \quad (0 < r < \infty).$$

(ii) Let  $T \in \mathcal{I}_b$  be normal. Then  $T \in [\mathcal{I}_b, \mathcal{M}]$  if and only if there is  $h \in \mu(\mathcal{I}_b)$  such that

$$(5.4) \quad |\tau(TE_{|T|}(\mu_s(T), \infty))| \leq sh(s), \quad (0 < s < \infty).$$

*Proof.* — Let us prove (i). If  $T \in [\mathcal{I}_{f_s}, \mathcal{M}]$ , then invoking Theorem 4.7 and letting  $s \rightarrow \infty$ , since  $\mu_s(T)$  and  $h(s)$  are eventually zero we obtain

$$|\tau(TE_{|T|}(0, \mu_r(T))| \leq rh(r),$$

which clearly implies (5.3). On the other hand, if (5.3) holds, then for  $0 < r < s < \infty$  we have

$$\begin{aligned} |\tau(TE_{|T|}(\mu_s(T), \mu_r(T))| &\leq |\tau(TE_{|T|}[0, \mu_r(T))| + |\tau(TE_{|T|}[0, \mu_s(T))| \\ &\leq rh(r) + sh(s), \end{aligned}$$

so  $T \in [\mathcal{I}_{f_s}, \mathcal{M}]$  by Theorem 4.7.

For (ii), if  $T \in [\mathcal{I}_b, \mathcal{M}]$ , then invoking Theorem 4.7 and letting  $r \rightarrow 0$ , we get

$$|\tau(TE_{|T|}(\mu_s(T), \infty))| = |\tau(TE_{|T|}(\mu_s(T), \|T\|))| \leq sh(s),$$

since  $h(r)$  stays bounded. Thus (5.4) holds. The argument that (5.4) implies  $T \in [\mathcal{I}_b, \mathcal{M}]$  is similar to the analogous one in case (i).  $\square$

Recall (Remark 3.3)  $\mathcal{F}$  denotes the submodule (in fact, the ideal of  $\mathcal{M}$ ) consisting of  $\tau$ -finite rank bounded operators:  $\mathcal{F} = \mathcal{M}_{f_s}$ .

COROLLARY 5.2. —  $[\mathcal{F}, \mathcal{M}] = \mathcal{F} \cap \ker \tau$ .

*Proof.* — It will suffice to show that if  $T = T^* \in \mathcal{F}$ , then  $T \in [\mathcal{F}, \mathcal{M}]$  if and only if  $\tau(T) = 0$ . Suppose  $T \in [\mathcal{F}, \mathcal{M}]$  and let  $h \in \mu(\mathcal{F})$  be such that (5.4) holds. Then  $h(s) = 0$  and  $\mu_s(T) = 0$  for some  $s > 0$  and therefore  $\tau(T) = \tau(TE_{|T|}(\mu_s(T), \infty)) = 0$ .

Suppose  $\tau(T) = 0$ . Then  $\mu_{s'}(T) = 0$  for some  $s' > 0$ . Let

$$h(s) = \begin{cases} \|T\| & s < s' \\ 0 & s \geq s'. \end{cases}$$

Then  $h \in \mu(\mathcal{F})$ . Using (2.2), we see that (5.4) holds when  $0 < s < s'$ , and it holds when  $s \geq s'$  because  $\tau(T) = 0$ .  $\square$

See Definition 2.4 for an explanation of the notation  $\oplus$  used below.

**THEOREM 5.3.** — *Let  $\mathcal{I} \subseteq \overline{\mathcal{M}}$  be a nonzero submodule and let  $T = T_{f_s} + T_b \in \mathcal{I}$ , where  $T_{f_s} \in \mathcal{I}_{f_s}$  and  $T_b \in \mathcal{I}_b$ . Then the following are equivalent:*

(a)  $T \in [\mathcal{I}, \mathcal{M}]$ .

(b) There is  $X \in \mathcal{F}$  such that

$$(5.5) \quad \begin{aligned} T_b \oplus X &\in [\mathcal{I}_b, \mathcal{M}] \\ T_{f_s} \oplus (-X) &\in [\mathcal{I}_{f_s}, \mathcal{M}]. \end{aligned}$$

(c) There is  $a \in \mathbb{C}$  such that whenever  $X, Y \in \mathcal{F}$ ,  $\tau(X) \neq 0$  and  $\tau(Y) \neq 0$ ,

$$(5.6) \quad T_b \oplus \frac{a}{\tau(X)}X \in [\mathcal{I}_b, \mathcal{M}]$$

$$(5.7) \quad T_{f_s} \oplus \frac{-a}{\tau(Y)}Y \in [\mathcal{I}_{f_s}, \mathcal{M}].$$

*Proof.* — We first prove (a)  $\implies$  (c). Suppose  $T \in [\mathcal{I}, \mathcal{M}]$ . From (5.2), we have  $T = \tilde{T}_{f_s} + \tilde{T}_b$  for some  $\tilde{T}_{f_s} \in [\mathcal{I}_{f_s}, \mathcal{M}]$  and  $\tilde{T}_b \in [\mathcal{I}_b, \mathcal{M}]$ . Then using Corollary 5.2,

$$\tilde{T}_b - T_b = T_{f_s} - \tilde{T}_{f_s} \in \mathcal{I}_b \cap \mathcal{I}_{f_s} = \mathcal{F}.$$

Let  $a = \tau(\tilde{T}_b - T_b)$  and let  $X \in \mathcal{F}$  with  $\tau(X) \neq 0$ . Then

$$\tilde{T}_b - T_b - \frac{a}{\tau(X)}X \in \mathcal{F} \cap \ker \tau = [\mathcal{F}, \mathcal{M}] \subseteq [\mathcal{I}_b, \mathcal{M}].$$

Thus

$$\begin{aligned} T_b \oplus \frac{a}{\tau(X)}X &\in (T_b \oplus \frac{a}{\tau(X)}X \oplus 0) + [\mathcal{I}_b, \mathcal{M}] \\ &= (T_b \oplus \frac{a}{\tau(X)}X \oplus (\tilde{T}_b - T_b - \frac{a}{\tau(X)}X)) + [\mathcal{I}_b, \mathcal{M}] \\ &= (T_b \oplus (\tilde{T}_b - T_b)) + [\mathcal{I}_b, \mathcal{M}] \\ &= \tilde{T}_b + [\mathcal{I}_b, \mathcal{M}] = [\mathcal{I}_b, \mathcal{M}] \end{aligned}$$

and (5.6) holds. Similarly, we have

$$\begin{aligned} T_{f_s} \oplus \frac{-a}{\tau(Y)}Y &\in (T_{f_s} \oplus \frac{-a}{\tau(Y)}Y \oplus 0) + [\mathcal{I}_{f_s}, \mathcal{M}] \\ &= (T_{f_s} \oplus \frac{-a}{\tau(Y)}Y \oplus (\tilde{T}_{f_s} - T_{f_s} + \frac{a}{\tau(Y)}Y)) + [\mathcal{I}_{f_s}, \mathcal{M}] \\ &= (T_{f_s} \oplus (\tilde{T}_{f_s} - T_{f_s})) + [\mathcal{I}_{f_s}, \mathcal{M}] \\ &= \tilde{T}_{f_s} + [\mathcal{I}_{f_s}, \mathcal{M}] = [\mathcal{I}_{f_s}, \mathcal{M}] \end{aligned}$$

and (5.7) holds.

The implication (c)  $\implies$  (b) is clear.

For (b)  $\implies$  (a), assuming (5.5), we have

$$T_{f_S} + T_b \in T_{f_S} \oplus T_b + [\mathcal{I}, \mathcal{M}] = T_{f_S} \oplus (-X) \oplus X \oplus T_b + [\mathcal{I}, \mathcal{M}] = [\mathcal{I}, \mathcal{M}]. \quad \square$$

LEMMA 5.4. — *Let  $\mathcal{I} \subseteq \overline{\mathcal{M}}$  be a nonzero submodule, let  $T \in \mathcal{I}_{f_S}$  be normal,  $T \neq 0$  and let  $a \in \mathbb{C}$ . Let  $P \in \mathcal{F}$  be a nonzero projection such that either  $T$  is unbounded or  $\frac{|a|}{\tau(P)} < \|T\|$ . Then*

$$T \oplus \frac{a}{\tau(P)}P \in [\mathcal{I}_{f_S}, \mathcal{M}]$$

if and only if there is  $h \in \mu(\mathcal{I}_{f_S})$  such that

$$(5.8) \quad \forall r \in (0, 1), \quad |a + \tau(TE_{|T|}[0, \mu_r(T)])| \leq rh(r).$$

Remark 5.5. — As will be apparent from the proof, for any  $r' > 0$  the existence of  $h \in \mu(\mathcal{I}_{f_S})$  such that (5.8) holds is equivalent to the existence of  $h' \in \mu(\mathcal{I}_{f_S})$  such that

$$\forall r \in (0, r'), \quad |a + \tau(TE_{|T|}[0, \mu_r(T)])| \leq rh'(r)$$

holds.

*Proof of Lemma 5.4.* — There is  $r' > 0$  such that  $\mu_r(T) > \frac{|a|}{\tau(P)}$  for all  $r \in (0, r')$ . Let  $T' = T \oplus \frac{a}{\tau(P)}P$ . Then (by Proposition 2.5), for  $r \in (0, r')$  we have  $\mu_r(T') = \mu_r(T)$ ,

$$\begin{aligned} E_{|T'|}[0, \mu_r(T')] &= E_{|T|}[0, \mu_r(T)] \oplus P \\ \tau(T'E_{|T'|}[0, \mu_r(T')]) &= a + \tau(TE_{|T|}[0, \mu_r(T))). \end{aligned}$$

If  $T' \in [\mathcal{I}_{f_S}, \mathcal{M}]$ , then by Lemma 5.1, there is  $h' \in \mu(\mathcal{I}_{f_S})$  such that

$$\forall r \in (0, r'), \quad |a + \tau(TE_{|T|}[0, \mu_r(T)])| \leq rh'(r).$$

Since  $d := \tau(E_{|T|}(0, \infty)) < \infty$  and

$$|\tau(TE_{|T|}[0, \mu_r(T)])| \leq \mu_r(T)d$$

for all  $r > 0$ , we can find  $h \in \mu(\mathcal{I}_{f_S})$  such that (5.8) holds.

Conversely, suppose  $h \in \mu(\mathcal{I}_{f_S})$  is such that (5.8) holds. Assume without loss of generality  $r' \leq 1$ . Then we have

$$|\tau(T'E_{|T'|}[0, \mu_r(T')])| \leq rh(r)$$

for all  $r \in (0, r')$ . Let  $r'' > r'$  be such that  $\mu_{r''}(T') = 0$ . Let  $d' = E_{|T'|}(0, \infty)$ . Then

$$|\tau(T'E_{|T'|}[0, \mu_r(T')])| \leq \begin{cases} 0 & \text{if } r \geq r'' \\ \mu_r(T')d' & \text{otherwise.} \end{cases}$$

Letting

$$h'(t) = \begin{cases} \max(h(t), \frac{\mu_{r'}(T')d'}{r'}) & \text{if } 0 < t < r' \\ \frac{\mu_{r'}(T')d'}{r'} & \text{if } r' \leq t < r'' \\ 0 & \text{if } r'' \leq t, \end{cases}$$

we have  $h' \in \mu(\mathcal{I}_{f_s})$  and

$$|\tau(T'E_{|T'|}[0, \mu_r(T')])| \leq rh'(r)$$

for all  $r > 0$ . Thus  $T' \in [\mathcal{I}_{f_s}, \mathcal{M}]$  by Lemma 5.1. □

LEMMA 5.6. — *Let  $\mathcal{I} \subseteq \overline{\mathcal{M}}$  be a nonzero submodule, let  $T \in \mathcal{I}_b$  be normal and let  $a \in \mathbb{C}$ . If  $a \neq 0$ , let  $P \in \mathcal{F}$  be a projection such that  $\frac{|a|}{\tau(P)} > \|T\|$ . If  $a = 0$ , let  $P \in \mathcal{F}$  have nonzero trace. Then*

$$T \oplus \frac{a}{\tau(P)}P \in [\mathcal{I}_b, \mathcal{M}]$$

if and only if there is  $h \in \mu(\mathcal{I}_b)$  such that

$$(5.9) \quad \forall s \in [1, \infty), \quad |a + \tau(T'E_{|T'|}(\mu_s(T), \infty))| \leq sh(s).$$

Remark 5.7. — As will be apparent from the proof, for any  $s' > 0$  the existence of  $h \in \mu(\mathcal{I}_b)$  such that (5.9) holds is equivalent to the existence of  $h' \in \mu(\mathcal{I}_b)$  such that

$$\forall s \in [s', \infty), \quad |a + \tau(T'E_{|T'|}(\mu_s(T), \infty))| \leq sh'(s)$$

holds.

Proof of Lemma 5.6. — Suppose  $a \neq 0$ . Let  $T' = T \oplus \frac{a}{\tau(P)}P$ . Then for all  $s > 0$ , we have, (by Proposition 2.5),  $\mu_{s+\tau(P)}(T') = \mu_s(T)$ ,

$$(5.10) \quad \begin{aligned} E_{|T'|}(\mu_{s+\tau(P)}(T'), \infty) &= E_{|T'|}(\mu_s(T), \infty) \oplus P, \\ \tau(T'E_{|T'|}(\mu_{s+\tau(P)}(T'), \infty)) &= a + \tau(T'E_{|T'|}(\mu_s(T), \infty)). \end{aligned}$$

If  $T' \in [\mathcal{I}_b, \mathcal{M}]$ , then it follows from (5.10) and Lemma 5.1 that there is  $h' \in \mu(\mathcal{I}_b)$  such that

$$\forall s \in (0, \infty), \quad |a + \tau(T'E_{|T'|}(\mu_s(T), \infty))| \leq (s + \tau(P))h'(s + \tau(P)).$$

Letting  $h(s) = (1 + \tau(P))h'(s + \tau(p))$ , we have  $h \in \mu(\mathcal{I}_b)$  and that (5.9) holds.

On the other hand, still taking  $a \neq 0$ , suppose  $h \in \mu(\mathcal{I}_b)$  and (5.9) holds. Using (5.10), we have

$$|\tau(T'E_{|T'|}(\mu_t(T'), \infty))| \leq (t - \tau(P))h(t - \tau(P))$$

for all  $t \geq 1 + \tau(P)$ . Using Proposition 2.2, we have

$$|\tau(T'E_{|T'|}(\mu_t(T'), \infty))| \leq \|T'\|t$$

for all  $t > 0$ . Therefore, letting

$$h'(t) = \begin{cases} \frac{1}{1 + \tau(P)}h(t - \tau(P)) & \text{if } t \geq 1 + \tau(P) \\ \max\left(\frac{|a|}{\tau(P)}, \frac{h(1)}{1 + \tau(P)}\right) & \text{if } 0 < t < 1 + \tau(P), \end{cases}$$

we get  $h' \in \mu(\mathcal{I}_b)$  and

$$|\tau(T'E_{|T'|}(\mu_t(T'), \infty))| \leq th'(t)$$

for all  $t > 0$ . Thus  $T' \in [\mathcal{I}_b, \mathcal{M}]$  by Lemma 5.1.

When  $a = 0$ , the existence of  $h \in \mu(\mathcal{I}_b)$  satisfying (5.9) follows from  $T \in [\mathcal{I}_b, \mathcal{M}]$  directly from Lemma 5.1, while proving that the existence of  $h \in \mu(\mathcal{I}_b)$  such that (5.9) holds implies  $T \in [\mathcal{I}_b, \mathcal{M}]$  is similar to the case  $a \neq 0$ , but easier. □

**THEOREM 5.8.** — *Let  $\mathcal{I} \subseteq \overline{\mathcal{M}}$  be a nonzero submodule and let  $T = T_{f_s} + T_b \in \mathcal{I}$ , where  $T_{f_s} \in \mathcal{I}_{f_s}$  and  $T_b \in \mathcal{I}_b$  are normal. Then  $T \in [\mathcal{I}, \mathcal{M}]$  if and only if there are  $a \in \mathbb{C}$ ,  $h_{f_s} \in \mu(\mathcal{I}_{f_s})$  and  $h_b \in \mu(\mathcal{I}_b)$  such that*

$$(5.11) \quad \forall r \in (0, 1), \quad |a - \tau(T_{f_s}E_{|T_{f_s}|}[0, \mu_r(T_{f_s})])| \leq rh_{f_s}(r)$$

$$(5.12) \quad \forall s \in [1, \infty), \quad |a + \tau(T_bE_{|T_b|}(\mu_s(T_b), \infty))| \leq sh_b(s).$$

*Proof.* — If  $T_{f_s} \neq 0$ , then the conclusion of the theorem follows from Theorem 5.3 and Lemmas 5.4 and 5.6. If  $T_{f_s} = 0$ , then we choose  $a = 0$  and apply Lemma 5.1. □

Let  $\omega_{f_s}, \omega_b \in D^+(0, \infty)$  be given by

$$\omega_{f_s}(t) = \begin{cases} 1/t & \text{if } t < 1 \\ 0 & \text{if } t \geq 1, \end{cases}$$

$$\omega_b(t) = \frac{1}{1 + t}.$$

COROLLARY 5.9. — Let  $\mathcal{I} \subseteq \overline{\mathcal{M}}$  be a nonzero submodule and let  $T = T_{f_s} + T_b \in \mathcal{I}$ , where  $T_{f_s} \in \mathcal{I}_{f_s}$  and  $T_b \in \mathcal{I}_b$  are normal.

(I) Suppose  $\omega_{f_s}, \omega_b \in \mu(\mathcal{I})$ . Then  $T \in [\mathcal{I}, \mathcal{M}]$  if and only if  $T_{f_s} \in [\mathcal{I}_{f_s}, \mathcal{M}]$  and  $T_b \in [\mathcal{I}_b, \mathcal{M}]$ .

(II) Suppose  $\omega_{f_s} \in \mu(\mathcal{I})$  and  $\omega_b \notin \mu(\mathcal{I})$ . Then  $T \in [\mathcal{I}, \mathcal{M}]$  if and only if  $T_{f_s} \in [\mathcal{I}_{f_s}, \mathcal{M}]$  and there are  $a \in \mathbb{C}$  and  $h_b \in \mu(\mathcal{I}_b)$  such that (5.12) holds.

(III) Suppose  $\omega_{f_s} \notin \mu(\mathcal{I})$  and  $\omega_b \in \mu(\mathcal{I})$ . Then  $T \in [\mathcal{I}, \mathcal{M}]$  if and only if  $T_b \in [\mathcal{I}_b, \mathcal{M}]$  and there are  $a \in \mathbb{C}$  and  $h_{f_s} \in \mu(\mathcal{I}_{f_s})$  such that (5.11) holds.

*Proof.* — If  $\omega_b \in \mu(\mathcal{I})$ , then for any  $a \in \mathbb{C}$ , the function

$$t \mapsto \begin{cases} |a|/t, & 0 < t < 1, \\ 0, & t \geq 1 \end{cases}$$

lies in  $\mu(\mathcal{I}_{f_s})$ , while if  $\omega_{f_s} \in \mu(\mathcal{I})$ , then for any  $a \in \mathbb{C}$ , the function

$$t \mapsto \begin{cases} |a|, & t \in (0, 1), \\ |a|/t, & t \geq 1 \end{cases}$$

lies in  $\mu(\mathcal{I}_b)$ . □

This seems like a convenient place to prove the following proposition, which will be needed in Section 6.

PROPOSITION 5.10. — Let  $\mathcal{I} \subseteq \overline{\mathcal{M}}$  be a nonzero submodule and suppose  $\mathcal{M} \subseteq \mathcal{I}$ . Let

$$\mathcal{I}_0 = \left\{ T \in \mathcal{I} \mid \lim_{t \rightarrow \infty} \mu_t(T) = 0 \right\}.$$

Then  $[\mathcal{I}, \mathcal{M}] \cap \mathcal{I}_0 = [\mathcal{I}_0, \mathcal{M}]$ .

*Proof.* — Since  $\supseteq$  is clear, we need only show  $\subseteq$ . Suppose  $T \in [\mathcal{I}, \mathcal{M}] \cap \mathcal{I}_0$  and  $T$  is normal. It will suffice to show  $T \in [\mathcal{I}_0, \mathcal{M}]$ . Let  $T = T_{f_s} + T_b$  where  $T_{f_s} \in (\mathcal{I}_0)_{f_s} = \mathcal{I}_{f_s}$  and  $T_b \in (\mathcal{I}_0)_b$  are normal. Note we have  $(\mathcal{I}_0)_b = \mathcal{M}_0 = \mathcal{K}$ , (see Remark 3.3). Since  $T \in [\mathcal{I}, \mathcal{M}]$ , by Corollary 5.9, if  $\omega_{f_s} \in \mu(\mathcal{I})$ , then  $T_{f_s} \in [\mathcal{I}_{f_s}, \mathcal{M}]$ , while if  $\omega_{f_s} \notin \mu(\mathcal{I})$ , then there are  $a \in \mathbb{C}$  and  $h_{f_s} \in \mu(\mathcal{I}_{f_s})$  such that (5.11) holds. Since  $\omega_b \in \mu(\mathcal{I}_0)$ , by Corollary 5.9 in order to show  $T \in [\mathcal{I}_0, \mathcal{M}]$  it will suffice to show  $T_b \in [(\mathcal{I}_0)_b, \mathcal{M}]$ . But  $[\mathcal{K}, \mathcal{M}] = \mathcal{K} \ni T_b$ . □



We will finish this section with a few observations relating  $[\mathcal{I}, \mathcal{M}]$  to  $[\mathcal{I}_b, \mathcal{M}]$  and  $[\mathcal{I}_{f_s}, \mathcal{M}]$ , and examples involving ideals of  $p$ -summable operators. Writing  $\mathcal{I} = \mathcal{I}_{f_s} + \mathcal{I}_b$ , we have  $[\mathcal{I}, \mathcal{M}] = [\mathcal{I}_{f_s}, \mathcal{M}] + [\mathcal{I}_b, \mathcal{M}]$ . Since  $\mathcal{I}_{f_s} \cap \mathcal{I}_b = \mathcal{F}$ , and (see Corollary 5.2)  $[\mathcal{F}, \mathcal{M}] = \mathcal{F} \cap \ker \tau$ , we have

$$[\mathcal{I}_{f_s}, \mathcal{M}] \cap \mathcal{I}_b = [\mathcal{I}_{f_s}, \mathcal{M}] \cap \mathcal{F} = \begin{cases} \mathcal{F} & \text{if } \omega_{f_s} \in \mu(\mathcal{I}) \\ \mathcal{F} \cap \ker \tau & \text{if } \omega_{f_s} \notin \mu(\mathcal{I}), \end{cases}$$

$$[\mathcal{I}_b, \mathcal{M}] \cap \mathcal{I}_{f_s} = [\mathcal{I}_b, \mathcal{M}] \cap \mathcal{F} = \begin{cases} \mathcal{F} & \text{if } \omega_b \in \mu(\mathcal{I}) \\ \mathcal{F} \cap \ker \tau & \text{if } \omega_b \notin \mu(\mathcal{I}). \end{cases}$$

So we have the following result.

PROPOSITION 5.11. — *Let  $\mathcal{I}$  be a nonzero submodule of  $\overline{\mathcal{M}}$ , for a  $\text{II}_\infty$  factor  $\mathcal{M}$ . Then*

- (i)  $\mathcal{F} + [\mathcal{I}, \mathcal{M}] = \mathcal{I}$  if and only if  $\mathcal{F} + [\mathcal{I}_{f_s}, \mathcal{M}] = \mathcal{I}_{f_s}$  and  $\mathcal{F} + [\mathcal{I}_b, \mathcal{M}] = \mathcal{I}_b$ ;
- (ii)  $[\mathcal{I}, \mathcal{M}] = \mathcal{I}$  if and only if at least one of the following holds:
  - (a)  $[\mathcal{I}_{f_s}, \mathcal{M}] = \mathcal{I}_{f_s}$  and  $\mathcal{F} + [\mathcal{I}_b, \mathcal{M}] = \mathcal{I}_b$ ;
  - (b)  $\mathcal{F} + [\mathcal{I}_{f_s}, \mathcal{M}] = \mathcal{I}_{f_s}$  and  $[\mathcal{I}_b, \mathcal{M}] = \mathcal{I}_b$ .

We now relate the commutator space  $[\mathcal{I}_b, \mathcal{M}]$  to its discrete analogue. Let  $\mathcal{B} \subseteq \mathcal{M}$  be any type  $\text{I}_\infty$  factor (i.e. a copy of  $B(\mathcal{H})$ ) such that the restriction of  $\tau$  to  $\mathcal{B}$  is semifinite. Let  $\mathcal{I}_d = \mathcal{I} \cap \mathcal{B}$  and let  $\mathcal{F}_d = \mathcal{F} \cap \mathcal{B}$ ; (the “d” is for “discrete”). Note that  $\mathcal{I}_d$  is an ideal of  $\mathcal{B}$  and  $\mathcal{F}_d$  is the ideal of finite rank operators in  $\mathcal{B}$ . In the notation used in [7], the characteristic set  $\mu(\mathcal{I}_d)$  of  $\mathcal{I}_d$ , consisting of the sequences of singular numbers of elements of  $\mathcal{I}_d$ , is naturally identified with the set of all functions  $f \in \mu(\mathcal{I})$  that are constant on the intervals  $[0, 1), [1, 2), [2, 3), \dots$ . The commutator space  $[\mathcal{I}_d, \mathcal{B}]$  of an ideal of a  $\text{I}_\infty$  factor has been extensively studied — see [7] and references contained therein, and see [18] for some further results.

LEMMA 5.12. — *Let  $T \in \mathcal{I}_b$  and assume  $\lim_{t \rightarrow \infty} \mu_t(T) = 0$ . Then there is  $A \in \mathcal{I}_d$  such that  $T - A \in [\mathcal{I}_b, \mathcal{M}]$ .*

*Proof.* — We may without loss of generality assume  $T = T^*$  and that  $\tau(\tilde{Q}) = 1$  for a minimal projection  $\tilde{Q}$  of  $\mathcal{B}$ . Let  $(P_t)_{t \geq 0}$  be a family of projections in  $\mathcal{M}$  obtained from Lemma 2.6. Let  $Q_k = P_k - P_{k-1}$ , ( $k \in \mathbb{N}$ ),  $\alpha_k = \tau(TQ_k)$  and  $A' = \sum_{k=1}^\infty \alpha_k Q_k$ . Then  $TQ_k - \alpha_k Q_k$  is an element of the  $\text{II}_1$ -factor  $Q_k \mathcal{M} Q_k$  of trace zero and with

$$\|(T - \alpha_k)Q_k\| \leq \|TQ_k\| + |\alpha_k| \leq 2\|TQ_k\| \leq 2\mu_k(T).$$

Using [11, Thm. 2.3] as in the proof of Lemma 4.2, one shows  $T - A' \in [\mathcal{I}_b, \mathcal{M}]$ . Let  $\tilde{Q}_1, \tilde{Q}_2, \dots \in \mathcal{B}$  be pairwise orthogonal projections, each of trace 1, and let  $U \in \mathcal{M}$  be a partial isometry such that  $U^*Q_jU = \tilde{Q}_j$ . Let  $A = \sum_{k=1}^\infty \alpha_k \tilde{Q}_k$ . Then  $A = U^*A'U \in \mathcal{I}_d$  and  $A' - A = [U, U^*A'] \in [\mathcal{I}_b, \mathcal{M}]$ . Thus  $T - A \in [\mathcal{I}_b, \mathcal{M}]$ .  $\square$

PROPOSITION 5.13.

- (i)  $\mathcal{B} \cap [\mathcal{I}_b, \mathcal{M}] = [\mathcal{I}_d, \mathcal{B}]$ .
- (ii)  $[\mathcal{I}_b, \mathcal{M}] = \mathcal{I}_b$  if and only if  $[\mathcal{I}_d, \mathcal{B}] = \mathcal{I}_d$ .
- (iii)  $\mathcal{F} + [\mathcal{I}_b, \mathcal{M}] = \mathcal{I}_b$  if and only if  $\mathcal{F}_d + [\mathcal{I}_d, \mathcal{B}] = \mathcal{I}_d$ .

*Proof.* — We may without loss of generality assume  $\tau(F) = 1$  for a minimal projection  $F$  of  $\mathcal{B}$ . The inclusion  $\supseteq$  in (i) is clear. To show  $\subseteq$ , it will suffice to show that  $T = T^* \in \mathcal{B} \cap [\mathcal{I}_b, \mathcal{M}]$  implies  $T \in [\mathcal{I}_d, \mathcal{B}]$ . By Lemma 5.1, there is  $h \in \mu(\mathcal{I}_b)$  satisfying (5.4). Since  $h$  is bounded, replacing  $h$  if necessary by a slightly greater function, we may without loss of generality assume  $h$  is constant on all intervals  $[0, 1), [1, 2), \dots$ . We may write  $T = \sum_{i=1}^\infty \lambda_i F_i$  for a sequence of pairwise orthogonal, minimal projections  $F_i$  of  $\mathcal{B}$  and for  $\lambda_i \in \mathbb{R}$  with  $|\lambda_1| \geq |\lambda_2| \geq \dots$ . If  $\lim_{n \rightarrow \infty} |\lambda_n| > 0$ , then  $\mathcal{I}_b = \mathcal{M}$  and  $\mathcal{I}_d = \mathcal{B}$ , so (i) holds. Hence we may without loss of generality assume  $\lim_{n \rightarrow \infty} |\lambda_n| = 0$ . Suppose  $k$  and  $n$  are nonnegative integers with  $k < n$ ,

$$|\lambda_{k+1}| = |\lambda_{k+2}| = \dots = |\lambda_n| > |\lambda_{n+1}|$$

and either  $k = 0$  or  $|\lambda_k| > |\lambda_{k+1}|$ . If  $s \in [k, n)$ , then  $\mu_s(T) = |\lambda_{k+1}|$ , so by (5.4),

$$|\lambda_1 + \dots + \lambda_k| = |\tau(TE|_{T|}(\mu_s(T), \infty))| \leq sh(s).$$

Thus, if  $\ell \in \{k, \dots, n - 1\}$  and  $\ell \neq 0$ , then

$$|\lambda_1 + \dots + \lambda_\ell| \leq |\lambda_1 + \dots + \lambda_k| + (\ell - k)|\lambda_\ell| \leq \ell h(\ell) + \ell |\lambda_\ell|$$

and

$$\frac{|\lambda_1 + \dots + \lambda_\ell|}{\ell} \leq h(\ell) + |\lambda_\ell|.$$

From this, the main result of [7] implies  $T \in [\mathcal{I}_d, \mathcal{B}]$ , and (i) is proved.

From (i), we have

$$[\mathcal{I}_b, \mathcal{M}] = \mathcal{I}_b \implies [\mathcal{I}_d, \mathcal{B}] = \mathcal{I}_d.$$

The reverse implication follows from Lemma 5.12. Hence (ii) is proved.

To prove (iii), we have  $\mathcal{F} = \mathcal{F}_d + (\mathcal{F} \cap \ker \tau) = \mathcal{F}_d + [\mathcal{F}, \mathcal{M}]$ , so

$$\mathcal{F} + [\mathcal{I}_b, \mathcal{M}] = \mathcal{F}_d + [\mathcal{I}_b, \mathcal{M}].$$

From (i) we thus obtain

$$\mathcal{F} + [\mathcal{I}_b, \mathcal{M}] = \mathcal{I}_b \implies \mathcal{F}_d + [\mathcal{I}_d, \mathcal{B}] = \mathcal{I}_d.$$

The reverse implication follows from Lemma 5.12. □

We now point out results relating  $[\mathcal{I}_{f_S}, \mathcal{M}]$  and commutator spaces of submodules of  $\text{II}_1$ -factors. Let  $P \in \mathcal{M}$  be a projection with  $\tau(P) = 1$  and consider the  $\text{II}_1$ -factor  $\mathcal{M}_1 = P\mathcal{M}P$ . Then  $P\overline{\mathcal{M}}P$  is equal to the module  $\overline{\mathcal{M}}_1$  of  $\tau$ -measurable operators affiliated to  $\mathcal{M}_1$ . Given a nonzero submodule  $\mathcal{I}$  of  $\overline{\mathcal{M}}$ , consider the submodule  $\mathcal{I}_1 = P\mathcal{I}P$  of  $\overline{\mathcal{M}}_1$ . Then the following result follows directly from the characterizations of commutator spaces found in Theorem 4.6 and Lemma 5.1.

PROPOSITION 5.14.

- (i)  $\overline{\mathcal{M}}_1 \cap [\mathcal{I}_{f_S}, \mathcal{M}] = [\mathcal{I}_1, \mathcal{M}_1]$ .
- (ii)  $[\mathcal{I}_{f_S}, \mathcal{M}] = \mathcal{I}_{f_S}$  if and only if  $[\mathcal{I}_1, \mathcal{M}_1] = \mathcal{I}_1$ .
- (iii)  $\mathcal{F} + [\mathcal{I}_{f_S}, \mathcal{M}] = \mathcal{I}_{f_S}$  if and only if  $\mathcal{M}_1 + [\mathcal{I}_1, \mathcal{M}_1] = \mathcal{I}_1$ .

For  $0 < p < \infty$ , let  $\mathcal{L}_p$  denote the submodule of  $\overline{\mathcal{M}}$  whose characterisic set  $\mu(\mathcal{L}_p)$  consists of all the  $p$ -integrable functions in  $D^+(0, \infty)$ . Thus

$$\mathcal{L}_p = \left\{ T \in \overline{\mathcal{M}} \mid \tau((T^*T)^{p/2}) < \infty \right\},$$

where we have extended  $\tau$  in the usual way to be a map from positive elements of  $\overline{\mathcal{M}}$  to  $[0, +\infty]$ . Also, let  $\mathcal{L}_\infty = \mathcal{M}$ .

PROPOSITION 5.15. — *If  $0 < p < 1$ , then*

$$(5.13) \quad [(\mathcal{L}_p)_{f_S}, \mathcal{M}] = (\mathcal{L}_p)_{f_S}$$

and

$$(5.14) \quad [(\mathcal{L}_p)_b, \mathcal{M}] = (\mathcal{L}_p)_b \cap \ker \tau,$$

so  $\mathcal{F} + [(\mathcal{L}_p)_b, \mathcal{M}] = (\mathcal{L}_p)_b$ .

With  $p = 1$ , we have

$$(5.15) \quad \mathcal{F} + [(\mathcal{L}_1)_{f_S}, \mathcal{M}] \neq (\mathcal{L}_1)_{f_S}$$

and

$$(5.16) \quad \mathcal{F} + [(\mathcal{L}_1)_b, \mathcal{M}] \neq (\mathcal{L}_1)_b.$$

If  $1 < p \leq \infty$ , then

$$(5.17) \quad [(\mathcal{L}_p)_{f_S}, \mathcal{M}] = (\mathcal{L}_p)_{f_S} \cap \ker \tau,$$

so  $\mathcal{F} + [(\mathcal{L}_p)_{f_S}, \mathcal{M}] = (\mathcal{L}_p)_{f_S}$ , and

$$(5.18) \quad [(\mathcal{L}_p)_b, \mathcal{M}] = (\mathcal{L}_p)_b.$$

*Proof.* — When  $p = \infty$ , we have  $(\mathcal{L}_p)_{f_S} = \mathcal{F}$  and  $(\mathcal{L}_p)_b = \mathcal{M}$ , and these special cases of (5.17) and (5.18) have been considered previously. For  $p < \infty$ , all of the relations (5.13)–(5.18) can be readily verified from properties of  $L^p$ -functions.

Moreover, (5.14), (5.16) and (5.18) follow from Proposition 5.13 and the corresponding discrete analogues, which follow readily from the main result of [7] and were originally proved in [1], [28] and [25], respectively. On the other hand, (5.13) and (5.17) follow from Proposition 5.14 and [13, Prop. 2.12].

As an example, let us verify (5.15) directly. Clearly  $[(\mathcal{L}_1)_{f_S}, \mathcal{M}] \subseteq \ker \tau$ , so it will suffice to find  $T = T^* \in (\mathcal{L}_1)_{f_S} \cap \ker \tau$  with  $T \notin [(\mathcal{L}_1)_{f_S}, \mathcal{M}]$ . Using Lemma 5.1, it will suffice to find  $f \in L^1[0, 1]$  such that  $\int_0^1 f = 0$  but the function

$$s \mapsto \frac{1}{s} \int_s^1 f(t) dt, \quad 0 < s < 1$$

is not integrable. Such a function is given by

$$f(t) = \begin{cases} \frac{1}{t(\log t)^2} & \text{if } 0 < t < 1/2 \\ \frac{-2}{\log 2} & \text{if } 1/2 \leq t < 1. \end{cases} \quad \square$$

Propositions 5.15 and 5.11 now yield the following examples.

*Examples 5.16.* — Let  $\mathcal{I} = (\mathcal{L}_p)_{f_S} + (\mathcal{L}_q)_b$ , for some  $0 < p, q \leq \infty$ .

- (i) If  $p < 1$  and  $q \neq 1$  or if  $p \neq 1$  and  $q > 1$ , then  $[\mathcal{I}, \mathcal{M}] = \mathcal{I}$ .
- (ii) If  $p > 1$  and  $q < 1$ , then  $[\mathcal{I}, \mathcal{M}] = \mathcal{I} \cap \ker \tau$  and  $\mathcal{F} + [\mathcal{I}, \mathcal{M}] = \mathcal{I}$ .
- (iii) If  $p = 1$  or  $q = 1$ , then  $\mathcal{F} + [\mathcal{I}, \mathcal{M}] \neq \mathcal{I}$ .

### 6. Spectral characterization of $[\mathcal{I}, \mathcal{M}]$ .

In this section,  $\mathcal{M}$  will be a  $\text{II}_\infty$ -factor with fixed normal, semifinite trace  $\tau$ .

Let  $\mathcal{L}_{\log}$  be the submodule of all  $T \in \overline{\mathcal{M}}$  such that

$$\int_0^\infty \log(1 + \mu_s(T)) ds < \infty.$$

As is usual, let  $\mathcal{L}_p$  be the submodule of all  $T \in \overline{\mathcal{M}}$  such that

$$\int_0^\infty \mu_s(T)^p ds < \infty.$$

If  $\mathcal{I}$  is a submodule of  $\overline{\mathcal{M}}$  we say that  $\mathcal{I}$  is *geometrically stable* if  $\mathcal{I} \subset \mathcal{M} + \mathcal{L}_{\log}$  and if whenever  $h \in \mu(\mathcal{I})$  then  $g \in \mu(\mathcal{I})$ , where

$$g(t) = \exp\left(t^{-1} \int_0^t \log h(s) ds\right), \quad t > 0.$$

Geometric stability is a relatively mild condition. For example let  $\mathcal{X}$  be a rearrangement-invariant quasi-Banach function space on  $(0, \infty)$  and suppose  $\mathcal{I} = \{T : (\mu_s(T))_{s>0} \in \mathcal{X}\} \subseteq \mathcal{K} + \mathcal{L}_{\log}$ , where  $\mathcal{K} \subseteq \mathcal{M}$  is the ideal of  $\tau$ -compact operators (see Remark 3.3); then  $\mathcal{I}$  is geometrically stable by Proposition 3.2 of [13]. A non-geometrically stable ideal in  $B(\mathcal{H})$  is constructed in [8], and from this a non-geometrically stable ideal of  $\mathcal{M}$  can be constructed.

Suppose  $T \in \mathcal{L}_1 \cap \mathcal{M}$ . Then the Fuglede-Kadison determinant [14] of  $I + T$  is defined by

$$\Delta(I + T) = \exp(\tau(\log |I + T|)).$$

Using [2] Remark 3.4 we note that  $T \mapsto \log \Delta(I + T)$  is plurisubharmonic on  $\mathcal{L}_1 \cap \mathcal{M}$ . In the Appendix of [2] the definition of  $\Delta(I + T)$  is extended to  $\mathcal{L}_{\log}$  and it is shown that  $T \mapsto \Delta(I + T)$  is upper-semicontinuous for the natural topology of  $\mathcal{L}_{\log}$ . It is not shown explicitly that  $T \mapsto \log \Delta(I + T)$  is plurisubharmonic on  $\mathcal{L}_{\log}$  but this follows trivially from the results of [2]:

LEMMA 6.1. — *Suppose  $S, T \in \mathcal{L}_{\log}$ . Then*

$$\log \Delta(I + S) \leq \frac{1}{2\pi} \int_0^{2\pi} \log \Delta(I + S + e^{i\theta} T) d\theta.$$

*Proof.* — Let  $S = H + iK$  and  $T = H' + iK'$  be the splitting of  $S, T$  into real and imaginary parts. Let  $R = |H| + |H'| + |K| + |K'|$ . Then

$R \in \mathcal{L}_{\log}$  and  $(I + S + zT)(I + R)^{-1} \in I + \mathcal{L}_1 \cap \mathcal{M}$  for all  $z$ . Using the fact that  $T \mapsto \log \Delta(I + T)$  is plurisubharmonic on  $\mathcal{L}_1 \cap \mathcal{M}$  and

$$\Delta((I + S + zT)(1 + R)^{-1}) = \Delta(I + S + zT)(\Delta(I + R))^{-1},$$

it is easy to deduce the Lemma. □

Let  $g_0(w) = (1 - w)$  and

$$g_k(w) = (1 - w) \exp\left(w + \dots + \frac{w^k}{k}\right)$$

for  $k \geq 1$ . If  $T \in \mathcal{L}_{\log}$  let  $k = 0$ ; if  $T \in \mathcal{M} \cap \mathcal{L}_p$  for some  $p > 0$ , let  $k$  be an integer such that  $k + 1 \geq p$ . Then, following [2], there is a unique  $\sigma$ -finite measure  $\nu = \nu_T$  on  $\mathbb{C} \setminus \{0\}$  such that

$$\log \Delta(g_k(wT)) = \int \log |g_k(wz)| d\nu_T(z), \quad w \in \mathbb{C}.$$

$\nu_T$  is called the *Brown measure* of  $T$ , and is independent of the choice of  $k$  when many choices are permissible. If  $T \in \mathcal{L}_{\log} \cup \bigcup_{p>0} (\mathcal{L}_p \cap \mathcal{M})$  we shall say that  $T$  admits a *Brown measure*. The measure  $\nu_T$  satisfies the following estimates. If  $T \in \mathcal{L}_{\log}$  and  $k = 0$  then

$$(6.1) \quad \int_{\mathbb{C}} \log(1 + |z|) d\nu_T(z) < \infty$$

while if  $T \in \mathcal{L}_p \cap \mathcal{M}$  and  $k + 1 \geq p$ , then

$$(6.2) \quad \int_{\mathbb{C}} |z|^p d\nu_T(z) < \infty.$$

We refer to [2, Theorem 3.6] and the remark on p. 29 of [2].

Of course if  $T$  is *normal* there is a projection-valued spectral measure  $B \rightarrow E_T(B)$  defined for Borel subsets  $B$  of the complex plane and we can define a spectral measure  $\nu_T$  by

$$\nu_T(B) = \tau(E_T(B)).$$

If  $T$  also admits a Brown measure, then  $\nu_T$  coincides with the Brown measure.

If  $T$  either admits a Brown measure or is normal and satisfies  $\lim_{t \rightarrow \infty} \mu_t(T) = 0$ , then for every  $0 < r < s < \infty$  we define

$$(6.3) \quad \Phi(r, s; T) = \int_{r < |z| \leq s} z d\nu_T(z).$$

If  $T$  is normal then we can rewrite (6.3) in the form

$$(6.4) \quad \Phi(r, s; T) = \tau(T E_{|T|}(r, s]).$$

Note that it is elementary that if  $|\alpha| = 1$  then  $\Phi(r, s; \alpha T) = \alpha\Phi(r, s; T)$ .

PROPOSITION 6.2. — *Let  $0 < r < s < \infty$ .*

(1) *Suppose  $T_1, \dots, T_N$  are normal with  $\lim_{t \rightarrow \infty} \mu_t(T_j) = 0$  and  $T_1 + \dots + T_N = 0$ . Then*

$$(6.5) \quad \left| \sum_{j=1}^N \Phi(r, s; T_j) \right| \leq 2N \sum_{j=1}^N (r\tau(E_{|T_j|}(r, \infty)) + s\tau(E_{|T_j|}(s, \infty))).$$

(2) *Suppose  $|\alpha| \leq 1$  and  $T$  is normal with  $\lim_{t \rightarrow \infty} \mu_t(T) = 0$ . Then*

$$(6.6) \quad |\Phi(r, s; \alpha T) - \alpha\Phi(r, s; T)| \leq |\tau(rE_{|T|}(r, \infty) + sE_{|T|}(s, \infty))|.$$

(3) *If  $T$  is normal with  $\lim_{t \rightarrow \infty} \mu_t(T) = 0$ , then*

$$(6.7) \quad |\Phi(r, s; \operatorname{Re} T) - \operatorname{Re} \Phi(r, s; T)| \leq \tau(rE_{|T|}(r, \infty) + sE_{|T|}(s, \infty))$$

and

$$(6.8) \quad |\Phi(r, s; \operatorname{Im} T) - \operatorname{Im} \Phi(r, s; T)| \leq \tau(rE_{|T|}(r, \infty) + sE_{|T|}(s, \infty))$$

*Proof.* — (1) Pick a projection  $P \geq E_{|T_j|}(s, \infty)$  for  $1 \leq j \leq N$  and such that  $\tau(P) \leq \sum_{j=1}^N \tau(E_{|T_j|}(s, \infty))$ . Then choose  $Q \geq P$  with  $Q \geq E_{|T_j|}(r, \infty)$  for  $1 \leq j \leq N$  and

$$\tau(Q) \leq \sum_{j=1}^N (\tau(E_{|T_j|}(r, \infty)) + \tau(E_{|T_j|}(s, \infty))) \leq 2 \sum_{j=1}^N \tau(E_{|T_j|}(r, \infty)).$$

Then

$$\|(Q - E_{|T_j|}(r, \infty))T_j\| \leq r, \quad \|(P - E_{|T_j|}(s, \infty))T_j\| \leq s, \quad 1 \leq j \leq N.$$

Hence

$$|\tau((Q - E_{|T_j|}(r, \infty))T_j)| \leq r\tau(Q)$$

and

$$|\tau((P - E_{|T_j|}(s, \infty))T_j)| \leq s\tau(P).$$

We thus have

$$\begin{aligned} \left| \sum_{j=1}^N \Phi(r, s; T_j) \right| &= \left| \sum_{j=1}^N \tau(T_j(Q - E_{|T_j|}(s, \infty)) - T_j(P - E_{|T_j|}(r, \infty))) \right| \\ &\leq N(r\tau(Q) + s\tau(P)). \end{aligned}$$

Now (6.5) follows.

For (2), we note that

$$|\Phi(r, s; \alpha T) - \alpha \Phi(r, s; T)| \leq |\alpha| \left( \int_{r < |z| \leq |\alpha|^{-1}r} |z| d\nu_T(z) + \int_{s < |z| \leq |\alpha|^{-1}s} |z| d\nu_T(z) \right).$$

Then (6.6) follows immediately.

Part (3) is similar to (2). For example we observe for (6.7) that

$$|\Phi(r, s; \operatorname{Re} T) - \operatorname{Re} \Phi(r, s; T)| \leq \int_{|\operatorname{Re} z| \leq r < |z|} |\operatorname{Re} z| d\nu_T(z) + \int_{|\operatorname{Re} z| \leq s < |z|} |\operatorname{Re} z| d\nu_T(z). \quad \square$$

PROPOSITION 6.3. — *Let  $\mathcal{I}$  be a submodule of  $\overline{\mathcal{M}}$ . Suppose  $T \in \mathcal{I}$  is normal and satisfies  $\lim_{t \rightarrow \infty} \mu_t(T) = 0$ . Then  $T \in [\mathcal{I}, \mathcal{M}]$  if and only if there exists a positive operator  $V \in \mathcal{I}$  such that*

$$(6.9) \quad |\Phi(r, s; T)| \leq r\tau(E_V(r, \infty)) + s\tau(E_V(s, \infty)), \quad 0 < r < s < \infty.$$

*Proof.* — Assume that (6.9) holds. By replacing  $V$  with  $V + |T|$ , if necessary, we may without loss of generality assume  $V \geq |T|$ . Let  $h(t) = \mu_t(V)$ . Then  $h(t) \geq \mu_t(T)$ . If  $0 < t < s < \infty$ , then from (2.2) we have

$$|\tau(TE_{|T|}(h(s), h(t)))| \leq sh(s) + th(t).$$

Now using (2.2) again, we get

$$\begin{aligned} & |\tau(TE_{|T|}(\mu_s(T), \mu_t(T))) - \tau(TE_{|T|}(h(s), h(t)))| \\ & \leq \int_{\mu_s(T) < |z| \leq h(s)} |z| d\nu_T(z) + \int_{\mu_t(T) < |z| \leq h(t)} |z| d\nu_T(z) \\ & \leq sh(s) + th(t). \end{aligned}$$

Hence

$$|\tau(TE_{|T|}(\mu_s(T), \mu_t(T)))| \leq 2sh(s) + 2th(t)$$

and we can apply Theorem 4.7 and (4.17) to conclude that  $T \in [\mathcal{I}, \mathcal{M}]$ .

Conversely, suppose  $T$  satisfies (4.17) for some  $h$ . Replacing  $h$  with

$$\tilde{h}(t) = \frac{2}{t} \int_{t/2}^t h(s) ds,$$



if necessary, we may without loss of generality assume  $h$  is continuous. Let  $V \in \mathcal{I}$  be a positive operator such that  $\mu_t(V) = h(t)$ . Given  $0 < r < s < \infty$ , choose  $0 < v < u$  so that  $h(2u) \leq r < h(u)$  and  $h(2v) \leq s < h(v)$ . Then

$$|\tau(TE_{|T|}(\mu_{2u}(T), \mu_{2v}(T)))| \leq 2ur + 2vs.$$

Now arguing as above,

$$\begin{aligned} &|\tau(TE_{|T|}(\mu_{2u}(T), \mu_{2v}(T))) - \tau(TE_{|T|}(r, s))| \\ &\leq \int_{\mu_{2u}(T) < |z| \leq r} |z| d\nu_T(z) + \int_{\mu_{2v}(T) < |z| \leq s} |z| d\nu_T(z) \\ &\leq 2ur + 2vs. \end{aligned}$$

Using Lemma 2.3, we have  $\tau(E_V(r, \infty)) \geq u$  and  $\tau(E_V(s, \infty)) \geq v$ . Combining gives

$$|\tau(TE_{|T|}(r, s))| \leq 4ur + 4vs \leq 4r\tau(E_V(r, \infty)) + 4s\tau(E_V(s, \infty)).$$

Replacing  $V$  by  $V \oplus V \oplus V \oplus V$ , (cf. Definition 2.4) we have (6.9). □

LEMMA 6.4. — *Let  $\psi : \mathbb{C} \rightarrow \mathbb{R}$  be a subharmonic function such that  $\psi$  vanishes in a neighborhood of 0, is harmonic outside some compact set, and for a suitable constant  $C$ , satisfies the estimate  $|\psi(z)| \leq C \log(1 + |z|)$  for all  $z$ . If  $T$  admits a Brown measure, then define*

$$\Psi(T) = \int_{\mathbb{C}} \psi(z) d\nu_T(z).$$

Suppose  $S, T \in \mathcal{L}_{\log}$  or  $S, T \in \mathcal{L}_p \cap \mathcal{M}$  for some  $p > 0$ . Then  $\Psi(S + e^{i\theta}T)$  is a Borel function of  $\theta$  and

$$(6.10) \quad \Psi(S) \leq \frac{1}{2\pi} \int_0^{2\pi} \Psi(S + e^{i\theta}T) d\theta.$$

*Proof.* — By an easy approximation argument it will suffice to consider the case when  $\psi$  is  $C^2$ . In this case for any choice of  $k \geq 0$  we have the formula ([2] Proposition 2.2)

$$\psi(z) = \int_{\mathbb{C}} \log |g_k(w^{-1}z)| \nabla^2 \psi(w) d\lambda(w), \quad z \in \mathbb{C}$$

where  $\lambda$  denotes area measure. Hence if  $T$  admits a Brown measure and  $k$  is suitably chosen,

$$(6.11) \quad \Psi(T) = \int_{\mathbb{C}} \left( \int_{\mathbb{C}} \log |g_k(w^{-1}z)| \nabla^2 \psi(w) d\lambda(w) \right) d\nu_T(z).$$

Now it can be checked that the function  $|\log g_k(w^{-1}z)| \nabla^2 \psi(w)$  is integrable for the product measure  $\lambda \times \nu_T$ . Indeed, let us first consider the case when

$T \in \mathcal{L}_p \cap \mathcal{M}$ , with  $k+1 \geq p$ . Estimates on the growth of  $\log |g_k(w)|$  (cf. p. 11 of [2]) give

$$\int_0^{2\pi} |\log |g_k(r^{-1}e^{-i\theta}z)|| d\theta \leq C \min(|z|^{k+1}|r|^{-k-1}, |z|^{k+\epsilon}|r|^{-k-\epsilon})$$

for suitable  $C$  and  $\epsilon > 0$ . Since  $\nabla^2\psi$  has compact support contained in some annulus away from the origin we need only observe that

$$\int \min(|z|^{k+1}, |z|^{k+\epsilon}) d\nu_T(z) < \infty$$

which follows from (6.2). On the other hand, if  $T \in \mathcal{L}_{\log}$  and thus  $k = 0$ , we use the estimate

$$\int_0^{2\pi} |\log |g_k(r^{-1}e^{-i\theta}z)|| d\theta \leq C \log(1 + |z|)$$

and (6.1). It follows we can use Fubini's theorem to rewrite (6.11) in the form

$$\begin{aligned} \Psi(T) &= \int_{\mathbb{C}} \left( \int_{\mathbb{C}} \log |g_k(w^{-1}z)| d\nu_T(z) \right) \nabla^2\psi(w) d\lambda(w) \\ &= \int_{\mathbb{C}} \log \Delta(g_k(w^{-1}T)) \nabla^2\psi(w) d\lambda(w). \end{aligned}$$

Now the result follows easily from the upper semicontinuity of  $\log \Delta$  and Lemma 6.1. □

**PROPOSITION 6.5.** — *Let  $\mathcal{I}$  be a geometrically stable submodule of  $\overline{\mathcal{M}}$ . If  $T \in \mathcal{I}$  admits a Brown measure, then there is a normal operator  $S \in \mathcal{I}$  with  $\nu_S = \nu_T$ . Furthermore,  $S$  admits a Brown measure.*

*Proof.* — It will suffice to show the existence of a positive operator  $V \in \mathcal{I}$  so that

$$\nu_T(|z| > r) \leq \nu_V(r, \infty), \quad 0 < r < \infty.$$

Let  $H = \operatorname{Re} T$ ,  $K = \operatorname{Im} T$  and then set  $P = |H| + |K|$ . Since  $\mathcal{I}$  is geometrically stable there exists a positive  $V \in \mathcal{I}$  with

$$\frac{1}{t} \int_0^t \log \mu_s(P) ds \leq \log \mu_t(V), \quad 0 < t < \infty.$$

Therefore,  $\mu_t(P) \leq \mu_t(V)$  and  $\nu_P(r, \infty) \leq \nu_V(r, \infty)$  for all  $0 < r < \infty$ .

Suppose for contradiction that for some  $0 < r < \infty$  we have  $t = \nu_T(|z| > r) > \nu_V(r, \infty)$ . Choose  $r_0 < r$  so that  $\nu_P[r_0, \infty) \geq t \geq \nu_P(r_0, \infty)$ . Let  $\psi(z) = \log_+ \frac{|z|}{r_0}$  and define  $\Psi$  as in Lemma 6.4. Then

$$\Psi(T) \leq \frac{1}{2\pi} \int_0^{2\pi} \Psi(T + e^{i\theta}T^*) d\theta.$$

Now  $T + e^{i\theta}T^* = 2e^{i\theta/2}(H \cos \frac{\theta}{2} + K \sin \frac{\theta}{2})$ . Hence  $|T + e^{i\theta}T^*| \leq 2(|H| + |K|) = P$  and it follows that  $\Psi(T + e^{i\theta}T^*) \leq \Psi(P)$  for  $0 \leq \theta \leq 2\pi$ .

$$\begin{aligned} t \log \frac{r}{r_0} &\leq \int \log_+ \frac{|z|}{r_0} d\nu_T(z) = \Psi(T) \\ &\leq \Psi(P) = \int_0^t \log_+ \frac{\mu_s(P)}{r_0} ds \leq t \log \frac{\mu_t(V)}{r_0}. \end{aligned}$$

Thus  $\mu_t(V) \geq r$  and hence  $\nu_V(r, \infty) \geq t$  contrary to assumption.

The inequalities (6.1) and (6.2) imply that  $S$  admits a Brown measure. □

Before proving our main result it will be convenient to introduce some notation. Let  $\mathcal{I}$  be any submodule of  $\overline{\mathcal{M}}$  not containing  $\mathcal{M}$ . Hence  $\lim_{t \rightarrow \infty} \mu_t(T) = 0$  for every  $T \in \mathcal{I}$ . Let  $F(r, s)$  be a function of two variables defined for  $0 < r < s < \infty$ . We write  $F \in \mathcal{F}(\mathcal{I})$  if there exists a positive operator  $V \in \mathcal{I}$  such that

$$|F(r, s)| \leq r\tau(E_V(r, \infty)) + s\tau(E_V(s, \infty)), \quad 0 < r < s < \infty.$$

We write  $F \in \mathcal{G}(\mathcal{I})$  if there if there is a positive operator  $V \in \mathcal{I}$  such that

$$|F(r, s)| \leq \int_{(0, \infty)} \left( r \log_+ \frac{x}{r} + s \log_+ \frac{x}{s} \right) d\nu_V(x), \quad 0 < r < s < \infty.$$

Both  $\mathcal{F}(\mathcal{I})$  and  $\mathcal{G}(\mathcal{I})$  are easily seen to be vector spaces. Also note that  $\mathcal{F}(\mathcal{I}) \subset \mathcal{G}(\mathcal{I})$  (replace  $V$  by  $eV$ .) Proposition 6.3 states that if  $T$  is normal then  $T \in [\mathcal{I}, \mathcal{M}]$  if and only if  $\Phi(r, s; T) \in \mathcal{F}(\mathcal{I})$ . We improve this for geometrically stable submodules.

**PROPOSITION 6.6.** — *Suppose  $\mathcal{I}$  is a geometrically stable submodule of  $\overline{\mathcal{M}}$  with  $\mathcal{M} \not\subseteq \mathcal{I}$ . If  $T \in \mathcal{I}$  is normal, then  $T \in [\mathcal{I}, \mathcal{M}]$  if and only if  $\Phi(r, s; T) \in \mathcal{G}(\mathcal{I})$ .*

*Proof.* — One direction is trivial from Proposition 6.3. For the other direction suppose  $\Phi(r, s; T) \in \mathcal{G}(\mathcal{I})$ . Choose  $V$  a positive operator in  $\mathcal{I}$  so that

$$(6.12) \quad |\Phi(r, s; T)| \leq \int_{(0, \infty)} \left( r \log_+ \frac{x}{r} + s \log_+ \frac{x}{s} \right) d\nu_V(x), \quad 0 < r < s < \infty.$$

We can assume  $V \geq |T|$ . Let  $h(t) = \mu_t(V)$  and let

$$g(t) = \exp\left(\frac{1}{t} \int_0^t \log h(s) ds\right), \quad 0 < t < \infty.$$

Suppose  $0 < t < s < \infty$ . Then similarly to in the proof of Proposition 6.3, we get

$$|\tau(TE_{|T|}(\mu_s(T), \mu_t(T)))| \leq |\tau(TE_{|T|}(\mu_s(V), \mu_t(V)))| + sh(s) + th(t).$$

Now from (6.12) we get

$$\begin{aligned} &|\tau(TE_{|T|}(\mu_s(V), \mu_t(V)))| \\ &\leq \int_{(0, \infty)} (\mu_s(V) \log_+ \frac{x}{\mu_s(V)} + \mu_t(V) \log_+ \frac{x}{\mu_t(V)}) d\nu_V(x) \\ &= \int_0^s h(s) \log \frac{h(u)}{h(s)} du + \int_0^t h(t) \log \frac{h(u)}{h(t)} du \\ &= sh(s) \log \frac{g(s)}{h(s)} + th(t) \log \frac{g(t)}{h(t)} \\ &\leq sg(s) + tg(t). \end{aligned}$$

Combining, we see that

$$|\tau(TE_{|T|}(\mu_s(T), \mu_t(T)))| \leq s(h(s) + g(s)) + t(h(t) + g(t))$$

and so by Theorem 4.7,  $T \in [\mathcal{I}, \mathcal{M}]$ . □

**THEOREM 6.7.** — Suppose  $\mathcal{I}$  is a submodule of  $\overline{\mathcal{M}}$  with  $\mathcal{M} \not\subseteq \mathcal{I}$  and  $T \in \mathcal{I}$  admits a Brown measure. Then

$$\operatorname{Re} \Phi(r, s; T) - \Phi(r, s; \operatorname{Re} T) \in \mathcal{G}(\mathcal{I}), \quad \operatorname{Im} \Phi(r, s; T) - \Phi(r, s; \operatorname{Im} T) \in \mathcal{G}(\mathcal{I}).$$

*Proof.* — Let  $H = \operatorname{Re} T$  and  $K = \operatorname{Im} T$ . We need only prove the statement concerning the real part, since the other half follows by considering  $iT$ . We also note that if  $s \leq 2r$  we have  $|\Phi(r, s, T)| \leq 2r\nu_T(|z| > r)$  and  $|\Phi(r, s; H)| \leq 2r\nu_{|H|}(r, \infty)$ . By Proposition 6.5, this implies an estimate

$$|\operatorname{Re} \Phi(r, s; T) - \Phi(r, s; H)| \leq 2r\nu_V(r, \infty), \quad 0 < r < s \leq 2r < \infty$$

for a suitable positive operator  $V \in \mathcal{I}$ . This means we need only consider estimates when  $s > 2r$ .

We first fix a smooth bump function  $b : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\operatorname{supp} b \subset (0, 1/2)$ ,  $b \geq 0$ ,  $\int b(x)dx = 1$ . Let  $\beta(t) = 2|b(t)| + |b'(t)|$ .

Now suppose  $0 < r < s < \infty$ , with  $s > 2r$ . We define

$$\varphi_{r,s}(\tau) = \int_{-\infty}^{\tau} b(t - \log r) - b(t - \log s) dt.$$

Notice that the two terms in the integrand are never simultaneously positive (since  $\log 2 > \frac{1}{2}$ ), and  $\varphi_{r,s}$  is a bump function which satisfies  $\varphi_{r,s}(\tau) = 0$  if  $\tau < \log r$  or  $\tau > \frac{1}{2} + \log s$ , while  $\varphi_{r,s}(\tau) = 1$  if  $\frac{1}{2} + \log r \leq \tau \leq \log s$  and  $0 \leq \varphi_{r,s}(\tau) \leq 1$  for all  $\tau$ .

Then let  $\rho_{r,s}$  be defined to be the function such that  $\rho_{r,s}(\tau) = 0$  if  $\tau < \log r$  and

$$\rho''_{r,s}(\tau) = e^\tau (2|\varphi'_{r,s}(\tau)| + |\varphi''_{r,s}(\tau)|).$$

In fact, this implies that

$$\rho''_{r,s}(\tau) = e^\tau (\beta(\tau - \log r) + \beta(\tau - \log s))$$

and then

$$\rho'_{r,s}(\tau) = \int_{-\infty}^\tau e^t (\beta(t - \log r) + \beta(t - \log s)) dt$$

and

$$\rho_{r,s}(\tau) = \int_{-\infty}^\tau (\tau - t) e^t (\beta(t - \log r) + \beta(t - \log s)) dt.$$

Thus, if we set

$$C_0 = \int_{-\infty}^\infty e^t \beta(t) dt,$$

then

$$\rho'_{r,s}(\tau) \leq C_0 (r\chi_{(\tau > \log r)} + s\chi_{(\tau > \log s)})$$

and so

$$(6.13) \quad 0 \leq \rho_{r,s}(\tau) \leq C_0 (r(\tau - \log r)_+ + s(\tau - \log s)_+).$$

Now we use the argument of Lemma 2.6 of [20]. We define

$$\psi_{r,s}(z) = \rho_{r,s}(\log |z|) - x\varphi_{r,s}(\log |z|), \quad z = x + iy \neq 0$$

and  $\psi(0) = 0$ . Then if  $z \neq 0$ ,

$$\nabla^2 \rho_{r,s}(\log |z|) = |z|^{-2} \rho''_{r,s}(\log |z|).$$

Similarly

$$\nabla^2 (x\varphi_{r,s}(\log |z|)) = \frac{x}{|z|^2} (2\varphi'_{r,s}(\log |z|) + \varphi''_{r,s}(\log |z|)).$$

Thus by construction,  $|\nabla^2 (x\varphi_{r,s}(\log |z|))| \leq \nabla^2 (\rho_{r,s}(\log |z|))$  and so  $\psi_{r,s}$  is subharmonic. Note that  $\psi_{r,s}$  also vanishes on a neighborhood of 0 and is harmonic outside a compact set. We note the estimates (from (6.13))

$$(6.14) \quad 0 \leq \rho_{r,s}(\log |z|) \leq C_0 \left( r \log_+ \frac{|z|}{r} + s \log_+ \frac{|z|}{s} \right)$$

and

$$(6.15) \quad 0 \leq \psi_{r,s}(z) \leq C_0 \left( r \log_+ \frac{|z|}{r} + s \log_+ \frac{|z|}{s} \right), \quad |z| \geq 2s.$$

Note of course that  $C_0$  is independent of  $r, s$ .

If  $A$  admits a Brown measure or is normal with  $\lim_{t \rightarrow \infty} \mu_t(A) = 0$ , let us define

$$\tilde{\Phi}(r, s; A) = \int_{\mathbb{C}} (\operatorname{Re} z) \varphi_{r,s}(\log |z|) d\nu_A(z)$$

$$\Omega(r, s; A) = \int_{\mathbb{C}} \rho_{r,s}(\log |z|) d\nu_A(z)$$

$$\Psi(r, s; A) = \int_{\mathbb{C}} \psi_{r,s}(z) d\nu_A(z).$$

Thus  $\Psi(r, s; A) = \Omega(r, s; A) - \tilde{\Phi}(r, s; A)$  and  $\Psi(r, s; -A) = \Omega(r, s; A) + \tilde{\Phi}(r, s; A)$ . We can apply Lemma 6.4 to  $\Psi(r, s; \cdot)$ , giving

$$\begin{aligned} \Psi(r, s; T) &\leq \frac{1}{2\pi} \int_0^{2\pi} \Psi(r, s; T + e^{i\theta} T^*) d\theta \\ \Psi(r, s; -T) &\leq \frac{1}{2\pi} \int_0^{2\pi} \Psi(r, s; -T - e^{i\theta} T^*) d\theta. \end{aligned}$$

Note that  $\theta \rightarrow \tilde{\Phi}(r, s; T + e^{i\theta} T^*)$  is a Borel function by using Lemma 6.4 and the equation

$$\tilde{\Phi}(r, s; A) = \frac{1}{2}(\Psi(r, s; A) - \Psi(r, s; -A)).$$

We have

$$(6.16) \quad \left| \frac{1}{2\pi} \int_0^{2\pi} \tilde{\Phi}(r, s; T + e^{i\theta} T^*) d\theta - \tilde{\Phi}(r, s; T) \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \Omega(r, s; T + e^{i\theta} T^*) d\theta.$$

We first estimate the right-hand side of (6.16). Note that  $\Omega(r, s; T + e^{i\theta} T^*) = \Omega(r, s; W_\theta)$  where  $W_\theta = 2(H \cos \frac{\theta}{2} + K \sin \frac{\theta}{2})$  is hermitian and hence from (6.14),

$$\Omega(r, s, W_\theta) \leq C_0 \left( r \int_0^\infty \log_+ \frac{\mu_t(|W_\theta|)}{r} dt + s \int_0^\infty \log_+ \frac{\mu_t(|W_\theta|)}{s} dt \right).$$

Hence for all  $0 \leq \theta \leq 2\pi$ ,

$$\Omega(r, s, W_\theta) \leq C_0 \left( r \int_0^\infty \log_+ \frac{\mu_t(P)}{r} dt + s \int_0^\infty \log_+ \frac{\mu_t(P)}{s} dt \right),$$

where  $P = 2(|H| + |K|)$ . Thus the right-hand side of (6.16) is estimated by

$$C_0 \left( r \int_0^\infty \log_+ \frac{\mu_t(P)}{r} dt + s \int_0^\infty \log_+ \frac{\mu_t(P)}{s} dt \right).$$

In other words the right-hand side of (6.16) belongs to  $\mathcal{G}(\mathcal{I})$ , and hence so does the left-hand side.

Now we turn to the left-hand side of (6.16). We note that

$$|\tilde{\Phi}(r, s; T) - \operatorname{Re} \Phi(r, s; T)| \leq \int_{r < |z| < 2r} |z| d\nu_T(z) + \int_{s < |z| < 2s} |z| d\nu_T(z).$$

Hence

$$|\tilde{\Phi}(r, s; T) - \operatorname{Re} \Phi(r, s; T)| \leq 2r\nu_T(|z| > r) + 2s\nu_T(|z| > s).$$

By Proposition 6.5 this implies that  $\tilde{\Phi}(r, s; T) - \operatorname{Re} \Phi(r, s; T) \in \mathcal{F}(\mathcal{I})$ .

By the same argument we also have

$$\sup_{0 \leq \theta \leq 2\pi} |\tilde{\Phi}(r, s; T + e^{i\theta}T^*) - \operatorname{Re} \Phi(r, s; T + e^{i\theta}T^*)| \in \mathcal{F}(\mathcal{I}).$$

Now, by using parts (1) and (3) of Proposition 6.2, we easily obtain that

$$\sup_{0 \leq \theta \leq 2\pi} |\operatorname{Re} \Phi(r, s; T + e^{i\theta}T^*) - (1 + \cos \theta)\Phi(r, s; H) + \sin \theta \Phi(r, s; K)| \in \mathcal{F}(\mathcal{I}).$$

So on integration we find that

$$\frac{1}{2\pi} \int_0^{2\pi} \tilde{\Phi}(r, s; T + e^{i\theta}T^*) d\theta - \Phi(r, s; H) \in \mathcal{F}(\mathcal{I}).$$

It follows that the left-hand side of (6.16) differs from  $|\operatorname{Re} \Phi(r, s; T) - \Phi(r, s; H)|$  by a function in class  $\mathcal{F}(\mathcal{I})$ . Combining we obtain:

$$\operatorname{Re} \Phi(r, s; T) - \Phi(r, s; H) \in \mathcal{G}(\mathcal{I}). \quad \square$$

Compare the following to Theorem 3 of [10].

**THEOREM 6.8.** — *Let  $\mathcal{I}$  be a geometrically stable submodule of  $\overline{\mathcal{M}}$ . Let  $T \in \mathcal{I}$  admit a Brown measure. Then  $T \in [\mathcal{I}, \mathcal{M}]$  if and only if there is a positive operator  $V \in \mathcal{I}$  with*

$$(6.17) \quad \left| \int_{r < |z| \leq s} z d\nu_T(z) \right| \leq r\tau(E_V(r, \infty)) + s\tau(E_V(s, \infty)), \quad 0 < r, s < \infty.$$

*Proof.* — First suppose  $\mathcal{M} \not\subseteq \mathcal{I}$ . Let  $H = \frac{1}{2}(T + T^*)$  and  $K = \frac{1}{2i}(T - T^*)$ . Note that  $T \in [\mathcal{I}, \mathcal{M}]$  if and only if  $H, K \in [\mathcal{I}, \mathcal{M}]$ . Then by

Theorem 6.7 we have  $\Phi(r, s; T) \in \mathcal{G}(\mathcal{I})$  if and only if  $\Phi(r, s; H), \Phi(r, s; K) \in \mathcal{G}(\mathcal{I})$ . By Proposition 6.6 this implies that  $\Phi(r, s; T) \in \mathcal{G}(\mathcal{I})$  if and only if  $T \in [\mathcal{I}, \mathcal{M}]$ .

Let  $S \in \mathcal{I}$  be a normal operator with  $\nu_S = \nu_T$  as given by Proposition 6.5. Then the same reasoning as above applies to  $S$ , yielding  $S \in [\mathcal{I}, \mathcal{M}]$  if and only if  $\Phi(r, s; S) \in \mathcal{G}(\mathcal{I})$ . By Proposition 6.3,  $S \in [\mathcal{I}, \mathcal{M}]$  if and only if  $\Phi(r, s; S) \in \mathcal{F}(\mathcal{I})$ . But  $\Phi(r, s; T) = \Phi(r, s; S)$ , so  $T \in [\mathcal{I}, \mathcal{M}]$  if and only if  $\Phi(r, s; T) \in \mathcal{F}(\mathcal{I})$ .

Now suppose  $\mathcal{M} \subseteq \mathcal{I}$ . If  $T \in [\mathcal{I}, \mathcal{M}]$ , then by Proposition 5.10,  $T \in [\mathcal{I}_0, \mathcal{M}]$ , so by the case just proved there is a positive operator  $V \in \mathcal{I}_0$  making (6.17) hold. On the other hand, suppose  $V \in \mathcal{I}$  is a positive operator making (6.17) hold. Let  $S \in \mathcal{I}$  be a normal operator with  $\nu_S = \nu_T$  as given by Proposition 6.5. Then

$$|\Phi(r, s; S)| = |\Phi(r, s; T)| \leq r\tau(E_V(r, \infty)) + s\tau(E_V(s, \infty)), \quad 0 < r, s < \infty.$$

Hence, by Proposition 6.3,  $S \in [\mathcal{I}, \mathcal{M}]$ . Invoking Propositions 5.10 and 6.3 again, we find a positive operator  $V' \in \mathcal{I}_0$  such that

$$|\Phi(r, s; S)| \leq r\tau(E_{V'}(r, \infty)) + s\tau(E_{V'}(s, \infty)), \quad 0 < r, s < \infty.$$

But then, since  $\mathcal{M} \not\subseteq \mathcal{I}_0$ , we get  $T \in [\mathcal{I}_0, \mathcal{M}]$  by the case proved above.  $\square$

Let us say that  $T$  is *approximately nilpotent* if  $T$  admits a Brown measure with support equal to  $\{0\}$ . This is equivalent to the statement that  $\Delta(g_k(wT)) = 1$  for all  $w \in \mathbb{C}$ .

**COROLLARY 6.9.** — *If  $\mathcal{I}$  is a geometrically stable submodule of  $\overline{\mathcal{M}}$  then every approximately nilpotent  $T \in \mathcal{I}$  belongs to  $[\mathcal{I}, \mathcal{M}]$ .*

A trace on a submodule  $\mathcal{I}$  of  $\overline{\mathcal{M}}$  is a linear functional  $\rho : \mathcal{I} \rightarrow \mathbb{C}$  such that  $\rho(AB) = \rho(BA)$  whenever  $A \in \mathcal{I}$  and  $B \in \mathcal{M}$ , (i.e. such that  $\rho$  vanishes on  $[\mathcal{I}, \mathcal{M}]$ ). The following is the analogue in the  $\text{II}_\infty$  case of Cor. 2.4 of [8].

**COROLLARY 6.10.** — *Let  $\mathcal{I}$  be a geometrically stable submodule of  $\overline{\mathcal{M}}$  and suppose  $\rho : \mathcal{I} \rightarrow \mathbb{C}$  is a trace. If  $T \in \mathcal{I}$  admits a Brown measure  $\nu_T$ , then  $\rho(T)$  depends only on  $\nu_T$ .*

*Proof.* — Suppose  $S \in \mathcal{I}$  admits a Brown measure  $\nu_S$  and  $\nu_S = \nu_T$ . We will show  $\rho(S) = \rho(T)$ . Consider  $R = S \oplus (-T)$  in the sense of Definition 2.4, namely,  $R = V_1 S V_1^* - V_2 T V_2^*$  for  $V_1$  and  $V_2$  in  $\mathcal{M}$  isometries



with  $V_1V_1^* + V_2V_2^* = I$ . By [2, Thm. 4.3],  $R$  admits a Brown measure  $\nu_R$  which is given by  $\nu_R(A) = \nu_S(A) + \nu_T(-A)$ . In particular,  $\nu_R$  is invariant under the transformation of  $\mathbb{C}$  described by multiplication by  $-1$ ; as a consequence, all of the integrals

$$\int_{r < |z| \leq s} z d\nu_R(z)$$

vanish, and then by Theorem 6.8, we get  $R \in [\mathcal{I}, \mathcal{M}]$ . From this we have  $\rho(R) = 0$ . But  $\rho(R) = \rho(S) - \rho(T)$ .  $\square$

We remark that in the case of an ideal  $\mathcal{I}$  of  $B(\mathcal{H})$  the relationship between the subspace  $[\mathcal{I}, B(\mathcal{H})]$  and the growth of the characteristic determinant is discussed further in [21], and it is possible that some analogous results can be obtained here for the Fuglede-Kadison determinant.

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