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HOLOMORPHIC RANK-2 VECTOR BUNDLES ON NON-KÄHLER ELLIPTIC SURFACES

by Vasile BRÎNZĂNESCU & Ruxandra MORARU

1. Introduction.

In this paper, we study the existence of holomorphic vector bundles on non-Kähler elliptic surfaces; their classification and stability properties are discussed in [BrMo1, BrMo2]. Let X be a smooth compact complex surface. The existence problem for vector bundles on X consists in determining which topological complex vector bundles admit holomorphic structures, or equivalently, in finding all triples (r, c_1, c_2) in $\mathbb{N} \times NS(X) \times \mathbb{Z}$ for which there exists a rank- r holomorphic vector bundle on X with Chern classes c_1 and c_2 . For projective surfaces, Schwarzenberger [S] proved that any triple (r, c_1, c_2) in $\mathbb{N} \times NS(X) \times \mathbb{Z}$ comes from a rank- r holomorphic (algebraic) vector bundle. In contrast, for non-projective surfaces, there is a natural necessary condition for the existence problem [BaL, BrF, LeP]:

$$\Delta(r, c_1, c_2) := \frac{1}{r} \left(c_2 - \frac{r-1}{2r} c_1^2 \right) \geq 0.$$

One can always construct filtrable bundles by using extensions of coherent sheaves; in fact, on a non-algebraic surface X , there exists a filtrable rank- r

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holomorphic vector bundle E with Chern classes c_1 and c_2 if and only if its discriminant $\Delta(E)$ satisfies the inequality

$$\Delta(E) := \Delta(r, c_1, c_2) \geq m(r, c_1),$$

where

$$m(r, c_1) := -\frac{1}{2r} \max \left\{ \sum_1^r \left(\frac{c_1}{r} - \mu_i \right)^2, \mu_1, \dots, \mu_r \in NS(X), \sum_1^r \mu_i = c_1 \right\}$$

(see [BaL, BrF, LeP]). Therefore, the only unknown situations occur for bundles of rank greater than one that have a discriminant in the interval $[0, m(r, c_1))$; vector bundles with such discriminants will, of course, be non-filtrable and the difficulty of the problem resides in the lack of a general method for constructing non-filtrable bundles. One is thus compelled to focus on particular classes of surfaces, to find specific construction methods.

The existence of bundles on non-projective surfaces is, in general, still an open question, which has been completely settled only in the case of primary Kodaira surfaces [ABrTo]. For rank-2 holomorphic vector bundles, the problem has been solved for complex 2-tori [To], as well as for K3 surfaces and the known surfaces of class VII [TTo]; moreover, the method used in [TTo] (Donaldson polynomials) seems to also work for (non-algebraic) Kähler elliptic surfaces. In this article, we consider general non-Kähler elliptic surfaces, giving necessary and sufficient conditions for the existence of holomorphic rank-2 vector bundles on non-Kähler elliptic surfaces.

Recall that a surface is said to be elliptic if it admits a holomorphic fibration over a curve with generic fibre an elliptic curve; for instance, non-Kähler elliptic surfaces are given by holomorphic fibrations without a section whose smooth fibres are isomorphic to a fixed elliptic curve. For vector bundles on any elliptic fibration $\pi : X \rightarrow B$, restriction to a fibre is a natural operation: there exists a divisor in the relative Jacobian $J(X)$ of X , called the *spectral curve* or *cover* of the bundle, that encodes the isomorphism class of the bundle over each fibre of π . This divisor is an important invariant of bundles on elliptic fibrations, which has proven very useful in their study (see [F1, FM, FMW, BJPS, D]) for projective fibrations, [DOPW1, DOPW2] for Calabi-Yau threefolds without a section, and [BH, Mo, T] for non-Kähler fibre bundles). The spectral construction presented in this paper is a modification of the Fourier-Mukai transform for

certain elliptic fibrations without a section, which will be used in [BrMo1] to define a twisted Fourier-Mukai transform that is specific to non-Kähler elliptic surfaces.

The paper is organised as follows. We begin by presenting and proving some topological and geometrical properties of non-Kähler elliptic surfaces; in particular, we show that if $\pi : X \rightarrow B$ is such a surface, then the restriction of *any* holomorphic vector bundle on X to a smooth fibre of π *always* has degree zero. Unlike the algebraic case [FM], the description of line bundles on non-Kähler elliptic surfaces is not straightforward; indeed, even though these surfaces have very few divisors (they are given by the fibres of π), there exist many line bundles on them. Nonetheless, we are able to establish a correspondence between line bundles on a non-Kähler elliptic surface and sections of its relative Jacobian; this follows from results of [Br1, Br2, Br3, BrU] regarding the Neron-Severi and Picard groups of these surfaces. In the third section, we extend the spectral construction of [BH, Mo] to the case of holomorphic rank-2 vector bundles on non-Kähler elliptic surfaces.

The last section of the article is devoted to the existence theorems, the principal one being the following. Consider a non-Kähler elliptic surface $\pi : X \rightarrow B$ and an element $c_1 \in NS(X)$. One can then construct a ruled surface $\mathbb{F}_{\bar{c}_1}$ over B , as the quotient of $J(X)$ by an involution, that is uniquely determined, up to isomorphism, by the class \bar{c}_1 of c_1 in $NS(X)$ modulo $2NS(X)$ (an explicit description of this ruled surface is given in section 3.2). We can now state the main result of the paper:

THEOREM . — *Let X be a minimal non-Kähler elliptic surface over a curve B of genus g and fix a pair (c_1, c_2) in $NS(X) \times \mathbb{Z}$. Let $m_{c_1} := m(2, c_1)$ and denote \bar{c}_1 the class of c_1 in $NS(X)$ modulo $2NS(X)$.*

- (i) *The class \bar{c}_1 uniquely determines, up to isomorphism, a ruled surface $\mathbb{F}_{\bar{c}_1}$ over B whose invariant $e_{\bar{c}_1} := e(\mathbb{F}_{\bar{c}_1})$ satisfies the inequality*

$$\max\{-g, -4m_{c_1}\} \leq e_{\bar{c}_1} \leq 0.$$

- (ii) *There exists a holomorphic rank-2 vector bundle on X with Chern classes c_1 and c_2 if and only if*

$$\Delta(2, c_1, c_2) \geq (m_{c_1} - d_{\bar{c}_1}/2),$$

where $d_{\bar{c}_1} := (e_{\bar{c}_1} + 4m_{c_1})/2$. Note that both $d_{\bar{c}_1}$ and $(m_{c_1} - d_{\bar{c}_1}/2)$ are non-negative numbers. Furthermore, if

$$(m_{c_1} - d_{\bar{c}_1}/2) \leq \Delta(2, c_1, c_2) < m_{c_1},$$

then the corresponding vector bundles are non-filtrable.

This theorem gives, in particular, sufficient conditions on topological invariants for the existence of non-filtrable bundles. Since non-filtrable bundles are always stable, this implies that if $(m_{c_1} - d_{\bar{c}_1}/2) \leq \Delta(2, c_1, c_2) < m_{c_1}$, then the moduli spaces of stable bundles with Chern classes c_1 and c_2 are non-empty. Nevertheless, there also exist non-empty moduli spaces for $\Delta(2, c_1, c_2) \geq m_{c_1}$ [BrMo2]. (For a definition of stability for vector bundles on compact complex manifolds, see [Bh, Kob].)

The theorem also reduces the existence problem of holomorphic rank-2 vector bundles on a non-Kähler elliptic surface $\pi : X \rightarrow B$ to the problem of computing the invariants of a specific class of ruled surfaces over B ; in fact, for a fixed $c_1 \in NS(X)$, the invariant $d_{\bar{c}_1}$ induced by $e_{\bar{c}_1}$ is an integer that corresponds to the maximal degree of subline bundles of a vector bundle V on B , of rank two and degree $4m_{c_1}$, such that $\mathbb{F}_{\bar{c}_1} = \mathbb{P}(V)$ (see Remark 3.9). The existence problem of holomorphic rank-2 vector bundles on non-Kähler elliptic surfaces thus boils down to an interesting complex geometric problem on the base curve: the maximal degree problem for subline bundles of a fixed holomorphic rank-2 vector bundle on the base curve.

We end by noting that the techniques developed here and in [BrMo1, BrMo2] can be used to study stable holomorphic vector bundles of arbitrary rank on non-Kähler elliptic and torus fibrations. Beyond their intrinsic mathematical interest, one of the motivations for studying stable bundles on such fibrations comes from recent developments in superstring theory, where six-dimensional non-Kähler manifolds occur in the context of $\mathcal{N} = 1$ supersymmetric heterotic and type II string compactifications with non-vanishing background H-field (see [BBDG, CCDLMZ, GP] and the references therein).

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2. Line bundles.

Let $X \xrightarrow{\pi} B$ be a minimal non-Kähler elliptic surface, with B a smooth compact connected curve; it is well-known that $X \xrightarrow{\pi} B$ is a quasi-bundle over B , that is, all the smooth fibres of π are isomorphic to a fixed elliptic curve T and the singular ones (if any) are isogeneous to multiples of T (see [Kod, Br3]). We begin by presenting several topological and geometric properties of these surfaces.

Let T^* denote the dual of T (we fix a non-canonical identification $T^* := \text{Pic}^0(T)$). In this case, the Jacobian surface associated to $X \xrightarrow{\pi} B$ is simply

$$J(X) = B \times T^* \xrightarrow{P_1} B$$

(see, for example, [Kod, BPV, Br1]) and the surface is obtained from its relative Jacobian by a finite number of logarithmic transformations [Kod, BPV, BrU]. Also, if X has multiple fibres T_1, \dots, T_r , with corresponding multiplicities m_1, \dots, m_r , then its canonical bundle is given by

$$K_X = \pi^* K_B \otimes \mathcal{O}_X \left(\sum_{i=1}^r (m_i - 1) T_i \right).$$

Finally, we have the following identification [Br1, Br2, BrU]:

$$NS(X)/\text{Tors}(NS(X)) \cong \text{Hom}(J_B, \text{Pic}^0(T)),$$

where $NS(X)$ is the the Neron-Severi group of the surface and J_B denotes the Jacobian variety of B ; the torsion of $H^2(X, \mathbb{Z})$ is generated by the classes of the fibres (both smooth and multiple). In the remainder, the class modulo $\text{Tors}(H^2(X, \mathbb{Z}))$ of an element $c \in H^2(X, \mathbb{Z})$ will be denoted \hat{c} . Given these considerations, we have:

LEMMA 2.1. — *Let $X \xrightarrow{\pi} B$ be a non-Kähler elliptic surface.*

- (i) *If $c \in NS(X)$, then $\pi_*(c) = 0$.*
 (ii) *For any element $c \in NS(X)$, $c^2 = -2 \deg(\widehat{c})$.*

Proof. — The lemma is certainly true for torsion classes. Let us then assume that $c \notin \text{Tors}(NS(X))$ and choose a line bundle L on X with first Chern class c . Then $\widehat{c} \neq 0$ and, by fixing a base-point in B , the cohomology class \widehat{c} can be considered as a covering map $\widehat{c} : B \rightarrow \text{Pic}^0(T)$ such that

$$\widehat{c}^{-1}(\lambda_0) = \{b \in B \mid L|_{F_b} \simeq \lambda_0\}.$$

Since $\widehat{c} \neq 0$, we have $\widehat{c}^{-1}(\mathcal{O}_T) \neq B$. Therefore, the stalk of π_*L is zero at the generic point in B and the direct image sheaf π_*L vanishes; furthermore, the higher direct image sheaf $R^1\pi_*L$ is a torsion sheaf supported on $\widehat{c}^{-1}(\mathcal{O}_T)$. In particular, $\pi^!L = -R^1\pi_*L$ and, by Grothendieck-Riemann-Roch, the pushdown $\pi_*(c)$ is equal to the rank of the torsion sheaf $R^1\pi_*L$, which is zero, proving (i). Combining the results of (i) with Grothendieck-Riemann-Roch, we obtain $c_1(R^1\pi_*L) = -\frac{1}{2}c^2 \cdot h$, where h is the positive generator of $H^2(B, \mathbb{Z})$. Hence, the degree of the map \widehat{c} is equal to $\#(\widehat{c}^{-1}(\mathcal{O}_T)) = -\frac{1}{2}c^2$ and we are done. \square

LEMMA 2.2. — *Let $\pi : X \rightarrow B$ be a non-Kähler elliptic surface and \mathcal{L} a line bundle on X . The restriction of \mathcal{L} to any smooth fibre of π has degree zero.*

Proof. — Let $m_1T_1, m_2T_2, \dots, m_\ell T_\ell$ be the multiple fibres of π and set $b_i = \pi(T_i)$. Denote m the least common multiple of m_1, m_2, \dots, m_ℓ and choose a non-negative integer e such that m divides $\ell + e$; next, take distinct points $b_{\ell+1}, \dots, b_{\ell+e}$, which are different from $b_i, i = 1, \dots, \ell$, and fix a point b with T_b smooth. Then, there exists at least one line bundle M on B with the property that

$$M^{\otimes m} \cong \mathcal{O}_B(b_1 + \dots + b_{\ell+e});$$

such a line bundle defines an m -cyclic covering $\varepsilon : B' \rightarrow B$ that is totally ramified at $b_1, \dots, b_{\ell+e}$ (see [BPV], Chapter I, Lemma 17.1). By Lemma 3.18 in [Br3], there exists a principal T -bundle $\pi' : X' \rightarrow B'$ and an m -cyclic covering $\psi : X' \rightarrow X$ over $\varepsilon : B' \rightarrow B$; let \tilde{T} be a connected component

of $\psi^{-1}(T_b)$. Then \tilde{T} is a fibre of π' and the restriction $\tilde{T} \rightarrow T_b$ of ψ is an isomorphism. Therefore, we have

$$c_1(\mathcal{L}|_{T_b}) = c_1(\psi^*(\mathcal{L})|_{\tilde{T}}) = 0,$$

because $\pi' : X' \rightarrow B'$ is a principal elliptic bundle [Br3, T]. □

Remark. — Similar results are stated in [ABrTo, T] for non-Kähler principal elliptic bundles, that is, non-Kähler elliptic surfaces without multiple fibres.

Referring to Lemma 2.2 and [BrU], we can therefore associate to any line bundle \mathcal{L} on X a holomorphic mapping $\varphi : B \rightarrow T^*$ such that

$$\mathcal{L}|_{T_b} = \varphi(b),$$

for any smooth fibre T_b , that is, a section of $J(X) = B \times T^*$. Conversely, one can associate to every section of $J(X)$ a line bundle on X , as stated in:

PROPOSITION 2.3. — *Let $\pi : X \rightarrow B$ be a non-Kähler elliptic surface, with general fibre T , and $J(X) = B \times T^*$ be the associated Jacobian surface of X . Then:*

- (i) *For any section $\Sigma \subset J(X)$, there exists a line bundle \mathcal{L} on X whose restriction to every smooth fibre T_b is the same as the line bundle Σ_b of degree zero on $T = T_b$.*
- (ii) *The set of all line bundles on X that restrict, on every smooth fibre of π , to the line bundle of degree zero determined by the section Σ is a principal homogeneous space over P_2 , where P_2 is the subgroup of line bundles on X generated by $\pi^* \text{Pic}(B)$ and the $\mathcal{O}_X(T_i)$ '.*

Proof. — Choose a general point $b \in B$ with T_b smooth and consider the natural restriction morphism $r : \text{Pic}(X) \rightarrow \text{Pic}(\pi^{-1}(b)) = \text{Pic}(T)$. Let (P_j) be the filtration of $\text{Pic}(X)$ defined by

$$P_0 = \text{Pic}(X), P_1 = \text{Ker}(r), \text{ and } P_2.$$

Set $N(X) := P_0/P_1$ and $\tilde{N}(X) := \{c_1(L) \mid L \in N(X)\}$. Referring to [Br1] and [BrU], we have $\tilde{N}(X) = 0$ and

$$NS(X)/\text{Tors } NS(X) \cong \text{Hom}(J_B, T^*) \cong P_1/P_2.$$

Consequently, $N(X) \subset \text{Pic}^0(T)$. Since any line bundle in $\text{Pic}^0(T)$ is invariant by translations, we obtain

$$N(X) = \text{Pic}^0(T)$$

by Lemma 2.2 and [BrU]. Let $\lambda = \Sigma_b \in T^*$ and let Σ^λ be the constant section $B \times \lambda \subset J(X)$. Following the construction in [BrU], the line bundle $\lambda \in T^*$ extends to a line bundle \mathcal{L}^λ on X that corresponds to the constant section Σ^λ . Let B_0 be the zero section of $J(X)$. Given the identification $P_1/P_2 \cong \text{Hom}(J_B, T^*)$, there exists a line bundle \mathcal{L}_1 in $P_1 = \text{Ker}(r)$ whose corresponding element in $\text{Hom}(J_B, T^*)$ is a section that is linearly equivalent to $\Sigma - \Sigma^\lambda + B_0$ (look at the addition law of the group $\text{Hom}(J_B, T^*)$). The line bundle $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}^\lambda$ is then such that its restriction to every smooth fibre T_b is the same as the line bundle $\Sigma_b \in T^*$, proving (i). If the line bundles \mathcal{L}' and \mathcal{L} on X both have the above property, then by the same isomorphism, $\mathcal{L}' \otimes \mathcal{L}^{-1} \in P_2$ and we are done. \square

We can now characterise the sections of the Jacobian surface as follows.

LEMMA 2.4. — *Let X be a non-Kähler elliptic surface. Then, any section Σ of the Jacobian surface $J(X)$ of X has trivial self-intersection. Furthermore, if \mathcal{L} is any line bundle on X corresponding to the section Σ of $J(X)$, then*

$$\Sigma \cdot B_0 = -c_1^2(\mathcal{L})/2,$$

where B_0 denotes the zero section of $J(X)$.

Proof. — The invariants of the Jacobian surface $J(X) = B \times T^*$ are

$$p_g(J(X)) = g, \quad q(J(X)) = g + 1, \quad \text{and} \quad K_{J(X)} = p_1^* K_B,$$

where g is the genus of the curve B ; the adjunction formula gives $\Sigma^2 = 0$. Let \hat{c}_1 be the class of $c_1(\mathcal{L})$ in $NS(X)/\text{Tors}(NS(X)) \cong \text{Hom}(J_B, T^*)$. As in the proof of Lemma 2.1, we can then think of \hat{c}_1 as being a covering map $\hat{c}_1 : B \rightarrow T^*$ of degree $-c_1^2(\mathcal{L})/2$; since the degree of \hat{c}_1 is also equal to $\Sigma \cdot B_0$, the lemma follows. \square

We end the section by giving a description of torsion line bundles on a principal elliptic bundle $X \xrightarrow{\pi} B$; the surface is now isomorphic to a quotient of the form

$$X = \Theta^* / \langle \tau \rangle,$$

where Θ is a line bundle on B with positive Chern class d , Θ^* is the complement of the zero section in the total space of Θ , and $\langle \tau \rangle$ is the multiplicative cyclic group generated by a fixed complex number τ , with $|\tau|$ greater than 1. The standard fibre of this bundle is

$$T \cong \mathbb{C}^* / \langle \tau \rangle \cong \mathbb{C} / (2\pi i\mathbb{Z} + \ln(\tau)\mathbb{Z}).$$

(We assume d to be positive so that the surface X is non-Kählerian.)

The set of all holomorphic line bundles on X with trivial Chern class is given by the zero component of the Picard group $\text{Pic}^0(X)$. Referring to Proposition 1.6 in [T], one has

$$\text{Pic}^0(X) \cong \text{Pic}^0(B) \times \mathbb{C}^*.$$

Any line bundle in $\text{Pic}^0(X)$ is therefore of the form $H \otimes L_\alpha$, where H is the pullback to X of an element of $\text{Pic}^0(B)$ and L_α is the line bundle corresponding to the constant automorphy factor $\alpha \in \mathbb{C}^*$. We illustrate this by constructing the restriction of the universal (Poincaré) line bundle \mathcal{U} over $X \times \text{Pic}^0(X)$ to

$$X \times \mathbb{C}^* := X \times \{0\} \times \mathbb{C}^*.$$

One starts with a trivial line bundle $\bar{\mathbb{C}}$ on $\Theta^* \times \mathbb{C}^*$ and applies to it the following \mathbb{Z} -action

$$\begin{aligned} \Theta^* \times \mathbb{C}^* \times \mathbb{Z} &\rightarrow \Theta^* \times \mathbb{C}^* \\ (z, \alpha, n) &\mapsto (\tau^n z, \alpha). \end{aligned}$$

Since this action is trivial on \mathbb{C}^* , the Poincaré line bundle \mathcal{U} is obtained by identifying $s \in \bar{\mathbb{C}}_{(z, \alpha)}$ with $\alpha s \in \bar{\mathbb{C}}_{(\tau z, \alpha)}$.

Notation. — In the remainder, we shall denote by L_α the line bundle corresponding to the automorphy factor $\alpha \in \mathbb{C}^*$.

Remark. — Although the line bundle L_{τ^m} is trivial over the fibres of π , one cannot define an action of \mathbb{Z} on \mathbb{C}^* that leaves the restriction of the Poincaré line bundle \mathcal{U} to $X \times \mathbb{C}^*$ invariant. Indeed, if \mathbb{Z} acts on \mathbb{C}^* , then multiplication by τ is defined on the fibres of $\bar{\mathbb{C}}$ by

$$(2.5) \quad \begin{aligned} \tau : \Theta^* \times \mathbb{C}^* \times \mathbb{C} &\rightarrow \Theta^* \times \mathbb{C}^* \times \mathbb{C} \\ (z, \alpha, t) &\mapsto (\tau z, \tau \alpha, \alpha t). \end{aligned}$$

On the surface X , z and τz define the same point x . However, (2.5) indicates that τ sends $\mathcal{U}_{(x,\alpha)}$ to $\mathcal{U}_{(x,\tau\alpha)} \otimes L_{\tau^{-1},x}$. Hence, the Poincaré line bundle is not invariant under such an action.

3. Holomorphic vector bundles.

Consider a pair (c_1, c_2) in $NS(X) \times \mathbb{Z}$. Its corresponding *discriminant* is then given by

$$\Delta(2, c_1, c_2) := \frac{1}{2} \left(c_2 - \frac{c_1^2}{4} \right) \geq 0.$$

Let E be a holomorphic rank-2 vector bundle on X , with $c_1(E) = c_1$ and $c_2(E) = c_2$. We fix the following notation:

$$\Delta(E) := \Delta(2, c_1, c_2) \text{ and } n_E := -ch_2(E),$$

where $ch_2(E) = c_1^2/2 - c_2$ is the second Chern character of E .

Remark 3.1. — Referring to Lemma 2.1, if $\Delta(2, c_1, c_2) \geq 0$, then $n_E \geq 0$.

To study bundles on X , one of our main tools will be restriction of the bundle to the smooth fibres $\pi^{-1}(b) \cong T$ of the fibration $\pi : X \rightarrow B$. Since the restriction of any bundle on X to a fibre T has first Chern class zero, we consider E as family of degree zero bundles over the elliptic curve T , parametrised by B . Given a rank two bundle over X , its restriction to a generic fibre of π is semistable. More precisely, we have:

PROPOSITION 3.2. — *Let E be a rank 2 holomorphic vector bundle over X . Then, $E|_{\pi^{-1}(b)}$ is unstable on at most an isolated set of points $b \in B$.*

Proof. — Suppose that $b \in B$ is a point such that $E|_{\pi^{-1}(b)}$ is unstable, splitting as $\lambda_b \oplus (\lambda'_b)^*$ for some line bundles λ_b and λ'_b in $\text{Pic}^{-k}(T)$, $k > 0$. Consider the elementary modification

$$0 \rightarrow E' \rightarrow E \rightarrow j_*\lambda_b \rightarrow 0,$$

where $j : T_b \rightarrow X$ is the natural inclusion. Referring to [F2] (Chapter II, Lemma 16), the discriminant of E' is given by

$$\Delta(E') = \Delta(E) + \frac{1}{2}j_*c_1(\lambda_b);$$

furthermore,

$$\Delta(E') < \Delta(E)$$

because $\deg(\lambda_b) = -k < 0$. Therefore, since the existence of E' implies that its discriminant is a non-negative number, the result follows. \square

Note. — These isolated points are called the *jumps* of the bundle E .

3.1. The spectral curve of a rank-2 vector bundle.

Let us assume for a moment that X does not have multiple fibres. Choose a line bundle L in $\text{Pic}^0(X)$ such that $h^0(\pi^{-1}(b), L^* \otimes E)$ is zero, for generic b . The direct image sheaf $R^1\pi_*(L^* \otimes E)$ is therefore a torsion sheaf supported on isolated points b such that $E|_{\pi^{-1}(b)}$ is semistable and has $L|_{\pi^{-1}(b)}$ as a subline bundle, or $E|_{\pi^{-1}(b)}$ is unstable; consequently, if h is the positive generator of $H^2(B, \mathbb{Z})$, then

$$c_1(R^1\pi_*(L^* \otimes E)) = -\pi_*(ch(E) \cdot td(X)) \cdot td(B)^{-1} = n_E h.$$

However, since the discriminant of E is a non-negative number, then so is the integer n_E (see remark 3.1): the sheaf $R^1\pi_*(L^* \otimes E)$ is supported on n_E points, counting multiplicity.

To obtain a complete description of the restriction of E to the fibres of π , this construction must be repeated for every line bundle on X ; this is done by taking the direct image $R^1\pi_*$ for all line bundles simultaneously. Let π also denote the projection $\pi := \pi \times id : X \times \text{Pic}^0(B) \times \mathbb{C}^* \rightarrow B \times \text{Pic}^0(B) \times \mathbb{C}^*$, where id is the identity map on $\text{Pic}^0(B) \times \mathbb{C}^*$, and let $s : X \times \text{Pic}^0(B) \times \mathbb{C}^* \rightarrow X$ be the projection onto the first factor. If \mathcal{U} is the universal (Poincaré) line bundle over $X \times \text{Pic}^0(B) \times \mathbb{C}^*$, one defines

$$\widetilde{\mathcal{L}} := R^1\pi_*(s^*E \otimes \mathcal{U}).$$

This sheaf is supported on a divisor \widetilde{S}_E that is defined with multiplicity. We have the following remarks:

- Let H be the pullback to X of a line bundle of degree zero on B . The restriction of H to any fibre T is then trivial, implying that the support of

$$R^1\pi_*(s^*E \otimes \mathcal{U} \otimes H)$$

is also \widetilde{S}_E . We can therefore restrict the above construction to $X \times \mathbb{C}^* := X \times \{0\} \times \mathbb{C}^*$. In the remainder, we will use the same notation for this restriction.

- Consider the \mathbb{Z} -action on $B \times \mathbb{C}^*$ induced from the one on $X \times \mathbb{C}^*$. For any (b, α) in $B \times \mathbb{C}^*$, multiplication by τ sends the stalk $\widetilde{\mathcal{L}}_{(x, \alpha)}$ to $\widetilde{\mathcal{L}}_{(x, \tau\alpha)} \otimes L_{\tau^{-1}, x}$, leaving the support of $\widetilde{\mathcal{L}}$ unchanged.

By the above remarks, since the quotient $\mathbb{C}^*/\langle \tau \rangle$ of \mathbb{C}^* by the \mathbb{Z} -action is isomorphic to T^* , the support \widetilde{S}_E of $\widetilde{\mathcal{L}}$ descends to a divisor S_E in $J(X) = B \times T^*$ of the form

$$S_E := \left(\sum_{i=1}^k \{x_i\} \times T^* \right) + \overline{C},$$

where \overline{C} is a bisection of $J(X)$ (that is, $S_E.T^* = 2$ for any fibre T^* of $J(X)$) and x_1, \dots, x_k are points (counted with multiplicities) in B that correspond to the jumps of E .

If the fibration π has multiple fibres, the spectral cover of a bundle E on X is then constructed as follows. Referring to the proof of Lemma 2.2, there exists a principal T -bundle $\pi' : X' \rightarrow B'$ over an m -cyclic covering $\varepsilon : B' \rightarrow B$. Note that the map ε induces natural m -cyclic coverings $\psi : X' \rightarrow X$ and $J(X') \rightarrow J(X)$. By replacing X with X' (which has no multiple fibres) in the above construction, we obtain the spectral cover S_{ψ^*E} of ψ^*E as a divisor in $J(X')$. We define the spectral cover S_E of E as the projection of S_{ψ^*E} in $J(X)$; one easily sees that S_E does indeed give the isomorphism type of E over each smooth fibre of π .

Remark. — The above construction can be defined for any rank- r vector bundle. In particular, for a line bundle, the spectral cover corresponds to the section of the Jacobian surface $J(X)$ defined in section 2.

3.2. The graph of a rank-2 vector bundle.

Let δ be the determinant line bundle of E . It then defines the following involution on the relative Jacobian $J(X) = B \times T^*$ of X :

$$\begin{aligned} i_\delta : J(X) &\rightarrow J(X) \\ (b, \lambda) &\mapsto (b, \delta_b \otimes \lambda^{-1}), \end{aligned}$$

where δ_b denotes the restriction of δ to the fibre $T_b = \pi^{-1}(b)$. For a fixed point b in B , the involution induced on the corresponding fibre of $p_1 : J(X) \rightarrow B$ has four fixed points (the solutions of $\lambda^2 = \delta_b$). Taking the quotient of $J(X)$ by this involution, each fibre of p_1 becomes $T^*/i_\delta \cong \mathbb{P}^1$ and the quotient $J(X)/i_\delta$ is isomorphic to a ruled surface \mathbb{F}_δ over B . Let $\eta : J(X) \rightarrow \mathbb{F}_\delta$ be the canonical map. By construction, the spectral curve S_E associated to E is invariant under the involution i_δ and descends to the quotient \mathbb{F}_δ ; it can therefore be considered as the pullback via η of a divisor on \mathbb{F}_δ of the form

$$(3.3) \quad \mathcal{G}_E := \sum_{i=1}^k f_i + A,$$

where f_i is the fibre of the ruled surface \mathbb{F}_δ over the point x_i and A is a section of the ruling such that $\eta^*A = \overline{C}$. The divisor \mathcal{G}_E is called the *graph* of the bundle E . We finish by noting that, although the section A is a smooth curve on \mathbb{F}_δ , its pullback need not be smooth: it may be reducible or multiple with multiplicity 2.

Remark 3.4. — The invariant of the ruled surfaces \mathbb{F}_δ will be used in Theorem 4.5 to give sharp lower bounds for the discriminant of rank-2 vector bundles on X with fixed first Chern class. Recall that for any c_1 in $NS(X)$ we have

$$m(2, c_1) = -\frac{1}{2} \sup_{\mu \in NS(X)} (c_1/2 - \mu)^2$$

(see for example Remark (1) of page 103 in [Br3]); this means that for any line bundle $a \in c_1 + 2NS(X)$ and $c_2 \in \mathbb{Z}$ the following hold:

$$m(2, c_1(a)) = m(2, c_1)$$

and

$$\Delta(2, c_1(a), c_2) = \Delta(2, c_1, c_2).$$

It is therefore sufficient to consider only the classes $c_1 + 2NS(X)$ of c_1 modulo $2NS(X)$ to determine the lower bound of $\Delta(2, c_1, c_2)$.

Remark 3.5. — If δ is the pullback of a line bundle on B , then its restriction to any fibre of π is trivial and the induced involution i_δ is given by $(b, \lambda) \mapsto (b, \lambda^{-1})$; in this case, we have $(B \times T^*)/i_\delta = B \times \mathbb{P}^1$. Furthermore, if there exist line bundles a and δ' on X such that $\delta' = a^2\delta$, then \mathbb{F}_δ is isomorphic to $\mathbb{F}_{\delta'}$; indeed, the map $a : J(X) \rightarrow J(X)$ defined by $(b, \lambda) \mapsto (b, a_b\lambda)$ is an isomorphism of the Jacobian surface that commutes with the involutions determined by δ and δ' . For example, if δ is an element of $2NS(X)$, then $\delta = a^2$ for some line bundle on X and \mathbb{F}_δ is isomorphic to $B \times \mathbb{P}^1$.

LEMMA 3.6. — *The class $c_1 + 2NS(X)$ determines a ruled surface that is unique up to isomorphism. We denote this ruled surface by $\mathbb{F}_{\bar{c}_1}$, where \bar{c}_1 represents the class of c_1 in $NS(X)$ modulo $2NS(X)$.*

Proof. — Consider two line bundles δ and δ' such that $c_1(\delta) = c_1(\delta') = c_1$. Then, there exists a line bundle λ in $\text{Pic}^0(X)$ such that $\delta' = \lambda\delta$; referring to section 2, there exist $H \in \pi^*\text{Pic}^0(B)$ and a factor of automorphy $\alpha \in \mathbb{C}^*$ such that $\lambda = H \otimes L_\beta$. Note that since the restriction of H to every fibre of π is trivial, the line bundles $H \otimes L_\beta$ and L_β define the same involution. Choose a square root of β , say α , and set $a = L_\alpha$, then $i_{\delta'} = i_{a^2\delta}$. Hence, referring to Remark 3.5, the involutions $i_{\delta'}$ and i_δ define isomorphic ruled surfaces. \square

Let us now give a more detailed description of the ruled surface $\mathbb{F}_{\bar{c}_1}$; in particular, we shall express its invariant explicitly in terms of data given by \bar{c}_1 and give upper and lower bounds for the invariant. For any c_1 in $NS(X)$, choose a line bundle δ on X such that $c_1(\delta) \in c_1 + 2NS(X)$. It then induces a ruled surface that is isomorphic to $\mathbb{F}_{\bar{c}_1}$. Let us first fix some notation that will be used in the remainder of the paper. Set $m_{c_1} := m(2, c_1)$. Moreover, denote B_0 the zero-section of $J(X)$ and Σ_δ the section in $J(X)$ corresponding to δ ; also, let $p_1 : J(X) \rightarrow B$ be the projection onto the first factor. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_{J(X)}(\Sigma_\delta) \rightarrow \mathcal{O}_{J(X)}(B_0 + \Sigma_\delta) \rightarrow \mathcal{O}_{B_0}(\Sigma_\delta) \rightarrow 0.$$

Pushing down to B , we obtain a new exact sequence

$$(3.7) \quad 0 \rightarrow \mathcal{O}_B \rightarrow V_\delta \rightarrow L \rightarrow 0,$$

where

$$V_\delta := p_{1*}(\mathcal{O}_{J(X)}(B_0 + \Sigma_\delta))$$

is a rank-2 vector bundle on the curve B and

$$L := p_{1*}(\mathcal{O}_{B_0}(\Sigma_\delta))$$

is a line bundle on B , given by the effective divisor that corresponds to the projection onto B of the intersection points $B_0 \cap \Sigma_\delta$ (counted with multiplicity); consequently, the bundles L and V_δ both have degree $-c_1^2(\delta)/2$. Note that $\mathbb{F}_{\bar{c}_1} = \mathbb{P}(V_\delta)$.

Remark. — Two lines bundles δ, δ' in the congruence class \bar{c}_1 cannot yield isomorphic rank-2 vector bundles $V_\delta, V_{\delta'}$ on B , unless $c_1^2(\delta) = c_1^2(\delta')$, otherwise $V_\delta, V_{\delta'}$ would not have the same degree. If $c_1^2(\delta) = c_1^2(\delta')$, then $V_{\delta'} \cong V_\delta \otimes L$, for some (possibly non-trivial) line bundle L on B of degree zero; furthermore, one can show that if $\delta' = a^2\delta$, where $a \in \pi^*\text{Pic}(B)$, then $V_{\delta'} \cong V_\delta$. In general, however, all we can say is that the vector bundles V_δ on B induced by the line bundles δ in the congruence class \bar{c}_1 in $NS(X)$ are all isomorphic up to tensoring by a line bundle on B .

Given the above notation, we have the following result.

LEMMA 3.8. — *The invariant $e_{\bar{c}_1} := e(\mathbb{F}_{\bar{c}_1})$ of the ruled surface $\mathbb{F}_{\bar{c}_1}$ over B satisfies the inequality*

$$\max\{-g, -4m_{c_1}\} \leq e_{\bar{c}_1} \leq 0,$$

where g is the genus of B .

Proof. — Let δ be a line bundle on X in the congruence class \bar{c}_1 ; consider the rank-2 vector bundle V_δ on B of degree $-c_1^2(\delta)/2$ defined above. Since $\mathbb{F}_{\bar{c}_1} = \mathbb{P}(V_\delta)$, the invariant $e_{\bar{c}_1}$ of the ruled surface is therefore given by

$$e_{\bar{c}_1} = \max\{2 \deg \lambda - \deg V_\delta : \text{there exists a nonzero map } \lambda \rightarrow V_\delta\},$$

where λ is a line bundle on B (see, for example, [F2]). Note that \mathcal{O}_B is a subline bundle of V_δ (see (3.7)); consequently, since $\deg(V_\delta) = -c_1^2(\delta)/2$, we have

$$e_{\bar{c}_1} \geq c_1^2(\delta)/2 \geq -4m_{c_1}.$$

We still have to verify that

$$-g \leq e_{\bar{c}_1} \leq 0.$$

The left-hand inequality follows from a theorem of Segre-Nagata [F2]; hence, there only remains to show that $e_{\bar{c}_1}$ is less than or equal to zero.

Let A be a section of the ruled surface $\mathbb{F}_{\bar{c}_1}$; the pullback η^*A is therefore a bisection of $J(X)$. If it is reducible, then its two components are sections C_1 and C_2 of $J(X)$, giving

$$2A^2 = (\eta^*A)^2 = (C_1 + C_2)^2 = 2C_1 \cdot C_2 \geq 0.$$

If the bisection $\bar{C} = \eta^*A$ is instead irreducible, we consider its normalization $C \rightarrow \bar{C}$ and let $\gamma : C \rightarrow B$ be the two-to-one map induced by $C \rightarrow \bar{C} \subset J(X)$. Note that the natural map $C \rightarrow J(X) \times_B C$ gives a section C_1 of the surface $C \times T^* \rightarrow C$; moreover, if we denote by $\tilde{\gamma} : C \times T^* \rightarrow J(X)$ the two-to-one map induced by γ , then the pullback $\tilde{\gamma}^*(\bar{C})$ is reducible, with components C_1 and C_2 , where C_2 is also section of $C \times T^* \rightarrow C$, and we have

$$4A^2 = (\tilde{\gamma}^*(\bar{C}))^2 = (C_1 + C_2)^2 = 2C_1 \cdot C_2 \geq 0.$$

Therefore, since

$$e_{\bar{c}_1} = -\min\{A^2 \mid A \text{ section of } \mathbb{F}_{\bar{c}_1}\}$$

(see [F2], Proposition 12, Chapter 5), it follows that $e_{\bar{c}_1}$ is non-positive. \square

Remark 3.9. — Let $d_{\bar{c}_1} := (e_{\bar{c}_1} + 4m_{c_1})/2$. Referring to the proof of Lemma 3.8, one can view $d_{\bar{c}_1}$ as the maximal degree of subline bundles of a rank-2 vector bundle V_δ of degree $4m_{c_1}$, where δ is a line bundle in $c_1 + 2NS(X)$ such that $-c_1^2(\delta)/2 = 4m_{c_1}$; consequently, the invariant $d_{\bar{c}_1}$ is a non-negative integer such that $(m_{c_1} - d_{\bar{c}_1}/2) \geq 0$. In Theorem 4.5, we shall see that the lower bound for the discriminant of rank-2 vector bundles on X with fixed first Chern class c_1 is $(m_{c_1} - d_{\bar{c}_1}/2)$. The existence problem for rank-2 vector bundles on X with first Chern class c_1 therefore boils down to the maximal degree problem for subline bundles of a fixed rank-2 vector bundle on B of degree $4m_{c_1}$ given by an extension of the form (3.7).

Remark. — For a generic curve B of genus greater than 1, the Neron-Severi group of an elliptic surface X over B is trivial and the ruled surface is $B \times \mathbb{P}^1$ for any δ in $\text{Pic}(X)$. Moreover, this is always true if B is rational: the sections of the ruled surface are given by rational maps $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ and the irreducible bisections of $J(X)$ are the pullbacks to $J(X)$ of non-constant rational maps (for details, see [Mo]).

We finish this section by determining the genus of irreducible bisections.

LEMMA 3.10. — *If the spectral cover of the bundle E is a smooth irreducible bisection \overline{C} of $J(X)$, then its genus is given by*

$$(3.11) \quad g(\overline{C}) = 4\Delta(E) + 2g - 1,$$

where g is the genus of B .

Proof. — We begin by noting that the pushforward $A_0 := \eta_*(B_0)$ of the zero section of $J(X)$ is a section of the ruled surface $\mathbb{F}_{\overline{c}_1}$ whose pullback η^*A_0 to $J(X)$ is the reducible bisection $B_0 + \Sigma_\delta$; consequently, it has self-intersection $A_0^2 = -c_1^2/2$. We now describe the ramification and branching divisors of η . Let R be the ramification divisor in $J(X)$, defined as the fixed point set of i_g ; referring to Lemma 2.4, we have

$$R \cdot B_0 = \#\{(b, t) : \delta_b = \mathcal{O}_T\} = \Sigma_\delta \cdot B_0 = -c_1^2/2.$$

The branching divisor G is a 4-section of $\mathbb{F}_{\overline{c}_1}$ such that $\eta^*G = 2R$; since

$$G \cdot A_0 = G \cdot \eta_*(B_0) = \eta_*(\eta^*G \cdot B_0) = -c_1^2,$$

it is equivalent to a divisor of the form $4A_0 + \mathfrak{b}f$, where \mathfrak{b} is a divisor on B of degree c_1^2 and f is a fibre of the ruled surface.

Let A be the graph of the bundle E , that is, the section of $\mathbb{F}_{\overline{c}_1}$ such that $\overline{C} = \eta^*A$. If we write $A \sim A_0 + \mathfrak{b}'f$, for some divisor \mathfrak{b}' on B , then $\overline{C} \sim (B_0 + \Sigma_\delta) + \mathfrak{b}'T^*$, where \mathfrak{b}' also denotes the pullback of the divisor to $J(X)$, and the intersection number $\overline{C} \cdot B_0$ is equal to $-c_1^2/2 + \text{deg } \mathfrak{b}'$. Recall that $\overline{C} \cdot B_0$ is, by construction, the number of points (counted with multiplicity) in the support of the torsion sheaf $R^1\pi_*(E \otimes \mathcal{O}_X)$, which is equal to $n_E = c_2 - c_1^2/2$ (see section 3); therefore, we have $\text{deg } \mathfrak{b}' = c_2$. Hence, the smooth bisection \overline{C} is a double cover of B of branching order $G \cdot A = 4c_2 - c_1^2$ and (3.11) follows by the Hurwitz formula. \square

4. Existence theorems.

Let E be a holomorphic rank-2 vector bundle on the non-Kähler elliptic surface X with determinant line bundle δ and Chern classes c_1 and c_2 . If we denote

$$\Delta(E) := \Delta(2, c_1, c_2)$$

the discriminant of E , then a well-known result states that $\Delta(E)$ cannot be negative [BaL, ElFo, BrF, Br3, LeP].

4.1. Rank-2 vector bundle as extensions.

By using Lemma 2.2, Proposition 2.3 and Lemma 2.4, one obtains the following result, whose proof is similar to that of Theorem 1.3, Chapter VII, [FM]:

THEOREM 4.1. — *Let $\pi : X \rightarrow B$ be a non-Kähler elliptic surface and E be a holomorphic rank-2 vector bundle on X with determinant line bundle δ . Then E satisfies one of the following two cases:*

(A) *There exists a line bundle \mathcal{D} on X and a locally complete intersection Z of codimension 2 in X such that E is given by an extension*

$$0 \rightarrow \mathcal{D} \rightarrow E \rightarrow \delta \otimes \mathcal{D}^{-1} \otimes I_Z \rightarrow 0.$$

In fact, Z is the set of points (counted with multiplicity) corresponding to the fibres of π over which the bundle E is unstable. Moreover, we have

$$\Delta(E) = \frac{1}{8} \overline{C}^2 + \frac{1}{2} \ell(Z).$$

(B) *There exists:*

- (i) *a smooth irreducible curve C and a birational map $C \rightarrow \overline{C} \subset J(X)$, where \overline{C} is a bisection that is invariant under the involution i_δ on $J(X)$ defined by the line bundle δ ;*
- (ii) *a line bundle $\tilde{\mathcal{D}}$ on the normalisation W of $X \times_B C$, whose restriction to a smooth fibre of $W \rightarrow C$ is the same as the one induced by the section of $J(W)$ that corresponds to the map $C \rightarrow J(X)$;*

(iii) a codimension 2 locally complete intersection \tilde{Z} in W , an exact sequence

$$0 \rightarrow \tilde{\mathcal{D}} \rightarrow \tilde{\gamma}^* E \rightarrow \tilde{\gamma}^* \delta \otimes \tilde{\mathcal{D}}^{-1} \otimes I_{\tilde{Z}} \rightarrow 0,$$

where $\tilde{\gamma} : W \rightarrow X$ is the natural map, and

$$\Delta(E) = \frac{1}{8} \overline{C}^2 + \frac{1}{4} \ell(\tilde{Z}).$$

This time, \tilde{Z} is the set of points corresponding to the fibres of $W \rightarrow C$ over which the bundle $\tilde{\gamma}^* E$ is unstable. □

Remark 4.2. — Suppose that the vector bundle E satisfies case (A) of Theorem 4.1. Let Σ_1 and Σ_2 be the sections of $J(X)$ determined by the line bundles \mathcal{D} and $\mathcal{D} \otimes \delta$, respectively. Then, one can easily verify that $\overline{C} = \Sigma_1 + \Sigma_2$, implying that the bisection associated to E is reducible or a section counted with multiplicity 2 (if $\Sigma_1 = \Sigma_2$).

We now have the following complete description of non-filtrable bundles:

PROPOSITION 4.3. — *Let E be any holomorphic 2-vector bundle over X . Suppose that the spectral cover of E includes the bisection \overline{C} of $J(X)$. Then E is non-filtrable if and only if \overline{C} is irreducible.*

Proof. — Suppose that there exists a line bundle \mathcal{D} on X that maps into E . After possibly tensoring \mathcal{D} by the pullback of a suitable line bundle on B , the rank-2 bundle E is then given as an extension

$$0 \rightarrow \mathcal{D} \rightarrow E \rightarrow \mathcal{D}^{-1} \otimes \delta \otimes I_Z \rightarrow 0,$$

where $Z \subset X$ is a locally complete intersection of codimension 2, that is, E satisfies case (A) of Theorem 4.1; referring to remark 4.2, the bisection is then not irreducible. Conversely, suppose that the bisection is not irreducible and that Σ is one of its components. If \mathcal{D} is a line bundle on X corresponding to Σ , then \mathcal{D} maps non-trivially into E , implying that E is filtrable. □

Note. — A partial characterisation of non-filtrable bundles is also given in [ATo].

4.2. Existence of rank-2 vector bundles.

A partial converse of Theorem 4.1 is the following result:

THEOREM 4.4. — *Let $\pi : X \rightarrow B$ be a non-Kähler elliptic surface and δ be a line bundle in $Pic(X)$. Furthermore, let $i_\delta : J(X) \rightarrow J(X)$ be the involution defined by δ and suppose that \overline{C} is a bisection of $J(X) \rightarrow B$ that is invariant with respect to the involution i_δ . Then, there exists a rank-2 holomorphic vector bundle E on X such that*

$$c_1(E) = c_1(\delta) \text{ and } \Delta(E) = \frac{1}{8}\overline{C}^2 = \frac{1}{4}A^2,$$

where A is a section of the ruled surface \mathbb{F}_δ with $\eta^*A = \overline{C}$.

Proof. — If the bisection \overline{C} is reducible, then its components are sections Σ_1 and Σ_2 of $J(X)$. Let \mathcal{D} be a line bundle on X corresponding to Σ_1 (see Proposition 2.3); if E is any extension of $\mathcal{D}^{-1} \otimes \delta$ by \mathcal{D} , then E is a rank-2 vector bundle on X that has determinant δ and spectral cover \overline{C} .

If the bisection \overline{C} is irreducible, then consider its normalisation $C \rightarrow \overline{C}$ and let $\gamma : C \rightarrow B$ be the double covering induced by $C \rightarrow \overline{C} \subset J(X)$. The normalisation W of the fibred product $X \times_B C$ is then a non-Kähler elliptic surface over C with relative Jacobian $J(W) = C \times T^*$; furthermore, the natural two-to-one map $\tilde{\gamma} : W \rightarrow X$ induces a covering $\gamma' : J(W) \rightarrow J(X)$. Note that the inclusion map $C \rightarrow J(X) \times_B C$ gives a section Σ_1 of $J(W) \rightarrow C$; the pullback $\gamma'^*\overline{C}$ is then reducible with components Σ_1 and Σ_2 , where Σ_2 is another section of $J(W)$. By Proposition 2.3, there exists a line bundle \mathcal{L} on W whose restriction to any smooth fibre T_c of W is Σ_{2c} . Let \mathcal{D} be the line bundle on W satisfying the equality

$$\mathcal{L} \cong \tilde{\gamma}^*\delta \otimes \mathcal{D}^{-1}$$

and define the holomorphic rank-2 vector bundle E on X by

$$E := \tilde{\gamma}_*(\mathcal{L});$$

we then have to show that E has first Chern class $c_1(\delta)$ and discriminant $\frac{1}{8}\overline{C}^2$.

Let \overline{i}_δ be the involution on W that interchanges the sheets of $\tilde{\gamma}$. If $G \subset X$ is the (smooth) branch divisor of the double covering $\tilde{\gamma} : W \rightarrow X$,

then there exists a line bundle L_0 on X such that $L_0^2 = \mathcal{O}_X(G)$; moreover, by Lemma 29, Chapter 2 of [F2] or by [Br4], there is an exact sequence:

$$0 \rightarrow \bar{i}_\delta^* \mathcal{L} \otimes \tilde{\gamma}^* L_0^{-1} \rightarrow \tilde{\gamma}^* \tilde{\gamma}_*(\mathcal{L}) \rightarrow \mathcal{L} \rightarrow 0.$$

Since the involution \bar{i}_δ on W is induced by interchanging the sheets of the double cover $C \rightarrow B$, the restriction of $\bar{i}_\delta^* \mathcal{L}$ to any smooth fibre T_c of W (which is not in the ramification locus of $\tilde{\gamma}$) is isomorphic to the restriction of \mathcal{D} to the same fibre, namely to Σ_{1c} . From the preceding exact sequence, we obtain

$$0 \rightarrow \mathcal{D} \otimes \mathcal{O}_W(F) \rightarrow \tilde{\gamma}^* E \rightarrow \tilde{\gamma}^* \delta \otimes \mathcal{D}^{-1} \rightarrow 0,$$

where F is a divisor on W (hence a combination of fibres of the non-Kähler elliptic surface $W \rightarrow C$). Referring to Theorem 4.1, we have

$$\Delta(E) = \frac{1}{8} \bar{C}^2 = \frac{1}{4} A^2,$$

where A is the section of the ruled surface \mathbb{F}_δ defined by the bisection \bar{C} . By [ABrTo], we also have

$$c_1(E) \equiv c_1(\delta) \pmod{\text{Tors}(NS(X))}.$$

To get rid of the torsion, we need to add multiples of classes of fibres. Then, as in [ABrTo], we can modify the line bundle \mathcal{L} , by tensoring it with line bundles of the form $\mathcal{O}_W(T_c)$ or $\mathcal{O}_W(T_i)$, to obtain the desired result

$$c_1(E) = c_1(\delta).$$

Note that the discriminant remains unchanged (see the formula in [ABrTo] for the direct image of a line bundle). □

The above result implies that the existence problem for vector bundles is equivalent to the existence problem of bisections of $J(X)$ that are invariant under a given involution. Let us fix an element c_1 in $NS(X)$ and a line bundle δ on X such that $c_1(\delta) \in c_1 + 2NS(X)$. Denote \bar{c}_1 the class of c_1 in $NS(X)$ modulo $2NS(X)$. Referring to section 3.2, the Jacobian surface $J(X)$ of X is thus endowed with an involution i_δ and the quotient is a ruled surface $\mathbb{F}_{\bar{c}_1}$ that has a non-positive invariant $e_{\bar{c}_1}$; moreover, there is a one-to-one correspondence between sections of $\mathbb{F}_{\bar{c}_1}$ and spectral curves of rank-2 vector bundles on X that have determinant δ and no jumps. Therefore, the minimum value of the discriminant of a vector bundle E on

X with first Chern class c_1 is equal to $-e_{\bar{c}_1}/4$. Conversely, one can show that for any integer c_2 such that $\Delta(2, c_1, c_2)$ is greater or equal to $-e_{\bar{c}_1}/4$, there exists a rank-2 vector bundle on X with Chern classes c_1 and c_2 . We can now state the main result of the paper:

THEOREM 4.5. — *Let X be a minimal non-Kähler elliptic surface over a curve B of genus g and fix a pair (c_1, c_2) in $NS(X) \times \mathbb{Z}$. Set $m_{c_1} := m(2, c_1)$ and denote \bar{c}_1 the class of c_1 in $NS(X)$ modulo $2NS(X)$; moreover, let $e_{\bar{c}_1}$ be the invariant of the ruled surface $\mathbb{F}_{\bar{c}_1}$ determined by \bar{c}_1 . Then, there exists a holomorphic rank-2 vector bundle on X with Chern classes c_1 and c_2 if and only if*

$$\Delta(2, c_1, c_2) \geq (m_{c_1} - d_{\bar{c}_1}/2),$$

where $d_{\bar{c}_1} := (e_{\bar{c}_1} + 4m_{c_1})/2$. Note that both $d_{\bar{c}_1}$ and $(m_{c_1} - d_{\bar{c}_1}/2)$ are non-negative numbers (see Lemma 3.8). Furthermore, if

$$(m_{c_1} - d_{\bar{c}_1}/2) \leq \Delta(2, c_1, c_2) < m_{c_1},$$

then the corresponding vector bundles are non-filtrable.

Proof. — Given the definition of $d_{\bar{c}_1}$, the invariant $e_{\bar{c}_1}$ of the ruled surface is equal to $2d_{\bar{c}_1} - 4m_{c_1}$. Let $\Delta_0 := -e_{\bar{c}_1}/4 = m_{c_1} - d_{\bar{c}_1}/2$ and consider $\Delta := \Delta(2, c_1, c_2) \geq \Delta_0$; note that $k = 2(\Delta - \Delta_0) \geq 0$ is an integer. Choose a line bundle δ on X such that $c_1(\delta) \equiv c_1 \pmod{2NS(X)}$. It is sufficient to prove the existence of a holomorphic rank-2 vector bundle E with first Chern class $c_1(\delta)$ and discriminant Δ . Let \bar{C}_0 be a bisection of $J(X)$ of minimal self-intersection $8\Delta_0$. If $k = 0$, choose a holomorphic rank-2 vector bundle E_0 corresponding to \bar{C}_0 , for example, any bundle determined by Theorem 4.4.

For $k > 0$, choose a smooth fibre $T := \pi^{-1}(b)$ of π , with $b \in B$, such that if the bisection \bar{C}_0 is irreducible, then the double cover $\bar{C}_0 \rightarrow B$ does not have a branch point over b . Set $\delta' := \delta \otimes \mathcal{O}_X(kT)$. The line bundles δ and δ' then both correspond to the same section in $J(X)$, inducing isomorphic ruled surfaces $\mathbb{F}_{\delta'}$ and \mathbb{F}_{δ} , respectively. Consequently, there exists a holomorphic rank-2 vector bundle E'_0 on X with first Chern class $c_1(\delta')$ and discriminant Δ_0 that is regular on the fibre T (over an elliptic curve, a bundle is said to be *regular* if its group of automorphisms is of the smallest possible dimension). Indeed, if \bar{C}_0 is reducible, then choose line bundles L_1 and L_2 on X associated to the components of \bar{C}_0 , with $L_1 \otimes L_2 = \delta'$, and let E'_0 be an extension of L_2 by L_1 that is regular on T . Moreover, if \bar{C}_0 is irreducible, then E'_0 can be any vector bundle given

by Theorem 4.4. Let $j : T \rightarrow X$ be the natural inclusion map; if λ is a line bundle on T of degree 1, then there exists a surjection $E'_0 \rightarrow j_*\lambda$. Consider the elementary modification

$$0 \rightarrow E_1 \rightarrow E'_0 \rightarrow j_*\lambda \rightarrow 0;$$

then, the bundle E_1 splits as $\lambda \oplus \lambda^*$ over T and there exists a surjection $E_1 \rightarrow j_*\lambda$. Hence, by performing $(k-1)$ successive elementary modifications on E_1 with respect to $j_*\lambda$, one obtains a holomorphic vector bundle E on X with first Chern class $c_1(\delta)$ and discriminant Δ . \square

Remark. — If the genus of the base curve B is less than 2, then the statement of the theorem becomes: there exists a holomorphic rank-2 vector bundle E on X with Chern classes c_1 and c_2 if and only if the discriminant $\Delta(2, c_1, c_2)$ is a non-negative number. (For an alternate proof in the case of primary Kodaira surfaces, see [ABrTo].) In contrast, if the genus of the base curve is greater than 1, there are "gaps" for the discriminant of holomorphic rank-2 vector bundles, whenever m_{c_1} is greater than $d_{c_1}/2$; thus, the existence of holomorphic vector bundles on X depends on the geometry of the base curve B . However, by the proof of Theorem 4.5, once there is an irreducible bisection of $J(X)$, one can construct infinitely many non-filtrable vector bundles.

Note. — Bundles with $\Delta(2, c_1, c_2) = 0$ have also been studied in [ABr].

BIBLIOGRAPHY

- [ABr] M. APRODU, V. BRÎNZĂNESCU, On the holomorphic rank-2 vector bundles with trivial discriminant over non-Kähler elliptic bundles, to appear in J. Math. Kyoto Univ. 42 (2002) no. 4, (2003) 617-623 .
- [ABrTo] M. APRODU, V. BRÎNZĂNESCU, M. TOMA, Holomorphic vector bundles on primary Kodaira surfaces, Math. Z. 242 (2002) 63-73.
- [ATo] M. APRODU, M. TOMA, Une note sur les fibrés holomorphes non-filtrables, Preprint 2002.
- [BaL] C. BĂNICĂ, J. LE POTIER, Sur l'existence des fibrés vectoriels holomorphes sur les surfaces non-algébriques, J. Reine Angew. Math. 378 (1987) 1-31.
- [BBDG] K. BECKER, M. BECKER, K. DASGUPTA, P. S. GREEN, Compactification of heterotic theory on non-Kähler complex manifolds: I, preprint arXiv:hep-th/0301161.
- [BPV] W. BARTH, C. PETERS, A. VAN DE VEN, Compact complex surfaces, Springer-Verlag, Berlin-Heidelberg-New York, 1984.

- [BJPS] M. BERSHADSKY, A. JOHANSEN, T. PANTEV, V. SADOV, On four-dimensional compactifications of F-theory, *Nuclear Phys. B* 505(1-2) (1997) 165-201.
- [BH] P. J. BRAAM, J. HURTUBISE, Instantons on Hopf surfaces and monopoles on solid tori, *J. Reine Angew. Math.* 400 (1989) 146-172.
- [Br1] V. BRÎNZĂNESCU, Néron-Severi group for non-algebraic elliptic surfaces I: elliptic bundle case, *Manuscripta Math.* 79 (1993) 187-195; II: non-Kählerian case, *Manuscripta Math.* 84 (1994) 415-420; III, *Rev. Roumaine Math. Pures Appl.* 43(1-2) (1998) 89-95.
- [Br2] V. BRÎNZĂNESCU, The Picard group of a primary Kodaira surface, *Math. Ann.* 296 (1993) 725-738.
- [Br3] V. BRÎNZĂNESCU, Holomorphic vector bundles over compact complex surfaces, *Lecture Notes in Mathematics* 1624, Springer, 1996.
- [Br4] V. BRÎNZĂNESCU, Double covers and vector bundles, *An. Stiint. Univ. Ovidius Constanta, Ser. Mat.* 9 (2001) no 1 21-26.
- [BrF] V. BRÎNZĂNESCU, P. FLONDOR, Holomorphic 2-vector bundles on non-algebraic 2-tori, *J. Reine Angew. Math.* 363 (1985) 47-58.
- [BrMo1] V. BRÎNZĂNESCU, R. MORARU, Twisted Fourier-Mukai transforms and bundles on non-Kähler elliptic surfaces, to appear in *Math. Res. Lett.*; preprint arXiv:math.AG/0309031.
- [BrMo2] V. BRÎNZĂNESCU, R. MORARU, Stable bundles on non-Kähler elliptic surfaces, *Comm. Math. Phys.* 254 (2005) 565-580.
- [BrU] V. BRÎNZĂNESCU, K. UENO, Néron-Severi group for torus quasi bundles over curves. *Moduli of vector bundles (Sanda, 1994; Kyoto, 1994)*, *Lecture Notes in Pure and Applied Mathematics* 179, Dekker, New York, 1996, 11-32.
- [Bh] N. P. BUCHDAHL, Hermitian-Einstein connections and stable vector bundles over compact complex surfaces, *Math. Ann.* 280 (1988) 625-648.
- [CCDLM2] G. L. CARDOSO, G. CURIÓ, G. DALL'AGATA, D. LÜST, P. MANOUSSELIS, G. ZOUPANOS, Non-Kähler string backgrounds and their five torsion classes, preprint arXiv:hep-th/0211118.
- [D] R. DONAGI, Principal bundles on elliptic fibrations, *Asian J. Math.* 1(2) (1997) 214-223.
- [DOPW1] R. DONAGI, B. OVRUT, T. PANTEV, D. WALDRAM, Standard-model bundles, *Adv. Theor. Math. Phys.* 5(3) (2001) 563-615.
- [DOPW2] R. DONAGI, B. OVRUT, T. PANTEV, D. WALDRAM, Standard models from heterotic M-theory, *Adv. Theor. Math. Phys.* 5(1) (2001) 93-137.
- [ElFo] G. ELENCSWAJG, O. FORSTER, Vector bundles on manifolds without divisors and a theorem of deformation, *Ann. Inst. Fourier* 32(4) (1982) 25-51.
- [F1] R. FRIEDMAN, Rank two vector bundles over regular elliptic surfaces, *Invent. Math.* 96 (1989) 283-332.
- [F2] R. FRIEDMAN, Algebraic surfaces and holomorphic vector bundles, *Universitext*, Springer-Verlag, 1998.
- [FM] R. FRIEDMAN, J. W. MORGAN, *Smooth Four-Manifolds and Complex Surfaces*, Springer-Verlag, 1994.
- [FMW] R. FRIEDMAN, J. MORGAN, E. WITTEN, Vector bundles over elliptic fibrations, *J. Algebraic Geom.* 2 (1999) 279-401.
- [GP] E. GOLDSTEIN, S. PROKUSHKIN, Geometric model for complex non-Kähler manifolds with $SU(3)$ structure, preprint arXiv:hep-th/0212307.

- [Kob] S. KOBAYASHI, Differential geometry of complex vector bundles, Princeton University Press, 1987.
- [Kod] K. KODAIRA, On the structure of compact complex analytic surfaces I, Amer. J. Math. 86 (1964) 751-798.
- [LeP] J. LE POTIER, Fibrés vectoriels sur les surfaces K3, Séminaire Lelong-Dolbeault-Skoda, Lecture Notes in Mathematics 1028, Springer, Berlin, 1983.
- [Mo] R. MORARU, Integrable systems associated to a Hopf surface, Canad. J. Math. 55(3) (2003) 609-635.
- [S] R. L. E. SCHWARZENBERGER, Vector bundles on algebraic surfaces, Proc. London Math. Soc. 3 (1961) 601-622.
- [T] A. TELEMAN, Moduli spaces of stable bundles on non-Kähler elliptic fibre bundles over curves, Expo. Math. 16 (1998) 193-248.
- [TTTo] A. TELEMAN, M. TOMA, Holomorphic vector bundles on non-algebraic surfaces, C. R. Acad. Sci. Paris 334 (2002) 1-6.
- [To] M. TOMA, Stable bundle with small c_2 over 2-dimensional complex tori, Math. Z. 232 (1999) 511-525.

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