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PROOF OF THE TREVES THEOREM ON THE KdV HIERARCHY

by Leonid A. DICKEY

1. Necessity of the Treves condition for KdV.

Here we give a shorter proof of the Treves theorem [1] and some addition to the theorem (Theorem 2 below). A discussion of the significance of the theorem, and a part of the present proof (necessity) one can find in [2] along with an attempt to generalize the theorem.

THEOREM 1 (Treves). — *A differential polynomial of u : $P[u] = P(u, u', u'', \dots)$ is, up to an exact derivative, a linear combination of $\text{res}_\partial L^{m/2}$ where $L = \partial^2 + u$ if and only if*

$$(1) \quad \text{res}_x P(\tilde{u}(x), \tilde{u}'(x), \tilde{u}''(x), \dots) = 0,$$

where $\tilde{u}(x)$ is an arbitrary formal Laurent series of the form

$$(2) \quad \tilde{u}(x) = -2x^{-2} + \sum_0^\infty u_i(x^i/i!), \quad u_1 = 0.$$

(The following notations are used: res_∂ symbolizes the coefficient in ∂^{-1} , and res_x the coefficient in x^{-1}).

We also prove the following addition to the Treves theorem:

THEOREM 2. — *A differential polynomial $P[u] = P(u, u', u'', \dots)$ is exactly a linear combination of $\text{res}_\partial L^{m/2}$ (without additional derivative*

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terms) if and only if $P(\tilde{u}(x), \tilde{u}'(x), \tilde{u}''(x), \dots)$ is a Laurent series with only one singular term $\text{const} \cdot x^{-2}$.

The beginning of the proof of the theorem 1. — In this section we prove the necessity of the Treves condition (1).

Let us try to “undress” the operator $L = \partial^2 + u$:

$$\partial^2 + u = w\partial^2 w^{-1} = (1 + w_1\partial^{-1} + w_2\partial^{-2} + \dots)\partial^2(1 + w_1\partial^{-1} + w_2\partial^{-2} + \dots)^{-1}.$$

Rewrite this as

$$(1 + w_1\partial^{-1} + w_2\partial^{-2} + \dots)\partial^2 = (\partial^2 + u)(1 + w_1\partial^{-1} + w_2\partial^{-2} + \dots)$$

which yields the recurrence relations

$$2w'_1 + u = 0$$

$$2w'_{k+1} + w''_k + uw_k = 0, \quad k > 0.$$

First, a lemma will be proven:

LEMMA 1. — *If a formal Laurent series $\tilde{u} = -2/x^2 + \sum_0^\infty u_i(x^i/i!)$, $u_1 = 0$, is taken for u , then all w_k can be found in the form of Laurent series.*

The necessity of the Treves condition immediately follows from this lemma. Indeed,

$$\begin{aligned} \text{res}_\partial L^{m/2} &= \text{res}_\partial w\partial^m w^{-1} = \text{res}_\partial [w\partial^m, w^{-1}] + \text{res}_\partial w^{-1}w\partial^m \\ &= \text{res}_\partial [w\partial^m, w^{-1}] = \partial(\) \end{aligned}$$

since the residue of the commutator of any two operators is an exact derivative. In this case this is an exact derivative of a Laurent series. Therefore, it cannot contain a term with x^{-1} , i.e., its residue with respect to the variable x is zero.

Proof of the lemma 1. — We have $2w'_1 + \tilde{u} = 0$ whence

$$w_1 = -\frac{1}{x} - \frac{1}{2} \sum_0^\infty u_i \frac{x^{i+1}}{(i+1)!} = -\frac{1}{x} - \sum_1^\infty b_i \frac{x^i}{i!}, \quad (b_2 = 0).$$

Further, $-w'_2 = w''_1 - 2w'_1 w_1$ and $-w_2 = w'_1 - w_1^2$. It is easy to calculate (taking into account that $b_2 = 0$) that $w_2 = (5b_3/6 + b_1^2)x^2 + O(x^3) = Ax^2 + O(x^3) = O(x^2)$. Here, $O(x^n)$ means a power series starting with the

term involving x^n . Using the recurrence formula, it is not difficult to show by induction that all the next terms have the same form:

$$\begin{aligned}
 -w'_{k+1} &= w''_k - 2w'_1 w_k = 2A + O(x) - 2(x^{-2} + b_1 + O(x^2))(Ax^2 + O(x^3)) \\
 &= O(x)
 \end{aligned}$$

whence $w_{k+1} = O(x^2)$. □

2. Proof of the sufficiency of the Treves condition.

There is a grading in the differential algebra \mathcal{A} of polynomials in symbols $u^{(k)}$, $P[u] = P(u, u', u'', \dots)$: $w(u^{(n)}) = n + 2$, $w(\partial) = 1$. If all terms of a polynomial P have the same weight k then

$$P(\lambda^2 u, \lambda^3 u', \lambda^4 u'' \dots) = \lambda^k P(u, u', u'', \dots).$$

LEMMA 2. — *If a differential polynomial P satisfies the Treves condition (1) then so does each homogeneous in weight component of this polynomial.*

Proof of the lemma 2. — Let $P = \sum P_\kappa$ where P_κ a homogeneous polynomial of weight κ . Since $\{u_n\}$ are arbitrary, we can replace them by $u_n \lambda^{n+2}$. Now,

$$\begin{aligned}
 \text{res}_x \sum P_\kappa &\left[-2/x^2 + \lambda^2 u_0 + \sum_2^\infty \lambda^{n+2} u_n \frac{x^n}{n!} \right] \\
 &= \text{res}_x \sum \lambda^\kappa P_\kappa \left[-2/(\lambda x)^2 + u_0 + \sum_2^\infty u_n \frac{(\lambda x)^n}{n!} \right] \\
 &= \sum \lambda^{\kappa-1} \text{res}_x P_\kappa \left[-2/x^2 + u_0 + \sum_2^\infty u_n \frac{x^n}{n!} \right].
 \end{aligned}$$

If this is zero, then each term is zero since λ is arbitrary. □

Therefore, we can consider each component of weight κ separately. The first integral $\text{res}_\partial L^{m/2}$ where $m = 2k - 1$ has the weight $2k$. It is possible to prove that it contains a term Cu^k with a non-zero coefficient C .

Indeed, dealing with the terms in $\text{res}_\partial L^{m/2}$ where u is not differentiated, one can consider ∂ and u as commuting. Then

$$\begin{aligned}
 \text{res}_\partial (\partial^2 + u)^{m/2} &= \text{res}_\partial \partial^m (1 + u\partial^{-2})^{m/2} = \text{res}_\partial \partial^m \sum_0^\infty \binom{m/2}{k} (u\partial^{-2})^k \\
 &= \binom{m/2}{(m+1)/2} u^{(m+1)/2}.
 \end{aligned}$$

If P is a differential polynomial satisfying the Treves condition (1), then subtracting from it a linear combination of first integrals $\text{res}_\partial L^{m/2}$, one can achieve that it does not contain terms Cu^k preserving the property to satisfy the condition (1). Then we reduce this polynomial. Namely, we reduce the order of the highest derivative involved in a differential monomial by “integration by parts” as much as possible: if a differential monomial $(u^{(i_1)})^{p_1}(u^{(i_2)})^{p_2} \dots (u^{(i_k)})^{p_k}$ where $i_1 < i_2 < \dots < i_k$ has $p_k = 1$ then the highest order of the derivative, i_k , can be reduced if $i_k \neq 0$ by addition of an exact derivative, for example $(u')^2 u''' = -2u'(u'')^2 + \partial((u')^2 u'')$. The second term is an exact derivative and the highest derivative involved in the first term is the second one. Another example: $uu'u'' = u(u'^2)' / 2 = \partial(uu'^2/2) - u'^3/2$. One can proceed doing this until all the monomials will contain their highest derivatives in power > 1 (with a possible exception: a term Cu). We call this the reduced form of a differential polynomial. It is unique.

The reduced polynomial preserves the property (1) and does not contain the terms Cu^k . It remains to prove the following:

LEMMA 3. — *A reduced differential polynomial homogeneous with respect to the weight which satisfies the condition (1) and does not contain the term Cu^k is zero.*

Suppose that it is not zero. Let us write Eq. (1) in more detail:

$$\text{res}_x Q \left(-2/x^2 + u_0 + \sum_2^\infty u_i(x^i/i!), 4/x^3 + \sum_2^\infty u_i \partial(x^i/i!), \right. \\ \left. -12/x^4 + \sum_2^\infty u_i \partial^2(x^i/i!), \dots \right) = 0.$$

This equality can be differentiated with respect to u_0 which is the same as $\text{res}_x \partial Q[\tilde{u}]/\partial u = 0$. This operation can be repeated until there will be no factor u at all, and, nevertheless, the polynomial is not zero since the term Cu^k is absent. More than that, the polynomial preserves all the properties assumed in Lemma 3. We have

$$\text{res}_x Q \left(4/x^3 + \sum_2^\infty u_i \partial(x^i/i!), -12/x^4 + \sum_2^\infty u_i \partial^2(x^i/i!), \dots \right) = 0.$$

Now let us take the derivative with respect to an arbitrary $u_k, k = 2, 3, \dots$:

$$\text{res}_x \sum_1^\infty \frac{\partial Q(4/x^3 + \sum_2^\infty u_i \partial(x^i/i!), -12/x^4 + \sum_2^\infty u_i \partial^2(x^i/i!), \dots)}{\partial u^{(n)}} \\ \times \partial^n(x^k/k!) = 0.$$

Integrating by parts, we get:

$$\text{res}_x \sum_1^\infty (-\partial)^{n-1} \times \frac{\partial Q(4/x^3 + \sum_2^\infty u_i \partial(x^i/i!), -12/x^4 + \sum_2^\infty u_i \partial^2(x^i/i!), \dots)}{\partial u^{(n)}} \cdot \frac{x^{k-1}}{(k-1)!} = 0$$

or

$$\text{res}_x \frac{\delta Q(4/x^3 + \sum_2^\infty u_i \partial(x^i/i!), -12/x^4 + \sum_2^\infty u_i \partial^2(x^i/i!), \dots)}{\delta u'} \cdot \frac{x^{k-1}}{(k-1)!} = 0$$

where $k = 2, 3, \dots$. Denoting the variational derivative $\delta Q/\delta u'$ as R , we have

$$\text{res}_x xR(4/x^3 + \sum_2^\infty u_i \partial(x^i/i!), -12/x^4 + \sum_2^\infty u_i \partial^2(x^i/i!), \dots) x^{k-2} = 0$$

whence

$$(3) \quad \left(xR(4/x^3 + \sum_2^\infty u_i \partial(x^i/i!), -12/x^4 + \sum_2^\infty u_i \partial^2(x^i/i!), \dots) \right)_- = 0.$$

Thus, the problem is now the following: to show that if $R[v]$ ($v = u'$) is a homogeneous polynomial satisfying Eq. (3), then $R = 0$. If this is proven, then $\delta Q/\delta u' = 0$ and Q is an exact derivative which is incompatible with the fact that it is a non-zero reduced polynomial.

We shall prove a slightly more general lemma, the generalization is needed in the proof of the theorem 2.

LEMMA 4. — *Let $R(u, u', u'', \dots)$ be a homogeneous polynomial satisfying*

$$\left(xR(-2/x^2 + \sum_0^\infty u_i(x^i/i!), 4/x^3 + \sum_2^\infty u_i \partial(x^i/i!), \dots) \right)_- = 0,$$

where $u_1 = 0$ and other u_i are arbitrary. Then $R = 0$.

Let R be of weight κ . From the homogeneity, it follows that the given equality can be written as

$$(4) \quad \left(x^{1-\kappa} R(-2 + \sum_0^\infty u_i x^{i+2}/i!, 4 + \sum_2^\infty u_i x^{i+2}/(i-1)!, \dots) \right)_- = 0.$$

Now, one must expand $R(-2 + \sum_0^\infty u_i x^{i+2}/i!, 4 + \sum_2^\infty u_i x^{i+2}/(i-1)!, \dots)$ in powers of x and write that all terms of power less than $\kappa - 1$

vanish. For that, it is more convenient to expand this expressions in powers of u_i , it automatically will be an expansion in powers of x .

The arguments of R : u, u', \dots we denote as ξ_0, ξ_1, \dots , thus, $R = R(\xi_0, \xi_1, \dots, \xi_\mu)$. We have (denoting $\partial_{u_j} = \partial/\partial u_j$ etc)

$$\begin{aligned} \partial_{u_j} R \left(-2 + \sum_0^\infty u_i \frac{x^{i+2}}{i!}, 4 + \sum_2^\infty u_i \frac{x^{i+2}}{(i-1)!}, \dots \right) &= \partial_{u_j} R(\xi_1, \xi_2, \dots) \\ &= D_j R(\xi_0, \xi_1, \dots) x^{j+2} \text{ where } D_j = \frac{1}{j!} \partial_{\xi_0} + \frac{1}{(j-1)!} \partial_{\xi_1} + \dots + \partial_{\xi_j} \end{aligned}$$

for $j = 0, 2, 3, \dots$. Now:

$$\begin{aligned} x^{1-\kappa} R \left(-2 + \sum_0^\infty \frac{x^{i+2}}{i!}, 4 + \sum_2^\infty u_i \frac{x^{i+2}}{(i-1)!}, -12 + \sum_2^\infty u_i \frac{x^{i+2}}{(i-2)!}, \dots \right) \\ (5) \qquad \qquad \qquad = \sum_{q_0, q_2, \dots} \frac{D_0^{q_0} D_2^{q_2} \dots R(\theta)}{q_0! q_2! \dots x^{\kappa-1-\lambda(q)}} u_0^{q_0} u_2^{q_2} \dots \end{aligned}$$

where

$$(\theta) = (-2 \cdot 1!, 2 \cdot 2!, -2 \cdot 3!, \dots, (-1)^{\mu+1} 2 \cdot (\mu + 1)!), \quad \lambda(q) = \sum (j + 2) q_j.$$

The condition that the expression (5) does not contain negative powers of x becomes

$$(6) \qquad \qquad \qquad D_0^{q_0} D_2^{q_2} \dots R(\theta) = 0 \text{ when } \lambda(q) < \kappa - 1.$$

LEMMA 5 ON HOMOGENEOUS POLYNOMIALS. — Let

$$R(\xi_0, \xi_1, \xi_2, \dots, \xi_\mu) = \sum_{(p)} a_{(p_0 p_1 p_2 \dots p_\mu)} \xi_0^{p_0} \xi_1^{p_1} \xi_2^{p_2} \dots \xi_\mu^{p_\mu}$$

where $2p_0 + 3p_1 + 4p_2 + \dots + (\mu + 2)p_\mu = \kappa \geq 2$. Suppose the equation (6) where $(\theta) = (\xi_0^*, \xi_1^*, \xi_2^*, \dots)$ is a fixed set of nonzero values of the corresponding variables is satisfied for all sets of integers $\{q_0, q_2, q_3, \dots\}$ such that $\lambda(q) \equiv \sum (2 + j) q_j \leq \kappa - 2$. If $\kappa \geq 4$, we consider only sets $\{q_i\}$ such that not all of q_i are zero. Then all coefficients $a_{(p)}$ are zero.

Notice that if $\kappa < 4$, the polynomial R contains only one term, and the proof is obvious.

3. Proof of the Lemma 5 and the end of the proof of the sufficiency of the Treves condition.

We use the induction with respect to the number of variables $\xi_0, \xi_1, \dots, \xi_\mu$. For $\mu = 0$ the statement is trivial. Let it be proven for $\mu - 1$.

The Euler formula for the weight-homogeneous polynomial R reads

$$2\xi_0\partial_{\xi_0}R + 3\xi_1\partial_{\xi_1}R + 4\xi_2\partial_{\xi_2}R + \dots + (\mu + 2)\xi_\mu\partial_{\xi_\mu}R = \kappa R.$$

Solving μ equations with μ unknowns, one can express $\partial_{\xi_0}, \dots, \partial_{\xi_{\mu-1}}$ in terms of D_0, D_2, \dots, D_μ and ∂_{ξ_μ} :

$$\partial_{\xi_j} = \tau_{j0}D_0 + \sum_{k=2}^{\mu} \tau_{jk}D_k + \sigma_j\partial_{\xi_\mu}, \quad j = 0, \dots, \mu - 1$$

with constant coefficients. Substituting this for ∂_{ξ_i} in the Euler equation, we get

$$\left(a_0(\xi)D_0 + \sum_2^{\mu} \alpha_i(\xi)D_i + \beta(\xi)\partial_{\xi_\mu} \right) R = \kappa R$$

where coefficients linearly depend on $\{\xi_i\}$. Hence,

$$\partial_{\xi_\mu}R = \left(b(\xi) + a_0(\xi)D_0 + \sum_2^{\mu} a_i(\xi)D_i \right) R$$

with coefficients which are rational functions of $\{\xi_i\}$. Iterating this formula, we have

$$(7) \quad \partial_{\xi_\mu}^m R = \sum_{j_0+j_2+\dots+j_\mu \leq m} \alpha_{j_0, j_2, \dots, j_\mu}(\xi) D_0^{j_0} D_2^{j_2} \dots D_\mu^{j_\mu} R.$$

One can write $R = \sum_0^d a_j(\xi_0, \dots, \xi_{\mu-1})\xi_\mu^j/j!$ where a_d is not the identical zero. We consider two cases: when a_d is not a constant and when it is a constant. If a_d is not a constant, its weight is at least 2 (since this is the smallest weight of variables ξ_i).

Now we can prove that the weight-homogeneous differential polynomial $a_d(\xi_0, \dots, \xi_{\mu-1})$ of weight $\kappa_1 = \kappa - (\mu + 2)d$, if $\kappa_1 \geq 2$, has the property $D_0^{q_0} D_2^{q_2} \dots D_{\mu-1}^{q_{\mu-1}} a_d(\theta) = 0$ if $\lambda(q) = 2q_0 + 4q_2 + \dots + (\mu + 1)q_{\mu-1} \leq \kappa_1 - 2$.

Indeed, using (7), we obtain

$$\begin{aligned} D_0^{q_0} D_2^{q_2} \dots D_{\mu-1}^{q_{\mu-1}} a_d(\theta) &= D_0^{q_0} D_2^{q_2} \dots D_{\mu-1}^{q_{\mu-1}} \partial_{\xi_\mu}^d R(\theta) \\ &= \sum_{j_0+j_2+\dots+j_\mu \leq d} \alpha_{j_0, j_2, \dots, j_\mu}(\xi) D_0^{j_0} D_2^{j_2} \dots D_\mu^{j_\mu} D_0^{q_0} D_2^{q_2} \dots D_{\mu-1}^{q_{\mu-1}} R(\theta). \end{aligned}$$

Since

$$\begin{aligned} (2j_0 + 4j_2 + \dots + (\mu + 2)j_\mu) + (2q_0 + 4q_2 + \dots + (\mu + 1)q_{\mu-1}) \\ \leq (\mu + 2)(j_0 + j_2 + \dots + j_\mu) + \kappa_1 - 2 \\ \leq (\mu + 2)d + \kappa - (\mu + 2)d - 2 = \kappa - 2, \end{aligned}$$

all terms of the sum vanish. Since a_d depends on $\mu - 1$ variables, the lemma 5 is supposed to be true for a_d , and a_d is identically zero. Therefore, $R = 0$.

Now, we discuss the second alternative: a_d is a constant. Then, instead of $a_d = \partial_{\xi_\mu}^d R$, we consider

$$\tilde{a}_d = \partial_{\xi_\mu}^{d-1} R = a_{d-1}(\xi_0, \dots, \xi_{\mu-1}) + c\xi_\mu, \quad c = \text{const.}$$

We have $\kappa_2 = w(\tilde{a}_d) = \mu + 2 \geq 2$. Moreover, if $a_{d-1} \neq 0$, $\kappa_2 \geq 4$ since a_{d-1} cannot be linear. Just as in the first case, we can prove that $D_0^{q_0} D_2^{q_2} \dots \tilde{a}_d(\theta) = 0$ if $\lambda(q) \leq \kappa_2 - 2$. If not all of q_i are zero, then $D_0^{q_0} D_2^{q_2} \dots \tilde{a}_d(\theta) = D_0^{q_0} D_2^{q_2} \dots a_{d-1}(\theta) = 0$, and we conclude that $a_{d-1} = 0$. Then $c = 0$ since $\partial_{\xi_\mu}^{d-1} R(\theta) = 0$, and $a_d = 0$. Thus, $R = 0$ and the lemmas 5, 4, and also the theorem 1 are proven. \square

4. Proof of the Theorem 2.

In the same way as before, one can restrict himself to homogeneous polynomials. Let $\text{res}_x R(\tilde{u}(x), \tilde{u}'(x), \tilde{u}''(x), \dots) = 0$ for all series (2), R being a differential polynomial. One can differentiate this equality with respect to u_0, u_2, u_3, \dots :

$$\text{res}_x \frac{\partial R}{\partial u} = 0, \quad \text{res}_x \left(\frac{\partial R}{\partial u} \cdot \frac{x^2}{2} + \frac{\partial R}{\partial u'} \cdot \partial \frac{x^2}{2} + \frac{\partial R}{\partial u''} \cdot \partial^2 \frac{x^2}{2} \right) = 0, \dots,$$

or

$$\text{res}_x \frac{\delta R}{\delta u} = 0, \quad \text{res}_x \frac{\delta R}{\delta u} \cdot \frac{x^2}{2!} = 0, \quad \text{res}_x \frac{\delta R}{\delta u} \cdot \frac{x^3}{3!}, \dots$$

which means that $P = \delta R / \delta u$, after the substitution (2), can have, as a Laurent polynomial, only one singular term ax^{-2} . It is well-known that any polynomial $\text{res}_\partial L^{m/2}$ is the variational derivative of the next one, $\text{res}_\partial L^{(m+2)/2}$ (see [3], Proposition 3.5.2.). The theorem is proven one way.

Conversely, let P be a homogeneous differential polynomial such that after the substitution (2) there is only one singular term, ax^{-2} . In particular, $\text{res}_x P(\tilde{u}, \tilde{u}', \dots) = 0$. Then P is a sum of $c \cdot \text{res}_\partial L^{m/2}$ and a derivative, $\partial S(u, u', \dots)$. Unless S is identically zero, it must contain,

after the substitution (2), singular terms of order x^{-2} or higher since if it were not so, i.e., $(xS(\tilde{u}, \tilde{u}', \dots))_- = 0$, then, according to the Lemma 4, S would be zero. Then ∂S has nonzero singular terms of order at least x^{-3} in contradiction to the assumption. Thus, $S = 0$, and the given differential polynomial is just $c \cdot \text{res}_\partial L^{m/2}$. \square

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