ANNALES

DE

L'INSTITUT FOURIER

Anton BARANOV

Stability of the bases and frames reproducing kernels in model spaces

Tome 55, n° 7 (2005), p. 2399-2422.

<http://aif.cedram.org/item?id=AIF_2005__55_7_2399_0>

Ann. Inst. Fourier, Grenoble
55, 7 (2005), 2399–2422

STABILITY OF BASES AND FRAMES OF REPRODUCING KERNELS IN MODEL SPACES

by Anton BARANOV

Introduction.

Let Θ be an inner function in the upper half-plane $\mathbb{C}^+$, that is, a bounded analytic function such that $\lim_{y\to 0+} |\Theta(x+iy)| = 1$ for almost all $x \in \mathbb{R}$ with respect to the Lebesgue measure. With each inner function Θ we associate the subspace

$$K^2_\Theta = H^2 \ominus \Theta H^2$$

of the Hardy class H^2 in the upper half-plane. These subspaces play an outstanding role both in function and operator theory (see [24, 25, 26]). In particular, by the P.D. Lax theorem, any subspace of H^2 coinvariant with respect to the semigroup of shifts $(U_t)_{t\geqslant 0}$, $U_t f(x) = e^{itx} f(x)$, is of the form K^2_Θ for some inner function Θ. The subspaces K^2_Θ are often called model subspaces due to their relation with the Sz.-Nagy–Foias model for contractions in a Hilbert space; in what follows we also use this term.

Keywords: Inner function, shift-coinvariant subspace, reproducing kernel, Riesz basis, frame, stability.
Math. classification: 46E22, 42C15, 30D55, 47B32.

For $\lambda \in \mathbb{C}^+$ the function

$$k_\lambda(z) = \frac{i}{2\pi} \cdot \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{z - \overline{\lambda}}$$

is the reproducing kernel of the space K^2_Θ corresponding to the point λ, that is, $k_\lambda \in K^2_\Theta$ and

$$f(\lambda) = \int_{\mathbb{R}} f(t)\overline{k_\lambda(t)}dt, \qquad f \in K^2_\Theta.$$

Recall that

$$\|k_\lambda\|_2^2 = \frac{1 - |\Theta(\lambda)|^2}{4\pi \Im \lambda}.$$

A system of vectors $\{h_n\}$ in a Hilbert space H is said to be a Riesz basis if $\{h_n\}$ is an image of an orthonormal basis under a bounded and invertible linear operator in H. An equivalent definition is that each $h \in H$ may be represented as an unconditionally convergent series $h = \sum_n c_n h_n$ and there exist positive constants A and B such that

$$A \sum_n |c_n|^2 \leqslant \left\| \sum_n c_n h_n \right\|_H^2 \leqslant B \sum_n |c_n|^2. \tag{1}$$

In the present paper we are concerned with the sets of complex numbers $\Lambda = \{\lambda_n\}$ such that the normalized kernels $k_{\lambda_n}/\|k_{\lambda_n}\|_2$ constitute a Riesz basis in K^2_Θ for a given Θ. In this case we say for short that $\{k_{\lambda_n}\}$ is a basis in K^2_Θ. To be exact, we are interested in stability of this property: given a basis $\{k_{\lambda_n}\}$ and the points $\mu_n \approx \lambda_n$, which are in a sense close to λ_n, whether the system $\{k_{\mu_n}\}$ is also a basis? The problem is to determine which small perturbations are admissible, that is, preserve the property to be a basis.

We start with one important example which motivates the interest to this problem. Consider the inner function $\Theta(z) = \exp(2\pi i z)$. Then, by the Paley–Wiener theorem, the Fourier image of the model subspace K^2_Θ coincides with the space $L^2(0, 2\pi)$. Moreover, a system of reproducing kernels k_{λ_n} in K^2_Θ corresponds to a system of complex exponentials $e^{i\lambda_n t}$ in $L^2(0, 2\pi)$. Thus, in this particular case, our problem is equivalent to the famous problem of nonharmonic Fourier series which has important

applications in control theory and signal processing [26]. Bases of reproducing kernels in general model subspaces have applications to differential operators; in particular, in string scattering theory (see [20]).

The first result on the nonharmonic Fourier series is due to R.C.E.A. Paley and N. Wiener: if $\lambda_n \in \mathbb{R}$, $n \in \mathbb{Z}$, and

$$\sup_{n\in\mathbb{Z}} |\lambda_n - n| \leqslant \delta$$

with $\delta < \pi^{-2}$ (that is, $\{e^{i\lambda_n t}\}$ is close to the standart orthogonal basis $\{e^{int}\}$), then $\{e^{i\lambda_n t}\}$ is a Riesz basis in $L^2(0, 2\pi)$. Later, M. Kadec [21] showed that the same is true for any $\delta < 1/4$, whereas, by a result of A. Ingham, the system $\{e^{i\lambda_n t}\}$ may fail to be a basis if $\delta = 1/4$.

The complete description of Riesz bases of exponentials was obtained by S.V. Hruscev, N.K. Nikolskii and B.S. Pavlov [20] in terms the Helson-Szegö condition. In [20] also the case of general inner functions is treated and a necessary and sufficient condition is obtained under the additional restriction

$$\sup_n |\Theta(\lambda_n)| < 1. \tag{2}$$

In this case the system $\{k_{\lambda_n}\}$ is a basis if and only if the sequence Λ satisfies the Carleson interpolation condition and the Toeplitz operator $T_{\Theta\overline{B}}$, where B is the Blaschke product with zeros λ_n, is invertible. The invertibility of $T_{\Theta\overline{B}}$ is, in its turn, equivalent to the representation $\Theta\overline{B} = \alpha h/\overline{h}$, where $\alpha \in \mathbb{C}$, $|\alpha| = 1$, and $h \in H^2$ is an outer function such that $|h|^2$ satisfies the Helson-Szegö condition. The problem of the description of bases of reproducing kernels in the general case is still open.

For the case when Θ and Λ satisfy (2) a result on stability of the bases under small perturbations of "frequencies" λ_n was obtained by E. Fricain [18]: if $\{k_{\lambda_n}\}$ is a basis in K^2_Θ, then there is $\varepsilon > 0$ such that $\{k_{\mu_n}\}$ is a basis whenever

$$\sup_n \rho(\lambda_n, \mu_n) < \varepsilon, \tag{3}$$

where $\rho(z, w)$ stands for the pseudo-hyperbolic distance, that is, $\rho(z, w) = \left|\frac{z-w}{z-\overline{w}}\right|$, z, $w \in \mathbb{C}^+$.

However, the condition (2) seems to be too restrictive. In many cases there exist bases of reproducing kernels such that (2) is violated. Moreover,

there is an important class of orthogonal bases of reproducing kernels corresponding to real points λ_n and, thus, $|\Theta(\lambda_n)| \equiv 1$.

Orthogonal bases of reproducing kernels were studied by L. de Branges [10] for meromorphic inner functions and by D.N. Clark [11] in the general case. They may be constructed by the following procedure. For any $\alpha \in \mathbb{C}$, $|\alpha| = 1$, the function $(\alpha + \Theta)/(\alpha - \Theta)$ has a positive real part in the upper half plane. Hence, there exist $p_\alpha \geqslant 0$ and a measure σ_α such that

$$\Re \frac{\alpha + \Theta(z)}{\alpha - \Theta(z)} = p_\alpha \Im z + \frac{\Im z}{\pi} \int\limits_{\mathbb{R}} \frac{d\sigma_\alpha(t)}{|t - z|^2}, \qquad z \in \mathbb{C}^+.$$

The Clark theorem states that if σ_α is purely atomic (that is,

$$\sigma_\alpha = \sum_n a_n \delta_{t_n}$$

where δ_x denotes the Dirac measure at the point x) and $p_\alpha = 0$, then the system $\{k_{t_n}\}$ is an orthogonal basis in K^2_Θ; in particular, $k_{t_n} \in K^2_\Theta$. Note that all measures σ_α are purely atomic if the boundary spectrum $\sigma(\Theta) \cap \mathbb{R}$ (see Section 1 for the definition) is at most countable, whereas $p_\alpha = 0$ for all α except at most one. Note also that, by the Ahern-Clark theorem [1], $k_x \in K^2_\Theta$, $x \in \mathbb{R}$, if and only if $|\Theta'(x)| < \infty$, where $|\Theta'(x)|$ is the modulus of the angular derivative of Θ, and $\|k_x\|_2^2 = |\Theta'(x)|/(2\pi)$.

We consider the following example. If Θ admits an analytic extension across the real axis (and, thus, is meromorphic in the whole complex plane), then there is a well-defined branch of the argument of Θ on the line, that is, an increasing differentiable function φ such that $\Theta(t) = \exp(i\varphi(t))$, $t \in \mathbb{R}$ (note, that $\varphi'(t) = |\Theta'(t)|$). In this case the points t_n may be defined by the equation

$$\varphi(t_n) = \arg \alpha + 2\pi n.$$

In view of this example it seems reasonable to consider perturbations s_n of the points t_n such that

$$\sup_n |\varphi(s_n) - \varphi(t_n)| = \sup_n \int\limits_{<t_n, s_n>} |\Theta'(t)| dt < \varepsilon, \tag{4}$$

that is, the perturbations which are small with respect to the change of the argument of Θ. Here, by $< s, t >$ we denote the interval with the endpoints s and t.

Perturbations of the form (4) were studied by W.S. Cohn for one special but important class of inner functions which generalizes the functions $\Theta(z) = \exp(iaz)$. Given $\delta \in (0,1)$ consider the level set

$$\Omega(\Theta, \delta) = \{z \in \mathbb{C}^+ : |\Theta(z)| < \delta\}.$$

We say that an inner function Θ satisfies the connected level set condition (CLS) if the set $\Omega(\Theta, \delta)$ is connected for some $\delta \in (0,1)$ (sometimes also the term "one-component inner function" is used). The following theorem is due to W.S. Cohn [13]:

THEOREM. — *Let Θ be a CLS inner function. Then there is $\varepsilon = \varepsilon(\Theta)$ such that if $\{k_{t_n}\}$ is a Clark basis and (4) holds, then $\{k_{s_n}/\|k_{s_n}\|_2\}$ is a Riesz basis in K^2_Θ.*

In the author's paper [5] counterexamples are constructed which show that the CLS condition is essential here. The result analogous to the Cohn's theorem is no longer true for general Θ-s even if we consider only meromorphic inner functions (we include some of these examples for the sake of completeness).

In conclusion we mention the sampling property of systems of reproducing kernels which is important for applications. Recall that a system $\{h_n\}$ in a Hilbert space H is said to be a frame if there are positive constants A and B such that

$$A\|f\|_H^2 \leqslant \sum_n |(f, h_n)_H|^2 \leqslant B\|f\|_H^2, \qquad f \in H; \tag{5}$$

by $(f,g)_H$ we denote the inner product in H. Clearly, if the system $\{h_n\}$ is a Riesz basis, then it is automatically a frame. Moreover, in this case (5) holds with the same constants A and B as the inequality (1).

If $\{k_{\lambda_n}/\|k_{\lambda_n}\|_2\}$ is a frame in K^2_Θ, then

$$A\|f\|_2^2 \leqslant \sum_n |f(\lambda_n)|^2/\|k_{\lambda_n}\|_2^2 \leqslant B\|f\|_2^2, \qquad f \in K^2_\Theta. \tag{6}$$

Any set Λ satisfying (6) is said to be sampling for K^2_Θ. Sampling sets $\Lambda \subset \mathbb{R}$ for K^2_Θ with $\Theta(z) = \exp(2\pi iz)$ or, in other words, frames of exponentials, were recently described by J. Ortega-Cerda and K. Seip [27]. Also, it was shown by Seip [28] that even for the systems of exponentials

there is an essential difference between frames and bases: there exist frames of exponentials containing no subsequence which is a Riesz basis.

Stability of frames of reproducing kernels was considered in [6] for a special class of inner functions Θ with $\Theta' \in L^\infty(\mathbb{R})$. The following related results also should be mentioned: in [19] stability of the completeness property for the reproducing kernels is discussed, whereas in [8] a description of uniformly minimal systems of reproducing kernels is obtained.

In the present paper we consider an approach to the problem of stability of bases and frames of reproducing kernels based on the estimates of derivatives (Bernstein-type inequalities) obtained recently by the author [7]. This approach makes it possible to give unified proofs and generalize essentially the results of Fricain and Cohn. It should be emphasized that in the proof of the first stability theorem of Paley and Wiener also certain Bernstein-type inequality was used.

In what follows, the letters C, C_1, C_2 denote different positive constants which may change their values in different inequalities. We write $f \asymp g$ if $C_1 f \leqslant g \leqslant C_2 f$.

1. Main results.

To state our main results we need some additional information on inner functions. Recall that each inner function Θ admits the factorization

$$\Theta(z) = \gamma \exp(iaz) B(z) I_\psi(z), \tag{7}$$

where $\gamma \in \mathbb{C}$, $|\gamma| = 1$, $a \geqslant 0$,

$$B(z) = \prod_n e^{i\alpha_n} \frac{z - z_n}{z - \overline{z}_n}$$

is the Blaschke product with zeros $z_n \in \mathbb{C}^+$ satisfying the Blaschke condition $\sum_n \Im z_n (1 + |z_n|^2)^{-1} < \infty$ and $\alpha_n \in \mathbb{R}$. Singular inner function I_ψ is defined by the formula

$$I_\psi = \exp\left(i \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{t^2 + 1} \right) d\psi(t) \right),$$

where ψ is a Borel measure on the line singular with respect to the Lebesgue measure and such that $\int_{\mathbb{R}}(t^2+1)^{-1}d\psi(t)<\infty$. Let $\sigma(\Theta)$ be the so-called spectrum of the inner function Θ, that is, the set of all $\zeta\in\overline{\mathbb{C}^+}\cup\{\infty\}$ such that $\liminf_{z\to\zeta,\,z\in\mathbb{C}^+}|\Theta(z)|=0$. Note that $\sigma(\Theta)$ is closed and Θ (and, moreover, any $f\in K^2_\Theta$) has an analytic extension across any interval of the set $\mathbb{R}\setminus\sigma(\Theta)$.

Clearly, if $x\in\mathbb{R}\setminus\sigma(\Theta)$, then the function k_x belongs to K^2_Θ and is the reproducing kernels corresponding to the point x. It is possible, however, that $k_x\in K^2_\Theta$ even when $x\in\sigma(\Theta)$: it was shown by P.R. Ahern and D.N. Clark [1] that $k_x\in K^2_\Theta$ if and only if the modulus of the angular derivative $\Theta'(x)$ is finite. Recall that

$$|\Theta'(x)|=a+\sum_n\frac{2\Im z_n}{|x-z_n|^2}+\int\limits_{\mathbb{R}}\frac{d\psi(t)}{(t-x)^2}\,. \tag{8}$$

A generalization of this result is due to Cohn [12]: the kernel k_x is in H^p, $1<p<\infty$, if and only if

$$S_p(x)=\sum_n\frac{\Im z_n}{|x-z_n|^p}+\int\limits_{\mathbb{R}}\frac{d\psi(t)}{|t-x|^p}<\infty.$$

A Borel measure ν in the closed upper half-plane $\overline{\mathbb{C}^+}$ is said to be a Carleson measure if there is a constant $M_\nu>0$ such that

$$\mu(S(x,h))\leqslant M_\nu h$$

for any square $S(x,h)=[x,x+h]\times[0,h]$, $x\in\mathbb{R}$, $h>0$, in the upper half-plane.

Let $\{k_{\lambda_n}\}$ be a basis in K^2_Θ. In what follows we consider perturbations within certain neighbourhoods of the points λ_n. By $\langle u,v\rangle$ we denote the interval with the endpoints u and v; $\delta_{\langle u,v\rangle}$ denotes the Lebesgue measure on $\langle u,v\rangle$. Let $G=\bigcup_n G_n\subset\overline{\mathbb{C}^+}$ satisfy the following properties:

(i) there exist positive constants c and C such that

$$c\leqslant\|k_{z_n}\|_2/\|k_{\lambda_n}\|_2\leqslant C,\qquad z_n\in G_n;$$

(ii) for any $z_n \in G_n$ the measure

$$\nu = \sum_n \delta_{\langle \lambda_n, z_n \rangle}$$

is a Carleson measure and, moreover, the Carleson constants M_ν of such measures are uniformly bounded with respect to z_n.

It should be noted that for $\lambda_n \in \mathbb{C}^+$ there always exist nontrivial sets G_n satisfying (i)-(ii). It is known that for any basis of reproducing kernels the sequence Λ satisfies the Carleson interpolation condition

$$\inf_n \prod_{m \neq n} \left| \frac{\lambda_m - \lambda_n}{\lambda_m - \overline{\lambda}_n} \right| > 0$$

(see [24] or [26], p. 308). In particular, the measure $\nu = \sum_n \Im\lambda_n \delta_{\lambda_n}$ is a Carleson measure. The same arguments show that for a frame of the form $\{k_{\lambda_n}/\|k_{\lambda_n}\|_2\}$ the sequence Λ is a finite union of interpolating sequences, and, thus, ν is again a Carleson measure (see Lemma 4.2 below).

Therefore, we can take $G_n = \{z : |z - \lambda_n| < r\Im\lambda_n\}$ for sufficiently small $r > 0$. At the same time, it is possible that for $\lambda_n \in \mathbb{R}$ the only admissible set G_n consists of the point λ_n.

Now we are able to state our main result on stability which is applicable to general inner functions Θ and sequences Λ.

THEOREM 1.1. — *Let $\{k_{\lambda_n}\}$ be a basis in K^2_Θ, $p \in (1,2)$, $1/p+1/q = 1$. Then for any set G satisfying (i)-(ii) there is $\varepsilon > 0$ such that the system of reproducing kernels $\{k_{\mu_n}\}$ is a basis whenever $\mu_n \in G_n$ and*

$$\sup_n \frac{1}{\|k_{\lambda_n}\|_2^2} \int\limits_{\langle \lambda_n, \mu_n \rangle} \|k_z^2\|_q^{\frac{2p}{p+1}} |dz| < \varepsilon. \tag{9}$$

The constant ε in Theorem 1.1 depends on p, on the constants involved in the definition of the set G and on the constant A from (1), but not on the function Θ. To be exact, $\varepsilon = A\varepsilon_1(p, G)$.

The quantity in (9) which measures the smallness of perturbations looks somewhat implicit. For the applications of Theorem 1.1 it is necessary

to have effective estimates of the norm $\|k_z^2\|_q$. We start with the following simple estimate:

$$\|k_z^2\|_q^q \leqslant C\frac{1-|\Theta(z)|^2}{(\Im z)^{2q-1}} \qquad z\in\mathbb{C}^+. \tag{10}$$

Indeed,

$$\|4\pi^2 k_z^2\|_q^q = \int_{\mathbb{R}} \left|\frac{1-\overline{\Theta(z)}\Theta(t)}{t-\overline{z}}\right|^{2q} dt$$

$$\leqslant \frac{2^{2q-2}}{(\Im z)^{2q-2}} \int_{\mathbb{R}} \left|\frac{1-\overline{\Theta(z)}\Theta(t)}{t-\overline{z}}\right|^{2} dt \leqslant C\frac{1-|\Theta(z)|^2}{(\Im z)^{2q-1}}.$$

Estimate (10) implies

COROLLARY 1.2. — *Let $\{k_{\lambda_n}\}$ be a basis in K_Θ^2 and $\gamma > 1/3$. Then there is $\varepsilon = \varepsilon(\gamma, A)$ such that the system $\{k_{\mu_n}\}$ is a basis whenever*

$$\rho(\lambda_n,\mu_n) < \varepsilon(1-|\Theta(\lambda_n)|)^\gamma. \tag{11}$$

Note that in the case when (2) is satisfied, (11) is equivalent to (3); thus, the theorem of Fricain follows immediately from Corollary 1.2.

If Θ belongs to some special class of inner functions with additional properties we may have sharper estimates for the norms of reproducing kernels than (10). For example, we show (making use of the results of A.B. Aleksandrov [3, 4]) that for a CLS inner function Θ and a basis $\{k_{\lambda_n}\}$ we have the estimate

$$\|k_{\lambda_n}^2\|_q^q \leqslant C\|k_{\lambda_n}\|_2^{2(2q-1)} \tag{12}$$

with $C = C(\Theta, \Lambda, q)$. Thus, we obtain

COROLLARY 1.3. — *Let Θ be a CLS inner function, $\lambda_n \in \mathbb{C}^+$, and let $\{k_{\lambda_n}\}$ be a basis in K_Θ^2. Then there exists $\varepsilon > 0$ such that if*

$$|\lambda_n - \mu_n| < \varepsilon\|k_{\lambda_n}\|_2^{-2} = 4\pi\varepsilon\Im\lambda_n(1-|\Theta(\lambda_n)|^2)^{-1}, \tag{13}$$

then the system $\{k_{\mu_n}\}$ is also a basis.

In particular, (13) implies that in the case of CLS inner functions the bases of reproducing kernels are stable with respect to hyperbolically small perturbations of the points λ_n.

Remarks . —

1. All our results on stability of bases have their analogs for frames of reproducing kernels. To get the corresponding statement one should replace "$\{k_{\lambda_n}\}$ is a basis in K^2_Θ" by "$\{k_{\lambda_n}/\|k_{\lambda_n}\|_2\}$ is a frame in K^2_Θ" in the formulations. Moreover, all the proofs are based on some frame-type estimate and the general fact that if a frame is close to a basis, then it is a basis itself.

2. All the statements remain valid if we are interested in the stability of Riesz sequences of reproducing kernels, that is, of systems of reproducing kernels which constitute Riesz bases in their closed linear spans.

Now, we consider an extension of the Cohn's theorem on stability of bases with real "frequencies" (in particular, the Clark bases) to the case of a general inner function. For $t \in \mathbb{R}$ let $d_0(t) = \operatorname{dist}(t, \sigma(\Theta))$ be the distance to the spectrum of Θ.

THEOREM 1.4. — *Let $t_n \in \mathbb{R}$ and let $\{k_{t_n}\}$ be a basis ($\{k_{t_n}/\|k_{t_n}\|_2\}$ be a frame) in K^2_Θ. Then there is $\varepsilon > 0$ such that $\{k_{s_n}\}$ is a basis ($\{k_{s_n}/\|k_{s_n}\|_2\}$ is a frame) whenever*

$$\int\limits_{<t_n,s_n>} [|\Theta'(t)| + |\Theta'(t)|^{-1} d_0^{-2}(t)]\, dt < \varepsilon \tag{14}$$

or

$$|s_n - t_n| < \varepsilon |\Theta'(t_n)| \min(d_0^2(t_n), |\Theta'(t_n)|^{-2}). \tag{15}$$

Remark. — It should be emphasized that the admissible perturbations in Theorems 1.1 and 1.4 depend essentially on the properties of the function Θ and density of its spectrum near the points under consideration. If we are far from the spectrum, then larger perturbations are admissible. For particular cases the distance function (9) leads to Euclidean (in the case $\Theta(z) = \exp(iaz)$ or, more generally, for $\Theta' \in L^\infty(\mathbb{R})$) or pseudo-hyperbolic (for functions satisfying (2) or CLS condition) metrics.

2. Preliminaries on bases and frames.

A frame $\{h_n\}$ is said to be an exact frame if it fails to be a frame after the removal of any of its vectors h_n. It turns out that exact frames are just one more characterization of the Riesz bases. We cite the following well-known theorem ([30], Ch. 4, Th. 12):

THEOREM 2.1. — *The system $\{h_n\}$ is a Riesz basis in a separable Hilbert space H if and only if $\{h_n\}$ is an exact frame.*

A system $\{h_n\}$ is said to be minimal if $h_m \notin \overline{Span}(h_n, n \neq m)$ for any h_m, where $\overline{Span}$ denotes the closed linear span.

COROLLARY 2.2. — *The following statements are equivalent:*

1. $\{h_n\}$ is a Riesz basis in H;

2. $\{h_n\}$ is both a frame and minimal.

Proof. — Implication $1 \Longrightarrow 2$ is trivial. Let $\{h_n\}$ be a frame and minimal. Then the system $\{h_n\}_{n\neq m}$ is not complete in H, and, consequently, is not a frame. Thus, $\{h_n\}$ is an exact frame and, by Theorem 2.1, is a Riesz basis. □

The following lemma plays the key role in our arguments. It states that if a system $\{h'_n\}$ is close to $\{h_n\}$ in a certain "frame" sense and $\{h_n\}$ is a Riesz basis, then the same is true for $\{h'_n\}$. Here A is the constant from (1) and (5).

LEMMA 2.3. — *Let*

$$\sum_n |(f, h_n - h'_n)_H|^2 \leqslant \varepsilon \|f\|_H^2, \qquad f \in H. \tag{16}$$

If $\{h_n\}$ is a frame and $\varepsilon < A$, then $\{h'_n\}$ is also a frame. If, moreover, $\{h_n\}$ is a Riesz basis, then $\{h'_n\}$ is a Riesz basis.

Proof. — The first statement is obvious. Indeed,

$$\|(f, h'_n)_H\|_{\ell^2} \geqslant \|(f, h_n)_H\|_{\ell^2} - \|(f, h_n - h'_n)_H\|_{\ell^2} \geqslant (\sqrt{A} - \sqrt{\varepsilon})\|f\|_H, \quad f \in H.$$

Now let $\{h_n\}$ be a Riesz basis. To prove that $\{h'_n\}$ is a Riesz basis it suffices to show that $\{h'_n\}$ is minimal. Consider finite linear combinations $g = \sum_n c_n h_n$ and $g' = \sum_n c_n h'_n$. Clearly,

$$\|g - g'\|_H^2 = \sum_n \overline{c_n}(g - g', h_n - h'_n)_H \leqslant \|\{c_n\}\|_{\ell^2} \|\{(g - g', h_n - h'_n)_H\}\|_{\ell^2}.$$

Hence, by the estimate (16), $\|g - g'\|_H^2 \leqslant \varepsilon \|\{c_n\}\|_{\ell^2}^2$. Therefore, (1) implies

$$\|g'\|_H \geqslant \|g\|_H - \|g - g'\|_H \geqslant (\sqrt{A} - \sqrt{\varepsilon})\|\{c_n\}\|_{\ell^2}.$$

In particular, for any finite $\{c_n\}$ and for any m we have

$$\left\| h'_m - \sum_{n \neq m} c_n h'_n \right\|_H \geqslant \sqrt{A} - \sqrt{\varepsilon} > 0.$$

Thus, $\{h'_n\}$ is minimal (the latter inequality coincides with the definition of a stronger property of uniform minimality). □

3. Weighted Bernstein-type inequalities. Proof of the main theorem.

The interest to the Bernstein-type inequalities in the model subspaces K_Θ^2 and their L^p analogs $K_\Theta^p = H^p \cap \Theta\overline{H^p}$ is partially motivated by the classical Bernstein's inequality for the Paley–Wiener space PW_a^p. The space PW_a^p consists of all entire functions of exponential type at most a which belong to $L^p(\mathbb{R})$. This space is closely related to the model subspace K_Θ^p, where $\Theta(z) = \exp(iaz)$; namely, $K_\Theta^p = PW_a^p \cap H^p$. It is well known that

$$\|f'\|_p \leqslant a\|f\|_p, \qquad f \in PW_a^p.$$

This inequality was a starting point for many generalizations. The problem of existence of the non-tangential boundary values for the derivatives of K_Θ^p-functions was solved in [1, 14]; the boundedness and compactness of the differentiation operator on the model subspace were studied in [15, 17]; certain weighted norm inequalities were obtained in [6, 9, 16, 22]. A survey of these results is presented in [7].

In the present paper we apply the following estimate obtained in [7] (we consider only the case of L^2-norms):

THEOREM 3.1. — *Let ν be a Carleson measure, $p \in (1,2)$, $1/p + 1/q = 1$. Put*

$$w_p(z) = \|k_z^2\|_q^{-\frac{p}{p+1}};$$

we assume $w_p(x) = 0$ whenever $x \in \mathbb{R}$ and $S_{2q}(x) = \infty$. Then the operator

$$(T_p f)(z) = f'(z) w_p(z), \qquad z \in \overline{\mathbb{C}^+},$$

is bounded as an operator from K^2_Θ to $L^2(\nu)$, that is, there is a constant $C = C(M_\nu, p)$ such that

$$\|f' w_p\|_{L^2(\nu)} \leqslant C\|f\|_2, \qquad f \in K^2_\Theta.$$

In [7] a number of estimates for the norm $\|k_z^2\|_q$ (which is comparable with the norm of the functional $f \mapsto f'(z)$, $f \in K^p_\Theta$) is presented. We mention here (besides (10) and (12)) the following:

$$C_1 \min(d_0(x), |\Theta'(x)|^{-1}) \leqslant \|k_x^2\|_q^{-\frac{p}{p+1}} \leqslant C_2 |\Theta'(x)|^{-1}, \qquad x \in \mathbb{R}, \tag{17}$$

where $d_0(x) = \operatorname{dist}(x, \sigma(\Theta))$. Moreover, the quantity

$$v_0(x) = \min(d_0(x), |\Theta'(x)|^{-1})$$

has a simple geometrical meaning related to the level sets $\Omega(\Theta, \delta)$. Namely, $v_0(x) \asymp \operatorname{dist}(x, \Omega(\Theta, \delta))$ with the constants depending only on $\delta \in (0,1)$.

Note that $v_0(x) = 0$ whenever $x \in \sigma(\Theta)$. Hence, the function $f' v_0$ is well-defined on $\mathbb{R}$.

COROLLARY 3.2. — There exists an absolute constant C such that $\|f' v_0\|_2 \leqslant C\|f\|_2$, $f \in K^2_\Theta$.

Proof of Theorem 1.1. — Let $h_n = k_{\lambda_n}/\|k_{\lambda_n}\|_2$ and $h'_n = k_{\mu_n}/\|k_{\lambda_n}\|_2$. Recall that $\|k_{\lambda_n}\|_2 \asymp \|k_{\mu_n}\|_2$, $\mu_n \in G_n$. In view of Lemma 2.3, it suffices to check the estimate (16). Note that for the case λ_n, $\mu_n \in \mathbb{R}$ it follows from (9) that $\operatorname{Int}(\langle \lambda_n, \mu_n \rangle) \cap \sigma(\Theta) = \emptyset$ (see [7], Lemma 6.1), where $\operatorname{Int} I$ denotes the interior of the interval I. Thus, any $f \in K^2_\Theta$ is differentiable in $\operatorname{Int}(\langle \lambda_n, \mu_n \rangle)$.

It is known that the functions continuous in $\overline{\mathbb{C}^+}$ are dense in K^2_Θ (it is obvious if Θ is a Blaschke product; see [2] for the general case). Let $f \in K^2_\Theta$ be continuous in $\overline{\mathbb{C}^+}$. Then

$$|(f, h_n - h_n')|^2 = \frac{|f(\lambda_n) - f(\mu_n)|^2}{\|k_{\lambda_n}\|_2^2} = \frac{1}{\|k_{\lambda_n}\|_2^2} \left| \int\limits_{\langle \lambda_n, \mu_n \rangle} f'(z)|dz| \right|^2 .$$

By the Hölder inequality,

$$|(f, h_n - h_n')|^2 \leqslant \frac{1}{\|k_{\lambda_n}\|_2^2} \int\limits_{\langle \lambda_n, \mu_n \rangle} |f'(z) w_p(z)|^2 |dz| \int\limits_{\langle \lambda_n, \mu_n \rangle} w_p^{-2}(z)|dz|.$$

By our assumptions,

$$\frac{1}{\|k_{\lambda_n}\|_2^2} \int\limits_{\langle \lambda_n, \mu_n \rangle} w_p^{-2}(z)|dz| < \varepsilon$$

and $\nu = \sum_n \delta_{\langle \lambda_n, \mu_n \rangle}$ is a Carleson measure with a constant M_ν which does not exceed some absolute constant depending only on G. Hence, by Theorem 3.1,

$$\sum_n |(f, h_n - h_n')|^2 \leqslant \varepsilon \|f' w_p\|^2_{L^2(\nu)} \leqslant C\varepsilon \|f\|_2^2$$

for a constant C which depends on G, Λ and p. Now, Lemma 2.3 implies that if h_n is a basis (frame) in K^2_Θ, then we can choose a sufficiently small $\varepsilon > 0$ such that h_n' is also a basis (frame). □

4. Proofs of the corollaries.

We begin with the following (apparently well-known) property of inner functions.

Lemma 4.1. — *There exist absolute constants $\varepsilon_0 \in (0,1)$, C_1, $C_2 > 0$ such that for any z, $w \in \mathbb{C}^+$ satisfying $\rho(z,w) < \varepsilon_0$ we have*

$$C_1 \leqslant \frac{1 - |\Theta(z)|}{1 - |\Theta(w)|} \leqslant C_2. \tag{18}$$

Proof. — By the Frostman theorem, each inner function may be uniformly approximated by Blaschke products. So, without loss of generality, Θ is a Blaschke product with the zeros z_m. It follows from the condition $\rho(z,w)<\varepsilon_0$ that $|z-w|\leqslant 2\varepsilon_0(1-\varepsilon_0)^{-1}\Im w$. Let $\varepsilon_0=1/9$. Then $|z-w|\leqslant(\Im w)/4$, whence $4/5\leqslant \Im z/\Im w\leqslant 5/4$ and $4/5\leqslant |w-\overline{z}_m|/|z-\overline{z}_m|\leqslant 5/4$.

Due to the symmetry, it suffices to prove the right-hand side inequality in (18). The estimate is obvious if $|\Theta(w)|<1/2$. Also, if $|z-z_m|/|z-\overline{z}_m|<1/2$ for some z_m, then

$$\left|\frac{w-\overline{z}_m}{w-\overline{z}_m}\right|\leqslant\left|\frac{w-z}{w-\overline{z}_m}\right|+\left|\frac{z-z_m}{z-\overline{z}_m}\right|\left|\frac{z-\overline{z}_m}{w-\overline{z}_m}\right|<\frac{7}{8},$$

and, consequently, $(1-|\Theta(z)|)/(1-|\Theta(w)|)<8$.

Assume that $|\Theta(w)|\geqslant 1/2$ and $|z-z_m|/|z-\overline{z}_m|\geqslant 1/2$. Then, applying the elementary inequalities $\log(1-u)<-u$, $u\in(0,1)$, and $\log(1-u)>-2u$, $u\in(0,1/2)$, to $u=1-|\Theta(z)|$ and $u=1-|\Theta(w)|$ respectively, we get

$$\frac{1-|\Theta(z)|}{1-|\Theta(w)|}\leqslant 2\,\frac{\log|\Theta(z)|}{\log|\Theta(w)|}.$$

Note that for $\zeta\in\mathbb{C}^+$

$$\log|\Theta(\zeta)|=\sum_m\log\left|\frac{\zeta-z_m}{\zeta-\overline{z}_m}\right|=\frac{1}{2}\sum_m\log\left(1-\frac{4\Im\zeta\Im z_m}{|\zeta-\overline{z}_m|^2}\right). \tag{19}$$

Hence,

$$-\log|\Theta(w)|>2\sum_m\frac{\Im w\Im z_m}{|w-\overline{z}_m|^2}.$$

On the other hand, since $|z-z_m|/|z-\overline{z}_m|\geqslant 1/2$ and $\log(1-u)>-4u$, $u\in(0,3/4)$, we have

$$-\log|\Theta(z)|<8\sum_m\frac{\Im z\Im z_m}{|z-\overline{z}_m|^2}.$$

Therefore, $1-|\Theta(z)|\asymp 1-|\Theta(w)|$. □

LEMMA 4.2. — *If $\{k_{\lambda_n}/\|k_{\lambda_n}\|_2\}$ is a frame in K^2_Θ, then $\nu=\sum_n\Im\lambda_n\delta_{\lambda_n}$ is a Carleson measure.*

Proof. — Note that

$$\left(\frac{k_{\lambda_m}}{\|k_{\lambda_m}\|_2}, \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|_2}\right) = \frac{1-\overline{\Theta(\lambda_m)}\Theta(\lambda_n)}{\lambda_n - \overline{\lambda}_m} \cdot \frac{2i\sqrt{\Im\lambda_m \Im\lambda_n}}{(1-|\Theta(\lambda_m)|^2)^{1/2}(1-|\Theta(\lambda_n)|^2)^{1/2}}.$$

Clearly, $|1-\alpha\overline{\beta}|^2 \geqslant (1-|\alpha|^2)(1-|\beta|^2)$ when $|\alpha|<1$ and $|\beta|<1$. Hence, by (6),

$$\sum_n \frac{4\Im\lambda_m \Im\lambda_n}{|\lambda_n - \overline{\lambda}_m|^2} \leqslant \sum_n \left|\left(\frac{k_{\lambda_m}}{\|k_{\lambda_m}\|_2}, \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|_2}\right)\right|^2 \leqslant B,$$

whence

$$\sup_m \sum_n \frac{\Im\lambda_m \Im\lambda_n}{|\lambda_n - \overline{\lambda}_m|^2} < \infty.$$

It is well known that the latter condition is fulfilled if and only if ν is a Carleson measure (see [24], p. 151). □

Proof of Corollary 1.2. — By Lemmas 4.1 and 4.2, the sets $G_n = \{z : |z-\lambda_n| \leqslant (\Im\lambda_n)/9\}$ satisfy the conditions (i)-(ii). If ε is sufficiently small, then the estimate (11) implies that $\mu_n \in G_n$ and, moreover, $|\lambda_n - \mu_n| < C\varepsilon\Im\lambda_n$. Without loss of generality we may assume also that $\gamma < 1$. Since $\gamma > 1/3$ there exists $p \in (1,2)$ such that $2\frac{p-1}{p+1} = 1-\gamma$. Let q be the conjugate exponent. Note that $\frac{p+1}{p} = \frac{2q-1}{q}$ and $\frac{2p}{(p+1)q} = 1-\gamma$. Then, by the inequality (10) and Lemma 4.1,

$$\|k_z^2\|_q^{\frac{2p}{p+1}} \leqslant C_1 \frac{(1-|\Theta(z)|)^{\frac{2p}{(p+1)q}}}{(\Im z)^2} \leqslant C_2 \frac{(1-|\Theta(\lambda_n)|)^{1-\gamma}}{(\Im\lambda_n)^2}$$

for $z \in \langle \lambda_n, \mu_n \rangle$. Hence,

$$\frac{1}{\|k_{\lambda_n}\|_2^2} \int\limits_{\langle \lambda_n, \mu_n\rangle} \|k_z^2\|_q^{\frac{2p}{p+1}} |dz| \leqslant C_3 \frac{|\lambda_n - \mu_n| \Im\lambda_n}{1-|\Theta(\lambda_n)|} \cdot \frac{(1-|\Theta(\lambda_n)|)^{1-\gamma}}{(\Im\lambda_n)^2} \leqslant C_4\varepsilon.$$

All the constants involved depend only on p. To complete the proof we should take a sufficiently small ε and apply Theorem 1.1. □

Remark. — It is an interesting problem to determine the smallest γ such that the bases of reproducing kernels are stable under perturbations of the form (11) (or, at least, to find out whether this critical exponent is positive).

In the proof of Corollary 1.3 we will use a few lemmas.

LEMMA 4.3. — *Let $\{k_{\lambda_n}/\|k_{\lambda_n}\|_2\}$ be a frame in K^2_Θ. Then*

$$\sum_n \|k_{\lambda_n}\|_2^{-2}(|\lambda_n|^2+1)^{-2} < \infty.$$

Proof. — Taking $z_0 \in \mathbb{C}^+$ such that $|\Theta(z_0)| < 1/2$ and applying (6) to the function k_{z_0} we get the desired estimate. An analogous statement is proved in [20] for the case of a unit disk. □

LEMMA 4.4. — *Let $\infty \notin \sigma(\Theta)$. Then for any frame of the form $\{k_{\lambda_n}/\|k_{\lambda_n}\|_2\}$ we have $\sup_n |\lambda_n| < \infty$.*

Proof. — If $\infty \notin \sigma(\Theta)$, then the sequence of the zeros of Θ is bounded. Also the support of the singular measure ψ in (7) is compact and $a = 0$. It is easy to see that in this case $\|k_z\|_2^2 \asymp |z|^{-2}$, $|z| \to \infty$, $z \in \overline{\mathbb{C}^+}$. Hence, by Lemma 4.3, $\sup_n |\lambda_n| < \infty$. □

Now we discuss some properties of CLS inner functions. Most of these results and arguments may be found in [3, 4] for the case of the unit disk; certain small modifications are required to adapt them to the case of the half-plane. In what follows the key role belongs to a theorem of A.L. Volberg and S.R. Treil concerning embeddings of the model subspaces. Let μ be a measure in $\overline{\mathbb{C}^+}$ satisfying the following property: there is $\delta \in (0,1)$ such that

$$\sup\{h^{-1}\mu(S(x,h)) : S(x,h) \cap \Omega(\Theta,\delta) \neq \emptyset\} < \infty, \tag{20}$$

that is, the Carleson estimate $\mu(S(x,h)) \leqslant Ch$ holds for sufficiently large squares intersecting the level set $\Omega(\Theta,\delta)$. If μ satisfies (20), then $K^2_\Theta \subset L^2(\mu)$ and $\|f\|_{L^2(\mu)} \leqslant C\|f\|_2$, $f \in K^2_\Theta$ [29]. We will make use of the converse result (see [4]):

LEMMA 4.5. — *Let Θ be a CLS inner function and either $\infty \in \sigma(\Theta)$ or the measure μ has a compact support. Then the embedding $K^2_\Theta \subset L^2(\mu)$ implies (20) for any $\delta \in (0,1)$.*

LEMMA 4.6. — *Let Θ be a CLS inner function and $\{k_{\lambda_n}/\|k_{\lambda_n}\|_2\}$ be a frame in K^2_Θ. Then there exist positive constants r, C_1, C_2, such that*

$$C_1\|k_{\lambda_n}\|_2 \leqslant \|k_z\|_2 \leqslant C_2\|k_{\lambda_n}\|_2, \quad z \in D(\lambda_n, r\|k_{\lambda_n}\|_2^{-2}) \cap \overline{\mathbb{C}^+},$$

where $D(w,R)$ denotes the ball with the center w and the radius R.

Proof. — In view of the Frostman theorem we may confine ourselves with the case of a Blaschke product. Let $\{z_m\}_{m\in\mathbb{N}}$ be the zeros of Θ and $b_m(z) = \frac{z-z_m}{z-\overline{z}_m}$. Then

$$\|k_z\|_2^2 = \frac{1-|\Theta(z)|^2}{4\pi\Im z} = \sum_{m=1}^{\infty} \frac{1-|b_m(z)|^2}{4\pi\Im z}\,|b_1(z)\dots b_{m-1}(z)|^2. \tag{21}$$

Since $z_m \in D(\lambda_n, |\lambda_n - \overline{z}_m|)$, the ball $D(\lambda_n, |\lambda_n - \overline{z}_m|)$ intersects any level set $\Omega(\Theta,\delta)$. On the other hand, for the measure $\mu = \sum_n \|k_{\lambda_n}\|_2^{-2}\delta_{\lambda_n}$ we have an embedding $K^2_\Theta \subset L^2(\mu)$ since $\{k_{\lambda_n}/\|k_{\lambda_n}\|_2\}$ is a frame. Note that, by Lemma 4.4, the support of μ is compact if $\infty \notin \sigma(\Theta)$. Hence, applying Lemma 4.5 to the measure μ, we get the inequality

$$\|k_{\lambda_n}\|_2^{-2} \leqslant C_0|\lambda_n - \overline{z}_m| \tag{22}$$

for some positive constant C_0 and for any λ_n and z_m. In particular, if $r < (2C_0)^{-1}$, then

$$|\lambda_n - \overline{z}_k|/2 \leqslant |z - \overline{z}_k| \leqslant 2|\lambda_n - \overline{z}_k|, \qquad z \in D(\lambda_n, r\|k_{\lambda_n}\|_2^{-2}). \tag{23}$$

If $|\Theta(\lambda_n)| \leqslant 1/2$, then $\|k_{\lambda_n}\|_2^{-2} \asymp \Im\lambda_n$ and the existence of r with required property follows from Lemma 4.1. Now, let $|\Theta(\lambda_n) > 1/2$; note that the same is true for any subproduct of Θ. Then it follows from (21) that

$$\|k_{\lambda_n}\|_2^2 \asymp \sum_{m=1}^{\infty} \frac{1-|b_m(\lambda_n)|^2}{\Im\lambda_n} = \sum_{m=1}^{\infty} \frac{4\Im z_m}{|\lambda_n - \overline{z}_m|^2}.$$

Making use of (23) and (19) it is easy to show that $|\Theta(z)| > 1/4$, $z \in D(\lambda_n, r\|k_{\lambda_n}\|_2^{-2})$, for sufficiently small r. Hence, by (21),

$$\|k_z\|_2^2 \asymp \sum_m \frac{\Im z_m}{|z-\overline{z}_m|^2} \asymp \sum_m \frac{\Im z_m}{|\lambda_n - \overline{z}_m|^2} \asymp \|k_{\lambda_n}\|_2^2. \qquad \square$$

Lemma 4.7. — *Let Θ be a CLS inner function, let $\{k_{\lambda_n}/\|k_{\lambda_n}\|_2\}$ be a frame, and let r be the constant from Lemma 4.6. Then for any $s \geqslant 2$ there exists $C = C(\Theta,\Lambda,s) > 0$ such that*

$$\|k_z\|_s^s \leqslant C\|k_{\lambda_n}\|_2^{2(s-1)}, \quad z \in D(\lambda_n, \frac{r}{2}\|k_{\lambda_n}\|_2^{-2}) \cap \overline{\mathbb{C}^+}.$$

Proof. — Let $z \in D(\lambda_n, \frac{r}{2}\|k_{\lambda_n}\|_2^{-2})$ and $w \in D(\lambda_n, r\|k_{\lambda_n}\|_2^{-2})$. Then, by the Cauchy inequality and Lemma 4.6,

$$|k_z(w)| \leqslant \|k_z\|_2 \|k_w\|_2 \leqslant C_1 \|k_{\lambda_n}\|_2^2.$$

If $w \in \overline{\mathbb{C}^+} \setminus D(\lambda_n, r\|k_{\lambda_n}\|_2^{-2})$, then $|z-w| \geqslant r\|k_{\lambda_n}\|_2^{-2}/2$ and

$$|k_z(w)| = \frac{1}{2\pi} \left| \frac{1 - \overline{\Theta(z)}\Theta(w)}{w - \overline{z}} \right| \leqslant \frac{2}{\pi r} \|k_{\lambda_n}\|_2^2.$$

Thus, $\|k_z\|_\infty \leqslant C_2 \|k_{\lambda_n}\|_2^2$. Therefore,

$$\|k_z\|_s^s \leqslant \|k_z\|_\infty^{s-2} \|k_z\|_2^2 \leqslant C_3 \|k_{\lambda_n}\|_2^{2(s-1)}. \qquad \square$$

Proof of Corollary 1.3. — Let $G_n = D(\lambda_n, \frac{r}{2}\|k_{\lambda_n}\|_2^{-2}) \cap \overline{\mathbb{C}^+}$. By Lemma 4.6, the sets G_n have the property (i), and applying Lemma 4.5 once more, one can show that (ii) is also satisfied. To apply Theorem 1.1 one should verify the estimate (9). Lemma 4.7 implies $\|k_z^2\|_q^q \leqslant C_1 \|k_{\lambda_n}\|_2^{2(2q-1)}$, $z \in G_n$. Note also that $(2q-1)/q = (p+1)/p$. Hence, $\|k_z^2\|_q^{\frac{2p}{p+1}} \leqslant C_2 \|k_{\lambda_n}\|_2^4$, $z \in \langle \lambda_n, \mu_n \rangle$, and

$$\frac{1}{\|k_{\lambda_n}\|_2^2} \int\limits_{\langle \lambda_n, \mu_n \rangle} \|k_z^2\|_q^{\frac{2p}{p+1}} |dz| \leqslant C_2 |\lambda_n - \mu_n| \cdot \|k_{\lambda_n}\|_2^2 \leqslant C_2 \varepsilon. \qquad \square$$

5. Proof of Theorem 1.4. Examples.

LEMMA 5.1. — *Let $t_n \in \mathbb{R}$ and $\{k_{t_n}/\|k_{t_n}\|_2\}$ be a frame. Put*

$$G_n = \{t \in \mathbb{R} : |t - t_n| \leqslant v_0(t_n)/2\},$$

where $v_0(t) = \min(d_0(t), |\Theta'(t)|^{-1})$. Then the sets G_n satisfy the conditions (i)-(ii).

Proof. — Consider the nontrivial case $v_0(t_n) > 0$. We have, in particular, $|t - t_n| \leqslant d_0(t_n)/2$, $t \in G_n$. Hence, $d_0(t_n)/2 \leqslant d_0(t) \leqslant 2d_0(t_n)$

and it follows immediately from (8) that $|\Theta'(t_n)|/4 \leqslant |\Theta'(t)| \leqslant 4|\Theta'(t_n)|$. Thus, we get the property (i). Note also that $G_n \cap \sigma(\Theta) = \emptyset$ and, consequently, Θ is analytic on G_n.

Now, we show that there is $N \in \mathbb{N}$ such that each point $t \in \mathbb{R}$ belongs to at most N of the sets G_n (and, thus, condition (ii) is also satisfied). Let φ be an increasing continuous branch of the argument of Θ on G_n, that is, $\Theta(s) = e^{i\varphi(s)}$, $s \in G_n$. Let $t \in G_n$. Then

$$|k_t(t_n)| = \left| \frac{1 - \overline{\Theta(t)}\Theta(t_n)}{2\pi(t_n - t)} \right| = \left| \frac{\sin \frac{1}{2}(\varphi(t_n) - \varphi(t))}{2\pi(t_n - t)} \right|.$$

Note that

$$\int\limits_{\langle t_n, t\rangle} \varphi'(s)ds = \int\limits_{\langle t_n, t\rangle} |\Theta'(s)|ds \leqslant 4|\Theta'(t_n)| \cdot |t - t_n| \leqslant 2|\Theta'(t_n)|v_0(t_n) \leqslant 2;$$

thus, $|\varphi(t_n) - \varphi(t)| \leqslant 2$. Making use of the estimate $\sin u \geqslant 2u/\pi$, $u \in (0, \pi/2)$, we get

$$|k_t(t_n)| \geqslant \left| \frac{\varphi(t_n) - \varphi(t)}{2\pi^2(t_n - t)} \right| \geqslant \frac{|\Theta'(t_n)|}{8\pi^2},$$

where the last inequality follows from the fact that $\varphi'(t) \geqslant \varphi'(t_n)/4$. Finally, applying the frame property (6) to the function k_t, we see that the number of integers n such that $t \in G_n$ is uniformly bounded. □

Proof of Theorem 1.4.— We consider only the case of nontrivial perturbations, that is, $s_n \neq t_n$. Both (14) and (15) imply that there is a point $u_n \in \langle t_n, s_n\rangle$ such that $|s_n - t_n| \leqslant \varepsilon v_0(u_n)$. If $\varepsilon < 1/2$, then $v_0(u_n) \leqslant 4v_0(t_n)$. Hence, $s_n \in G_n$ if $\varepsilon < 1/8$.

Let us verify the estimate (9). Fix $p \in (1, 2)$. By (17),

$$\frac{1}{|\Theta'(t_n)|} \int\limits_{\langle t_n, s_n\rangle} \|k_t^2\|_q^{\frac{2p}{p+1}} dt \leqslant C \int\limits_{\langle t_n, s_n\rangle} |\Theta'(t)|^{-1} \max(d_0^{-2}(t), |\Theta'(t)|^2)dt$$

$$\leqslant C \int\limits_{\langle t_n, s_n\rangle} [|\Theta'(t)| + |\Theta'(t)|^{-1} d_0^{-2}(t)]\, dt < C\varepsilon.$$

An application of Theorem 1.1 completes the proof. □

Remarks. — 1. If Θ is a CLS inner function and $\{k_{t_n}/\|k_{t_n}\|_2\}$ is a frame, then it follows from (22) that $|\Theta'(t)|^{-1} \leqslant Cd_0(t)$ for $t \in G_n$. Thus, in this case (14) is equivalent to (4) and Theorem 1.4 coincides with the Cohn's theorem.

2. The estimates (14) - (15) imply that

$$|s_n - t_n| < \varepsilon v_0(t_n) = \varepsilon \min(d_0(t_n), |\Theta'(t_n)|^{-1}).$$

A question arises whether the property to be a frame or a basis is stable under such larger perturbations.

Now we produce examples which show that the natural analog of the Cohn's theorem may fail when the generating inner function is not CLS, that is, the Clark bases are not stable with respect to perturbations s_n such that

$$\sup_n |\varphi(s_n) - \varphi(t_n)| = \sup_n \int\limits_{<t_n, s_n>} |\Theta'(t)|dt < \varepsilon. \tag{24}$$

Consider the measure

$$\nu = \sum_n |\Theta'(s_n)|^{-1} \delta_{s_n}, \tag{25}$$

If $\{k_{s_n}/\|k_{s_n}\|_2\}$ is a Riesz basis or a frame, then the measure ν defines a norm on K^2_Θ equivalent to the usual L^2-norm. We show that even the estimate from above, which is equivalent to the embedding $K^2_\Theta \subset L^2(\nu)$, may not hold for perturbations of the form (24).

In these examples Θ is a meromorphic Blaschke product with very sparse zeros (geometrically it means that for any $\delta \in (0,1)$ there are components of $\Omega(\Theta, \delta)$ around infinitely many of the zeros).

Example 5.2. — Let Θ be the Blaschke product with the zeros $z_n = 2^n + i$, $n \in \mathbb{N}$, and let φ be an increasing branch of the argument of Θ on $\mathbb{R}$ such that $\lim\limits_{t \to -\infty} \varphi(t) = 0$. Put $s_n = 2^n + 2^{n-1}$. Then

$$\varphi'(s_n) = \sum_m \frac{1}{|2^n + 2^{n-1} - 2^m|^2} \asymp \frac{n}{2^{2n}}.$$

A simple calculation shows also that $\varphi(s_n) - 2\pi n = O(n2^{-n})$. Thus, $\{k_{s_n}\}$ is a perturbation of the Clark basis $\{k_{t_n}\}$, where $\varphi(t_n) = 2\pi n$.

Now we show that the embedding $K^2_\Theta \subset L^2(\nu)$, where ν is defined by (25), does not take place. Let $f(z) = (z - z_1)^{-1}$; then $|f(s_n)|^2/|\Theta'(s_n)| \asymp n^{-1}$. Thus, $f \notin L^2(\nu)$ and, consequently, $\{k_{s_n}/\|k_{s_n}\|_2\}$ is not a frame (the same statement follows from Lemma 4.3).

Example 5.3. — Let $z_n = n + iy_n$, $n \in \mathbb{Z}$, and $y_n \in (0,1)$. Fix $\varepsilon \in (0,1)$ and set $s_n = n + \varepsilon$. We assume that

$$\varphi'(s_n) = \sum_m \frac{2y_m}{|s_n - z_m|^2} \asymp \frac{y_n}{y_n^2 + \varepsilon^2} \asymp y_n, \tag{26}$$

that is, $\varphi'(s_n)$ is approximately equal to the summand corresponding to the nearest zero. Note that $\inf_n(\varphi(s_{n+1}) - \varphi(s_n)) > 0$. Hence, if the Clark bases are stable with respect to the perturbations (24), then for the measure ν defined by (25) we have the embedding $K^2_\Theta \subset L^2(\nu)$.

The zero set $\{z_m\}$ satisfies the Carleson interpolation condition. Hence, by the Shapiro-Shields theorem (see [24]), the system of functions $\frac{\sqrt{y_m}}{z - \overline{z}_m}$ is a Riesz basis in K^2_Θ. Thus, each function $f \in K^2_\Theta$ may be represented as an unconditionally convergent series

$$f(z) = \sum_m \frac{c_m \sqrt{y_m}}{z - \overline{z}_m}$$

and $\|f\|_2 \asymp \|\{c_m\}\|_{\ell^2}$. It is easy to see that under the condition (26) the embedding $K^2_\Theta \subset L^2(\nu)$ is equivalent to the boundedness in $\ell^2(\mathbb{Z})$ of the operator defined by the infinite matrix

$$a_{nm} = \sqrt{\frac{y_m}{y_n}} \cdot \frac{1}{n - m}, \qquad n \neq m.$$

It is well known that the boundedness of the discrete Hilbert transform in a weighted space $\ell^2(\{y_n\})$ is equivalent to the discrete Muckenhoupt's (A_2) condition [23]. One can easily construct a sequence $\{y_n\}$ satisfying (26), whereas the Muckenhoupt's condition is not satisfied (see [5] for details), and this gives us one more example of the situation when the Clark bases are not stable under perturbations small with respect to the change of the argument of Θ.

Acknowledgements. — The author is grateful to N.K. Nikolski for many valuable discussions and helpful suggestions. The work is partially supported by RFBR grant 03-01-00377 and by the grant NSH-2266.2003.1.

BIBLIOGRAPHY

[1] P. R. Ahern, D. N. Clark, Radial limits and invariant subspaces, Amer. J. Math., 92, 2 (1970), 332-342.

[2] A. B. Aleksandrov, Invariant subspaces of shift operators. An axiomatic approach, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 113 (1981), 7-26; English transl.: J. Soviet Math. 22 (1983), 1695-1708.

[3] A. B. Aleksandrov, A simple proof of the Volberg–Treil theorem on the embedding of coinvariant subspaces of the shift operator, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 217 (1994), 26-35; English transl.: J. Math. Sci., 85 (1997), 2, 1773-1778.

[4] A. B. Aleksandrov, Embedding theorems for coinvariant subspaces of the shift operator. II, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 262 (1999), 5-48; English transl.: J. Math. Sci., 110, (2002), 5, 2907-2929.

[5] A. D. Baranov, The Bernstein inequality in the de Branges spaces and embedding theorems, Proc. St. Petersburg Math. Soc., 9 (2001), 23-53; English transl.: Amer. Math. Soc. Transl. Ser. 2, 209 (2003), 21-49.

[6] A. D. Baranov, Weighted Bernstein-type inequalities and embedding theorems for shift-coinvariant subspaces, Algebra i Analiz, 15, 5 (2003), 138-168; English transl.: St. Petersburg Math. J., 15 (2004), 5, 733-752.

[7] A. D. Baranov, Bernstein-type inequalities for shift-coinvariant subspaces and their applications to Carleson embeddings, J. Funct. Anal., 223 (2005), 1, 116-146.

[8] I. Borichеva, Geometric properties of projections of reproducing kernels on z^*-invariant subspaces of H^2, J. Funct. Anal., 161 (1999), 2, 397-417.

[9] P. Borwein, T. Erdelyi, Sharp extensions of Bernstein's inequality to rational spaces, Mathematika, 43 (1996), 2, 413-423.

[10] L. De Branges, Hilbert spaces of entire functions, Prentice Hall, Englewood Cliffs (NJ), 1968.

[11] D. N. Clark, One-dimensional perturbations of restricted shifts, J. Anal. Math., 25 (1972), 169-191.

[12] W. S. Cohn, Radial limits and star invariant subspaces of bounded mean oscillation, Amer. J. Math., 108 (1986), 3, 719-749.

[13] W. S. Cohn, Carleson measures and operators on star-invariant subspaces, J. Oper. Theory, 15 (1986), 1, 181-202.

[14] W. S. Cohn, On fractional derivatives and star invariant subspaces, Michigan Math. J., 34 (1987), 3, 391-406.

[15] K. M. Dyakonov, Entire functions of exponential type and model subspaces in H^p, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 190 (1991), 81-100; English transl.: J. Math. Sci., 71 (1994), 1, 2222-2233.

[16] K. M. Dyakonov, Smooth functions in the range of a Hankel operator, Indiana Univ. Math. J., 43 (1994), 805-838.

[17] K. M. Dyakonov, Differentiation in star-invariant subspaces I, II, J. Funct. Anal., 192 (2002), 2, 364-409.

[18] E. Fricain, Bases of reproducing kernels in model spaces, J. Oper. Theory, 46 (2001), 3 (suppl.), 517-543.

[19] E. Fricain, Complétude des noyaux reproduisants dans les espaces modèles, Ann. Inst. Fourier (Grenoble), 52 (2002), 2, 661-686.

[20] S. V. Hruscev, N. K. Nikolskii, B. S. Pavlov, Unconditional bases of exponentials and of reproducing kernels, Lecture Notes in Math., 864 (1981), 214-335.

[21] M. I. Kadec, The exact value of the Paley–Wiener constant, Dokl. Akad. Nauk SSSR, 155 (1964), 1253-1254; English transl.: Sov. Math. Dokl., 5 (1964), 559-561.

[22] M. B. Levin, An estimate of the derivative of a meromorphic function on the boundary of domain, Sov. Math. Dokl., 15 (1974), 3, 831-834.

[23] Yu. I. Lyubarskii, K. Seip, Complete interpolating sequences for Paley–Wiener spaces and Muckenhoupt's (A_p) condition, Rev. Mat. Iberoamericana, 13 (1997), 2, 361-376.

[24] N. K. Nikolski, Treatise on the shift operator, Springer-Verlag, Berlin-Heidelberg, 1986.

[25] N. K. Nikolski, Operators, functions, and systems: an easy reading. Vol. 1. Hardy, Hankel, and Toeplitz, Mathematical Surveys and Monographs, 92, AMS, Providence, RI, 2002.

[26] N. K. Nikolski, Operators, functions, and systems: an easy reading. Vol. 2. Model operators and systems, Mathematical Surveys and Monographs, 93, AMS, Providence, RI, 2002.

[27] J. Ortega-Cerda, K. Seip, Fourier frames, Ann. of Math. (2), 155 (2002), 3, 789-806.

[28] K. Seip, On the connection between exponential bases and certain related sequences in $L^2(-\pi,\pi)$, J. Funct. Anal., 130 (1995), 1, 131-160.

[29] A. L. Volberg, S. R. Treil, Embedding theorems for invariant subspaces of the inverse shift operator, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 149 (1986), 38-51; English transl.: J. Soviet Math., 42 (1988), 2, 1562-1572.

[30] R. M. Young, An Introduction to Nonharmonic Fourier Series, Academic Press, New-York, 1980.

Manuscrit reçu le 30 avril 2004,
accepté le 3 février 2005.

Anton BARANOV,
St. Petersburg State University
Department of Mathematics and Mechanics
28, Universitetskii pr.
St. Petersburg 198504 (Russia)

Université Bordeaux 1
Laboratoire d'Analyse et Géométrie
351, Cours de la Libération
33405 Talence (France)

antonbaranov@netscape.net