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HOMOLOGY AND MODULAR CLASSES OF LIE ALGEBROIDS

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ABSTRACT. — For a Lie algebroid, divergences chosen in a classical way lead to a uniquely defined homology theory. They define also, in a natural way, modular classes of certain Lie algebroid morphisms. This approach, applied for the anchor map, recovers the concept of modular class due to S. Evens, J.-H. Lu, and A. Weinstein.

RÉSUMÉ. — Pour un algébroïde de Lie, le choix des divergences à la mode classique donne une théorie de l'homologie unique. Elles définissent aussi naturellement les classes modulaires de quelques morphismes des algébroïdes de Lie. Cette méthode, appliquée à l'application d'ancre, nous permet de retrouver la classe modulaire due à S. Evens, J.-H. Lu, et A. Weinstein.

1. Introduction

Homology of a Lie algebroid structure on a vector bundle E over M are usually considered as homology of the corresponding Batalin-Vilkovisky algebra associated with a chosen generating operator ∂ for the Schouten-Nijenhuis bracket on multisections of E. The generating operators that are homology operators, i.e. $\partial^2 = 0$, can be identified with flat E-connections on $\bigwedge^{\mathrm{top}} E$ (see [18]) or divergence operators (flat right E-connections on $M \times \mathbb{R}$, see [8]). The problem is that the homology group depends on the choice of the generating operator (flat connection, divergence) and no one seems to be privileged. For instance, if a Lie algebroid on T^*M associated

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with a Poisson tensor P on M is concerned, then the traditional Poisson homology is defined in terms of the Koszul-Brylinski homology operator $\partial_P = [d, i_P]$. However, the Poisson homology groups may differ from the homology groups obtained by means of 1-densities on M. The celebrated modular class of the Poisson structure [16] measures this difference. Analogous statement is valid for triangular Lie bialgebroids [10].

The concept of a Lie algebroid divergence, so a generating operator, associated with a 'volume form', i.e. nowhere-vanishing section of $\bigwedge^{\mathrm{top}} E^*$, is completely classical (see [10], [18]). Less-known seems to be the fact that we can use 'odd-forms' instead of forms (cf. [2]) with same formulas for divergence and that such nowhere-vanishing volume odd-forms always exist. The point is that the homology groups obtained in this way are all isomorphic, independently on the choice of the volume odd-form. This makes the homology of a Lie algebroid a well-defined notion. From this point of view the Poisson homology is not the homology of the associated Lie algebroid T^*M but a deformed version of the latter, exactly as the exterior differential $d^\phi\mu = d\mu + \phi\wedge\mu$ of Witten [17] is a deformation of the standard de Rham differential.

In this language, the modular class of a Lie algebroid morphism $\kappa : E_1 \to E_2$ covering the identity on M is defined as the class of the difference between the pull-back of a divergence on E_2 and a divergence on E_1, both associated with volume odd-forms. In the case when $\kappa : E \to TM$ is the anchor map, we recognize the standard modular class of a Lie algebroid [3] but it is clear that other (canonical) morphisms will lead to other (canonical) modular classes.

2. Divergences and generating operators

2.1. Lie algebroids and their cohomology

Let $\tau : E \to M$ be a vector bundle. Let $\mathcal{A}^i(E) = \mathrm{Sec}(\bigwedge^i E)$ for $i = 0, 1, 2, \dots$, let $A^i(E) = \{0\}$ for $i < 0$, and denote by $\mathcal{A}(E) = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}^i(E)$ the Grassmann algebra of multisections of E. It is a graded commutative associative algebra with respect to the wedge product.

There are different ways to define a Lie algebroid structure on E. We prefer to see it as a linear graded Poisson structure on $\mathcal{A}(E)$ (see [7]), i.e., a graded bilinear operation $[\cdot,\cdot]$ on $\mathcal{A}(E)$ of degree -1 with the following properties:

(a) Graded anticommutativity:
$$[a,b] = -(-1)^{(|a|-1)(|b|-1)}\,[b,a].$$
(b) The graded Jacobi identity:
$$\big[a,[b,c]\big] = \big[[a,b],c\big] + (-1)^{(|a|-1)(|b|-1)}\,\big[b,[a,c]\big].$$
(c) The graded Leibniz rule:
$$[a, b\wedge c] = [a,b]\wedge c + (-1)^{(|a|-1)|b|}\, b\wedge [a,c].$$

This bracket is just the Schouten bracket associated with the the standard Lie algebroid bracket on sections of E. It is well known that such brackets are in bijective correspondence with de Rham differentials d on the Grassmann algebra $\mathcal{A}(E^*)$ of multisections of the dual bundle E^* which are described by the formula

$$\begin{aligned}(2.1)\quad d\mu(X_0,\dots,X_n) = {}&\sum_i (-1)^i\big[X_i, \mu(X_0,\dots\hat{\imath}\dots,X_n)\big]\\ &+\sum_{k<l}(-1)^{k+l}\mu\big([X_k,X_l],X_0,\dots\hat{k}\dots\hat{l}\dots,X_n\big)\end{aligned}$$

where the X_i are sections of E. We will refer to elements of $\mathcal{A}(E^*)$ as *forms*. Since d is a derivation on $\mathcal{A}(E^*)$ of degree 1 with $d^2 = 0$, it defines the corresponding de Rham cohomology $H^*(E,d)$ of the Lie algebroid in the obvious way.

2.2. Generating operators and divergences

The definition of the homology of a Lie algebroid is more delicate than that of cohomology. The standard approach is via generating operators for the Schouten bracket $[\cdot,\cdot]$. By this we mean an operator ∂ of degree -1 on $\mathcal{A}(E)$ which satisfies

$$(2.2)\qquad [a,b] = (-1)^{|a|}\big(\partial(a\wedge b) - \partial(a)\wedge b - (-1)^{|a|}a\wedge\partial(b)\big).$$

The idea of a generating operator goes back to the work by Koszul [13]. A generating operator which is a homology operator, i.e. $\partial^2 = 0$, gives rise to the so called Batalin-Vilkovisky algebra. Remark that the leading sign $(-1)^{|a|}$ serves to produce graded antisymmetry with respects to the degrees shifted by -1 out of graded symmetry. One could equally well use $(-1)^{|b|}$ instead of $(-1)^{|a|}$, or one could use the obstruction for ∂ to be a graded right derivation in the parentheses instead of a graded left one as we did. We shall stick to the standard conventions.

It is clear from Eq. (2.2) and from the properties of the Schouten bracket that ∂ is then a second order differential operator on the graded commutative associative algebra $\mathcal{A}(E)$, which is completely determined by its restriction to $\mathrm{Sec}(E)$. In fact, it is easy to see (cf. [8]) that

$$(2.3)\quad \partial(X_1 \wedge \cdots \wedge X_n) = \sum_i (-1)^{i+1}\partial(X_i) X_1 \wedge \dots \hat{i} \cdots \wedge X_n \\ + \sum_{k<l} (-1)^{k+l} [X_k, X_l] \wedge X_1 \wedge \dots \hat{k} \dots \hat{l} \cdots \wedge X_n$$

for $X_1, \dots, X_n \in \mathrm{Sec}(E)$, which looks completely dual to Eq. (2.1). From Eq. (2.2) we get the following property of ∂:

$$(2.4)\qquad -\partial(fX) = -f\partial(X) + [X, f] \text{ for } X \in \mathrm{Sec}(E), f \in C^\infty(M).$$

Since $[X, f] = \rho(X)(f)$, where $\rho : E \to TM$ is the anchor map of the Lie algebroid structure on E, the operator $-\partial$ has the algebraic property of a divergence. Conversely, Eq. (2.3) defines a generating operator for $[\cdot,\cdot]$ if only Eq. (2.4) is satisfied, i.e., generating operators can be identified with divergences. We may express this by $\mathrm{div} \leftrightarrow \partial_{\mathrm{div}}$. But a true divergence $\mathrm{div} : \mathrm{Sec}(E) \to C^\infty(M)$ satisfies besides Eq. (2.4) a cocycle condition

$$(2.5)\qquad \mathrm{div}([X, Y]) = [\mathrm{div}(X), Y] + [X, \mathrm{div}(Y)], \quad X, Y \in \mathrm{Sec}(E),$$

which is equivalent (see [8]) to the fact that the corresponding generating operator ∂_{div} is a homology operator: $(\partial_{\mathrm{div}})^2 = 0$. Note that divergences can be used in construction of generating operators also in the supersymmetric case (cf. [12]).

From now on we will fix the Lie algebroid structure on E, and we will denote by $\mathrm{Gen}(E)$ the set of generating operators for $[\cdot,\cdot]$ which are homology operators, and by $\mathrm{Div}(E)$ the canonically isomorphic (by Eq. (2.3)) set of divergences for the Lie algebroid satisfying Eq. (2.4) and Eq. (2.5). The problem is that there does not exist a canonical divergence, thus no canonical generating operator.

The set $\mathrm{Div}(E)$ can be identified with the set of all flat E-connections on $\bigwedge^{\mathrm{top}} E^*$, i.e., operators $\nabla : \mathrm{Sec}(E) \times \mathrm{Sec}(\bigwedge^{\mathrm{top}}(E^*)) \to \mathrm{Sec}(\bigwedge^{\mathrm{top}}(E^*))$ which satisfy

(i) $\nabla_{fX}\mu = f\nabla_X\mu$,
(ii) $\nabla_X(f\mu) = f\nabla_X\mu + \rho(X)(f)\mu$,
(iii) $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]}$.

The identification is via

$$(2.6)\qquad \mathcal{L}_X\mu - \nabla_X\mu = \mathrm{div}(X)\mu$$

(cf. [10, (50)]), where $\mathcal{L}_X = d\,i_X + i_X\,d$ is the Lie derivative. Note that Eq. (2.6) is independent of the choice of the section $\mu \in \mathrm{Sec}(\bigwedge^{\mathrm{top}}(E^*))$. We can use $\bigwedge^{\mathrm{top}}(E)$ instead of $\bigwedge^{\mathrm{top}}(E^*)$ and get the identification of $\mathrm{Div}(E)$ with the set of flat E-connection on $\bigwedge^{\mathrm{top}}(E)$ by (see [18])

$$\mathcal{L}_X\Lambda - \nabla_X\Lambda = \mathrm{div}(X)\Lambda. \tag{2.7}$$

Of course, additional structures on E as, e.g., a Riemannian metric (smoothly arranged scalar products on fibers of E), may furnish a distinguished divergence on E. Fixing a metric we can distinguish a canonical torsionfree connection ∇ on E—the Levi-Civita connection for the Lie algebroid—in the standard way. It satisfies the standard Bianchi and Ricci identities (see [15]) and induces a connection on $\bigwedge^{\mathrm{top}}(E)$ for which the generating operator ∂_∇ has the local form (see [18]) $\partial_\nabla(a) = -\sum_k i(\alpha^k)\nabla_{X_k}a$, where the X_k and α^k are dual local frames for E and E^*, respectively. Since

$$\partial_\nabla^2 = \sum_{k,j} i(\alpha^j)\nabla_{X_j} i(\alpha^k)\nabla_{X_k} = \sum_{k,j} i(\alpha^j)i(\alpha^k)(\nabla_{X_j}\nabla_{X_k} - \nabla_{\nabla_{X_j}X_k}),$$

$\partial_\nabla^2 = 0$ is equivalent to

$$\sum_{j,k} i(\alpha^j)i(\alpha^k)R(X_j,X_k) = 0, \tag{2.8}$$

where R is the curvature tensor of ∇. For a Levi-Civita connection ∇ the generating operator ∂_∇ is really a homology operator due to the following lemma.

Lemma 2.1. — *A torsionfree connection ∇ on E satisfies simultaneously the Bianchi and the Ricci identity if and only if Eq.* (2.8) *holds for dual local frames X_k and α^k of E and E^*, respectively.*

Proof. — Eq. (2.8) is equivalent to $\sum_{j,k} R(X_j,X_k)^*(\alpha^k \wedge \alpha^j \wedge \omega) = 0$ for all forms ω. It suffices to check this for ω a function or a 1-form due to the derivation property of contractions. For ω a function f we have

$$\begin{aligned}
&\sum_{j,k} R(X_j,X_k)^*(f\alpha^k \wedge \alpha^j) = \\
&\qquad = \sum_{j,k} f\Big(R(X_j,X_k)^*(\alpha^k) \wedge \alpha^j + \alpha^k \wedge R(X_j,X_k)^*(\alpha^j)\Big) \\
&\qquad = 2f \sum_{s,j,k} R^k_{jks}\alpha^s \wedge \alpha^j
\end{aligned}$$

and this vanishes for all f if and only if R^k_{jks} is symmetric in (j,s), i.e., if the Ricci identity holds. For ω a 1-form, say α^i, we have

$$\begin{aligned}\sum_{j,k} R(X_j,X_k)^*(\alpha^k\wedge\alpha^j\wedge\alpha^i) = \\ &= \sum_{j,k}\Big(R(X_j,X_k)^*(\alpha^k\wedge\alpha^j)\wedge\alpha^i+\alpha^k\wedge\alpha^j\wedge R(X_j,X_k)^*(\alpha^i)\Big)\\ &= 0+\sum_{j,k,s} R^i_{jks}\alpha^k\wedge\alpha^j\wedge\alpha^s\end{aligned}$$

and this vanishes for all i if and only if $\sum_{\mathrm{cycl}(j,k,s)} R^i_{jks}=0$, i.e., if the first Bianchi identity holds.

Corollary 2.2. — *Any Levi-Civita connection for a Riemannian metric on a Lie algebroid E induces a flat connection on $\bigwedge^{\mathrm{top}} E$, thus also on $\bigwedge^{\mathrm{top}} E^*$.*

3. Homology of the Lie algebroid

3.1. Getting divergences from odd forms

There is no distinguished divergence for the Lie algebroid structure on E, but there is a distinguished subset of divergences which we may obtain in a classical way. Firstly, suppose that the line bundle $\bigwedge^{\mathrm{top}} E^*$ is trivializable. So we can choose a vector volume, i.e., a nowhere vanishing section $\mu\in\mathrm{Sec}(\bigwedge^{\mathrm{top}} E^*)$. Then the formula

$$\mathcal{L}_X\mu=\mathrm{div}_\mu(X)\mu,\ \text{ where } X\in\mathrm{Sec}(E), \tag{3.1}$$

defines a divergence div_μ. We observe that $\mathrm{div}_{-\mu}=\mathrm{div}_\mu$. Thus for the non-orientable case we look for sections of a bundle over M which locally consists of non-ordered pairs $\{\mu_\alpha,-\mu_\alpha\}$ for an open cover $M=\bigcup_\alpha U_\alpha$ such that the sets $\{\mu_\alpha,-\mu_\alpha\}$ and $\{\mu_\beta,-\mu_\beta\}$ coincide when restricted to $U_\alpha\cap U_\beta$. The fundamental observation is that such global sections always exist and define global divergences. This is because they can be viewed as sections of the bundle $|\mathrm{Vol}|_E=(\bigwedge^{\mathrm{top}} E^*)_0/\mathbb{Z}_2$, where $(\bigwedge^{\mathrm{top}} E^*)_0/\mathbb{Z}_2$ is the bundle $\bigwedge^{\mathrm{top}} E^*$ with the zero section removed and divided by the obvious $\mathbb{Z}_2$-action of passing to the opposite vector. The bundle $|\mathrm{Vol}|_E$ is a 1-dimensional affine bundle modelled on the vector bundle $M\times\mathbb{R}$, and also a principal $\mathbb{R}$-bundle where $t\in\mathbb{R}$ acts by scalar multiplication with e^t. Since it has a contractible fiber, sections always exist. Note that sections

$|\mu|$ of $|\operatorname{Vol}|_E$ are particular cases of odd forms, [2]: Let $p : \widetilde{M} \to M$ be the two-fold covering of M on which p^*E is oriented, namely the set of vectors of length 1 in the line bundle over M with cocycle of transition functions $\operatorname{sign}\det(\phi_{\alpha\beta})$, where $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$ is the cocycle of transition functions for the vector bundle E. Then the odd forms are those forms on p^*E which are in the -1 eigenspace of the natural vector bundle isomorphism which covers the decktransformation of $\widetilde{M}$. So odd forms are certain sections of a line bundle over a two-fold covering of the base manifold M. This is related but complementary to the construction of the line bundle (over M) of densities which involve the cocycle of transition functions $|\det(\phi_{\alpha\beta})|$. For example, any Riemannian metric g on the vector bundle E induces an odd volume form $|\mu|_g \in Sec(|\operatorname{Vol}|_E) \simeq Sec(|\operatorname{Vol}|_{E^*})$ which locally is represented by the wedge product of any orthonormal basis of local sections of E (thus E^*). Note that such product is independent on the choice of the basis modulo sign, so our odd volume is well defined.

For the definition of a divergence $\operatorname{div}_{|\mu|}$ associated to $|\mu| \in \operatorname{Sec}(|\operatorname{Vol}|_E)$ we will write simply

$$\mathcal{L}_X|\mu| = \operatorname{div}_{|\mu|}(X)|\mu| \text{ for } X \in \operatorname{Sec}(E). \tag{3.2}$$

Note that the distinguished set $\operatorname{Div}_0(E)$ of divergences obtained in this way from sections of $|\operatorname{Vol}|_E$ corresponds (in the sense of Eq. (2.6)) to the set of those flat connections on $\bigwedge^{\text{top}} E^*$ whose holonomy group equals $\mathbb{Z}_2$: Associate the horizontal leaf $|\mu|$ to such a connection, and note that a positive multiple of $|\mu|$ gives rise to the same divergence.

In the case of a vector bundle Riemannian metric g on E a natural question arises about the relation between the divergence $\operatorname{div}_{|\mu|_g}$ associated with the odd volume $|\mu|_g$ induced by the metric g and the divergence $\operatorname{div}_{\nabla_g}$ induced by the flat Levi-Civita connection ∇_g on $\bigwedge^{\text{top}} E^* \simeq \bigwedge^{\text{top}} E$.

Theorem 3.1. — *For any vector bundle Riemannian metric g on E*

$$\operatorname{div}_{|\mu|_g} = \operatorname{div}_{\nabla_g}.$$

Proof. — Let $X_1, \dots, X_n$ be an orthonormal basis of local sections of E and $\alpha^k = g(X_k, \cdot)$ be the dual basis of local sections of E^*, so that $|\mu|_g$ is locally represented by $\alpha^1 \wedge \cdots \wedge \alpha^n$.

For any local section X of E

$$\begin{aligned}\operatorname{div}_{|\mu|_g}(X) &= -\langle \mathcal{L}_X(\alpha^1 \wedge \cdots \wedge \alpha^n), X_1 \wedge \cdots \wedge X_n\rangle \\ &= \langle \alpha^1 \wedge \cdots \wedge \alpha^n, \mathcal{L}_X(X_1 \wedge \cdots \wedge X_n)\rangle \\ &= \sum_k \langle \alpha^k, [X, X_k]\rangle \\ &= \sum_k \langle \alpha^k, \nabla_X X_k - \nabla_{X_k} X\rangle \\ &= \sum_k g(X_k, \nabla_X X_k) - \sum_k i(\alpha^k)\nabla_{X_k} X.\end{aligned}$$

But $-\sum_k i(\alpha^k)\nabla_{X_k} X = \operatorname{div}_{\nabla_g}(X)$ and

$$2\sum_k g(X_k, \nabla_X X_k) = \sum_k \rho(X) g(X_k, X_k) - \sum_k \nabla_X(g)(X_k, X_k) = 0,$$

where $\rho : E \to TM$ is the anchor of the Lie algebroid on E, since ∇ is Levi-Civity ($\nabla g = 0$).

3.2. The generating operator for an odd form

The corresponding generating operator $\partial_{|\mu|}$ for the divergence of a non-vanishing odd form $|\mu|$ can be defined explicitly by

$$\mathcal{L}_a|\mu| = -i(\partial_{|\mu|}(a))|\mu|,$$

where $\mathcal{L}_a = i_a\, d - (-1)^{|a|} d\, i_a$ is the Lie differential associated with $a \in \mathcal{A}^{|a|}(E)$ so that

$$i(\partial_{|\mu|}(a))|\mu| = (-1)^{|a|} d\, i_a |\mu|. \tag{3.3}$$

In other words, locally over U we have

$$\partial_{|\mu|}(a) = (-1)^{|a|} *_\mu^{-1}\ d\ *_\mu (a), \tag{3.4}$$

where $*_\mu$ is the isomorphism of $\mathcal{A}(E)|_U$ and $\mathcal{A}(E^*)|_U$ given by $*_\mu(a) = i_a\mu$, for a representative μ of $|\mu|$. Note that the right hand side of Eq. (3.4) depends only on $|\mu|$ and not on the choice of the representative, since $*_\mu\ d\ *_\mu = *_{-\mu}\ d\ *_{-\mu}$. Formula Eq. (3.4) gives immediately $\partial^2_{|\mu|} = 0$, which also follows from the remark on flat connections above. So $\partial_{|\mu|}$ is a homology operator.

Moreover, it is also a generating operator. Namely, using standard calculus of Lie derivatives we get

$$\mathcal{L}_{a\wedge b} = i_b\, \mathcal{L}_a - (-1)^{|a|} i_{[a,b]} + (-1)^{|a|\,|b|} i_a\, \mathcal{L}_b$$

which can be rewritten in the form

$$i_{[a,b]} = (-1)^{|a|}\Big(-\mathcal{L}_{a\wedge b} + i_b\,\mathcal{L}_a + (-1)^{|a|(|b|+1)} i_a\,\mathcal{L}_b\Big). \tag{3.5}$$

When we apply Eq. (3.5) to $|\mu|$ we get

$$i_{[a,b]}|\mu| = (-1)^{|a|}\Big(i(\partial_{|\mu|}(a\wedge b)) - i(\partial_{|\mu|}(a)\wedge b) - (-1)^{|a|} i(a\wedge \partial_{|\mu|}(b))\Big)|\mu|$$

which proves Eq. (2.2). Thus we get:

Theorem 3.2. — *For any $|\mu| \in \mathrm{Sec}(|\mathrm{Vol}|_E)$ the formula*

$$\mathcal{L}_a|\mu| = -i(\partial_{|\mu|}(a))|\mu| \tag{3.6}$$

defines uniquely a generating operator $\partial_{|\mu|} \in \mathrm{Gen}(E)$.

We remark that formula Eq. (3.6) in the case of trivializable $\bigwedge^{\mathrm{top}} E^*$ has been already found in [10]. In this sense the formula is well known. What is stated in Theorem 3.2 is that Eq. (3.6) serves in general, as if the bundle $\bigwedge^{\mathrm{top}} E^*$ were trivial, if we replace ordinary forms with odd volume forms.

3.3. Homology of the Lie algebroid

The homology operator of the form $\partial_{|\mu|}$ will be called the homology operator for the Lie algebroid E. The crucial point is that they all define the same homology. This is due to the fact that $\partial_{|\mu_1|}$ and $\partial_{|\mu_2|}$ differ by contraction with an exact 1-form.

In general, two divergences differ by contraction with a closed 1-form. Indeed, $(\mathrm{div}_1 - \mathrm{div}_2)(fX) = f(\mathrm{div}_1 - \mathrm{div}_2)(X)$, so $(\mathrm{div}_1 - \mathrm{div}_2)(X) = i_\phi X$ for a unique 1-form ϕ. Moreover, Eq. (2.5) implies that $i_\phi[X,Y] = [i_\phi X, Y] + [X, i_\phi Y]$, so ϕ is closed. Since both sides are derivations we have

$$\partial_{\mathrm{div}_2} - \partial_{\mathrm{div}_1} = i_\phi. \tag{3.7}$$

But for any $|\mu_1|, |\mu_2| \in \mathrm{Sec}(|\mathrm{Vol}|_E)$ there exists a positive function $F = e^f$ such that $|\mu_2| = F\,|\mu_1|$. Then

$$\mathcal{L}_X|\mu_2| = \mathcal{L}_X(F\,|\mu_1|) = \mathcal{L}_X(F)\,|\mu_1| + F\,\mathcal{L}_X(|\mu_1|)$$

so that

$$\mathrm{div}_{|\mu_2|}(X)|\mu_2| = \mathcal{L}_X(f)\,|\mu_2| + \mathrm{div}_{|\mu_1|}(X)|\mu_2|,$$

i.e.,

$$\mathrm{div}_{|\mu_2|} - \mathrm{div}_{|\mu_1|} = i(df).$$

To see that the homology of $\partial_{|\mu_1|}$ and $\partial_{|\mu_2|}$ are the same, note first that $\partial_{|\mu_2|} = \partial_{|\mu_1|}a + i_{df}a$. And then let us gauge $\mathcal{A}(E)$ by multiplication with $F = e^f$. This is an isomorphism of graded vector spaces and we have

$$e^f\, \partial_{|\mu_1|}\, e^{-f}\, a = \partial_{|\mu_1|}a + i_{df}a = \partial_{|\mu_2|}a,$$

so $\partial_{|\mu_1|}$ and $\partial_{|\mu_2|}$ are graded conjugate operators.

This is just the dual picture of the well-known gauging of the de Rham differential by Witten [17], see also [7] for consequences in the theory of Lie algebroids. Thus we have proved (cf. [10, p.120]):

Theorem 3.3. — *All homology operators for a Lie algebroid generate the the same homology: $H_*(E, \partial_{|\mu_1|}) = H_*(E, \partial_{|\mu_2|})$. In the case of trivializable $\bigwedge^{\mathrm{top}} E^*$, Eq.* (3.4) *gives Poincaré duality*

$$H^*(E, d) \cong H_{\text{top-}*}(E, \partial_{|\mu|}).$$

3.4. Remark

We got a well-defined Lie algebroid homology, in contrast with the standard approach when all generating operators are admitted. It is clear that adding a term i_ϕ with ϕ a closed 1-from which is not exact, as in Eq. (3.7), will probably change the homology. But this could be understood as an a priori deformation, like in the case of the deformed de Rham differential of Witten [17]:

$$d^\phi \eta = d\eta + \phi \wedge \eta. \tag{3.8}$$

Indeed, $i(i_\phi a)\mu = -(-1)^{|a|}\phi \wedge i_a\mu$ implies $*_\mu\, i_\phi(a) = -(-1)^{|a|}\, e_\phi\, *_\mu\, (a)$, where $e_\phi\eta = \phi \wedge \eta$. Thus we get $(-1)^{|a|}\, *_\mu^{-1}\, (d + e_\phi)\, *_\mu\, (a) = (\partial_{|\mu|} - i_\phi)(a)$, so, at least in the the trivializable case, there is the Poincaré duality

$$H^*(E, d + e_\phi) \cong H_{\text{top}-*}(E, \partial_\mu - i_\phi).$$

Note that the differentials d^ϕ appear as part of the Cartan differential calculus for Jacobi algebroids, see [9], [6], [7], so that there is a relation between generating operators for a Lie algebroid and the Jacobi algebroid structures associated with it.

4. Modular classes

4.1. The modular class of a morphism

As we have shown, every Lie algebroid E has a distinguished class $\mathrm{Div}_0(E)$ of divergences obtained from sections of $|\operatorname{Vol}|_E$. Such divergences differ by contraction with an exact 1-form. Let now $\kappa : E_1 \to E_2$ be a morphism of Lie algebroids.

There is the induced map $\kappa^* : \mathrm{Div}(E_2) \to \mathrm{Div}(E_1)$ defined by

$$\kappa^*(\mathrm{div}_2)(X_1) = \mathrm{div}_2(\kappa(X_1)).$$

The fact that κ^* maps divergences into divergences follows from $\kappa(fX) = f\kappa(X)$ and the fact that the Lie algebroid morphism respects the anchors, $\rho_1 = \rho_2 \circ \kappa$. The space $\kappa^*(\mathrm{Div}_0(E_2)) \subset \mathrm{Div}(E_1)$ consists of divergences which differ by insertion of an exact 1-form. Therefore, the cohomology class of the 1-form ϕ which is defined by the equation

$$\kappa^*(\mathrm{div}_{E_2}) - \mathrm{div}_{E_1} = i_\phi, \text{ for } \mathrm{div}_{E_i} \in \mathrm{Div}_0(E_i), \quad i = 1, 2, \tag{4.1}$$

does not depend on the choice of div_{E_1} and div_{E_2}. We will call it the *modular class* of κ and denote it by $\mathrm{Mod}(\kappa)$. Thus we have:

Theorem 4.1. — *For every Lie algebroid morphism*

$$\begin{tikzcd} E_1 \arrow[rr, "\kappa"] \arrow[dr, "\tau_1"'] & & E_2 \arrow[dl, "\tau_2"] \\ & M & \end{tikzcd}$$

the cohomology class $\mathrm{Mod}(\kappa) = [\phi] \in H^1(E_1, d_{E_1})$ *defined by* ϕ *in Eq.* (4.1) *is well defined independently of the choice of* $\mathrm{div}_{E_1} \in \mathrm{Div}_0(E_1)$ *and* $\mathrm{div}_{E_2} \in \mathrm{Div}_0(E_2)$.

4.2. The modular class of a Lie algebroid

In the case when the morphism $\kappa = \rho : E \to TM$ is the anchor map of a Lie algebroid E, the modular class $\mathrm{Mod}(\rho)$ is called the *modular class of the Lie algebroid* E and it is denoted by $\mathrm{Mod}(E)$. The idea that the modular class is associated with the difference between the Lie derivative action on $\bigwedge^{\mathrm{top}}(E^*)$ and on $\bigwedge^{\mathrm{top}} T^*M$ via the anchor map is, in fact, already present in [3]. Also the interpretation of the modular class as certain secondary characteristic class of a Lie algebroid, present in [4], is a quite similar. In [4] the trace of the difference of some connections is used instead of the difference of two divergences. We have

Theorem 4.2. — $\mathrm{Mod}(E)$ *is the modular class* Θ_E *in the sense of* [3].

Proof. — The modular class Θ_E in the sense of [3] is defined as the class $[\phi]$ where ϕ is given by

$$\mathcal{L}_X(a) \otimes \mu + a \otimes \mathcal{L}_{\rho(X)}\mu = \langle X, \phi\rangle a \otimes \mu \tag{4.2}$$

for all sections a of $\bigwedge^{\mathrm{top}}(E)$ and μ of $\bigwedge^{\mathrm{top}}(T^*M)$, respectively. Let us take $|a^*| \in \mathrm{Sec}(|\operatorname{Vol}|_E)$ and $|\mu| \in \mathrm{Sec}(|\operatorname{Vol}|_{TM})$, locally represented by $a^* \in \mathrm{Sec}(\bigwedge^{\mathrm{top}}(E^*|_U))$ and $\mu \in \mathrm{Sec}(\bigwedge^{\mathrm{top}}(T^*M|_U))$. Let a be a local section of $\bigwedge^{\mathrm{top}} E$ dual to a^*. Then $\mathcal{L}_X(a) = -\operatorname{div}_{|a^*|}(X)\, a$ and $\mathcal{L}_X(\mu) = \rho^*(\operatorname{div}_{|\mu|})(X)\, \mu$ so that Eq. (4.2) yields $i_\phi = \rho^*(\operatorname{div}_{|\mu|}) - \operatorname{div}_{|a^*|}$.

Note that in our approach the modular class $\mathrm{Mod}(TM)$ of the canonical Lie algebroid TM is trivial by definition. It is easy to see that the modular class of a base preserving morphism can be expressed in terms of the modular classes of the corresponding Lie algebroids.

Theorem 4.3. — *For a base preserving morphism $\kappa : E_1 \to E_2$ of Lie algebroids*

$$\mathrm{Mod}(\kappa) = \mathrm{Mod}(E_1) - \kappa^*(\mathrm{Mod}(E_2)).$$

Proof. — Let $\rho_l : E_l \to TM$ be the anchor of E_l, $l = 1, 2$. Take $\operatorname{div}_{E_l} \in \mathrm{Div}_0(E_l)$, $l = 1, 2$, and $\operatorname{div}_{TM} \in \mathrm{Div}_0(TM)$. Since $\mathrm{Mod}(E_l)$ is represented by η_l, $i_{\eta_l} = \operatorname{div}_{E_l} - \rho_l^*(\operatorname{div}_{TM})$ and $\rho_1 = \rho_2 \circ \kappa$, we can write

$$\begin{aligned} i_{\eta_1} &= \operatorname{div}_{E_1} - \rho_1^*(\operatorname{div}_{TM}) \\ &= \operatorname{div}_{E_1} - \kappa^*(\operatorname{div}_{E_2}) + \kappa^*(\operatorname{div}_{E_2} - \rho_2^*(\operatorname{div}_{TM})) \\ &= i_{\eta_\kappa} + i_{\kappa^*(\eta_2)}, \end{aligned}$$

where η_κ represents $\mathrm{Mod}(\kappa)$. Thus $\eta_1 = \eta_\kappa + \eta_2$.

4.3. The universal Lie algebroid

For any vector bundle $\tau : E \to M$ there exists a universal Lie algebroid $\mathrm{QD}(E)$ whose sections are the quasi-derivations on E, i.e., mappings $D : \mathrm{Sec}(E) \to \mathrm{Sec}(E)$ such that $D(fX) = f\,D(X) + \hat{D}(f)\,X$ for $f \in C^\infty(M)$ and $X \in \mathrm{Sec}(E)$, where $\hat{D}$ is a vector field on M; see the survey article [5]. Quasi-derivations are known in the literature under various names: covariant differential operators [14], module derivations [15], derivative endomorphisms [11], etc. The Lie algebroid $\mathrm{QD}(E)$ can be described as the Atiyah algebroid associated with the principal $GL(n,\mathbb{R})$-bundle $\mathrm{Fr}(E)$ of frames in E, and quasi-derivations can be identified with the $GL(n,\mathbb{R})$-invariant vector fields on $\mathrm{Fr}(E)$. The corresponding short exact Atiyah sequence in this case is

$$0 \to \mathrm{End}(E) \to \mathrm{QD}(E) \to TM \to 0.$$

This observation shows that there is a modular class associated to every vector bundle E, namely the modular class $\mathrm{Mod}(\mathrm{QD}(E))$, which is a vector bundle invariant.

It is also obvious that, viewing a flat E_0-connection (representation) in a vector bundle E over M for a Lie algebroid E_0 over M as a Lie algebroid morphism $\nabla : E_0 \to QD(E)$, one can define the modular class $\mathrm{Mod}(\nabla)$.

QUESTION. — *How is* $\mathrm{Mod}(\mathrm{QD}(E))$ *related to other invariants of* E *(e.g. characteristic classes)?*

4.4. Remark

One can interpret the modular class $\mathrm{Mod}(E)$ of the Lie algebroid E as a "trace" of the adjoint representation. Indeed, if we fix local coordinates u^a on $U \subset M$ a local frame X_i of local sections of E over U, and the dual frame α^i of E^*, then the Lie algebroid structure is encoded in the "structure functions"

$$[X_i, X_j] = \sum_k c_{ij}^k \, X_k, \qquad \rho(X_i) = \sum_a \rho_i^a \, \partial_{u^a}.$$

PROPOSITION. — *The modular class* $\mathrm{Mod}(E)$ *is locally represented by the closed 1-form*

$$\phi = \sum_i \Big(\sum_k c_{ik}^k + \sum_a \frac{\partial \rho_i^a}{\partial u^a} \Big) \alpha^i. \tag{4.3}$$

Proof. — We insert into Eq. (4.2) the elements $a = X_1 \wedge \cdots \wedge X_n$ and $\mu = du^1 \wedge \cdots \wedge du^m$. Since

$$\mathcal{L}_{X_i} a = \sum_k c_{ik}^k a \text{ and } \mathcal{L}_{X_i} \mu = \sum_a \frac{\partial \rho_i^a}{\partial u^a} \mu,$$

we get

$$\langle X_i, \phi \rangle a \otimes \mu = \Big(\sum_k c_{ik}^k + \sum_a \frac{\partial \rho_i^a}{\partial u^a} \Big) a \otimes \mu.$$

One could say that representing cohomology locally does not make much sense, e.g. the modular class $\mathrm{Mod}(TM)$ is trivial so locally trivial. However, remember that for a general Lie algebroid the Poincaré lemma does not hold: closed forms need not be locally exact. In particular, for a Lie algebra (with structure constants), Eq. (4.3) says that the modular class is just the trace of the adjoint representation. In any case, Eq. (4.3) gives us a closed form, which is not obvious on first sight. If E is a trivial bundle, Eq. (4.3) gives us a globally defined modular class in local coordinates.

4.5. Remark

As we have already mentioned, the modular class of a Lie algebroid is the first characteristic class of R. L. Fernandes [4]. There are also higher classes, shown in [1] to be characteristic classes of the anchor map, interpreted as a representation "up to homotopy". It is interesting if our idea can be adapted to describe these higher characteristic classes as well.

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