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Polyhedral realisation of hyperbolic metrics with conical singularities on compact surfaces

POLYHEDRAL REALISATION OF HYPERBOLIC METRICS WITH CONICAL SINGULARITIES ON COMPACT SURFACES

by François FILLASTRE

ABSTRACT. — A Fuchsian polyhedron in hyperbolic space is a polyhedral surface invariant under the action of a Fuchsian group of isometries (i.e. a group of isometries leaving globally invariant a totally geodesic surface, on which it acts cocompactly). The induced metric on a convex Fuchsian polyhedron is isometric to a hyperbolic metric with conical singularities of positive singular curvature on a compact surface of genus greater than one. We prove that these metrics are actually realised by exactly one convex Fuchsian polyhedron (up to global isometries). This extends a famous theorem of A.D. Alexandrov.

RÉSUMÉ. — Un polyèdre fuchsien de l'espace hyperbolique est une surface polyédrale invariante sous l'action d'un groupe fuchsien d'isométries (c.a.d. un groupe d'isométries qui laissent globalement invariante une surface totalement géodésique et sur laquelle il agit de manière cocompacte). La métrique induite sur un polyèdre fuchsien convexe est isométrique à une métrique hyperbolique avec des singularités coniques de courbure singulière positive sur une surface compacte de genre plus grand qu'un. On démontre que ces métriques sont en fait réalisées par un unique polyèdre fuchsien convexe (modulo les isométries globales). Ce résultat étend un théorème célèbre de A.D. Alexandrov.

1. Definitions and statements

Metrics with conical singularities and convex polyhedra. Let M_K^+ be the simply connected (Riemannian) space of dimension 3 of constant curvature K, $K \in \{-1, 0, 1\}$. A *convex polyhedron* is an intersection of half-spaces of M_K^+. The number of half-spaces may be infinite, but the intersection is asked to be locally finite: each face must be a polygon with a finite number of vertices, and the number of edges at each vertex must be

finite. A *polyhedron* is a connected union of convex polyhedra. A *polyhedral surface* is the boundary of a polyhedron and a *convex polyhedral surface* is the boundary of a convex polyhedron. A *convex (polyhedral) cone* in M_K^+ is a convex polyhedral surface with only one vertex. Note that the sum of the angles between the edges is strictly between 0 and 2π.

A *metric of curvature K with conical singularities with positive singular curvature* on a compact surface S is a (Riemannian) metric of constant curvature K on S minus n points $(x_1, \ldots, x_n)$ such that the neighbourhood of each x_i is isometric to the induced metric on the neighbourhood of the vertex of a convex cone in M_K^+. The x_i are called the *singular points*. By definition the set of singular points is discrete, hence finite since the surface is compact.

An *invariant polyhedral surface* is a pair (P, F), where P is a polyhedral surface in M_K^+ and F a discrete group of isometries of M_K^+ such that $F(P) = P$ and F acts freely on P. The group F is called the *acting group*.

If there exists an invariant polyhedral surface (P, F) in M_K^+ such that the induced metric on P/F is isometric to a metric h of curvature K with conical singularities on a surface S, we say that P *realises* the metric h (obviously the singular points of h correspond to the vertices of P, and F is isomorphic to the fundamental group of S). In this case we say that h is *realised by a unique invariant polyhedral surface* (P, F) if P is unique up to isometries of M_K^+.

Let P be the boundary of a convex compact polyhedron in M_K^+. The induced metric on P is isometric to a metric of constant curvature K with conical singularities of positive singular curvature on the sphere.

A famous theorem of A.D. Alexandrov asserts that each such metric on the sphere is realised by the boundary of a unique convex compact polyhedron of M_K^+ [1, 3, 14] - in this case the acting group F is the trivial one.

In this paper we prove

THEOREM 1.1. — *A hyperbolic metric with conical singularities of positive singular curvature on a compact surface S of genus > 1 is realised by a unique convex Fuchsian polyhedron in hyperbolic space (up to global isometries).*

A *Fuchsian polyhedron* is a polyhedral surface invariant under the action of a Fuchsian group of hyperbolic space $\mathbb{H}^3$. A *Fuchsian group of hyperbolic space* is a discrete group of orientation-preserving isometries leaving globally invariant a totally geodesic surface, on which it acts cocompactly

and without fixed points. The idea to use them comes from Gromov [7]. Analog statement can be found in [20], see further.

The general outline of the proof of Theorem 1.1 is very classical and has been used in several other cases, starting from A.D. Alexandrov's works. Roughly speaking, the idea is to endow with suitable topology both the space of convex Fuchsian polyhedra with n vertices and the space of corresponding metrics, and to show that the map from one to the other given by the induced metric is a homeomorphism.

The difficult step is to show the local injectivity of the map "induced metric". This is equivalent to a statement on infinitesimal rigidity of convex Fuchsian polyhedra. The section 2 is devoted to this result.

Example of convex Fuchsian polyhedra. Consider the set of points at constant (hyperbolic) distance of a totally geodesic surface $P_{\mathbb{H}^2}$ and denote by M the subset of points which are on one side of $P_{\mathbb{H}^2}$.

Obviously M is globally invariant under the action of any Fuchsian group F leaving $P_{\mathbb{H}^2}$ invariant. Moreover, M has the properties of being strictly convex and umbilic (an *umbilic surface* is a surface such that its principal curvatures are the same for all points. There are other kinds of umbilic surfaces in the hyperbolic space, but in all this text, the expression "umbilic surface" means a complete surface at constant distance of $P_{\mathbb{H}^2}$ and contained in one of the half-spaces bounded by $P_{\mathbb{H}^2}$).

Take n points $(x_1, \ldots, x_n)$ on M, and let F act on these points. We denote by E the boundary of the convex hull of the points fx_i, for all $f \in F$ and $i = 1 \ldots n$. By construction, the convex polyhedral surface E is globally invariant under the action F: it is a convex Fuchsian polyhedron.

Global rigidity of convex Fuchsian polyhedron. A polyhedral surface is called *globally rigid* if any polyhedral surface that is isometric to it is in fact congruent. A direct consequence of the uniqueness of the convex Fuchsian polyhedron realising the induced metric is

THEOREM 1.2. — *Convex Fuchsian polyhedra in hyperbolic space are globally rigid among convex Fuchsian polyhedra.*

Hyperbolic manifolds with polyhedral boundary. Take a convex Fuchsian polyhedron (P, F) and consider the Fuchsian polyhedron (P', F) obtained by the reflection on the invariant surface $P_{\mathbb{H}^2}$ of F. Then cut the hyperbolic space along P and P', and keep the component bounded by P and P'. The quotient of this manifold by F is a kind of hyperbolic manifold called *Fuchsian manifold* (with convex polyhedral boundary): these are

compact hyperbolic manifolds with boundary with an isometric involution fixing a compact hyperbolic surface (the symmetry relative to $P_{\mathbb{H}^2}/F$), see Figure 1.1. In this case we obtain a Fuchsian manifold with convex polyhedral boundary, and all the Fuchsian manifolds with convex polyhedral boundary can be obtained in this way: the lifting to the universal cover of a component of the boundary of the Fuchsian manifold gives a convex Fuchsian polyhedron in the hyperbolic space. Then Theorem 1.1 says exactly that for a choice of the metric h on the boundary, there exists a unique metric on the manifold such that it is a Fuchsian manifold with convex polyhedral boundary and the induced metric on the boundary is isometric to h:

THEOREM 1.3. — *The metric on a Fuchsian manifold with convex polyhedral boundary is determined by the induced metric on its boundary.*
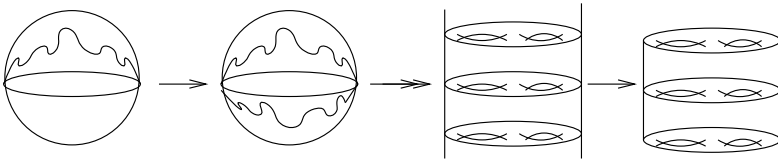

Figure 1.1. From a Fuchsian polyhedron to a Fuchsian manifold.

This is a part of

CONJECTURE 1.4. — *Let h be a hyperbolic metric on a compact manifold M of dimension 3 such that ∂M is polyhedral and convex. Then the induced metric on ∂M is a hyperbolic metric with conical singularities with positive singular curvature. Each hyperbolic metric with conical singularities with positive singular curvature on ∂M is induced on ∂M for a unique choice of h.*

The statement of Conjecture 1.4 in the case where the boundary is smooth and strictly convex as been proved in [21] (the existence part was found in [9]). Remark that A.D. Alexandrov's theorem is a part of Conjecture 1.4 for the case of the hyperbolic ball.

The smooth analog of Conjecture 1.4 provides a smooth version of Theorem 1.1. The existence part of this smooth statement was done in [7].

Towards a general result. The Lorentzian space-forms of dimension 3 are the de Sitter space dS^3 (with curvature 1), the Minkowski space $\mathbb{R}^3_1$

(with curvature 0) and the anti-de Sitter space AdS^3 (with curvature -1), see e.g. [13]. We denote them by M_K^-, where K is the curvature.

Such spaces contain surfaces on which the induced metric is Riemannian (think about the hyperbolic plane in the Minkowski space), and these surfaces are called *space-like*. A space-like convex polyhedral cone in a Lorentzian space-form has a negative singular curvature at its vertex (i.e. the sum of the angles between its edges is $> 2\pi$).

A theorem of Rivin–Hodgson [16, 17] says that each metric of curvature 1 on the sphere with conical singularities with negative singular curvature such that its closed geodesics have lengths $> 2\pi$ is realised by a unique space-like convex polyhedral surface homeomorphic to the sphere in the de Sitter space (beware that $\pi_2(dS^3) \neq 0$).

In [20] it is stated that each metric of curvature 1 on a compact surface of genus > 1 with conical singularities with negative singular curvature such that its contractible geodesics have lengths $> 2\pi$ is realised by a unique space-like convex Fuchsian polyhedron in the de Sitter space (the definition of a Fuchsian polyhedron in a Lorentzian space-form is the same as in the hyperbolic space, after replacing "totally geodesic surface" by "umbilic hyperbolic surface").

We think that each constant curvature K metric with conical singularities with constant sign singular curvature $\epsilon \in \{-,+\}$ on a compact surface can be realised in M_K^ϵ by a unique (space-like) convex polyhedral surface invariant under the action of a representation of the fundamental group of the surface in a group of isometries of dimension 3 - with a condition on contractible geodesics in the cases $K = 1, \epsilon = -$. The images of these representations are trivial for genus 0, parabolic for genus 1 and Fuchsian for genus > 1 (a parabolic isometry fixes a point on the boundary at infinity).

The present paper proves this assertion for hyperbolic metrics with conical singularities with positive singular curvature on compact surfaces of genus > 1.

All the combinations with genus, curvature and sign of the singular curvature are not possible because of Gauss–Bonnet Formulas [24]. The complete list of results would be, if g is the genus of the compact surface (we imply that the polyhedral surfaces are (space-like) convex):

$g = 0$ $K = -1$, $\epsilon = +$: boundary of a compact polyhedron in $\mathbb{H}^3$ (Alexandrov);

$K = 0$, $\epsilon = +$: boundary of a compact polyhedron in $\mathbb{R}^3$ (Alexandrov);

$K = 1$,

$\epsilon = +$: boundary of a compact polyhedron in $\mathbb{S}^3$ (Alexandrov);

$\epsilon = -$ and lengths of the contractible geodesics $> 2\pi$: polyhedral surface homeomorphic to the sphere in dS^3 (Rivin–Hodgson);

$g = 1$ $K = -1$, $\epsilon = +$: parabolic polyhedron in $\mathbb{H}^3$;

$K = 1$, $\epsilon = -$ and lengths of the contractible geodesics $> 2\pi$: parabolic polyhedron in dS^3;

$g > 1$ $K = -1$,

$\epsilon = +$: Fuchsian polyhedron in $\mathbb{H}^3$ (this paper);

$\epsilon = -$: Fuchsian polyhedron in AdS^3;

$K = 0$, $\epsilon = -$: Fuchsian polyhedron in $\mathbb{R}_1^3$;

$K = 1$, $\epsilon = -$ and lengths of the contractible geodesics $> 2\pi$: Fuchsian polyhedron in dS^3 (Schlenker).

The proofs of the others cases for $g > 1$ would be close to the one presented here [5].

2. Infinitesimal rigidity

2.1. Background about infinitesimal isometric deformations

A *Killing field* of a constant curvature space M_K^+ is a vector field of M_K^+ such that the elements of its local 1-parameter group are isometries (see e.g. [6]). An *infinitesimal isometric deformation* of a polyhedral surface consists of

- a triangulation of the polyhedral surface given by a triangulation of each face, such that no new vertex arises,
- a Killing field on each face of the triangulation such that two Killing fields on two adjacent triangles are equal on the common edge.

The edges of the triangulation which were not edges of the polyhedral surface are called *additional edges*.

An infinitesimal isometric deformation is called *trivial* if it is the restriction to the polyhedral surface of a global Killing field. If all the infinitesimal isometric deformations of a polyhedral surface are trivial, then the polyhedral surface is said to be *infinitesimally rigid*.

2.1.1. Infinitesimal rigidity of polyhedral convex caps which may have infinitely many vertices accumulating at the boundary

A *polyhedral convex cap* is a convex polyhedral surface C, with boundary ∂C, in the Euclidean space homeomorphic to the (closed) disc, such that ∂C lies in a totally geodesic plane, and such that the orthogonal projection onto this plane is a bijection between C and the domain of the plane inside ∂C (up to global isometries, we suppose that ∂C lies in the horizontal plane).

This paragraph is dedicated to the proof of:

PROPOSITION 2.1. — *If the vertical component of an infinitesimal isometric deformation of a polyhedral convex cap vanishes on the boundary, then the deformation is trivial.*

Note that our definitions allow polyhedral convex caps with an infinite number of vertices which accumulate at the boundary, and also infinitesimal deformations which diverge on the boundary.

A smooth version of Proposition 2.1 has been known for a long time [14, Thm 1 Chap IV ğ 7] - this reference contains also polyhedral results.

DEFINITION 2.2. — *Let C be a polyhedral convex cap and p a vertex of C. Consider the spherical polygon which is the intersection of a triangulation of C with a little sphere centred at p (such that it intersects only edges incident with p). The link of C at p is the image of this polygon by a homothety sending the little sphere to the unit sphere.*

As C is convex, the link is a convex spherical polygon (with some interior angles may be equal to π because of the additional edges of C).

We denote by u the vertical component of an infinitesimal isometric deformation of C. The definition of an infinitesimal isometric deformation implies that it is a continuous vector field defined on the interior of the convex cap. It follows that the function u is continuous on the interior of the convex cap.

Up to a Euclidean isometry, we can consider that at a vertex p, $u(p) = 0$. Then, if the value of u is positive (resp. negative) at a point on an edge joining p to another vertex, it means that the vertical component of the deformation at this point is greater (resp. less) than at p. This doesn't depend on the choice of the point on the edge. In particular, it is the case for the point which is sent by a homothety to a vertex of the link. In this case, we say that the corresponding vertex of the link *goes up* (resp. *goes down*).
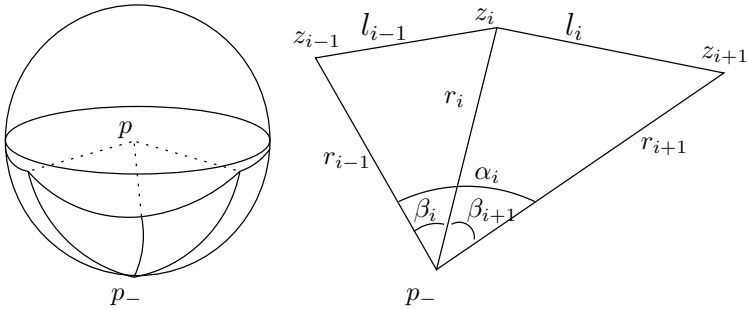
Figure 2.1. Some notations for the link at p.

PROPOSITION 2.3. — *The link cannot go up or go down, i.e. all the vertices of the link cannot go up or go down at the same time.*

We build a spherical polyhedral convex cone from the link by joining all the vertices of the link with the south pole - denoted by p_- - of the sphere containing the link. We denote by $(z_1, \ldots, z_n)$ the vertices of the link, β_i the angle at p_- between the (spherical) segments $p_- z_i$ and $p_- z_{i+1}$, α_i the angle at p_- between the (spherical) segments $p_- z_i$ and $p_- z_{i+2}$ and r_i the length of the (spherical) segment between p_- and z_i (see Figure 2.1).

Without loss of generality, we consider that all the vertices of the link go up. By definition the infinitesimal isometric deformation is isometric on the faces of the polyhedral convex cap, it implies that the lengths $(l_1, \ldots, l_n)$ of the edges of the link don't change under the deformation: each β_i is a function of r_{i-1} and r_i.

We recall the well known "spherical law of cosines (for the sides)". Let a, b, c be the length of the edges of a spherical triangle, and α the angle at the opposite vertex from the edge of length a. Then

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

LEMMA 2.4 (Corollary of the Cauchy Lemma, [2, 18.7.16],[19]). — *If a (convex) quadrilateral of the sphere is deformed such that the lengths of the edges remain constant under the deformation and such that two opposite angles increase, then the two others angles decrease (an angle may be equal to π).*

LEMMA 2.5. — *If r_{i-1} and r_{i+1} are fixed, then*

$$\frac{\partial \beta_i}{\partial r_i}(r_{i-1}, r_i) + \frac{\partial \beta_{i+1}}{\partial r_i}(r_i, r_{i+1}) < 0.$$

Proof. — We call θ the angle at z_{i-1} of the triangle (p_-, z_i, z_{i-1}). The "spherical law of cosines" applied to this triangle says that

$$\cos r_i = \cos l_{i-1} \cos r_{i-1} + \sin l_{i-1} \sin r_{i-1} \cos \theta.$$

As l_{i-1} and r_{i-1} are supposed to be fixed, and as the sines are positive, we deduce from this formula that θ is a strictly increasing function of r_i.

In the same way, the angle at z_{i+1} of the triangle (p_-, z_{i+1}, z_i) is a strictly increasing function of r_i.

In this case, the corollary of the Cauchy Lemma says that the angles at p_- and z_i of the quadrilateral $(p_-, z_{i+1}, z_i, z_{i-1})$ decrease. The first of these angles was called α_i, and it is the sum of β_i and β_{i+1}. It shows that this sum (strictly) decreases when r_i (and only r_i) increases, and this is another way to state the lemma. □

Proof of Proposition 2.3. — We consider $(\dot{r}_1, \ldots, \dot{r}_n)$, where $\dot{r}_i$ means $\frac{d}{dt} r_i(t)|_{t=0}$, a deformation (defined on each vertex) of the link such that all the vertices go up, i.e. $\dot{r}_i > 0 \ \forall i$. As the sum of the angles β_j goes around p we have

$$\sum_{j=1}^n \beta_j = 2\pi$$

and this remains true under the deformation:

(2.1) $$\sum_{j=1}^n \dot{\beta}_j = 0.$$

But on other hand

$$\dot{\beta}_j = \frac{\partial \beta_j}{\partial r_{j-1}} \dot{r}_{j-1} + \frac{\partial \beta_j}{\partial r_j} \dot{r}_j,$$

and we get a contradiction from Lemma 2.5, by replacing in Equation (2.1) (with a cyclic notation $\beta_{n+1} = \beta_1$):

$$0 = \sum_{j=1}^n \left(\frac{\partial \beta_j}{\partial r_{j-1}} \dot{r}_{j-1} + \frac{\partial \beta_j}{\partial r_j} \dot{r}_j \right) = \sum_{k=1}^n \frac{\partial \beta_k}{\partial r_{k-1}} \dot{r}_{k-1} + \sum_{i=1}^n \frac{\partial \beta_i}{\partial r_i} \dot{r}_i$$

$$\overset{i=k-1}{=} \sum_{i=-1}^{n-1} \frac{\partial \beta_{i+1}}{\partial r_i} \dot{r}_i + \sum_{i=1}^n \frac{\partial \beta_i}{\partial r_i} \dot{r}_i = \sum_{i=1}^n \underbrace{\left(\frac{\partial \beta_i}{\partial r_i} + \frac{\partial \beta_{i+1}}{\partial r_i} \right)}_{<0} \underbrace{\dot{r}_i}_{>0}.$$

□

COROLLARY 2.6. — *The function u does not attain a local extremum in the interior of the convex cap.*

Proof. — Suppose that a local extremum is reached at a vertex p (as the deformation is isometric, it is clear that a local extremum can't be reached at a point on a face or an edge). That means that the values of u at points on the edges from p are all greater (or less) than $u(p)$. It implies that the link at p goes up (or down), that's impossible by the preceding proposition. □

If we make the hypothesis that u vanishes at the boundary, this corollary implies that u vanishes for all the vertices of the polyhedral convex cap, and that proves the triviality of the infinitesimal isometric deformation associated to u. Effectively, if the vertical component of an infinitesimal isometric deformation in Euclidean space vanishes, then the deformation is trivial. (It comes from the fact that in this case the rotation field associated to the infinitesimal isometric deformation is constant, that is equivalent to the triviality of the deformation, see e.g. [22, Lemma 4, p. 256]). Proposition 2.1 is now proved.

2.1.2. Infinitesimal Pogorelov map

The following construction is an adaptation of a map invented by Pogorelov [14], which allows to transport deformation problems in a constant curvature space to deformation problems in a flat space, see for example [10, 18, 21].

We view hyperbolic space as a quadric in the Minkowski space of dimension 4, that is

$$\mathbb{H}^3 = \{x \in \mathbb{R}^4 | x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1, x_4 > 0\}.$$

We denote by φ the projective map sending $\mathbb{H}^3$ to the Klein projective model. This map is known to be a homeomorphism between the hyperbolic space and the unit open ball of the Euclidean space of dimension 3, and it sends geodesics to straight lines. Moreover, it sends convex sets to convex sets. In particular, convex polyhedral surfaces are sent to convex polyhedral surfaces.

Let $Z(x)$ be a vector of $T_x\mathbb{H}^3$. The *radial component* of $Z(x)$ is the projection of $Z(x)$ on the radial direction, which is given by the derivative at x of the geodesic l_x in $\mathbb{H}^3$ between $x_c := (0,0,0,1)$ (as a point of the Minkowski space) and x. The *lateral component* of $Z(x)$ is the component orthogonal to the radial one. We denote by μ the length of the geodesic l_x, and by Z_r and Z_l the radial and lateral components of Z. The definitions are the same in Euclidean space, taking the origin instead of x_c (the projective map sends x_c to the origin).

The *infinitesimal Pogorelov map* Φ is a map sending a vector field Z of hyperbolic space to a vector field $\Phi(Z)$ of Euclidean space, defined as follow: the radial component of $\Phi(Z)(\varphi(x))$ has same direction and same norm as $Z_r(x)$, and the lateral component of $\Phi(Z)(\varphi(x))$ is $d_x\varphi(Z_l)$.

If we see a polyhedral surface P in the Klein projective model, then the infinitesimal Pogorelov map is a map sending a vector field on P to another vector field on P.

We have

$$(2.2) \qquad \|Z_r\|_{\mathbb{H}^3} = \|\Phi(Z)_r\|_{\mathbb{R}^3} \;;\; \|Z_l\|_{\mathbb{H}^3} = \cosh\mu \, \|\Phi(Z)_l\|_{\mathbb{R}^3} \,.$$

The first one is the definition, the second one comes from a direct computation or an elementary property of the geometry of the plane (sometimes called the Thales Theorem, see Figure 2.2).
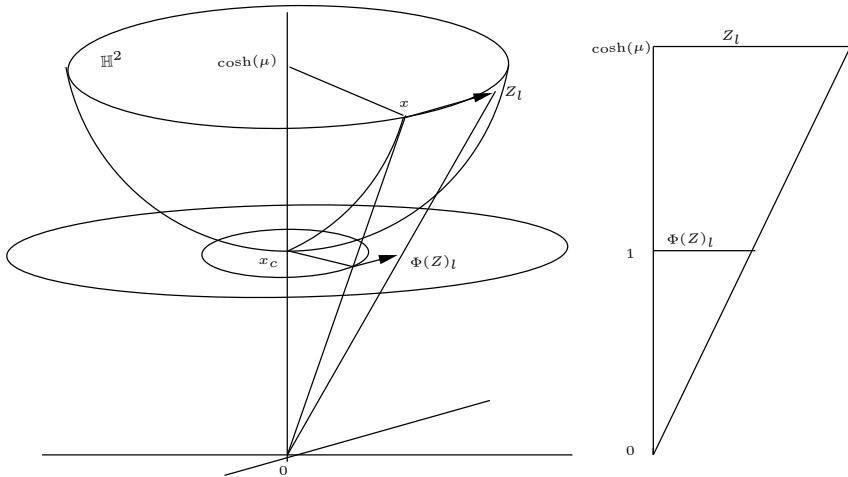
Figure 2.2. $\|Z_l\|_{\mathbb{H}^3} = \cosh\mu \, \|\Phi(Z)_l\|_{\mathbb{R}^3}$.

We will sometimes make the confusion consisting to miss out the point at which we evaluate a vector field.

The infinitesimal Pogorelov map has the following remarkable property:

LEMMA 2.7 (Fundamental property of the infinitesimal Pogorelov map [21, 1.9]). — *Let V be a vector field on $\mathbb{H}^3$, then V is a Killing field if and only if $\Phi(V)$ is a Killing field of the Euclidean space.*

As an infinitesimal isometric deformation of a polyhedral surface is the data of a Killing field on each triangle of a triangulation, this lemma says

that the image of an infinitesimal isometric deformation of a polyhedral surface P by the infinitesimal Pogorelov map is an infinitesimal isometric deformation of the image of P by the projective map. And one is trivial when the other is.

2.2. Fuchsian infinitesimal rigidity

We want to show that convex Fuchsian polyhedra are infinitesimally rigid among Fuchsian polyhedra. As we will consider the Fuchsian polyhedra up to isometries of hyperbolic space, we consider that the invariant totally geodesic surface in the definition is always the same. We choose the horizontal plane in the Klein projective model, and we will denote it by $P_{\mathbb{H}^2}$. By $p_{\mathbb{H}^2}$ we mean the orthogonal projection in hyperbolic space onto the plane $P_{\mathbb{H}^2}$.

Global form of convex Fuchsian polyhedra. Let (P, F) be a convex Fuchsian polyhedron in the hyperbolic space.

LEMMA 2.8. — *The polyhedral surface P has a boundary at infinity which is the same as the one of $P_{\mathbb{H}^2}$.*

Proof. — It is a general property of discrete subgroups of isometries of hyperbolic space that, for a point $x \in \mathbb{H}^3$, the set of accumulation points of $(fx)_{f \in F}$ doesn't depend on the choice of the point x. As F acts on P and on $P_{\mathbb{H}^2}$, we deduce from this that the boundary at infinity of P is included in the boundary at infinity of $P_{\mathbb{H}^2}$.

It remains to check that the fact that F acts cocompactly on $P_{\mathbb{H}^2}$ implies that all the points of the boundary at infinity of $P_{\mathbb{H}^2}$ are reached by its action, and this is left to the reader. □

Then, by convexity of P, P and $P_{\mathbb{H}^2}$ have no intersection point in hyperbolic space (if we glue them along their common infinite boundary, they bound a convex body - recall that in the Klein projective model, $P_{\mathbb{H}^2}$ is the intersection of a Euclidean plane with the unit ball) and, again by convexity of P, the orthogonal projection on $P_{\mathbb{H}^2}$ gives a homeomorphism between $P_{\mathbb{H}^2}$ and P. Note that this implies that for a convex Fuchsian polyhedron (P, F), the genus of the surface P/F is inevitably > 1.

By cocompactness of the action of F on $P_{\mathbb{H}^2}$, the distance between points of P and points of $P_{\mathbb{H}^2}$ (given by the orthogonal projection on $P_{\mathbb{H}^2}$) is bounded: P is between two umbilic surfaces, realising the extrema d_{min} and d_{max} of the distance between P and $P_{\mathbb{H}^2}$.

This leads to the fact that, if we see P in the Klein projective model, it is a convex polyhedral cap (with infinite number of vertices accumulating on the boundary), lying between two half-ellipsoids of radii $(1, 1, r_{min})$ and $(1, 1, r_{max})$, with $0 < r_{min} < r_{max} < 1$ (a direct computation shows that in this model, umbilic surfaces are half-ellipsoids of radius $(1, 1, r)$ where $r = \tanh(d) < 1$, with d the hyperbolic distance between the umbilic surface and $P_{\mathbb{H}^2}$).

Fuchsian polyhedral embeddings. We will need to introduce another way to describe Fuchsian polyhedra. In all the following, S is a compact surface of genus > 1.

DEFINITION 2.9. — *A polyhedral embedding of S in hyperbolic space is a cellulation of S together with a homeomorphism from S to a polyhedral surface of the hyperbolic space $\mathbb{H}^3$, sending polygons of the cellulation to geodesic polygons of $\mathbb{H}^3$.*

A Fuchsian polyhedral embedding of S in the hyperbolic space $\mathbb{H}^3$ is a couple (ϕ, ρ), where

- *ϕ is a polyhedral embedding of the universal cover $\widetilde{S}$ of S in $\mathbb{H}^3$,*
- *ρ is a representation of the fundamental group Γ of S in the group of orientation-preserving isometries of $\mathbb{H}^3$,*

such that ϕ is equivariant under the action of $\Gamma := \pi_1(S)$:

$$\forall \gamma \in \Gamma, \forall x \in \widetilde{S}, \; \phi(\gamma x) = \rho(\gamma)\phi(x),$$

and $\rho(\Gamma)$ leaves globally invariant a totally geodesic surface in $\mathbb{H}^3$, on which it acts cocompactly (without fixed points).

The number of vertices of the Fuchsian polyhedral embedding is the number of vertices of the cellulation of S.

The Fuchsian polyhedral embedding is convex if its image is a convex polyhedral surface of the hyperbolic space.

We consider the Fuchsian polyhedral embeddings up to homeomorphisms and up to global isometries: let (ϕ_1, ρ_1) and (ϕ_2, ρ_2) be two Fuchsian polyhedral embeddings of two surfaces S_1 and S_2. We say that (ϕ_1, ρ_1) and (ϕ_2, ρ_2) are equivalent if there exists a homeomorphism h between S_1 and S_2 and a hyperbolic isometry I such that, for a lift $\widetilde{h}$ of h to $\widetilde{S}_1$ we have

$$\phi_2 \circ \widetilde{h} = I \circ \phi_1.$$

As two lifts of h only differ by conjugation by elements of $\Gamma := \pi_1(S)$, using the equivariance property of the embedding, it is easy to check that the

definition of the equivalence relation doesn't depend on the choice of the lift.

As we see the Fuchsian polyhedral embeddings up to global isometries, we consider that the invariant surface is always $P_{\mathbb{H}^2}$.

DEFINITION 2.10. — *The* genus of a Fuchsian group F of hyperbolic space *is the genus of the quotient of the invariant totally geodesic surface (for the action of F) by the restriction of F to it.*

The genus of a Fuchsian polyhedron *is the genus of the Fuchsian group of hyperbolic space acting on it.*

The number of vertices of a Fuchsian polyhedron (P, F) *is the number of vertices of P in a fundamental domain for the action of F.*

As S is a compact surface of genus $g > 1$, it can be endowed with hyperbolic metrics, and each of them provides a cocompact representation of Γ in the group of orientation-preserving isometries of the hyperbolic plane. The images of such representations are usually called *Fuchsian groups* (of $\mathbb{H}^2$), that explains the terminology used. Moreover, there is a bijection between the cocompact representations of Γ in $\mathrm{Isom}^+(\mathbb{H}^2)$ and the Fuchsian groups of $\mathbb{H}^2$ of genus g.

LEMMA 2.11. — *There is a bijection between the cocompact representations of the fundamental group of S in $\mathrm{Isom}^+(\mathbb{H}^2)$ and the Fuchsian groups of $\mathbb{H}^3$ of genus g (which leave invariant $P_{\mathbb{H}^2}$).*

Proof. — It suffices to prove that there is a bijection between the Fuchsian groups of $\mathbb{H}^2$ and the Fuchsian groups of $\mathbb{H}^3$ (with same genus).

The restriction of a Fuchsian group of $\mathbb{H}^3$ to $P_{\mathbb{H}^2}$ obviously gives a Fuchsian group of $\mathbb{H}^2$.

Reciprocally, a Fuchsian group F acting on $P_{\mathbb{H}^2}$ canonically gives a Fuchsian group of the hyperbolic space: for a point x in the hyperbolic space, an element $f \in F$ sends $p_{\mathbb{H}^2}(x)$ on a point $z \in P_{\mathbb{H}^2}$. The image y of x by the element of $\mathrm{Isom}^+(\mathbb{H}^3)$ extending f is the unique point of the hyperbolic space such that its projection on $P_{\mathbb{H}^2}$ is z (and is in the same half-space delimited by $P_{\mathbb{H}^2}$ than x). And there is no other such subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$ (because if there is, each elements of both groups sends a geodesic segment orthogonal to $P_{\mathbb{H}^2}$ to the same geodesic segment, then they are equal).   □

LEMMA 2.12. — *There is a bijection between the convex Fuchsian polyhedra of genus g with n vertices and the convex Fuchsian polyhedral embeddings with n vertices of S.*

Proof. — Obviously, the image of S by a convex Fuchsian polyhedral embedding is a convex Fuchsian polyhedron. Reciprocally, the canonical

embedding in $\mathbb{H}^3$ of a convex Fuchsian polyhedron P invariant under the action of a group F gives a convex Fuchsian polyhedral embedding of the surface P/F in $\mathbb{H}^3$. We have seen that this surface is homeomorphic to $P_{\mathbb{H}^2}/F$, which is homeomorphic to S. □

Fuchsian deformations. Let (S, ϕ, ρ) a convex polyhedral Fuchsian embedding.

Let $(\phi_t)_t$ be a path of convex polyhedral embeddings of $\widetilde{S}$ in $\mathbb{H}^3$, such that:

- $\phi_0 = \phi$,
- the induced metric is preserved at the first order at $t = 0$,
- there are representations ρ_t of $\Gamma = \pi_1(S)$ in $\mathrm{Isom}^+(\mathbb{H}^3)$

such that

$$\phi_t(\gamma x) = \rho_t(\gamma)\phi_t(x)$$

and each $\rho_t(\Gamma)$ leaves globally invariant a totally geodesic surface, on which it acts cocompactly without fixed points (up to global isometries, we consider that the surface is always $P_{\mathbb{H}^2}$).

We denote by

$$Z(\phi(x)) := \frac{d}{dt}\phi_t(x)_{|t=0} \in T_{\phi(x)}\mathbb{H}^3$$

and

$$\dot{\rho}(\gamma)(\phi(x)) = \frac{d}{dt}\rho_t(\gamma)(\phi(x))_{|t=0} \in T_{\rho(\gamma)\phi(x)}\mathbb{H}^3.$$

The vector field Z has a property of equivariance under $\rho(\Gamma)$:

(2.3) $$Z(\rho(\gamma)\phi(x)) = \dot{\rho}(\gamma)(\phi(x)) + d\rho(\gamma).Z(\phi(x)).$$

This can be written

(2.4) $$Z(\rho(\gamma)\phi(x)) = d\rho(\gamma).(d\rho(\gamma)^{-1}\dot{\rho}(\gamma)(\phi(x)) + Z(\phi(x)))$$

and $d\rho(\gamma)^{-1}\dot{\rho}(\gamma)$ is a Killing field of $\mathbb{H}^3$, because it is the derivative of a path in the group of isometries of $\mathbb{H}^3$ (we must multiply by $d\rho(\gamma)^{-1}$, because $\dot{\rho}(\gamma)$ is not a vector field). We denote this Killing field by $\vec{\rho}(\gamma)$. Equation (2.4) can be written, if $y = \phi(x)$,

(2.5) $$Z(\rho(\gamma)y) = d\rho(\gamma).(\vec{\rho}(\gamma) + Z)(y).$$

A *Fuchsian deformation* is an infinitesimal isometric deformation Z on a Fuchsian polyhedron which satisfies Equation (2.5), where $\vec{\rho}(\gamma)$ is a *Fuchsian Killing field*, that is a Killing field of the hyperbolic plane extended to the hyperbolic space along geodesics orthogonal to the plane. More precisely, for a point $x \in \mathbb{H}^3$, let d be the distance between x and $p_{\mathbb{H}^2}(x)$. We denote by p_d the orthogonal projection onto $P_{\mathbb{H}^2}$ of the umbilic surface

which is at constant distance d from $P_{\mathbb{H}^2}$ (passing through x). Then the Killing field K at $p_{\mathbb{H}^2}(x)$ is extended as $dp_d^{-1}(K)$ at the point x.

A Fuchsian polyhedron is *Fuchsian infinitesimally rigid* if all its Fuchsian deformations are trivial (i.e. are restriction to the Fuchsian polyhedron of Killing fields of hyperbolic space).

We want to prove

THEOREM 2.13. — *Convex Fuchsian polyhedra are Fuchsian infinitesimally rigid.*

By the fundamental Property of the infinitesimal Pogorelov map, to prove Theorem 2.13 it suffices to prove that, for each Fuchsian deformation Z, the vertical component of the image of Z by the infinitesimal Pogorelov map vanishes at the boundary, because in this case Proposition 2.1 provides the result.

Remarks about the method employed. There is a non-direct way to prove the statement of Theorem 2.13, using another infinitesimal Pogorelov map. This way is pointed out in [5].

Moreover, it may be possible that the method employed to prove Proposition 2.1 leads to a direct proof of Theorem 2.13 in hyperbolic space, without using the infinitesimal Pogorelov map.

But we think that the method used here can be extended to prove infinitesimal rigidity results for ideal and hyperideal Fuchsian polyhedra (that means that some vertices could be on the sphere or out of the ball for the Klein projective model).

Note that the following proof is also true without any change for strictly convex smooth Fuchsian surfaces (using the smooth analog of Proposition 2.1).

Proof of Theorem 2.13. Recall that $p_{\mathbb{H}^2}$ is the orthogonal projection on the totally geodesic surface $P_{\mathbb{H}^2}$. At a point x of a convex Fuchsian polyhedron P, the derivative of the geodesic which realises this projection is called the *vertical direction* at x, and the directions orthogonal to this one are *horizontal directions*. So a Fuchsian deformation Z can be decomposed in a vertical component Z_v and in a horizontal component Z_h. We denote by $(Z_r)_h$ the horizontal component of the radial component of Z, etc. We have

$$Z_r = (Z_r)_h + (Z_r)_v = (Z_h)_r + (Z_v)_r.$$

The first one is obvious and the second one comes from the linearity of the orthogonal projection.

PROPOSITION 2.14. — *The vector field Z_v is invariant under the action of $\rho(\Gamma)$, i.e. $\forall x \in P$:*

$$Z_v(\rho(\gamma)x) = d\rho(\gamma).Z_v(x).$$

Proof. — From Equation (2.5), it suffices to check that $\bar{\rho}(\gamma)$ has no vertical component, that is true by definition of a Fuchsian Killing field.   □

And as $\rho(\Gamma)$ acts cocompactly on P

COROLLARY 2.15. — *There exists a constant c_v such that, for all $x \in P$,*

$$\|Z_v(x)\| \leqslant c_v.$$

Moreover

COROLLARY 2.16. — *The vector field Z_h is equivariant under the action of $\rho(\Gamma)$.*

Proof. — Equation (2.3) says that

$$Z_v(\phi(\gamma x)) + Z_h(\phi(\gamma x)) = \dot{\rho}(\phi(x)) + d\rho(\gamma).Z_v(\phi(x)) + d\rho(\gamma).Z_h(\phi(x)),$$

and the preceding proposition gives the result.   □

Recall that p_d is the orthogonal projection on $P_{\mathbb{H}^2}$ of the umbilic surface which is at constant distance d from $P_{\mathbb{H}^2}$ (passing through x). We call *radial-horizontal* the component of Z (at x) in the direction $dp_d^{-1}(r(p_{\mathbb{H}^2}(x)))$, where $r(p_{\mathbb{H}^2}(x))$ is the radial direction of $P_{\mathbb{H}^2}$ at the point $p_{\mathbb{H}^2}(x)$. This component is noted Z_{rh}, and it's a horizontal vector.

We denote by W the projection on $P_{\mathbb{H}^2}$ of the horizontal component of Z (it is equivariant under the action of $\rho(\Gamma)$). We denote by W_r its radial component. Then $dp_d^{-1}(W_r)$ is the radial-horizontal component of the horizontal component of Z.

The determining fact is:

PROPOSITION 2.17. — *Let H be a vector field of $\mathbb{H}^2$ equivariant under the action of $\rho(\Gamma)$. Then there exists a constant $c_{\dot{h}}$ such that*

$$\|H_r(x)\|_{\mathbb{H}^2} \leqslant c_{\dot{h}} d_{\mathbb{H}^2}(x_c, x).$$

The point x_c is always the origin in the Klein projective model.

Proof. — We deform the hyperbolic metric h along H, that is $\dot{h} = L_H h$, where L is the Lie derivative:

$$\begin{aligned}
\dot{h}(X,Y) = (L_H h)(X,Y) &= h(\nabla_X H, Y) + h(\nabla_Y H, X) \\
&= H.h(X,Y) + h([X,H],Y) + h(X,[Y,H]).
\end{aligned}$$

Let $c : [0,\eta] \to \mathbb{H}^2$, $c(0) = x_c$, $,c(\eta) = x$, $\|c'\| = 1$ a geodesic (then $\eta = d_{\mathbb{H}^2}(x_c, x)$ and $c'(\eta)$ is the radial direction at the point x). Up to adding a Killing field, we can suppose that $H(x_c) = 0$. Then

$$
\begin{aligned}
\int_0^\eta \dot{h}(c'(t), c'(t))dt &= 2\int_0^\eta h(\nabla_{c'(t)} H(c(t)), c'(t))dt \\
&= 2\int_0^\eta c'(t).h(H, c'(t))dt \\
&= 2h(H, c'(\eta)) \\
&= 2h(H_r + H_l, c'(\eta)) \\
&= 2h(H_r, c'(\eta)) \\
&= 2h(H_r, \pm\frac{H_r}{\|H_r\|_{\mathbb{H}^2}}),
\end{aligned}
$$

that leads to

$$
(2.6) \qquad \int_0^\eta \dot{h}(c'(t), c'(t))dt = \pm 2\|H_r\|_{\mathbb{H}^2} .
$$

As H is equivariant under the action of $\rho(\Gamma)$, the elements of $\rho(\Gamma)$ preserve the bilinear form $\dot{h}$, because, up to an isometry, $H(\rho(\gamma)x)$ is written as $H(x)$ plus a Killing field.

Formally, using Equation (2.5)

$$
H(\rho(\gamma)x) = d\rho(\gamma).(\vec{\rho}(\gamma) + H)(x),
$$

and we develop

$$
\begin{aligned}
\dot{h}&(d\rho(\gamma)X(x), d\rho(\gamma)Y(x)) \\
&:= h(\nabla_{d\rho(\gamma)X(x)} H(\rho(\gamma)x), d\rho(\gamma)Y(x)) \\
&\quad + h(\nabla_{d\rho(\gamma)Y(x)} H(\rho(\gamma)x), d\rho(\gamma)X(x)) \\
&= h(\nabla_{d\rho(\gamma)X(x)} d\rho(\gamma)(\vec{\rho}(\gamma)(x) + H(x)), d\rho(\gamma)Y(x)) \\
&\quad + h(\nabla_{d\rho(\gamma)Y(x)} d\rho(\gamma)(\vec{\rho}(\gamma)(x) + H(x)), d\rho(\gamma)X(x)) \\
&= h(\nabla_{X(x)}(\vec{\rho}(\gamma)(x) + H(x)), Y(x)) \\
&\quad + h(\nabla_{Y(x)}(\vec{\rho}(\gamma)(x) + H(x)), X(x)) \\
&= h(\nabla_{X(x)} H(x), Y(x)) + h(\nabla_{Y(x)} H(x), X(x)) \\
&\quad + h(\nabla_{X(x)}\vec{\rho}(\gamma)(x), Y(x)) + h(\nabla_{Y(x)}\vec{\rho}(\gamma)(x), X(x)) \\
&= h(\nabla_{X(x)} H(x), Y(x)) + h(\nabla_{Y(x)} H(x), X(x)) \\
&\quad + L_{\vec{\rho}(\gamma)} h(X(x), Y(x)) \\
&= \dot{h}(X(x), Y(x))
\end{aligned}
$$

$(L_{\vec{\rho}(\gamma)}h(X,Y) = 0$ because $\vec{\rho}(\gamma)$ is a Killing field).

Then $\dot{h}$ is a bilinear form on $P_{\mathbb{H}^2}/\rho(\Gamma)$, which is compact, then $|\dot{h}|$ is bounded by a constant $c_{\dot{h}}$, and by Equation (2.6):

$$\|H_r(x)\|_{\mathbb{H}^2} \leqslant c_{\dot{h}}\eta = c_{\dot{h}}d_{\mathbb{H}^2}(x_c, x).$$

$\square$

COROLLARY 2.18. — *There exists a constant c_{rh} such that:*

$$\|(Z_h)_{rh}(x)\|_{\mathbb{H}^3} \leqslant c_{rh}d_{\mathbb{H}^3}(x_c, x).$$

Proof. — A simple computation in Minkowski space shows that the induced metric on the umbilic surface at constant distance d of $P_{\mathbb{H}^2}$ (passing through x) is $\cosh(d)h$ where h is the hyperbolic metric, then

$$\|(Z_h)_{rh}(x)\|_{\mathbb{H}^3} = \cosh(d) \|W_r(p_{\mathbb{H}^2}(x))\|_{\mathbb{H}^2} ,$$

and as d is bounded for all $x \in P$, and with the preceding proposition, there exists a constant c_{rh} such that:

(2.7) $$\|(Z_h)_{rh}(x)\|_{\mathbb{H}^3} \leqslant c_{rh}d_{\mathbb{H}^2}(x_c, p_{\mathbb{H}^2}(x)).$$

Classical hyperbolic trigonometry applied to the rectangular triangle with edge length $d_{\mathbb{H}^3}(x_c, x)$ (the long edge), $d_{\mathbb{H}^3}(x, p_{\mathbb{H}^2}(x))$ and $d_{\mathbb{H}^2}(x_c, p_{\mathbb{H}^2}(x))$ gives:

$$\cosh(d_{\mathbb{H}^3}(x_c, x)) = \cosh(d_{\mathbb{H}^3}(x, p_{\mathbb{H}^2}(x))) \cosh(d_{\mathbb{H}^2}(x_c, p_{\mathbb{H}^2}(x))),$$

and as the values of the hyperbolic cosine are greater than 1:

$$\cosh(d_{\mathbb{H}^3}(x_c, x)) \geqslant \cosh(d_{\mathbb{H}^2}(x_c, p_{\mathbb{H}^2}(x))),$$

and the hyperbolic cosine is an increasing function for positive values, then:

$$d_{\mathbb{H}^3}(x_c, x) \geqslant d_{\mathbb{H}^2}(x_c, p_{\mathbb{H}^2}(x)).$$

This together with Equation (2.7) gives the result. $\square$

The *vertical plane* is the vector space of $T_x\mathbb{H}^3$ spanned by the orthogonal vectors Z_v and Z_{rh} (Z_{rh} is in the horizontal plane by definition, then it is orthogonal to the vertical direction).

We can see the vertical plane as the tangent plane (at x) to the totally geodesic surface passing through x_c and x and orthogonal to $P_{\mathbb{H}^2}$.

We denote by Z_{lv} the image of the projection on the vertical plane of the lateral component of Z.

The vector Z_r belongs to the vertical plane, because it can be decomposed in a horizontal component, which is in the radial-horizontal direction, and a vertical component.

The lateral component is orthogonal to the radial component, then the vector Z_{lv} is orthogonal to Z_r in the vertical plane.

In all the following, we denote by $\mu := d_{\mathbb{H}^3}(x_c, x)$, and vector fields are evaluated at the point $x \in P$.

LEMMA 2.19. — *Let V be the projection of a component of Z on the vertical plane. Then there exists a constant c such that*

$$\|V\|_{\mathbb{H}^3} \leqslant c(1 + \mu).$$

Proof. — We denote by Π_V the projection on the vertical plane, considered as spanned by the orthogonal vectors Z_{rh} and Z_v. We can write $\Pi_V(Z) = Z_{rh} + Z_v$. As V is already in the vertical plane, and as we project a component of Z, we can write:

$$\begin{aligned}
\|V\|_{\mathbb{H}^3} = \|\Pi_V(V)\|_{\mathbb{H}^3} &\leqslant \|\Pi_V(Z)\|_{\mathbb{H}^3} \\
&\leqslant \|Z_{rh}\|_{\mathbb{H}^3} + \|Z_v\|_{\mathbb{H}^3} \\
&\leqslant \|(Z_h)_{rh}\|_{\mathbb{H}^3} + \|(Z_v)_{rh}\|_{\mathbb{H}^3} + \|Z_v\|_{\mathbb{H}^3} \\
&\leqslant \|(Z_h)_{rh}\|_{\mathbb{H}^3} + 2\,\|Z_v\|_{\mathbb{H}^3}\,,
\end{aligned}$$

and as the overestimation of these two last norms are known (by Corollaries 2.15 and 2.18) we get

$$\|V\|_{\mathbb{H}^3} \leqslant c_{rh}\mu + 2c_v,$$

that is, if c is greater than c_{rh} and $2c_v$,

$$\|V\|_{\mathbb{H}^3} \leqslant c(1 + \mu).$$

$\square$

For convenience, we denote by u be the image of Z by the infinitesimal Pogorelov map.

The decompositions of tangent vectors defined above (vertical, horizontal,...) occur in Euclidean space. We want to show that the vertical component of u goes to 0 at the boundary. As u_v is in the vertical plane, if α is the angle between u_v and u_{lv}, we get

$$(2.8) \qquad\qquad u_v = \cos(\alpha)u_{lv} + \sin(\alpha)u_r,$$

because, for the same reason as for Z, u_r and u_{lv} give an orthogonal basis of the vertical plane.

By Lemma 2.19 we get, as Z_r and Z_{lv} are in the vertical plane,

$$\|Z_r\|_{\mathbb{H}^3} \leqslant c(1 + \mu),$$
$$\|Z_{lv}\|_{\mathbb{H}^3} \leqslant c(1 + \mu),$$

and as the infinitesimal Pogorelov map preserves the norm of the radial component and crushes by a coefficient $\cosh(\mu)$ the norm of the lateral direction (Equation (2.2)), these two inequalities rise to

$$\|u_r\|_{\mathbb{R}^3} \leqslant c(1 + \mu),$$
$$\|u_{lv}\|_{\mathbb{R}^3} \leqslant \cosh(\mu)^{-1} c(1 + \mu).$$

Starting from (2.8) we get

$$\|u_v\|_{\mathbb{R}^3} \leqslant \|u_{lv}\|_{\mathbb{R}^3} + \sin(\alpha) \|u_r\|_{\mathbb{R}^3}$$
$$\leqslant c\cosh(\mu)^{-1}(1 + \mu) + c\sin(\alpha)(1 + \mu)$$
$$\leqslant c(1 + \mu)(\cosh(\mu)^{-1} + \sin(\alpha)).$$

We call δ the Euclidean distance between a point x on P and $P_{\mathbb{H}^2}$ (in the Klein projective model), and we see easily that we have $\sin(\alpha) \underset{0}{\approx} \delta$ when we go near the boundary of the ball:

$$c(1 + \mu)(\cosh(\mu)^{-1} + \sin(\alpha)) \underset{0}{\approx} c(1 + \mu)(\cosh(\mu)^{-1} + \delta).$$

We denote by S_{max} the umbilic surface which realises the maximum of the distance between P and $P_{\mathbb{H}^2}$, x_{max} the intersection of S_{max} with the geodesic joining x and x_c, μ_{max} for $d_{\mathbb{H}^3}(x_c, x_{max})$, δ_{max} the distance in $\mathbb{R}^3$ between x_{max} and $P_{\mathbb{H}^2}$. Guess analog definitions of $S_{min}, x_{min}, \mu_{min}, \delta_{min}$ for the surface realising the minimum of the distance between P and $P_{\mathbb{H}^2}$.

LEMMA 2.20. — *Near the boundary of the ball, we have the approximations*

$$\mu_{max} \underset{\delta_{max} \to 0}{\approx} -c_{max} \ln(\delta_{max}),$$
$$\mu_{min} \underset{\delta_{min} \to 0}{\approx} -c_{min} \ln(\delta_{min}),$$

where c_{max} and c_{min} are positive constants.

Proof. — We prove the lemma in the case *max*, the proof for *min* is the same. It is easy to check that $\mu_{max} = d_{\mathbb{H}^3}(x_c, x_{max}) = \tanh^{-1}(\|x_{max}\|_{\mathbb{R}^3})$, that is

$$\mu_{max} = \ln\left(\frac{1 + \|x_{max}\|_{\mathbb{R}^3}}{1 - \|x_{max}\|_{\mathbb{R}^3}}\right) \underset{\|x_{max}\|_{\mathbb{R}^3} \to 1}{\approx} -\ln(1 - \|x_{max}\|_{\mathbb{R}^3}).$$

As the image of S_{max} in the projective Klein model is an ellipsoid, δ_{max} satisfies the equation

$$(x_{max})_1^2 + (x_{max})_2^2 + \frac{\delta_{max}^2}{r^2} = 1,$$

where r is a positive constant strictly less than 1. Adding and removing a δ_{max}^2 and reordering we get:

$$1 - \|x_{max}\|_{\mathbb{R}^3} = \delta_{max}^2 \frac{1 - r^2}{r^2},$$

as $\delta_{max}^2 = (x_{max})_3^2$, and this gives the result.   □

As obviously

$$\mu_{min} \leqslant \mu \leqslant \mu_{max} \text{ and } \delta_{min} \leqslant \delta \leqslant \delta_{max},$$

we get

$$\|u_v\|_{\mathbb{R}^3} \leqslant c(1 + \mu)(\cosh(\mu)^{-1} + \delta)$$
$$\leqslant c(1 + \mu_{max})(\cosh(\mu_{min})^{-1} + \delta),$$

and when x goes near the boundary, $\cosh(f(x)) \underset{\infty}{\approx} \exp(f(x))$, where f is a function going to ∞ when x goes near the boundary. Then:

$$c(1 + \mu_{max})(\cosh(\mu_{min})^{-1} + \delta)$$
$$\underset{0}{\approx} \quad c(1 - c_{max} \ln(\delta_{max}))(\cosh(-c_{min} \ln(\delta_{min}))^{-1} + \delta)$$
$$\underset{0}{\approx} \quad c'(1 - \ln(\delta_{max}))(\delta_{min} + \delta).$$

At the end, as

$$c'(1 - \ln(\delta_{max}))(\delta_{min} + \delta) \leqslant c'(1 - \ln(\delta_{max}))(\delta_{max} + \delta_{max})$$

and as when δ goes to 0, δ_{max} goes to 0, then $\|u_v\|_{\mathbb{R}^3}$ goes to 0.

Theorem 2.13 is now proved.

3. Realisation of metrics

3.1. Set of Fuchsian polyhedra

We denote by $\mathcal{P}(n)$ the set of convex Fuchsian polyhedral embeddings with n vertices of a compact surface S in $\mathbb{H}^3$, modulo isotopies of S fixing the vertices of the cellulation and modulo the isometries of $\mathbb{H}^3$.

More precisely, the equivalence relation is the following: let (ϕ_1, ρ_1) and (ϕ_2, ρ_2) be two convex Fuchsian polyhedral embeddings of S. We say that (ϕ_1, ρ_1) and (ϕ_2, ρ_2) are equivalent if there exists

- a homeomorphism h of S isotopic to the identity, such that if h_t is the isotopy (i.e. $t \in [0, 1]$, $h_0 = h$ and $h_1 = id$), then h_t fixes the vertices of the cellulation for all t,
- a hyperbolic isometry I,

such that, for a lift $\widetilde{h}$ of h to $\widetilde{S}$ we have

$$\phi_2 \circ \widetilde{h} = I \circ \phi_1.$$

Here again, the definition of the equivalence relation doesn't depend on the choice of the lift.

Z-V-C coordinates for Teichmüller space. For more details about Z-V-C coordinates (Z-V-C stands for Zieschang–Vogt–Coldewey, [25]) we refer to [4, 6.7].

DEFINITION 3.1. — *Let $g \geqslant 2$. A (geodesically convex) polygon of the hyperbolic plane with edges (in the direct order) $b_1, b_2, \overline{b}_1, \overline{b}_2, b_3, b_4, \ldots, \overline{b}_{2g}$ and with interior angles $\theta_1, \overline{\theta}_1, \ldots, \theta_{2g}, \overline{\theta}_{2g}$ is called* (normal) canonical *if, with $l(c)$ the length of the geodesic c,*

    i) $l(b_k) = l(\overline{b}_k), \forall k$;
    ii) $\theta_1 + \ldots + \overline{\theta}_{2g} = 2\pi$;
    iii) $\theta_1 + \theta_2 = \overline{\theta}_1 + \theta_2 = \pi$.

Two canonical polygons P and P' with edges $b_1, \ldots, \overline{b}_{2g}$ and $b'_1, \ldots, \overline{b}'_{2g}$ are said *equivalent if there exists an isometry from P to P' such that the edge b_1 is sent to the edge b'_1 and b_2 is sent to b'_2.*

If we identify the edges b_i with the edges $\overline{b}_i$, we get a compact hyperbolic surface of genus g. This surface could also be written $\mathbb{H}^2/F$, where F is the sub-group of $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$ generated by the translations along the edges b_i (the translation length is the length of b_i). The interior of the polygon is a fundamental domain for the action of F. This leads to a description of the Teichmüller space T_g:

PROPOSITION 3.2 ([4, 6.7.7]). — *Let $\mathcal{P}_g$ be the set of equivalence classes of canonical polygons. An element of $\mathcal{P}_g$ is described by the $(6g - 6)$ real numbers (the Z-V-C coordinates):*

$$(b_3, \ldots, b_{2g}, \theta_3, \overline{\theta}_3, \ldots, \theta_{2g}, \overline{\theta}_{2g}).$$

Endowed with this topology, $\mathcal{P}_g$ is in analytic bijection with T_g.

Surjection on the Teichmüller space with marked points. The Teichmüller space of $F_{g,n}$ (i.e. of a compact surface of genus $g > 1$ with $n > 0$ marked points), denoted by $T_g(n)$, can be defined as the set of hyperbolic metrics on $F_{g,n}$ modulo isotopies of $F_{g,n}$, such that each isotopy h_t fixes the marked points for all t.

Let (ϕ_1, ρ_1) and (ϕ_2, ρ_2) be two equivalent convex Fuchsian polyhedral embeddings of S with n vertices. Recall that h is an isotopy of S and $\widetilde{h}$ its

lift to $\widetilde{S}$. As h is homotopic to the identity, $\forall x \in \widetilde{S}, \forall \gamma \in \Gamma$, we get:

$$\phi_2(\widetilde{h}(\gamma x)) = I(\phi_1 \gamma x)$$
$$\Leftrightarrow \phi_2((h)_*(\gamma)x) = I(\rho_1(\gamma)(\phi_1(x)))$$
$$\Leftrightarrow \phi_2(\gamma x) = I(\rho_1(\gamma)(\phi_1(x)))$$
$$\Leftrightarrow \rho_2(\gamma)(\phi_2(x)) = I(\rho_1(\gamma)(\phi_1(x)))$$
$$\Leftrightarrow \rho_2(\gamma)(I(\phi_1(x))) = I(\rho_1(\gamma)(\phi_1(x))).$$

But if two orientation-preserving isometries of the hyperbolic space are equal on an open set of a totally geodesic surface (a face of the Fuchsian polyhedron), they are equal, then for all $\gamma \in \Gamma$, $\rho_2(\gamma) = I \circ \rho_1(\gamma) \circ I^{-1}$. As ρ_1 and ρ_2 are also representations of Γ in $\mathrm{PSL}(2,\mathbb{R})$ (modulo conjugation by an element of $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$), we deduce that ρ_1 and ρ_2 are the same element of $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$.

Then in the hyperbolic plane $P_{\mathbb{H}^2}$, the canonical polygons associated to these two representations are equal, and, up to an isometry, the projection of the vertices of the Fuchsian polyhedra $\phi_1(\tilde{S})$ and $\phi_2(\tilde{S})$ on $P_{\mathbb{H}^2}$ gives the same n marked points in this canonical polygon: we have described a map $\mathcal{S}$ which to each element of $\mathcal{P}(n)$ associates an element of $T_g(n)$.

And as we have seen that from any Fuchsian group and any n points on the plane, we can build a convex Fuchsian polyhedron with n vertices (it is enough to take n points at same distance from the plane), $\mathcal{S}$ is surjective.

Manifold structure on $\mathcal{P}(n)$.

LEMMA 3.3. — *Let $[h] \in T_g(n)$. Then $\mathcal{S}^{-1}([h])$ is diffeomorphic to the open unit ball of $\mathbb{R}^n$.*

Proof. — We will show that $\mathcal{S}^{-1}([h])$ is a contractible open subset of $(\mathbb{R}_+)^n$.

We fix an element of $T_g(n)$, that is the action of a cocompact Fuchsian group F on $P_{\mathbb{H}^2}$ and n points $(y_1, \ldots, y_n)$. We denote by $(x_1, \ldots, x_n)$ n points of the hyperbolic space such that their projection on the plane $P_{\mathbb{H}^2}$ is exactly $(y_1, \ldots, y_n)$. We look at the Klein projective model of the hyperbolic space, and the coordinates of the points are those of $\mathbb{R}^3$. The orthogonal projection, Euclidean or hyperbolic, of a point on the horizontal plane gives the same point.

We want the vertices of the polyhedron obtained as the boundary of the closure of the convex hull of the points $f x_i$, for any $f \in F$ and $i = 1 \ldots n$, to be exactly the set of points of the form $f x_i$, for any $f \in F$ and $i = 1 \ldots n$. This is the same as saying that no point of the form $f x_i$ is in the interior

of the convex hull of the others points of this form; that means, for all fx ($x \in \{x_1, \ldots, x_n\}$), and for all $f_i x_i, f_j x_j, f_k x_k$ such that $p_{\mathbb{H}^2}(fx)$ is contained in the triangle formed by the points $p_{\mathbb{H}^2}(f_i x_i)$, $p_{\mathbb{H}^2}(f_j x_j)$ and $p_{\mathbb{H}^2}(f_k x_k)$, then the plane generated by $f_i x_i, f_j x_j, f_k x_k$ (strictly) separates fx from $P_{\mathbb{H}^2}$.

Actually, it suffices to verify this condition only for the points $(x_1, \ldots, x_n)$, because if the plane generated by

$$f_i x_i, f_j x_j, f_k x_k$$

separates fx from $P_{\mathbb{H}^2}$, then the plane generated by

$$f^{-1} f_i x_i, f^{-1} f_j x_j, f^{-1} f_k x_k$$

separates x from $P_{\mathbb{H}^2}$ (as F acts by isometries, it sends the convex polyhedron with vertices

$$f_i x_i, f_j x_j, f_k x_k, fx, p_{\mathbb{H}^2}(f_i x_i), p_{\mathbb{H}^2}(f_j x_j), p_{\mathbb{H}^2}(f_i x_i), p_{\mathbb{H}^2}(fx)$$

on the convex polyhedron of vertices

$$f^{-1} f_i x_i, f^{-1} f_j x_j, f^{-1} f_k x_k, x, p_{\mathbb{H}^2}(f^{-1} f_i x_i),$$
$$p_{\mathbb{H}^2}(f^{-1} f_j x_j), p_{\mathbb{H}^2}(f^{-1} f_i x_i), p_{\mathbb{H}^2}(x)).$$

Then, if x is projected in the triangle $(p_{\mathbb{H}^2}(f_i x_i), p_{\mathbb{H}^2}(f_j x_j), p_{\mathbb{H}^2}(f_k x_k))$, we have a condition which can be written

$$\det(f_i x_i - f_j x_j, f_i x_i - f_k x_k, f_i x_i - x) > 0.$$

(we choose an orientation such that this det is > 0, but we could take the other orientation, it will change nothing, the important fact is that the condition is open).

For each equation, the set of solution is an open half-space delimited by an affine hyperplane: the possible set of heights for the vertices $(x_1, \ldots, x_n)$ is given by the intersection (of an infinite number) of open half-spaces.

This is a contractible open set. We want the vertices to stay in the unit ball, then we must add for each vertex the condition that its height must be (strictly) greater than 0 and less than $(1 - a_i^2 - b_i^2)$, where a_i and b_i are the horizontal coordinates of each x_i (they are fixed by hypothesis): we intersect the contractible open set with other open half-spaces, and the intersection remains a contractible open set. This set is non-empty, as we built examples of convex Fuchsian polyhedra. □

An open set of $T_g(n)$ is parameterised by a (small) deformation of a canonical polygon in the hyperbolic plane and a displacement of the marked points inside this polygon. With a fixed height for the vertices, a small

displacement of a convex Fuchsian polyhedron (corresponding to a path in $T_g(n)$), is always convex (the convexity is a property preserved by a little displacement of the vertices) and Fuchsian (by construction).

So we can endow $\mathcal{P}(n)$ with the topology which makes it a fiber space based on $T_g(n)$, with fibers homeomorphic to the open unit ball of $\mathbb{R}^n$:

PROPOSITION 3.4. — *The space $\mathcal{P}(n)$ is a contractible manifold of dimension $(6g - 6 + 3n)$.*

Because $T_g(n)$ is contractible manifold of dimension $(6g - 6 + 2n)$, see e.g. [12].

Triangulations.

DEFINITION 3.5. — *A (generalised) triangulation of a compact surface S is a decomposition of S by images by homeomorphisms of triangles of Euclidean space, with possible identification of the edges or the vertices, such that the interiors of the faces (resp. of the edges) are disjoint.*

This definition allows triangulations of the surface with only one or two vertices. For example, take a canonical polygon such as defined in the preceding section. Take a vertex of this polygon, and join it with the other vertices of the polygon. By identifying the edges of the polygon, we have a triangulation of the resulting surface with only one vertex.

We want to know the number of edges e for such a triangulation with n vertices of a compact surface of genus g. As the Euler characteristic is $\chi(g) = (2 - 2g)$ we have $f - e + 2 = 2 - 2g$, where f is the number of faces. As the faces are supposed to be triangles, $f = \dfrac{2}{3}a$ and then

$$a = 6g - 6 + 3n.$$

Local description of the space of polyhedra. Take a subdivision of each face of a convex Fuchsian polyhedron P in triangles (such that the resulting triangulation has no more vertices than the polyhedron, and is invariant under the action of $\rho(\Gamma)$). For such a triangulation on P, we get a map Ed_P which sends each convex Fuchsian polyhedron in a neighbourhood of P in $\mathcal{P}(n)$ to the square of the length of the edges of the triangulation in a fundamental domain for the Fuchsian group action. As this triangulation of P provides a triangulation of the surface S, the map Ed_P has its values in $\mathbb{R}^{6g-g+3n}$.

The map Ed_P associates to each Fuchsian polyhedron a set of $(6g - g + 3n)$ real numbers among all the $d_{\mathbb{H}^3}(\gamma x_i, \mu x_j)^2$, where $\gamma, \mu \in \rho(\Gamma)$, $i, j = 1, \ldots, n$ and $(x_1, \ldots, x_n)$ are the vertices of the polyhedron. It is in

particular a C^1 map (the description of the topology of $\mathcal{P}(n)$ says that for a neighbourhood of P in $\mathcal{P}(n)$ the vertices belong to open sets of the hyperbolic space).

By the local inverse theorem, Theorem 2.13 says exactly that Ed_P is a local homeomorphism.

3.2. Set of metrics with conical singularities

By standard methods involving Voronoi regions and Delaunay cellulations, it is known [15, 23, 8] that for each constant curvature metric with conical singularities on S with constant sign singular curvature there exists a geodesic triangulation such that the vertices of the triangulation are exactly the singular points. This allows us to see such a metric as a gluing of (geodesic) hyperbolic triangles.

(Actually, we don't need this result, because in the following we could consider only the metrics given by the induced metric on convex Fuchsian polyhedra. In this case, the geodesic triangulation of the metric is given by a triangulation of the faces of the polyhedron).

We denote by

- $\mathcal{M}(n)$ the space of Riemannian metrics on S minus n points. It is endowed with the following C^k topology: two metrics are close if their coefficients until those of their kth derivative in any local chart are close (we don't care which $k > 2$);
- $\widetilde{\mathrm{Cone}}(n) \subset \mathcal{M}(n)$ the set of hyperbolic metrics with n conical singularities of positive singular curvature on S, seen as Riemannian metrics after removing the singular points;
- $\mathrm{Cone}(n)$ the quotient of $\widetilde{\mathrm{Cone}}(n)$ by the isotopies of S minus the n marked points;
- $\widetilde{M}^T$ - where T is a geodesic triangulation of an element of $\widetilde{\mathrm{Cone}}(n)$ - the set of the metrics of $\widetilde{\mathrm{Cone}}(n)$ which admit a geodesic triangulation homotopic to T;
- $\mathrm{Conf}(n)$ the set of conformal structures on S with n marked points.

Topology of the set of metrics. A Theorem of Mc Owen–Troyanov [11][24, Theorem A] says that there is a bijection between $\widetilde{\mathrm{Cone}}(n)$ and $\mathrm{Conf}(n) \times]0, 2\pi[^n$. More precisely, for a conformal structure on S, n points on S and n real numbers α_i between 0 and 2π, there exists a unique conformal hyperbolic metric with cone singularities with angles α_i on S (the singular curvature is $(2\pi - \alpha_i)$).

As the Teichmüller space $T_g(n)$ is the quotient of $\mathrm{Conf}(n)$ by the isotopies of S minus its marked points, $\mathrm{Cone}(n)$ is in bijection with $T_g(n) \times]0, 2\pi[^n$, and we endow $\mathrm{Cone}(n)$ with the topology which makes this bijection a homeomorphism:

PROPOSITION 3.6. — *The set $\mathrm{Cone}(n)$ is a contractible manifold of dimension $(6g - 6 + 3n)$.*

Local description of the set of metrics. We call $\widetilde{Ed}_T$ the map from $\widetilde{M}^T$ to $\mathbb{R}^{6g-6+3n}$ which associates to each element of $\widetilde{M}^T$ the square of the length of the edges of the triangulation. The (square of) the distance between two points of S is a continuous function from $\mathcal{M}(n)$ to $\mathbb{R}$.

Remark that $\widetilde{M}^T$ is non empty if we consider a metric given by the induced metric on a convex Fuchsian polyhedron, on which we fix a triangulation. Moreover, around a point of $\widetilde{M}^T$, $\widetilde{Ed}_T$ has its values in an open set of $\mathbb{R}^{6g-6+3n}$: if we change a little the length of the $(6g - 6 + 3n)$ edges, the resulting metric will be still in $\widetilde{M}^T$, because the conditions to remain a hyperbolic triangle and that the sum of the angles around each vertex don't exceed 2π are open conditions.

Let i_T be the canonical inclusion of $\widetilde{M}^T$ (endowed with the topology of $\mathcal{M}(n)$) in $\widetilde{\mathrm{Cone}}(n)$ (endowed with the topology of $\mathrm{Conf}(n) \times]0, 2\pi[^n$).

The composition of i_T with the projection onto $\mathrm{Conf}(n)$ is the map which associates to each metric its conformal structure, and it is continuous mapping as by definition $\mathrm{Conf}(n)$ is the quotient of $\mathcal{M}(n)$ by the set of positive real-valued functions on S minus its marked points; and the composition of i_T with the projection on $]0, 2\pi[^n$ is obviously continuous.

Then i_T is continuous and injective: it is a local homeomorphism and then $\widetilde{Ed}_T$ is a continuous map on $\widetilde{M}^T \subset \widetilde{\mathrm{Cone}}(n)$. Moreover, modulo the isotopies of the surface, the map $\widetilde{Ed}_T$ becomes an injective mapping Ed_T from $M^T \subset \mathrm{Cone}(n)$ (the quotient of $\widetilde{M}^T$ by the isotopies) to $\mathbb{R}^{6g-6+3n}$. And as it has its values in an open set of $\mathbb{R}^{6g-6+3n}$ and as the dimension of $\mathrm{Cone}(n)$ is $(6g - 6 + 3n)$, Ed_T is a local homeomorphism from $\mathrm{Cone}(n)$ to $\mathbb{R}^{6g-6+3n}$ (defined around the images of Fuchsian polyhedra).

Realisation of metrics. We denote by $\mathcal{I}(n)$ the map from $\mathcal{P}(n)$ to $\mathrm{Cone}(n)$, which associates to each convex Fuchsian polyhedral embedding in hyperbolic space its induced metric. This map is well defined, because the induced metric on the quotient of a convex Fuchsian polyhedron (by $\rho(\Gamma)$) is isometric to a hyperbolic metric with conical singularities with positive singular curvature on S.

Let P be a convex Fuchsian polyhedron, and m its induced metric. We consider a triangulation of m given by a subdivision of the faces of P in triangles. Obviously, the (square of) the lengths of the edges of the triangulation of P are the same that the (square of) the lengths of the edges of the triangulation of $m := \mathcal{I}(n)(P)$. It means that locally (recall that the maps Ed_P and Ed_T are defined only locally around P and m):

$$Ed_T \circ \mathcal{I}(n) \circ Ed_P^{-1} = id.$$

$$\begin{array}{ccc} \mathcal{P}(n) & \xrightarrow{\mathcal{I}(n)} & \mathrm{Cone}(n) \\ {\scriptstyle Ed_P} \downarrow & & \downarrow {\scriptstyle Ed_T} \\ \mathbb{R}^{6g-6+3n} & \!\!\!=\!\!\! & \mathbb{R}^{6g-6+3n} \end{array}$$

From this we deduce immediately that $\mathcal{I}(n)$ is continuous and locally injective. Moreover, the map $\mathcal{I}(n)$ is proper: this will be proved in the next paragraph.

Then $\mathcal{I}(n)$ is a covering map. But as $\mathcal{P}(n)$ and $\mathrm{Cone}(n)$ are connected and simply connected, it is a homeomorphism.

Let $\mathrm{Mod}(n)$ be the quotient of the group of the homeomorphisms of S minus n points by its subgroup of isotopies.

Then the homeomorphism $\mathcal{I}(n)$ gives a bijection between $\mathcal{P}(n)/\mathrm{Mod}(n)$ and $\mathrm{Cone}(n)/\mathrm{Mod}(n)$, and this is exactly the statement of Theorem 1.1.

Properness of $\mathcal{I}(n)$. We will use the following characterisation of a proper map: $\mathcal{I}(n)$ is proper if, for each sequence $(P_k)_k$ in $\mathcal{P}(n)$ such that the sequence $(m_k)_k$ converges in $\mathrm{Cone}(n)$ (with $m_k := \mathcal{I}(n)(P_k)$), then $(P_k)_k$ converges in $\mathcal{P}(n)$ (maybe up to the extraction of a sub-sequence).

Suppose that $(m_k)_k$ converges to $m_\infty \in \mathrm{Cone}(n)$. As each m_k is a hyperbolic metric with n conical singularities with positive singular curvature on S, the convergence implies:

 i) a uniform bound on the distance between every pair of singular points of the surface for all k;
 ii) a uniform bound on the values of the angles at the singular points for all k, strictly between 0 and 2π;
 iii) a uniform bound on the lengths of the closed geodesics for all k;
 iv) a uniform bound on the areas of the metrics for all k.

From this we deduce the following assertions (all supposed for a k sufficiently big, and recall that we call the height of a vertex its distance from $P_{\mathbb{H}^2}$):

If the height of one vertex of the polyhedra goes to infinity, the heights of all the vertices go to infinity.

Suppose there exists a vertex which height doesn't go to infinity, incident to a vertex which height goes to infinity. Then the length of the geodesic between them goes to infinity, that is impossible by i).

The heights of all the vertices can't go to infinity.

By iv), on each P_k there is a subset R, homeomorphic to a closed disc, bounding a fundamental domain for the action of $\rho_k(\Gamma)$ and which area is fixed for all k. We consider the projection D_k of R on the umbilic surface realising the minimum d_k of the distance between P_k and $P_{\mathbb{H}^2}$. By orthogonality of the projection, the area of D_k is less than the area of R. The projection of D_k on $P_{\mathbb{H}^2}$ is a closed set $\overline{D}_k$ bounding a fundamental domain for the action of $\rho_k(\Gamma)$ on $P_{\mathbb{H}^2}$. The area of $\overline{D}_k$ is the area of D_k times $(\cosh^2)^{-1}(d_k)$, then it is less than the area of R, which is constant, times $(\cosh^2)^{-1}(d_k)$ (recall that the induced metric on an umbilic surface at distance d from $P_{\mathbb{H}^2}$ is $\cosh^2(d)\mathrm{can}_{\mathbb{H}^2}$, where $\mathrm{can}_{\mathbb{H}^2}$ is the hyperbolic metric induced on $P_{\mathbb{H}^2}$).

If the heights of all the vertices goes to infinity, it implies that d_k goes to infinity, then the area of $\overline{D}_k$ goes to zero, and so the area of the fundamental domain for the action of $\rho_k(\Gamma)$ in $P_{\mathbb{H}^2}$ goes to zero. This is impossible by the Gauss–Bonnet Theorem: the area of a fundamental domain on $P_{\mathbb{H}^2}$ for the action of a $\rho_k(\Gamma)$ is constant for all k (equal to minus the Euler characteristic of S).

The lengths of the edges of the canonical polygons associated to the P_k and the distances between the marked points inside it don't diverge.

The distance between two vertices of the canonical polygon associated to P_k in $P_{\mathbb{H}^2}$ is less than the distance on P_k between the points which are projected onto these vertices (because by orthogonality of the projection the distance in $P_{\mathbb{H}^2}$ is smaller than the distance between the points of P_k in $\mathbb{H}^3$, which is itself smaller than the induced distance on P_k), and this distance is bounded by i), so it converges (maybe after extracting a sub-sequence). The same argument shows that the distance between two marked points inside the canonical polygon converges.

The distance between two marked points inside a canonical polygon can't go to 0.

If it occurs, it implies that the two vertices s_k and s'_k of P_k go to two points on the geodesic joining s_k and $p_{\mathbb{H}^2}(s_k)$ (because we have seen that the heights of the vertices are bounded). These limit points are distinct, because the distance between two singular points is bounded. Then one is lower than the other, say s_k for example. As there is an infinite number of vertices on each polyhedron accumulating on the entire boundary at

infinity of $P_{\mathbb{H}^2}$, there exists at least three vertices lower than s_k and such that their projection on $P_{\mathbb{H}^2}$ forms a triangle containing the projection of s_k: s_k is contained in the interior of the convex hull of these vertices. Then there exists a k' such that for each $k > k'$, s_k is in the convex hull of the others vertices: P_k would be not convex, that is false.

No length of the edges of the canonical polygon goes to 0.

It is as above: if it occurs, there are two vertices of the canonical polygon which collapse. The corresponding points on the polyhedra can't collapse, because the geodesic (on the polyhedron) between them corresponds to a closed geodesic for the induced metric on S and the lengths of closed geodesics are bounded by *iii*). Then these two points go to two points on the same horizontal geodesic, and again it is in contradiction with the convexity of the polyhedra.

These assertions prove that the sequence of canonical polygons associated to the P_k converges to a compact polygon of $P_{\mathbb{H}^2}$, with exactly $4g$ edges and n marked points. As the definition of a canonical polygon shows that the limit of a converging sequence of canonical polygons is a canonical polygon, then $\mathcal{S}(P_k)$ converges in $T_g(n)$. We denote by $[h]$ its limit.

For k sufficiently big, we can consider that all the $\mathcal{S}(P_k)$ are in a neighbourhood of $[h]$ sufficiently small to trivialise the fibration $\mathcal{P}(n)$. That means that we can write the P_k as couples $([h]_k, H_k)$, where H_k is the heights of the vertices of P_k, and we have already seen that these heights converge.

Then $(P_k)_k$ converges to a Fuchsian polyhedron. It remains to check that it is convex with n vertices: by convexity of the P_k, no vertex converges to a point inside the convex hull of the other vertices, and no one converges to a point on the boundary of the convex hull of the other vertices, because the angles at the singularities of m_∞ are all less than 2π by *ii*).

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