*In memory of Alain Dufresnoy, a friend
and an inspiring and generous Mathematician.*

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Jean-Pierre ROSAY

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EXTENSION OF HOLOMORPHIC BUNDLES TO THE DISC (AND SERRE'S PROBLEM ON STEIN BUNDLES)

by Jean-Pierre ROSAY (*)

ABSTRACT. — Holomorphic bundles, with fiber $\mathbb{C}^n$, defined on open sets in $\mathbb{C}$ by locally constant transition automorphisms, are shown to extend to holomorphic bundles on the Riemann sphere. In particular, it allows us to give an example of a non-Stein holomorphic bundle on the unit disc, with polynomial transition automorphisms.

RÉSUMÉ. — On montre que des fibrés holomorphes, à fibre $\mathbb{C}^n$, définis sur des ouverts de $\mathbb{C}$ par des automorphismes de transition localement constants se prolongent en fibrés holomorphes définis sur la sphère de Riemann. Ceci permet en particulier d'obtenir un exemple de fibré non de Stein sur le disque, avec automorphismes de transition polynomiaux.

1. Introduction

We shall study (locally trivial) holomorphic bundles over open sets in $\mathbb{C}$ (in which, the variable will be denoted by ζ), with fiber $\mathbb{C}^n$. We shall assume that, in appropriate local trivializations, the bundle is given by gluing (transition) fiber automorphisms that are locally independent of the base point and that belong to a group $\mathbf{G}$ of automorphisms of $\mathbb{C}^n$.

Of course, such a bundle is trivial over any simply connected region. If U is an open set in $\mathbb{C}$ covered by simply connected domains (U_j), over U any such fiber bundle is obtained by gluing the trivial bundles $U_j \times \mathbb{C}^n$, by using fiber automorphisms belonging to $\mathbf{G}$ and locally independent of the base point.

Our goal is to extend such bundles to bundles defined over larger open sets in $\mathbb{C}$, and in fact over the whole Riemann sphere. For the extended

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bundles, we must allow the fiber automorphisms to (even locally) depend on the base point.

Definition 1.1. — *We shall say that a group* $\mathbf{G}$ *of automorphisms of* $\mathbb{C}^n$ *is generated by one parameter groups if there are one (complex) parameter groups* $(S^t)_{t\in\mathbb{C}}$ *(i.e.,* $S^{s+t} = S^s \circ S^t$*) of automorphisms of* $\mathbb{C}^n$ *whose elements generate* $\mathbf{G}$*.*

Examples to have in mind are:

(A) The group of all polynomial automorphisms of $\mathbb{C}^2$.

(B) For any n, the group of polynomial automorphisms generated by the invertible affine maps and the polynomial shears, *i.e.*, maps of the type

$$(z_1, \cdots, z_n) \mapsto (z_1, \cdots, z_{n-1}, z_n + Q(z_1, \cdots, z_{n-1})),$$

where Q is an arbitrary polynomial in $(n-1)$ variables. A corresponding one parameter group is

$$(z_1, \cdots, z_n) \mapsto S^t(z_1, \cdots, z_n) = (z_1, \cdots, z_{n-1}, z_n + tQ(z_1, \cdots, z_{n-1})).$$

If $n = 2$, this group (generated by affine maps and shears) is the group of all polynomial automorphisms of $\mathbb{C}^2$, by Jung's Theorem [11].

(C) The group of automorphisms of $\mathbb{C}^n$ generated by the invertible affine maps, the shears (as in (B) but with Q entire) and the 'over-shears' (in the terminology of Andersén-Lempert), *i.e.*, maps of the type

$$(z_1, \cdots, z_n) \mapsto (z_1, \cdots z_{n-1}, z_n e^{Q(z_1,\cdots,z_{n-1})}).$$

In that case, set

$$S^t(z_1, \cdots, z_n) = (z_1, \cdots z_{n-1}, z_n e^{tQ(z_1,\cdots,z_{n-1})}).$$

One parameter groups of automorphisms of $\mathbb{C}^n$ are given by the flow of complete (*i.e.*, integrable for all time) holomorphic vector fields on $\mathbb{C}^n$. There are several notions of completeness for holomorphic vector fields: in positive time, in real time or in complex time. In $\mathbb{C}^n$, these three notions coincide [1]. On the topic of Example (C) and of complete holomorphic vector fields, see [3] and [9].

Proposition 1.2. — *Let* D *be a bounded domain in* $\mathbb{C}$ *bounded by a Jordan curve, and for* $j = 1, \cdots N$*, let* D_j *be a region bounded by a Jordan curve. Assume that* $\overline{D}_j \subset D$*,* $\overline{D}_j \cap \overline{D}_k = \emptyset$*, if* $j \neq k$*. Let* $\Lambda = D \setminus \cup_{j=1}^N \overline{D}_j$ *. Let* S *denote the Riemann sphere* $S = \mathbb{C} \cup \{\infty\}$*.*

Let $\Pi : X \to \Lambda$ *be a (locally trivial) holomorphic fiber bundle with fiber* $\mathbb{C}^n$ *and (in appropriate local trivializations) with gluing automorphisms of the fibers locally independent of the base point and in a group* $\mathbf{G}$ *of* *automorphisms of* $\mathbb{C}^n$ *generated by one parameter groups. Then there exists a holomorphic fiber bundle* $\tilde{X} \to S$ *with fiber* $\mathbb{C}^n$ *and (in appropriate local trivializations) with gluing automorphisms of the fibers in* $\mathbf{G}$ *(allowed to depend on the base point), whose restriction over* Λ *is* X.

COROLLARY 1.3. — (a) *The fiber bundle in Skoda's original example of non-Stein bundle* [13] *(and see* [12]*) extends to a holomorphic bundle over the Riemann sphere (and so to a holomorphic bundle over the disc).*

(b) *The possibly non-Stein fiber bundles constructed by Demailly* [6] *on annuli, with fiber* $\mathbb{C}^2$ *and the gluing automorphism*

$$(z_1, z_2) \mapsto (z_2, -z_1 + z_2^k) \ ,$$

extend to holomorphic bundles on the Riemann sphere with polynomial automorphisms of the fibers.

The Corollary is immediate, and the consequences for the problem of Stein bundles are as follows.

(α) There are two versions of Skoda's example. In the first, more transparent, one (Theorem in [13]), the base is made of the unit disc with 8 discs removed. In the second one (final remark in [13] and Theorem 1 in [12]) the base is the disc with 2 discs removed. In the first example, around each deleted disc the bundle is defined by a gluing automorphism which, up to a possible permutation of the coordinates, is an over-shear as in Example C. In the second example, around one of the deleted disc the gluing is given by the automorphism $(z_1, z_2) \mapsto (z_1, z_2 e^{z_1})$, and for the other deleted disc by $(z_1, z_2) \mapsto (iz_2, z_1)$.

So, starting from Skoda's example, one gets a new construction of a non-trivial, non-Stein, holomorphic bundle on the unit disc (and on $\mathbb{C}$). This construction seems to me to be much less technical than the original example of Demailly [6] and [7]. Furthermore the construction relies only on the simplest case in the proof of Section II, with a trivial filling of each hole, since around each hole the gluing is made by automorphisms that are elements of a one parameter group of automorphisms.

(β) The maps used by Demailly are of Henon type. The map $(z_1, z_2) \mapsto (z_2, -z_1 + z_2^k)$ is the composition of the linear map $(z_1, z_2) \mapsto (z_2, -z_1)$, and of the shear $(z_1, z_2) \mapsto (z_1, z_2 + z_1^k)$. Each of these two maps is an element of a one parameter group of polynomial automorphisms. But, if $k \geqslant 2$, the map $(z_1, z_2) \mapsto (z_2, -z_1 + z_2^k)$ itself is not even an element of a one parameter group of (possibly non-polynomial) automorphisms of $\mathbb{C}^2$, since the proof of Case 1 in Section II would then show that the Demailly bundles would be trivial. Following Section II, the extension of the bundle from the annulus to the disc is then obtained by first extending the bundle to the disc with 2 (smaller) holes, and then extending trivially to each hole.

Starting from the polynomial examples of Demailly (non-Stein for a given annulus if k is large enough, and non-Stein for a given $k \geqslant 2$ if the annulus is thick enough), one gets examples of holomorphic bundles on the unit disc (and on $\mathbb{C}$), with fiber $\mathbb{C}^2$, and with gluing polynomial automorphisms, that are non trivial and non-Stein. This answers a question in [6]. The question was asked again recently by H. Skoda, who considered it to be the last question in the Serre Problem on Stein bundles, left open after the counterexamples of Skoda [13] [12], Demailly [6] [7], and Coeuré-Loeb [5].

If one takes $k = 2$, all the polynomial automorphisms to be used in the proof of the Proposisiton can be chosen of degree $\leqslant 2$. So, there are examples of non-Stein bundles on the disc, with polynomial gluing automorphisms of the fiber of degree $\leqslant 2$, while all bundles on the disc with affine gluing automorphisms are trivial.

2. Proof of the Proposition

It is of course enough to extend the given bundle X over each "hole" D_j, and over the component D_0 of $S \setminus \overline{D}$, containing ∞. Fix $j \in \{0, 1, \cdots, N\}$, and let V be a neighborhood of bD_j in Λ that is topologically an annulus. We shall decompose V into the union $V^+ \cup V^-$ of two simply connected regions with intersection $V^+ \cap V^- = \omega \cup \omega'$, where ω and ω' are disjoint connected open sets. We wish to extend the bundle $X|\Pi^{-1}(V)$ to a bundle over $\overline{D}_j \cup V$, defined by automorphisms of the fibers in the group $\mathbf{G}$. Unneeded for the easy Case 1 below, un-necessary but helpful for the general case, we shall be more specific. For $r > 0$, let Δ_r be the open disc in $\mathbb{R}^2$ of radius r centered at the origin. We can take V and $V^\pm$ so that for some homeomorphism χ of Δ_2 into S: $D_j = \chi(\Delta_1)$, and $V = \chi(\Delta_2 \setminus \overline{\Delta}_1)$, V^+ and V^- being respectively the image of the regions $\operatorname{Im} \zeta > -\frac{1}{2}$ and $\operatorname{Im} \zeta < \frac{1}{2}$.

Since we start with locally constant gluing fiber automorphisms, and since V^+ and V^- are simply connected, the fiber bundle X is trivial over V^+ and over V^-. Over V, it can be defined by gluing $V^+ \times \mathbb{C}^n$ and $V^- \times \mathbb{C}^n$ over $\omega \cup \omega'$, with the identification of $(\zeta, z) \in V^+ \times \mathbb{C}^n$ with $(\zeta, T_0 z) \in V^- \times \mathbb{C}^n$ if $\zeta \in \omega$, and with $(\zeta, T_1 z) \in V^- \times \mathbb{C}^n$ if $\zeta \in \omega'$, where T_0 and T_1 are automorphisms of $\mathbb{C}^n$ that belong to $\mathbf{G}$. Of course we could take $\mathbb{T}_0 = \mathbf{1}$ (but we don't, in order to simplify the exposition in the second case below). We now split the proof in two cases.

2.1. Case 1

This is the easy case when there is in $\mathbf{G}$ a one (complex) parameter group of automorphisms $(S^t)_{t\in\mathbb{C}}$ such that $T_1 \circ T_0^{-1} = S^1$. In that case, the bundle extends trivially over D_j, and the trivialization is by means of fiber automorphisms in $\mathbf{G}$. (If one wished to get a version of the Proposition for bundles defined by gluing automorphisms depending on the base point, one would need here that both T_0 and T_1 be defined on $V^+ \cup V^-$, as holomorphic function of the base point, and the possibity of a holomorphic choice of the one parameter group).

For identifying the trivial bundle $(\overline{D}_j \cup V) \times \mathbb{C}^n$ with the restriction of X over V, we need to define fiber automorphisms:

$$\Phi^+ : V^+ \times \mathbb{C}^n \to V^+ \times \mathbb{C}^n$$

and

$$\Phi^- : V^- \times \mathbb{C}^n \to V^- \times \mathbb{C}^n \ ,$$

such that

$$(*)\ (\Phi^-)^{-1} \circ \tilde{T}_0 \circ \Phi^+ = \mathbf{1} \text{ on } \omega \times \mathbb{C}^n \ ,$$
$$(**)\ (\Phi^-)^{-1} \circ \tilde{T}_1 \circ \Phi^+ = \mathbf{1} \text{ on } \omega' \times \mathbb{C}^n \ ,$$

where $\tilde{T}_j(\zeta, z) = (\zeta, T_j(z))$. Let L^+ and L^- be holomorphic functions defined respectively on V^+ and V^-, such that $L^+ + L^- = 0$ on ω, $L^+ + L^- = -1$ on ω'. This is a classical Cousin problem that here is elementary to solve (use logarithms!). Set

$$\Phi^+(\zeta, z) = \big(\zeta, T_0^{-1} S^{L^+(\zeta)}(z)\big)$$
$$\Phi^-(\zeta, z) = \big(\zeta, S^{-L^-(\zeta)}(z)\big) \ .$$

(Here comes the local dependence on ζ.)

For $\zeta \in \omega$:

$$[S^{-L^-(\zeta)}]^{-1} \circ T_0 \circ (\mathbb{T}_0^{-1} \circ S^{L^+(\zeta)}) = S^{L^-(\zeta)+L^+(\zeta)} = S^0 = \mathbf{1} \ ,$$

so $(*)$ is satisfied.

For $\zeta \in \omega'$:

$$\begin{aligned}(2.1)\quad [S^{-L^-(\zeta)}]^{-1} \circ T_1 \circ (\mathbb{T}_0^{-1} \circ S^{L^+(\zeta)}) &= S^{L^-(\zeta)} \circ S^1 \circ S^{L^+(\zeta)} \\ &= S^{L^-(\zeta)+1+L^+(\zeta)} = S^0 = \mathbf{1},\end{aligned}$$

so $(**)$ is satisfied.

That ends the proof of Case 1. Notice that in case a trivial extension is possible over each D_j, it does not give a global trivial extension, even just on $\mathbb{C}$ or D. Skoda's example illustrates that.

2.2. General Case

The general case can be reduced to the previous case by introducing a higher connectivity (replacing D_j by finitely many smaller holes) before filling each hole individually trivially. Recall that the 'hole' D_j is parameterized by Δ_1, under the map χ, and that $\overline{D}_j \cup V$ is parameterized by Δ_2. In general one has

$$T_1 \circ T_0^{-1} = E_k \circ \cdots \circ E_1 \ ,$$

where each E_p $(p = 1, \cdots, k)$ is an element of a one parameter group of automorphisms in $\mathbf{G}$.

Fix points

$$\text{(2.2)} \quad b_0 = -2 \ , \quad a_1 = -1 < b_1 < a_2 < b_2 < \cdots < a_p < b_p < a_{p+1} < \cdots < b_k = 1 \ , \quad a_{k+1} = 2.$$

Let W_0 be the complement in Δ_2 of the circles of diameter $[a_p, b_p]$, $p = 1, \cdots, k$, and let $W = \chi(W_0)$. So, W is a multiply connected region in S that contains V and that is contained in $V \cup \overline{D}_j$. Set $W^+ = \chi(W_0 \cap \{\mathrm{Im}\,\zeta \geqslant 0\})$, and $W^- = \chi(W_0 \cap \{\mathrm{Im}\ \zeta \leqslant 0\})$. We first extend the restriction of the given fiber bundle X over V, to a fiber bundle X' over W. For that purpose, consider the fiber bundle X' obtained by gluing $W^+ \times \mathbb{C}^n$ with $W^- \times \mathbb{C}^n$ over each component $\chi([b_p, a_{p+1}])$ of the intersection, by using the automorphism $E_p \circ \cdots \circ E_1 \circ T_0$ (T_0 if $p = 0$, T_1 if $p = k$). Now, for the extended bundle X', over each hole of W (the images under χ of the discs of diameter $[a_p, b_p]$), we are in Case 1, since

$$[E_p \circ \cdots E_1 \circ T_0] \circ [E_{p-1} \circ \cdots E_1 \circ T_0]^{-1} = E_p \ .$$

For $p = 1$, read $[E_1 \circ T_0] \circ T_0^{-1} = E_1$. Therefore, according to Case 1, X' extends to a bundle over $\Lambda \cup \overline{D}_j$, with gluing automorphisms in $\mathbf{G}$ (trivially over each hole of W, but not trivially over D_j as illustrated by Demailly's example).

Remark 2.1. — It has been pointed out that the map $(z_1, z_2) \mapsto (z_2, -z_1 + z_2^k)$, for $k \geqslant 2$, is not an element of a one parameter group of automorphisms of $\mathbb{C}^2$, *i.e.,* it is not the time-1 map of a complete vector field on $\mathbb{C}^2$. The reason given was rather indirect. For a more direct approach to this question, see [2] and [4]. F. Forstnerič pointed out to me the interesting fact that, although they are far from being trivial since they can even be non-Stein, all holomorphic fibers bundles with fiber C^n over a Stein base have holomorphic sections [10], [8], and they satisfy the "Oka Principle".

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Jean-Pierre ROSAY
University of Wisconsin
Department of Mathematics
Madison WI 53706 (USA)
jrosay@math.wisc.edu