

ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 16, n° 1 (1966), p. 145-157

http://www.numdam.org/item?id=AIF_1966__16_1_145_0

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ON Φ -BOUNDED HARMONIC FUNCTIONS

by MITSURU NAKAI

1. Throughout this paper, we denote by $\Phi(t)$ a *non-negative* real-valued function defined on the half real line $[0, \infty) = (t; 0 \leq t < \infty)$. A harmonic function u on a Riemann surface R is called Φ -bounded if the composite function $\Phi(|u|)$ admits a harmonic majorant on R , i. e. there exists a harmonic function h such that $\Phi(|u|) \leq h$ on R . We denote by

$$H\Phi = H\Phi(R)$$

the totality of Φ -bounded harmonic functions on a Riemann surface R and by $O_{H\Phi}$ the class of all Riemann surfaces on which every Φ -bounded harmonic function reduces to a constant. In our study, the following two quantities will play an important role :

$$\bar{d}(\Phi) = \limsup_{t \rightarrow \infty} \Phi(t)/t$$

$$\underline{d}(\Phi) = \liminf_{t \rightarrow \infty} \Phi(t)/t$$

The properties of $H\Phi$ -functions on Riemann surfaces and the class $O_{H\Phi}$ are first investigated by Parreau [3] for the special $\Phi(t)$ which is increasing and convex ⁽¹⁾. In the present paper we shall investigate the same problem for general $\Phi(t)$. Our conclusion is, roughly speaking, that Parreau's result about $O_{H\Phi}$ holds essentially for general $\Phi(t)$ and his result about properties of $H\Phi$ -functions can be derived by assuming $\underline{d}(\Phi) > 0$ instead of increasingness and convexity which is, in a sense, the weakest condition.

2. As for the class $O_{H\Phi}$, Parreau [3] showed that the class $O_{H\Phi}$ for

(1) For such a function, it is well-known that $\bar{d}(\Phi) = \underline{d}(\Phi) > 0$.

increasing and convex $\Phi(t)$ coincides with O_{HP} or O_{HB} ⁽²⁾ according to $\bar{d}(\Phi) < \infty$ or $\bar{d}(\Phi) = \infty$, respectively. Now we ask what can be said about $O_{H\Phi}$ for general $\Phi(t)$. The answer is given by

THEOREM 1. — *If $\bar{d}(\Phi) < \infty$ (resp. $\bar{d}(\Phi) = \infty$), then $O_{H\Phi} \subset O_{HP}$ (resp. $O_{H\Phi} \supset O_{HB}$).*

This was proved implicitly in our former paper [2] by using Wiener's compactification of Riemann surfaces. We shall again give an alternating elementary proof in § 1. In this theorem, we cannot replace the inclusion relation by the equality in general. But the function $\Phi(t)$, by which the equality does not hold in the above theorem, is very singular and trivial one from the view point of $H\Phi$ -functions as the following shows :

THEOREM 2. — (i) *If $\Phi(t)$ is bounded on $[0, \infty)$, then $O_{H\Phi}$ consists of all closed Riemann surfaces;*

(ii) *If $\Phi(t)$ is completely unbounded ⁽³⁾ on $[0, \infty)$, then $O_{H\Phi}$ consists of all open Riemann surfaces;*

(iii) *If $\Phi(t)$ is not bounded and not completely unbounded, then $O_{H\Phi} = O_{HP}$ or O_{HB} according to $\bar{d}(\Phi) < \infty$ or $\bar{d}(\Phi) = \infty$, respectively.*

This was proved in [2] and determines the class $O_{H\Phi}$ completely for any possible $\Phi(t)$. This is easily proved by using Theorem 1. We will do this also in § 1.

Observing Theorem 2, we are tempted to conclude that $H\Phi$ -property is closely related to positiveness or boundedness properties except trivial Φ 's as in (i) or (ii). Next we consider this problem. To state the problem formally, let us recall three notions for harmonic functions : essentially positive, quasi-bounded and singular.

3. A harmonic function u on a Riemann surface R is called *essentially positive* if u can be represented as a difference of two HP-functions on R , or equivalently, if u admits a harmonic majorant on R . We denote the totality of essentially positive harmonic functions on R by

$$HP' = HP'(R).$$

⁽²⁾ As usual, $HP(R)$ (resp. $HB(R)$) denotes the totality of non-negative (resp. bounded) harmonic functions on R . The meaning of O_{HP} and O_{HB} is similar to that of $O_{H\Phi}$.

⁽³⁾ We say that $\Phi(t)$ is *completely unbounded* on $[0, \infty)$ if $\Phi(t)$ is not bounded at any neighbourhood of any point in $[0, \infty)$.

Clearly $HP'(\mathbb{R}) \supset HP(\mathbb{R})$. For two functions u and v in $HP'(\mathbb{R})$, there always exists the least harmonic majorant (resp. the greatest harmonic minorant) of u and v , which we denote by $u \vee v$ (resp. $u \wedge v$). Then $HP'(\mathbb{R})$ forms a vector lattice with lattice operations \vee and \wedge . For u in $HP'(\mathbb{R})$, we denote by Mu the function $u \vee 0 + (-u) \vee 0$, which is the least harmonic majorant of $|u|$. Next first for u in $HP(\mathbb{R})$, we denote by Bu the HP-function defined by $\sup (v(p); u \geq v \in HB(\mathbb{R}))$ on \mathbb{R} . Clearly B is order-preserving, linear and $B^2 = B$ on $HP(\mathbb{R})$ (see Ahlfors-Sario [1], p. 210). Next for u in $HP'(\mathbb{R})$, we put $Bu = Bu_1 - Bu_2$, where $u = u_1 - u_2$ and u_1 and u_2 are in $HP(\mathbb{R})$. Here, by the linearity of B on $HP(\mathbb{R})$, Bu does not depend on the special decomposition of u into HP-functions. Again the operator B is order-preserving, linear and $B^2 = B$ on $HP'(\mathbb{R})$ and moreover B commutes with M , \vee , and \wedge . This is clear on $HP(\mathbb{R})$ by definitions of B , \vee , \wedge and M . For the general case, we have only to show that $B(u \vee 0) = (Bu) \vee 0$. Since

$$Bu = B(u \vee 0) - B((-u) \vee 0)$$

and

$$\begin{aligned} B(u \vee 0) \wedge B((-u) \vee 0) &= B((u \vee 0) \wedge ((-u) \vee 0)) \\ &= B0 = 0, \end{aligned}$$

$B(u \vee 0)$ is the positive part of the Jordan decomposition of Bu .

An HP'-function u is called *quasi-bounded* (resp. *singular*) if $Bu = u$ (resp. $Bu = 0$). These notions were introduced by Parreau [3]. We denote the totality of quasi-bounded harmonic functions on \mathbb{R} by

$$HB' = HB'(\mathbb{R}).$$

Clearly $HB' \supset HB$. Since B commutes with M , \vee and \wedge , we see that $Bu = u$ is equivalent to $BMu = Mu$. Hence we can also define

$$HB'(\mathbb{R}) = (u \in HP'(\mathbb{R}); BMu = Mu).$$

4. Parreau [3] showed that, for increasing and convex function $\Phi(t)$, $H\Phi \subset HP'$ and if moreover $\bar{d}(\Phi) = \infty$, then $H\Phi \subset HB'$. Our next problem is to investigate whether such relations hold or not for general $\Phi(t)$. The answer is negative in general: we shall single out in § 4 an increasing continuous unbounded function $\Phi(t)$ with $\bar{d}(\Phi) < \infty$ and $\underline{d}(\Phi) = 0$ and an $H\Phi$ -function in the open unit disc which is not an HP' -function there (*Example 2*). This shows the invalidity of $H\Phi \subset HP'$

in general. Only for this aim, we may take bounded $\Phi(t)$. But we are interested in unbounded $\Phi(t)$. We shall also construct in § 3 an increasing continuous function $\Phi(t)$ with $\bar{d}(\Phi) = \infty$ and $\underline{d}(\Phi) = 0$ and an $H\Phi$ -function in the open unit disc which is not an HP' -function there (*Example 1*). This shows the invalidity of the relation $H\Phi \subset HP'$ and so of the relation $H\Phi \subset HB'$ even if $\bar{d}(\Phi) = \infty$.

Then there arises the question when can we conclude the relation $H\Phi \subset HP'$ or HB' . Both examples above show that unboundedness, not completely unboundedness, increasingness, continuity or all of them cannot give the condition. In both examples above, $\underline{d}(\Phi) = 0$. This suggests us that the required condition may be $\underline{d}(\Phi) > 0$. This is really the case. Firstly the answer for $H\Phi \subset HP'$ is given completely by the following which includes Parreau's case :

THEOREM 3. — *In order that $H\Phi(\mathbb{R}) \subset HP'(\mathbb{R})$ for any Riemann surface \mathbb{R} , it is necessary and sufficient that $\underline{d}(\Phi) > 0$ (no matter whether $\bar{d}(\Phi)$ is finite or infinite).*

The proof of this will be given in § 5. Similarly we ask about the condition which assures the relation $H\Phi \subset HB'$. In this case, even in the Parreau's case, we must assume that $\bar{d}(\Phi) = \infty$ as the following simple example shows : $\Phi(t) = t$, $\mathbb{R} = (z ; 0 < |z| < 1)$ and $u(z) = -\log |z|$. The best possible general conclusion is as follows :

THEOREM 4. — *If $\bar{d}(\Phi) = \infty$, then $H\Phi(\mathbb{R}) \cap HP'(\mathbb{R}) \subset HB'(\mathbb{R})$.*

Here we cannot drop $HP'(\mathbb{R})$ in the above relation as *Example 1* shows. The above theorem will be proved in § 6. Now assume that $\underline{d}(\Phi) > 0$, then by Theorems 3 and 4, $H\Phi(\mathbb{R}) \subset HB'(\mathbb{R})$. Conversely if $H\Phi(\mathbb{R}) \subset HB'(\mathbb{R})$ for any \mathbb{R} , then $H\Phi(\mathbb{R}) \subset HP'(\mathbb{R})$ for any \mathbb{R} and by Theorem 3, $\underline{d}(\Phi) > 0$. Thus we get the following which includes Parreau's case :

THEOREM 5. — *Assume that $\underline{d}(\Phi) = \infty$. In order $H\Phi(\mathbb{R}) \subset HB'(\mathbb{R})$ for any Riemann surface \mathbb{R} , it is necessary and sufficient that $\underline{d}(\Phi) > 0$.*

1. Proofs of Theorems 1 and 2.

1. Proof of Theorem 1. — **I.** The case $\bar{d}(\Phi) = \infty$: Assume that there exists a non-constant $H\Phi$ -function u on \mathbb{R} . By the definition of

Φ -boundedness, there exists an HP-function h such that $\Phi(|u|) \leq h$ on R . We have to show that $R \notin O_{HB}$. Contrary to the assertion, assume that $R \in O_{HB}$. Since $\bar{d}(\Phi) = \infty$, we can find a strictly increasing sequence $(r_n)_{n=1}^\infty$ of positive numbers r_n such that $\lim_{n \rightarrow \infty} r_n = \infty$, $\Phi(r_n) > 0$, $G_n = \{p \in R; |u(p)| < r_n\} \neq \emptyset$ and $\lim_{n \rightarrow \infty} a_n = 0$, where $a_n = r_n/\Phi(r_n)$. Then clearly

$$G_1 \subset G_2 \subset \dots \subset G_n \subset \dots, \quad R = \bigcup_{n=1}^\infty G_n.$$

First we show that $G_n \notin SO_{HB}$ for some n on (4) . If this is not the case, then $G_n \in SO_{HB}$ for all n 's. Then since $a_n h - |u|$ is superharmonic and bounded from below on G_n and

$$a_n h - |u| \geq a_n \Phi(|u|) - |u| = a_n \Phi(r_n) - r_n = 0$$

on ∂G_n , we can conclude that $a_n h - |u| \geq 0$ on G_n . Since $a_n \searrow 0$, we must have $u \equiv 0$ on R , which is clearly a contradiction. Hence we may assume that $G_n \notin SO_{HB}$ ($n = 1, 2, 3, \dots$) by choosing a suitable subsequence of (r_n) , if necessary.

Next we assert that $G_n - \bar{G}_1 \in SO_{HB}$ ($n = 1, 2, 3, \dots$). For, if there exists a G_n with $G_n - \bar{G}_1 \notin SO_{HB}$, then there would exist two disjoint non-empty open sets G_1 and $G_n - \bar{G}_1$ not belonging to SO_{HB} . By the so called "two domains criterion", we must have that $R \notin O_{HB}$ (see Ahlfors-Sario [1], p. 213). But this contradicts our assumption $R \in O_{HB}$.

Now consider the function $w_n = a_n h + r_1 - |u|$ on G_n , which is superharmonic and bounded from below on G_n and so on $G_n - \bar{G}_1$. By the similar manner as before, we see that $w_n \geq a_n h - |u| = 0$ on ∂G_n . Clearly $w_n \geq r_1 - |u| = 0$ on ∂G_1 . Hence $w_n \geq 0$ on $\partial(G_n - \bar{G}_1)$. Since $G_n - \bar{G}_1 \in SO_{HB}$, we can conclude that $w_n \geq 0$ on G_n or $|u| \leq a_n h + r_1$ on G_n . Hence by the fact that $a_n \searrow 0$, we get that $|u| \leq r_1$ on R . This contradicts our assumption that $R \in O_{HB}$. Thus we must have $R \notin O_{HB}$.

II. The case $\bar{d}(\Phi) \leq \infty$: Assume that there exists a non-constant HP-function u on R . By $\bar{d}(\Phi) < \infty$, we can find a point s in $[0, \infty)$ such that there exists a finite positive constant C with $\Phi(t) \leq Ct$ ($s \leq t < \infty$). Let $v = s + u$. Clearly v is a non-constant HP-function on R with

(4) An open subset G of a Riemann surface R with smooth relative boundary ∂G is said to belong to SO_{HB} if every HB-function on G with continuous boundary value zero at ∂G reduces to a constant zero.

$v \geq s$ on R . Hence $\Phi(|v|) = \Phi(v) \leq Cv$ on R . Thus v is a non-constant $H\Phi$ -function on R and so $R \notin O_{H\Phi}$.

2. Proof of Theorem 2. — *Ad (i)*: If $\Phi(t)$ is bounded, then every non-constant harmonic function is an $H\Phi$ -function. Thus $O_{H\Phi}$ consists of all Riemann surfaces carrying no non-constant harmonic function, which are closed Riemann surfaces.

Ad (ii): For any non-constant harmonic function u on R , since u is open map of R into $[0, \infty)$ by the maximum principle, $\Phi(|u|)$ is completely unbounded on R along with $\Phi(t)$ and so u is not $H\Phi$ -function. Thus there exists no non-constant $H\Phi$ -function on any Riemann surface and $O_{H\Phi}$ consists of all Riemann surfaces.

Ad (iii): Assume that $\bar{d}(\Phi) = \infty$ and that there exists a non-constant HB -function u on R . As $\Phi(t)$ is not completely unbounded, so there exists an interval (a, b) in which $\Phi(t) < c = \text{const}$. Let

$$v = (a + b)/2 + ((b - a)/2) (\sup_R |u|)^{-1} u.$$

Then v is a non-constant HB -function and $\Phi(|v|) = \Phi(v) < c$ on R . Thus $O_{H\Phi} \subset O_{HB}$. This with Theorem 1 gives $O_{H\Phi} = O_{HB}$.

Next assume that $\bar{d}(\Phi) < \infty$. By Theorem 1, $O_{HP} \supset O_{H\Phi}$. Contrary to the assertion, assume that there exists an R in $O_{HP} - O_{H\Phi}$. Let u be a non-constant $H\Phi$ -function on R . Then $\Phi(|u|) \leq c = \text{constant}$ on R . Since $\Phi(t)$ is unbounded and $|u|(R)$ is connected in $[0, \infty)$ and contains 0, u must be bounded on R . Then $\sup_R |u| + u$ is a non-constant HP -function on R , which contradicts the assumption that $R \in O_{HP}$. Hence $O_{H\Phi} = O_{HP}$.

2. Preparations for Examples.

Let $U = \{z; |z| < 1\}$ and A be an arc in $\partial U = \{z; |z| = 1\}$. We denote by $w(z; A)$ the harmonic measure of A calculated at z in U with respect to U . It is well known that

$$(1) \quad w(z; A) = (2\beta - \alpha)/2\pi,$$

where α is the length of A and β is the angle seeing the arc A from z .

We denote by L_A the line segment connecting both end points of A . Then from (1), we easily see that

- (2) $w(0; A) = \alpha;$
- (3) $w(z; A) = 1 - \alpha/2\pi$ on L_A .

Next let A_j be the arc in $\partial U = \{z; |z| = 1\}$ with end points 1 and $e^{i\alpha_j}$ ($j = 1, 2$) such that $0 < \alpha_1 < \alpha_2, \alpha_1 < \pi/2, \alpha_2 < \pi/2$. We denote by \tilde{A}_j (resp. A'_j) the arc with end points 1 and $e^{-i\alpha_j}$ (resp. $A'_j = A_j \cup \tilde{A}_j$). For simplicity, we set $L_2 = L_{A_2}$, i.e. L_2 is the line segment connecting two end points of A'_2 . Then we get the following inequality which plays an important role in our forth-coming examples : there exists a universal constant $s_0 (\leq 4^{-1} \pi^4)$ such that

$$(4) \quad |w(z; A_1) - w(z; \tilde{A}_1)| \leq s_0 \alpha_1^2 / (\alpha_2^2 - \alpha_1^2)^2 \quad \text{on } L_2.$$

Proof. — We denote the points $e^{i\alpha_1}, e^{-i\alpha_1}, (e^{i\alpha_1} + e^{-i\alpha_1})/2, 1, (e^{i\alpha_2} + e^{-i\alpha_2})/2$ and z on L_2 with $\text{Im}(z) \geq 0$ by D, E, F, G, H and P respectively. We set $DF = FE = d, FH = k, DP = a, PF = b$ and $PE = c$. By (1), $w(z; A_1) - w(z; \tilde{A}_1) = (\angle DPG - \angle GPE)/\pi$. Let $\angle DPF = \theta_1$ and $\angle FPE = \theta_2$. Then clearly $\angle DPG \leq \theta_1$ and $\angle GPE \geq \theta_2$. Hence we have $0 \leq w(z; A_1) - w(z; \tilde{A}_1) \leq (\theta_1 - \theta_2)/\pi$. Applying the cosine theorem to triangles $\triangle DPF$ and $\triangle FPE$ and then Pappos' identity to the triangle $\triangle DPE$, we have

$$\sin 2^{-1}(\theta_1 - \theta_2) = (c - a) (8abc \sin 2^{-1}(\theta_1 + \theta_2))^{-1} (4d^2 - (a - c)^2).$$

Here we have

$$\begin{aligned} ac \sin 2^{-1}(\theta_1 + \theta_2) &\geq ac \sin 2^{-1}(\theta_1 + \theta_2) \cos 2^{-1}(\theta_1 + \theta_2) \\ &= 2^{-1} ac \sin \angle DPE = \Delta DPE \\ &= \Delta DHE = dk. \end{aligned}$$

By the triangle inequality applied for $\triangle DPE, c - a \leq 2d$. Thus by noticing $b \geq k$, we have $\sin 2^{-1}(\theta_1 - \theta_2) \leq d^2 k^{-2}$. As

$$\sin \theta \geq (2/\pi) \theta \quad (0 \leq \theta \leq 2^{-1} \pi),$$

so $\theta_1 - \theta_2 \leq \pi d^2 k^{-2}$. Now we have $d = \sin \alpha_1 \leq \alpha_1$ and

$$\begin{aligned} k &= \cos \alpha_1 - \cos \alpha_2 \\ &= 2 \sin^{-1}(\alpha_1 + \alpha_2) \sin 2^{-1}(\alpha_2 - \alpha_1) \geq 2 \pi^{-2} (\alpha_2^2 - \alpha_1^2). \end{aligned}$$

Hence

$$0 \leq w(z; A_1) - w(z; \tilde{A}_1) \leq 4^{-1} \pi^4 \alpha_1^2 / (\alpha_2^2 - \alpha_1^2)^2. \quad Q.E.D.$$

We shall use (4) in the particular case where $0 < \alpha_1 < \alpha_2/\sqrt{2}$. In this case, by using universal constant $s (\leq \pi^4)$, we get

$$(5) \quad |w(z; A_1) - w(z; \tilde{A}_1)| \leq s(\alpha_1^2 / \alpha_2^2) \quad \text{on } L_2.$$

3. Example 1.

We are now able to construct an example of a function Φ which is continuous, increasing, $\bar{d}(\Phi) = \infty$ and $\underline{d}(\Phi) = 0$; and an $H\Phi$ -function u in then open unit disc $U = (z; |z| < 1)$ which is not an HP' -function.

EXAMPLE 1. Let p be a constant such that $0 < p < \min(1/4, 1/4s)$, where s is the constant in (5) in § 2, and $(p_n)_{n=1}^\infty$ be a sequence defined by $p_n = (p^{4\nu})^{2\nu + \mu}$ for $n = 2^\nu + \mu$ ($\nu = 0, 1, 2, \dots; \mu = 1, 2, 3, \dots, 2^\nu$). Let A_n and \tilde{A}_n be arcs on the unit circumference such that

$$A_n = (e^{i\theta}; 0 \leq \theta \leq 2 p_n \pi/n)$$

and

$$\tilde{A}_n = (e^{i\theta}; -2 p_n \pi/n \leq \theta \leq 0).$$

Let $(r_\nu)_{\nu=1}^\infty$ and $(b_\nu)_{\nu=1}^\infty$ be two sequences of positive numbers defined by $r_\nu = 2/(p^{4\nu-1})^{2\nu}$ and $b_\nu = 2^{\nu/2} \cdot r_\nu$. Define the function $\Phi(t)$ on $[0, \infty]$ by

$$\Phi(t) = \begin{cases} 0, & t \in [0, r_1]; \\ b_1(t - r_1), & t \in [r_1, r_1 + 1]; \\ b_\nu, & t \in [r_\nu + 1, r_{\nu+1}] (\nu = 1, 2, \dots); \\ b_\nu + (b_{\nu+1} - b_\nu)(t - r_{\nu+1}), & t \in [r_{\nu+1}, r_{\nu+1} + 1] (\nu = 1, 2, \dots) \end{cases}$$

and the function $u(z)$ in U by

$$u(z) = \sum_{n=1}^\infty (w(z; A_n) - w(z; \tilde{A}_n))/p_n.$$

Then the following hold :

- (a) $\Phi(t)$ is continuous, increasing, $\bar{d}(\Phi) = \infty$ and $\underline{d}(\Phi) = 0$;
- (b) $u(z)$ is well defined in U and harmonic there;
- (c) $u(z) \in H\Phi(U)$;
- (d) $u(z) \notin HP'(U)$.

Proof of (a). — Is immediate by the definition of $\Phi(t)$.

Proof of (b). — For each $n = 1, 2, \dots$, set

$$v_n(z) = w(z; A_n) - w(z; \tilde{A}_n), \quad u_n(z) = \sum_{k=1}^n v_k(z)/p_k.$$

Then v_n and u_n are harmonic in U , positive in the upper half of U and $v_n(-z) = -v_n(z)$ and $u_n(-z) = -u_n(z)$ in U . Hence to show that the series defining $u(z)$ is convergent in U and defines a harmonic function there, we have only to prove that $(u_n(i/2))_{n=1}^\infty$ is convergent. By (5) in § 2, we have that

$$0 < v_n(i/2) \leq s (2 p_n \pi/n)^2 / (\pi/2)^4 \leq s' p_n^2,$$

where s' is a constant independent of $n \geq 1$. Thus

$$0 < u_{n+m}(i/2) - u_n(i/2) = \sum_{k=n+1}^{n+m} v_k(i/2)/p_k \leq s' \sum_{k=n+1}^{n+m} p_k < s' p^n / (1 - p).$$

This shows that $(u_n(i/2))_{n=1}^\infty$ is convergent.

Proof of (c). — For each $\nu = 1, 2, \dots$, we denote by L_ν the line segment $L_{A_{2^\nu}}$, i.e. the line segment connecting two end points of $A_{2^\nu} = A_{2^\nu} \cup \tilde{A}_{2^\nu}$. Since $|v_k(z)| < 1$ in U , we have

$$|v_k(z)/p_k| \leq 1/p_k \leq 1/(p^{4^{\nu-1}})^k \quad (1 \leq k \leq 2^\nu)$$

on U and so on L_ν . Next for $k = 2^\nu + \mu$ ($\mu = 1, 2, \dots$) and $z \in L_\nu$, by (5) in § 2, we have that

$$\begin{aligned} v_k(z)/p_k &\leq s(2 p_k \pi/k)^2 / (2 p_{2^\nu} \pi/2^\nu)^4 p_k \\ &= s(2^{4^\nu} / 4 \pi^2 k^2) [p_k/p_k^{\frac{1}{2}}] \\ &\leq s(2^{4^\nu} / 4 \pi^2 k^2) [(p^{4^\nu})^k / ((p^{4^{\nu-1}})^{2^\nu})^4] \\ &= s(2^{4^\nu} / 4 \pi^2 k^2) p^{4^{\nu\mu}} \leq p^{4^{\nu(\mu-1)}}. \end{aligned}$$

Hence for z in L_ν , we get that

$$\begin{aligned} |u(z)| &\leq \sum_{k=1}^{2^\nu} |v_k(z)/p_k| + \sum_{k=2^{\nu+1}}^\infty |v_k(z)/p_k| \\ &\leq \sum_{k=1}^{2^\nu} 1/(p^{4^{\nu-1}})^k + \sum_{\mu=1}^\infty p^{4^{\nu(\mu-1)}} \\ &\leq 2/(p^{4^{\nu-1}})^{2^\nu} = r_\nu. \end{aligned}$$

Since $u(z)$ is quasi-bounded in the upper half of U and in the lower half of U respectively, we have, for $e^{i\theta}$ in $U - A_{2^\nu}$, that

$$|u(e^{i\theta})| = \sum_{k=1}^{2^\nu} |v_k(e^{i\theta})| \leq \sum_{k=1}^{2^\nu} 1/p_k$$

$$= \sum_{k=1}^{2^v} 1/(p^{4^{v-1}})^k \leq r_\nu.$$

Hence by the maximum principle, $0 \leq u(z) \leq r_\nu$ in the intersection of the upper half of U and the left side of L_ν in U . Hence $|u(z)| \leq r_\nu$ in the left side of L_ν in U . By (3) in § 2, we see that $w(z; A'_{2^v}) \geq 1 - p_{2^v}/2^v$ on L_ν and so on the right side of L_ν in U . Hence if z lies between L_ν and $L_{\nu+1}$ in U , $b_\nu w(z; A'_{2^v}) \geq b_\nu - 2^{-\nu/2+2} \geq \Phi(|u(z)|) - 2^{-\nu/2+2}$, or $\Phi(|u(z)|) \leq b_\nu w(z; A'_{2^v}) + 2^{-\nu/2+2}$, since $\Phi(t) \leq b_\nu$ if $t \leq r_{\nu+1}$. On the other hand,

$$2\pi b_\nu w(0; A'_{2^v}) = b_\nu(4p_{2^v}\pi/2^v) = 8\pi 2^{-\nu/2}.$$

Hence if we set $w(z) = \sum_{\nu=1}^{\infty} (b_\nu w(z; A'_{2^v}) + 2^{-\nu/2+2})$, then $w(0) = 8 \cdot \sum_{\nu=1}^{\infty} 2^{-\nu/2} < \infty$ and so $w(z) \in \text{HP}(U)$. Thus

$$\Phi(|u(z)|) \leq b_\nu w(z; A'_{2^v}) + 2^{-\nu/2+2} \leq w(z)$$

between L_ν and $L_{\nu+1}$ in U . As ν is arbitrary, so $\Phi(|u(z)|) \leq w(z)$ in U ⁽⁶⁾. This shows that $u \in H\Phi(U)$.

Proof of (d). — Contrary to the assertion, assume that $u \in \text{HP}'(U)$. Then $|u(z)|$ has a harmonic majorant $h(z)$ on U . As $u(z)$, $u_n(z)$ and $v_n(z)$ are positive in the upper half of U and antisymmetric with respect to the real line (i.e. $u(z) = -u(-z)$ etc.), so $h(z) \geq |u(z)| \geq |u_n(z)|$ in U . Clearly $|u_n(z)| = \sum_{k=1}^n |w(z; A_k) - w(z; \tilde{A}_k)|/p_k$ and the least harmonic majorant of the subharmonic function $|u_n(z)|$ is $\sum_{k=1}^n w(z; A'_k)/p_k$, where $A'_k = A_k \cup \tilde{A}_k$ as before. Hence

$$\sum_{k=1}^n w(z; A'_k)/p_k \leq h(z)$$

on U for any $n = 1, 2, \dots$. Thus in particular, $\sum_{k=1}^{\infty} w(0; A'_k)/p_k \leq h(0)$, which gives the following contradiction :

$$\infty = 2 \sum_{k=1}^{\infty} 1/k = \frac{1}{2\pi} \sum_{k=1}^{\infty} (4p_k\pi/k)/p_k \leq h(0).$$

⁽⁶⁾ Notice that if z lies in the left of L_1 in U , then $|u(z)| \leq r_1$ and so $0 = \Phi(|u(z)|) \leq w(z)$ there.

4. Example 2.

Consider functions

$$\begin{cases} \Phi(t) = \log^+ t = \max(\log t, 0) & \text{on } [0, \infty); \\ u(z) = r^{-1} \cos \theta \quad (z = r e^{i\theta}) & \text{on } U_0 = \{z; 0 < |z| < 1\}. \end{cases}$$

Then $\Phi(t)$ is unbounded, increasing, continuous and $\bar{d}(\Phi) = \underline{d}(\Phi) = 0$. We can also easily see that $u(z)$ is an $H\Phi$ -function in U_0 but not an HP' -function in U_0 . But this example deeply depends on the weakness of the special boundary point 0 of U_0 . However, without using such a special boundary property, we can construct such an example in the open unit disc $U = \{z; |z| < 1\}$ by the aid of Example 1.

EXAMPLE 2. Let $\Phi(t)$ and $u(z)$ be as in Example 1. Let

$$\Phi_a(t) = \min(\Phi(t), at) \quad (0 < a < \infty).$$

Then the followings hold :

- (a) $\Phi_a(t)$ is increasing, continuous, $\bar{d}(\Phi_a) = a$ and $\underline{d}(\Phi_a) = 0$;
- (b) $u(z) \in H\Phi_a(U)$;
- (c) $u(z) \notin HP'(U)$.

5. Proof of Theorem 3.

First we prove that $H\Phi(R) \subset HP'(R)$ for any R if $\underline{d}(\Phi) > 0$. Let $u \in H\Phi(R)$ and $\underline{d}(\Phi) = 2c > 0$. Then there exists a point t_0 in $[0, \infty)$ such that $\Phi(t) > ct$ ($t \geq t_0$). Then for any t in $[0, \infty)$, $\Phi(t) + ct_0 \geq ct$. As $\Phi(|u|)$ possesses a harmonic majorant h on R , so

$$h + ct_0 \geq \Phi(|u|) + ct_0 > c|u|$$

on R . Thus u possesses a harmonic majorant $(h + ct_0)/c$, i.e. $u \in HP'(R)$.

Conversely, if $H\Phi(R) \subset HP'(R)$ for any R , then Examples 1 and 2 show that $\underline{d}(\Phi) > 0$ no matter whether $\bar{d}(\Phi)$ is finite or infinite.

6. Proof of Theorem 4.

Let $u \in H\Phi(\mathbb{R}) \cap HP'(\mathbb{R})$. We have to show that $u \in HB'(\mathbb{R})$. As $u \in H\Phi(\mathbb{R})$, so there exists an HP-function h such that $\Phi(|u|) \leq h$ on \mathbb{R} . Since $u \in HP'(\mathbb{R})$, we can consider $Mu = u \vee 0 + (-u) \wedge 0 \geq |u|$ and Bu . To show that $u \in HB'(\mathbb{R})$, we have to prove that $Bu = u$ or equivalently, $BMu = Mu$ (see 3 in the introductory part of this paper).

By the assumption that $\bar{d}(\Phi) = \infty$, we can find an increasing sequence $(r_n)_{n=1}^{\infty}$ of positive numbers converging to ∞ such that $\Phi(r_n) > 0$ and $\lim_{n \rightarrow \infty} a_n = 0$, $a_n = r_n / \Phi(r_n)$. Let $G_n = \{p \in \mathbb{R}; |u(p)| < r_n\}$ ($n = 1, 2, \dots$). Clearly

$$G_1 \subset G_2 \subset \dots \subset G_n \subset \dots, \quad \mathbb{R} = \bigcup_{n=1}^{\infty} G_n.$$

Let $(R_m)_{m=1}^{\infty}$ be an exhaustion of \mathbb{R} and w_m be a harmonic function on $R_m \cap G_n$ with the boundary value

$$w_m = \begin{cases} \min(Mu - BMu, r_n) & \text{on } (\partial R_m) \cap G_n; \\ 0 & \text{on } \partial G_n. \end{cases}$$

Since $\min(Mu - BMu, r_n)$ is superharmonic on \mathbb{R} , w_m is subharmonic on R_m if we define $w_m = 0$ in $R_m - G_n$, and $w_m \geq w_{m+1}$ on R_m . Let w'_m be harmonic in R_m with the boundary value

$$w'_m = \begin{cases} \min(Mu - BMu, r_n) & \text{on } (\partial R_m) \cap G_n; \\ 0 & \text{on } \partial R_m - G_n. \end{cases}$$

Then clearly $(w'_m)_{m=1}^{\infty}$ is a bounded sequence and $0 \leq w'_m \leq Mu - BMu$, $n = 1, 2, \dots$. If w' is any limiting harmonic function of a convergent subsequence of $(w'_m)_{m=1}^{\infty}$, then $0 \leq w' \leq Mu - BMu$. By applying the operator B , we get

$$0 \leq Bw' \leq B(Mu - BMu) = BMu - B^2Mu = BMu - BMu = 0.$$

Since w' is bounded and positive, $Bw' = w'$. Hence $w' \equiv 0$ on \mathbb{R} . Thus $\lim_{m \rightarrow \infty} w'_m = 0$ on \mathbb{R} . As we have $w'_m \geq w_m \geq 0$ on R_m , so we conclude that $\lim_{m \rightarrow \infty} w_m = 0$ on \mathbb{R} .

On $(\partial R_m) \cap G_n$, $|u| \leq r_n$ and $|u| \leq Mu = BMu + (Mu - BMu)$ or $|u| - BMu \leq Mu - BMu$. Hence on $(\partial R_m) \cap G_n$, $|u| - BMu \leq \min(Mu - BMu, r_n)$ or $|u| \leq BMu + w_m$. On ∂G_n , we have $|u| = r_n = a_n \Phi(|u|) \leq a_n h$. Thus we conclude that $|u| \leq a_n h +$

+ $BMu + w_m$ on $\partial(R_m \cap G_n)$. Since $|u|$ is subharmonic and $a_n h + BMu + w_m$ is harmonic on $R_m \cap G_n$, we can conclude that

$$|u| \leq a_n h + BMu + w_m \quad \text{on } R_m \cap G_n.$$

By letting $m \nearrow \infty$ and then $n \nearrow \infty$, we conclude that $|u| \leq BMu$ on R . By the definition of Mu , we must have $Mu \leq BMu$ and hence $BMu = Mu$.

BIBLIOGRAPHY

- [1] L. V. AHLFORS and L. SARIO, *Riemann Surfaces*, Princeton (1960).
- [2] M. NAKAI, Φ -bounded harmonic functions and classification of Riemann surfaces, *Pacific J. Math.*, 15 (1965), 1329-1335.
- [3] M. PARREAU, Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann, *Ann. Inst. Fourier*, 3, (1952), 103-197.

Manuscrit reçu le 8 juin 1965.

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