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MITSURU NAKAI

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## ON $\Phi$ -BOUNDED HARMONIC FUNCTIONS

by MITSURU NAKAI

1. Throughout this paper, we denote by  $\Phi(t)$  a *non-negative* real-valued function defined on the half real line  $[0, \infty) = (t; 0 \leq t < \infty)$ . A harmonic function  $u$  on a Riemann surface  $R$  is called  $\Phi$ -bounded if the composite function  $\Phi(|u|)$  admits a harmonic majorant on  $R$ , i. e. there exists a harmonic function  $h$  such that  $\Phi(|u|) \leq h$  on  $R$ . We denote by

$$H\Phi = H\Phi(R)$$

the totality of  $\Phi$ -bounded harmonic functions on a Riemann surface  $R$  and by  $O_{H\Phi}$  the class of all Riemann surfaces on which every  $\Phi$ -bounded harmonic function reduces to a constant. In our study, the following two quantities will play an important role :

$$\bar{d}(\Phi) = \limsup_{t \rightarrow \infty} \Phi(t)/t$$

$$\underline{d}(\Phi) = \liminf_{t \rightarrow \infty} \Phi(t)/t$$

The properties of  $H\Phi$ -functions on Riemann surfaces and the class  $O_{H\Phi}$  are first investigated by Parreau [3] for the special  $\Phi(t)$  which is increasing and convex <sup>(1)</sup>. In the present paper we shall investigate the same problem for general  $\Phi(t)$ . Our conclusion is, roughly speaking, that Parreau's result about  $O_{H\Phi}$  holds essentially for general  $\Phi(t)$  and his result about properties of  $H\Phi$ -functions can be derived by assuming  $\underline{d}(\Phi) > 0$  instead of increasingness and convexity which is, in a sense, the weakest condition.

2. As for the class  $O_{H\Phi}$ , Parreau [3] showed that the class  $O_{H\Phi}$  for

(1) For such a function, it is well-known that  $\bar{d}(\Phi) = \underline{d}(\Phi) > 0$ .

increasing and convex  $\Phi(t)$  coincides with  $O_{HP}$  or  $O_{HB}$  <sup>(2)</sup> according to  $\bar{d}(\Phi) < \infty$  or  $\bar{d}(\Phi) = \infty$ , respectively. Now we ask what can be said about  $O_{H\Phi}$  for general  $\Phi(t)$ . The answer is given by

**THEOREM 1.** — *If  $\bar{d}(\Phi) < \infty$  (resp.  $\bar{d}(\Phi) = \infty$ ), then  $O_{H\Phi} \subset O_{HP}$  (resp.  $O_{H\Phi} \supset O_{HB}$ ).*

This was proved implicitly in our former paper [2] by using Wiener's compactification of Riemann surfaces. We shall again give an alternating elementary proof in § 1. In this theorem, we cannot replace the inclusion relation by the equality in general. But the function  $\Phi(t)$ , by which the equality does not hold in the above theorem, is very singular and trivial one from the view point of  $H\Phi$ -functions as the following shows :

**THEOREM 2.** — (i) *If  $\Phi(t)$  is bounded on  $[0, \infty)$ , then  $O_{H\Phi}$  consists of all closed Riemann surfaces;*

(ii) *If  $\Phi(t)$  is completely unbounded <sup>(3)</sup> on  $[0, \infty)$ , then  $O_{H\Phi}$  consists of all open Riemann surfaces;*

(iii) *If  $\Phi(t)$  is not bounded and not completely unbounded, then  $O_{H\Phi} = O_{HP}$  or  $O_{HB}$  according to  $\bar{d}(\Phi) < \infty$  or  $\bar{d}(\Phi) = \infty$ , respectively.*

This was proved in [2] and determines the class  $O_{H\Phi}$  completely for any possible  $\Phi(t)$ . This is easily proved by using Theorem 1. We will do this also in § 1.

Observing Theorem 2, we are tempted to conclude that  $H\Phi$ -property is closely related to positiveness or boundedness properties except trivial  $\Phi$ 's as in (i) or (ii). Next we consider this problem. To state the problem formally, let us recall three notions for harmonic functions : essentially positive, quasi-bounded and singular.

3. A harmonic function  $u$  on a Riemann surface  $R$  is called *essentially positive* if  $u$  can be represented as a difference of two HP-functions on  $R$ , or equivalently, if  $u$  admits a harmonic majorant on  $R$ . We denote the totality of essentially positive harmonic functions on  $R$  by

$$HP' = HP'(R).$$

<sup>(2)</sup> As usual,  $HP(R)$  (resp.  $HB(R)$ ) denotes the totality of non-negative (resp. bounded) harmonic functions on  $R$ . The meaning of  $O_{HP}$  and  $O_{HB}$  is similar to that of  $O_{H\Phi}$ .

<sup>(3)</sup> We say that  $\Phi(t)$  is *completely unbounded* on  $[0, \infty)$  if  $\Phi(t)$  is not bounded at any neighbourhood of any point in  $[0, \infty)$ .

Clearly  $HP'(\mathbb{R}) \supset HP(\mathbb{R})$ . For two functions  $u$  and  $v$  in  $HP'(\mathbb{R})$ , there always exists the least harmonic majorant (resp. the greatest harmonic minorant) of  $u$  and  $v$ , which we denote by  $u \vee v$  (resp.  $u \wedge v$ ). Then  $HP'(\mathbb{R})$  forms a vector lattice with lattice operations  $\vee$  and  $\wedge$ . For  $u$  in  $HP'(\mathbb{R})$ , we denote by  $Mu$  the function  $u \vee 0 + (-u) \vee 0$ , which is the least harmonic majorant of  $|u|$ . Next first for  $u$  in  $HP(\mathbb{R})$ , we denote by  $Bu$  the HP-function defined by  $\sup (v(p); u \geq v \in HB(\mathbb{R}))$  on  $\mathbb{R}$ . Clearly  $B$  is order-preserving, linear and  $B^2 = B$  on  $HP(\mathbb{R})$  (see Ahlfors-Sario [1], p. 210). Next for  $u$  in  $HP'(\mathbb{R})$ , we put  $Bu = Bu_1 - Bu_2$ , where  $u = u_1 - u_2$  and  $u_1$  and  $u_2$  are in  $HP(\mathbb{R})$ . Here, by the linearity of  $B$  on  $HP(\mathbb{R})$ ,  $Bu$  does not depend on the special decomposition of  $u$  into HP-functions. Again the operator  $B$  is order-preserving, linear and  $B^2 = B$  on  $HP'(\mathbb{R})$  and moreover  $B$  commutes with  $M$ ,  $\vee$ , and  $\wedge$ . This is clear on  $HP(\mathbb{R})$  by definitions of  $B$ ,  $\vee$ ,  $\wedge$  and  $M$ . For the general case, we have only to show that  $B(u \vee 0) = (Bu) \vee 0$ . Since

$$Bu = B(u \vee 0) - B((-u) \vee 0)$$

and

$$\begin{aligned} B(u \vee 0) \wedge B((-u) \vee 0) &= B((u \vee 0) \wedge ((-u) \vee 0)) \\ &= B0 = 0, \end{aligned}$$

$B(u \vee 0)$  is the positive part of the Jordan decomposition of  $Bu$ .

An  $HP'$ -function  $u$  is called *quasi-bounded* (resp. *singular*) if  $Bu = u$  (resp.  $Bu = 0$ ). These notions were introduced by Parreau [3]. We denote the totality of quasi-bounded harmonic functions on  $\mathbb{R}$  by

$$HB' = HB'(\mathbb{R}).$$

Clearly  $HB' \supset HB$ . Since  $B$  commutes with  $M$ ,  $\vee$  and  $\wedge$ , we see that  $Bu = u$  is equivalent to  $BMu = Mu$ . Hence we can also define

$$HB'(\mathbb{R}) = (u \in HP'(\mathbb{R}); BMu = Mu).$$

4. Parreau [3] showed that, for increasing and convex function  $\Phi(t)$ ,  $H\Phi \subset HP'$  and if moreover  $\bar{d}(\Phi) = \infty$ , then  $H\Phi \subset HB'$ . Our next problem is to investigate whether such relations hold or not for general  $\Phi(t)$ . The answer is negative in general: we shall single out in § 4 an increasing continuous unbounded function  $\Phi(t)$  with  $\bar{d}(\Phi) < \infty$  and  $\underline{d}(\Phi) = 0$  and an  $H\Phi$ -function in the open unit disc which is not an  $HP'$ -function there (*Example 2*). This shows the invalidity of  $H\Phi \subset HP'$

in general. Only for this aim, we may take bounded  $\Phi(t)$ . But we are interested in unbounded  $\Phi(t)$ . We shall also construct in § 3 an increasing continuous function  $\Phi(t)$  with  $\bar{d}(\Phi) = \infty$  and  $\underline{d}(\Phi) = 0$  and an  $H\Phi$ -function in the open unit disc which is not an  $HP'$ -function there (*Example 1*). This shows the invalidity of the relation  $H\Phi \subset HP'$  and so of the relation  $H\Phi \subset HB'$  even if  $\bar{d}(\Phi) = \infty$ .

Then there arises the question when can we conclude the relation  $H\Phi \subset HP'$  or  $HB'$ . Both examples above show that unboundedness, not completely unboundedness, increasingness, continuity or all of them cannot give the condition. In both examples above,  $\underline{d}(\Phi) = 0$ . This suggests us that the required condition may be  $\underline{d}(\Phi) > 0$ . This is really the case. Firstly the answer for  $H\Phi \subset HP'$  is given completely by the following which includes Parreau's case :

**THEOREM 3.** — *In order that  $H\Phi(R) \subset HP'(R)$  for any Riemann surface  $R$ , it is necessary and sufficient that  $\underline{d}(\Phi) > 0$  (no matter whether  $\bar{d}(\Phi)$  is finite or infinite).*

The proof of this will be given in § 5. Similarly we ask about the condition which assures the relation  $H\Phi \subset HB'$ . In this case, even in the Parreau's case, we must assume that  $\bar{d}(\Phi) = \infty$  as the following simple example shows :  $\Phi(t) = t$ ,  $R = (z ; 0 < |z| < 1)$  and  $u(z) = -\log |z|$ . The best possible general conclusion is as follows :

**THEOREM 4.** — *If  $\bar{d}(\Phi) = \infty$ , then  $H\Phi(R) \cap HP'(R) \subset HB'(R)$ .*

Here we cannot drop  $HP'(R)$  in the above relation as Example 1 shows. The above theorem will be proved in § 6. Now assume that  $\underline{d}(\Phi) > 0$ , then by Theorems 3 and 4,  $H\Phi(R) \subset HB'(R)$ . Conversely if  $H\Phi(R) \subset HB'(R)$  for any  $R$ , then  $H\Phi(R) \subset HP'(R)$  for any  $R$  and by Theorem 3,  $\underline{d}(\Phi) > 0$ . Thus we get the following which includes Parreau's case :

**THEOREM 5.** — *Assume that  $\underline{d}(\Phi) = \infty$ . In order  $H\Phi(R) \subset HB'(R)$  for any Riemann surface  $R$ , it is necessary and sufficient that  $\underline{d}(\Phi) > 0$ .*

## 1. Proofs of Theorems 1 and 2.

**1. Proof of Theorem 1.** — **I.** The case  $\bar{d}(\Phi) = \infty$  : Assume that there exists a non-constant  $H\Phi$ -function  $u$  on  $R$ . By the definition of

$\Phi$ -boundedness, there exists an HP-function  $h$  such that  $\Phi(|u|) \leq h$  on  $R$ . We have to show that  $R \notin O_{HB}$ . Contrary to the assertion, assume that  $R \in O_{HB}$ . Since  $\bar{d}(\Phi) = \infty$ , we can find a strictly increasing sequence  $(r_n)_{n=1}^\infty$  of positive numbers  $r_n$  such that  $\lim_{n \rightarrow \infty} r_n = \infty$ ,  $\Phi(r_n) > 0$ ,  $G_n = \{p \in R; |u(p)| < r_n\} \neq \emptyset$  and  $\lim_{n \rightarrow \infty} a_n = 0$ , where  $a_n = r_n/\Phi(r_n)$ . Then clearly

$$G_1 \subset G_2 \subset \dots \subset G_n \subset \dots, R = \bigcup_{n=1}^\infty G_n.$$

First we show that  $G_n \notin SO_{HB}$  for some  $n$  on  $(4)$ . If this is not the case, then  $G_n \in SO_{HB}$  for all  $n$ 's. Then since  $a_n h - |u|$  is superharmonic and bounded from below on  $G_n$  and

$$a_n h - |u| \geq a_n \Phi(|u|) - |u| = a_n \Phi(r_n) - r_n = 0$$

on  $\partial G_n$ , we can conclude that  $a_n h - |u| \geq 0$  on  $G_n$ . Since  $a_n \searrow 0$ , we must have  $u \equiv 0$  on  $R$ , which is clearly a contradiction. Hence we may assume that  $G_n \notin SO_{HB}$  ( $n = 1, 2, 3, \dots$ ) by choosing a suitable subsequence of  $(r_n)$ , if necessary.

Next we assert that  $G_n - \bar{G}_1 \in SO_{HB}$  ( $n = 1, 2, 3, \dots$ ). For, if there exists a  $G_n$  with  $G_n - \bar{G}_1 \notin SO_{HB}$ , then there would exist two disjoint non-empty open sets  $G_1$  and  $G_n - \bar{G}_1$  not belonging to  $SO_{HB}$ . By the so called "two domains criterion", we must have that  $R \notin O_{HB}$  (see Ahlfors-Sario [1], p. 213). But this contradicts our assumption  $R \in O_{HB}$ .

Now consider the function  $w_n = a_n h + r_1 - |u|$  on  $G_n$ , which is superharmonic and bounded from below on  $G_n$  and so on  $G_n - \bar{G}_1$ . By the similar manner as before, we see that  $w_n \geq a_n h - |u| = 0$  on  $\partial G_n$ . Clearly  $w_n \geq r_1 - |u| = 0$  on  $\partial G_1$ . Hence  $w_n \geq 0$  on  $\partial(G_n - \bar{G}_1)$ . Since  $G_n - \bar{G}_1 \in SO_{HB}$ , we can conclude that  $w_n \geq 0$  on  $G_n$  or  $|u| \leq a_n h + r_1$  on  $G_n$ . Hence by the fact that  $a_n \searrow 0$ , we get that  $|u| \leq r_1$  on  $R$ . This contradicts our assumption that  $R \in O_{HB}$ . Thus we must have  $R \notin O_{HB}$ .

**II.** The case  $\bar{d}(\Phi) \leq \infty$ : Assume that there exists a non-constant HP-function  $u$  on  $R$ . By  $\bar{d}(\Phi) < \infty$ , we can find a point  $s$  in  $[0, \infty)$  such that there exists a finite positive constant  $C$  with  $\Phi(t) \leq Ct$  ( $s \leq t < \infty$ ). Let  $v = s + u$ . Clearly  $v$  is a non-constant HP-function on  $R$  with

(4) An open subset  $G$  of a Riemann surface  $R$  with smooth relative boundary  $\partial G$  is said to belong to  $SO_{HB}$  if every HB-function on  $G$  with continuous boundary value zero at  $\partial G$  reduces to a constant zero.

$v \geq s$  on  $R$ . Hence  $\Phi(|v|) = \Phi(v) \leq Cv$  on  $R$ . Thus  $v$  is a non-constant  $H\Phi$ -function on  $R$  and so  $R \notin O_{H\Phi}$ .

**2. Proof of Theorem 2.** — *Ad (i)*: If  $\Phi(t)$  is bounded, then every non-constant harmonic function is an  $H\Phi$ -function. Thus  $O_{H\Phi}$  consists of all Riemann surfaces carrying no non-constant harmonic function, which are closed Riemann surfaces.

*Ad (ii)*: For any non-constant harmonic function  $u$  on  $R$ , since  $u$  is open map of  $R$  into  $[0, \infty)$  by the maximum principle,  $\Phi(|u|)$  is completely unbounded on  $R$  along with  $\Phi(t)$  and so  $u$  is not  $H\Phi$ -function. Thus there exists no non-constant  $H\Phi$ -function on any Riemann surface and  $O_{H\Phi}$  consists of all Riemann surfaces.

*Ad (iii)*: Assume that  $\bar{d}(\Phi) = \infty$  and that there exists a non-constant  $HB$ -function  $u$  on  $R$ . As  $\Phi(t)$  is not completely unbounded, so there exists an interval  $(a, b)$  in which  $\Phi(t) < c = \text{const}$ . Let

$$v = (a + b)/2 + ((b - a)/2) \left( \sup_R |u| \right)^{-1} u.$$

Then  $v$  is a non-constant  $HB$ -function and  $\Phi(|v|) = \Phi(v) < c$  on  $R$ . Thus  $O_{H\Phi} \subset O_{HB}$ . This with Theorem 1 gives  $O_{H\Phi} = O_{HB}$ .

Next assume that  $\bar{d}(\Phi) < \infty$ . By Theorem 1,  $O_{HP} \supset O_{H\Phi}$ . Contrary to the assertion, assume that there exists an  $R$  in  $O_{HP} - O_{H\Phi}$ . Let  $u$  be a non-constant  $H\Phi$ -function on  $R$ . Then  $\Phi(|u|) \leq c = \text{constant}$  on  $R$ . Since  $\Phi(t)$  is unbounded and  $|u|(R)$  is connected in  $[0, \infty)$  and contains 0,  $u$  must be bounded on  $R$ . Then  $\sup_R |u| + u$  is a non-constant  $HP$ -function on  $R$ , which contradicts the assumption that  $R \in O_{HP}$ . Hence  $O_{H\Phi} = O_{HP}$ .

## 2. Preparations for Examples.

Let  $U = (z; |z| < 1)$  and  $A$  be an arc in  $\partial U = (z; |z| = 1)$ . We denote by  $w(z; A)$  the harmonic measure of  $A$  calculated at  $z$  in  $U$  with respect to  $U$ . It is well known that

$$(1) \quad w(z; A) = (2\beta - \alpha)/2\pi,$$

where  $\alpha$  is the length of  $A$  and  $\beta$  is the angle seeing the arc  $A$  from  $z$ .

We denote by  $L_A$  the line segment connecting both end points of  $A$ . Then from (1), we easily see that

$$(2) \quad w(0; A) = \alpha;$$

$$(3) \quad w(z; A) = 1 - \alpha/2\pi \quad \text{on} \quad L_A.$$

Next let  $A_j$  be the arc in  $\partial U = \{z; |z| = 1\}$  with end points 1 and  $e^{i\alpha_j}$  ( $j = 1, 2$ ) such that  $0 < \alpha_1 < \alpha_2, \alpha_1 < \pi/2, \alpha_2 < \pi/2$ . We denote by  $\tilde{A}_j$  (resp.  $A'_j$ ) the arc with end points 1 and  $e^{-i\alpha_j}$  (resp.  $A'_j = A_j \cup \tilde{A}_j$ ). For simplicity, we set  $L_2 = L_{A_2}$ , i.e.  $L_2$  is the line segment connecting two end points of  $A'_2$ . Then we get the following inequality which plays an important role in our forth-coming examples : there exists a universal constant  $s_0 (\leq 4^{-1} \pi^4)$  such that

$$(4) \quad |w(z; A_1) - w(z; \tilde{A}_1)| \leq s_0 \alpha_1^2 / (\alpha_2^2 - \alpha_1^2)^2 \quad \text{on} \quad L_2.$$

*Proof.* — We denote the points  $e^{i\alpha_1}, e^{-i\alpha_1}, (e^{i\alpha_1} + e^{-i\alpha_1})/2, 1, (e^{i\alpha_2} + e^{-i\alpha_2})/2$  and  $z$  on  $L_2$  with  $\text{Im}(z) \geq 0$  by D, E, F, G, H and P respectively. We set  $DF = FE = d, FH = k, DP = a, PF = b$  and  $PE = c$ . By (1),  $w(z; A_1) - w(z; \tilde{A}_1) = (\angle DPG - \angle GPE)/\pi$ . Let  $\angle DPF = \theta_1$  and  $\angle FPE = \theta_2$ . Then clearly  $\angle DPG \leq \theta_1$  and  $\angle GPE \geq \theta_2$ . Hence we have  $0 \leq w(z; A_1) - w(z; \tilde{A}_1) \leq (\theta_1 - \theta_2)/\pi$ . Applying the cosine theorem to triangles  $\triangle DPF$  and  $\triangle FPE$  and then Pappos' identity to the triangle  $\triangle DPE$ , we have

$$\sin 2^{-1}(\theta_1 - \theta_2) = (c - a) (8abc \sin 2^{-1}(\theta_1 + \theta_2))^{-1} (4d^2 - (a - c)^2).$$

Here we have

$$\begin{aligned} ac \sin 2^{-1}(\theta_1 + \theta_2) &\geq ac \sin 2^{-1}(\theta_1 + \theta_2) \cos 2^{-1}(\theta_1 + \theta_2) \\ &= 2^{-1} ac \sin \angle DPE = \triangle DPE \\ &= \triangle DHE = dk. \end{aligned}$$

By the triangle inequality applied for  $\triangle DPE, c - a \leq 2d$ . Thus by noticing  $b \geq k$ , we have  $\sin 2^{-1}(\theta_1 - \theta_2) \leq d^2 k^{-2}$ . As

$$\sin \theta \geq (2/\pi) \theta \quad (0 \leq \theta \leq 2^{-1} \pi),$$

so  $\theta_1 - \theta_2 \leq \pi d^2 k^{-2}$ . Now we have  $d = \sin \alpha_1 \leq \alpha_1$  and

$$\begin{aligned} k &= \cos \alpha_1 - \cos \alpha_2 \\ &= 2 \sin^{-1}(\alpha_1 + \alpha_2) \sin 2^{-1}(\alpha_2 - \alpha_1) \geq 2 \pi^{-2} (\alpha_2^2 - \alpha_1^2). \end{aligned}$$

Hence

$$0 \leq w(z; A_1) - w(z; \tilde{A}_1) \leq 4^{-1} \pi^4 \alpha_1^2 / (\alpha_2^2 - \alpha_1^2)^2. \quad Q.E.D.$$



We shall use (4) in the particular case where  $0 < \alpha_1 < \alpha_2/\sqrt{2}$ . In this case, by using universal constant  $s (\leq \pi^4)$ , we get

$$(5) \quad |w(z; A_1) - w(z; \tilde{A}_1)| \leq s(\alpha_1^2 / \alpha_2^2) \quad \text{on } L_2.$$

### 3. Example 1.

We are now able to construct an example of a function  $\Phi$  which is continuous, increasing,  $\bar{d}(\Phi) = \infty$  and  $\underline{d}(\Phi) = 0$ ; and an  $H\Phi$ -function  $u$  in then open unit disc  $U = (z; |z| < 1)$  which is not an  $HP'$ -function.

**EXAMPLE 1.** Let  $p$  be a constant such that  $0 < p < \min(1/4, 1/4s)$ , where  $s$  is the constant in (5) in § 2, and  $(p_n)_{n=1}^\infty$  be a sequence defined by  $p_n = (p^{4\nu})^{2\nu+\mu}$  for  $n = 2\nu + \mu$  ( $\nu = 0, 1, 2, \dots$ ;  $\mu = 1, 2, 3, \dots, 2\nu$ ). Let  $A_n$  and  $\tilde{A}_n$  be arcs on the unit circumference such that

$$A_n = (e^{i\theta}; 0 \leq \theta \leq 2 p_n \pi/n)$$

and

$$\tilde{A}_n = (e^{i\theta}; -2 p_n \pi/n \leq \theta \leq 0).$$

Let  $(r_\nu)_{\nu=1}^\infty$  and  $(b_\nu)_{\nu=1}^\infty$  be two sequences of positive numbers defined by  $r_\nu = 2/(p^{4\nu-1})^{2\nu}$  and  $b_\nu = 2^{\nu/2} \cdot r_\nu$ . Define the function  $\Phi(t)$  on  $[0, \infty]$  by

$$\Phi(t) = \begin{cases} 0, & t \in [0, r_1]; \\ b_1(t - r_1), & t \in [r_1, r_1 + 1]; \\ b_\nu, & t \in [r_\nu + 1, r_{\nu+1}] (\nu = 1, 2, \dots); \\ b_\nu + (b_{\nu+1} - b_\nu)(t - r_{\nu+1}), & t \in [r_{\nu+1}, r_{\nu+1} + 1] (\nu = 1, 2, \dots) \end{cases}$$

and the function  $u(z)$  in  $U$  by

$$u(z) = \sum_{n=1}^{\infty} (w(z; A_n) - w(z; \tilde{A}_n))/p_n.$$

Then the following hold :

- (a)  $\Phi(t)$  is continuous, increasing,  $\bar{d}(\Phi) = \infty$  and  $\underline{d}(\Phi) = 0$ ;
- (b)  $u(z)$  is well defined in  $U$  and harmonic there;
- (c)  $u(z) \in H\Phi(U)$ ;
- (d)  $u(z) \notin HP'(U)$ .

*Proof of (a).* — Is immediate by the definition of  $\Phi(t)$ .

*Proof of (b).* — For each  $n = 1, 2, \dots$ , set

$$v_n(z) = w(z; A_n) - w(z; \tilde{A}_n), \quad u_n(z) = \sum_{k=1}^n v_k(z)/p_k.$$

Then  $v_n$  and  $u_n$  are harmonic in  $U$ , positive in the upper half of  $U$  and  $v_n(-z) = -v_n(z)$  and  $u_n(-z) = -u_n(z)$  in  $U$ . Hence to show that the series defining  $u(z)$  is convergent in  $U$  and defines a harmonic function there, we have only to prove that  $(u_n(i/2))_{n=1}^\infty$  is convergent. By (5) in § 2, we have that

$$0 < v_n(i/2) \leq s(2p_n\pi/n)^2/(\pi/2)^4 \leq s'p_n^2,$$

where  $s'$  is a constant independent of  $n \geq 1$ . Thus

$$0 < u_{n+m}(i/2) - u_n(i/2) = \sum_{k=n+1}^{n+m} v_k(i/2)/p_k \leq s' \sum_{k=n+1}^{n+m} p_k < s'p^n/(1-p).$$

This shows that  $(u_n(i/2))_{n=1}^\infty$  is convergent.

*Proof of (c).* — For each  $\nu = 1, 2, \dots$ , we denote by  $L_\nu$  the line segment  $L_{A_{2^\nu}}$ , i.e. the line segment connecting two end points of  $A_{2^\nu} = A_{2^\nu} \cup \tilde{A}_{2^\nu}$ . Since  $|v_k(z)| < 1$  in  $U$ , we have

$$|v_k(z)/p_k| \leq 1/p_k \leq 1/(p^{4^{\nu-1}})^k \quad (1 \leq k \leq 2^\nu)$$

on  $U$  and so on  $L_\nu$ . Next for  $k = 2^\nu + \mu$  ( $\mu = 1, 2, \dots$ ) and  $z \in L_\nu$ , by (5) in § 2, we have that

$$\begin{aligned} v_k(z)/p_k &\leq s(2p_k\pi/k)^2/(2p_{2^\nu}\pi/2^\nu)^4 p_k \\ &= s(2^{4^\nu}/4\pi^2 k^2) [p_k/p_{2^\nu}^4] \\ &\leq s(2^{4^\nu}/4\pi^2 k^2) [(p^{4^\nu})^k/((p^{4^{\nu-1}})^{2^\nu})^4] \\ &= s(2^{4^\nu}/4\pi^2 k^2) p^{4^{\nu\mu}} \leq p^{4^\nu(\mu-1)}. \end{aligned}$$

Hence for  $z$  in  $L_\nu$ , we get that

$$\begin{aligned} |u(z)| &\leq \sum_{k=1}^{2^\nu} |v_k(z)/p_k| + \sum_{k=2^\nu+1}^\infty |v_k(z)/p_k| \\ &\leq \sum_{k=1}^{2^\nu} 1/(p^{4^{\nu-1}})^k + \sum_{\mu=1}^\infty p^{4^\nu(\mu-1)} \\ &\leq 2/(p^{4^{\nu-1}})^{2^\nu} = r_\nu. \end{aligned}$$

Since  $u(z)$  is quasi-bounded in the upper half of  $U$  and in the lower half of  $U$  respectively, we have, for  $e^{i\theta}$  in  $U - A_{2^\nu}$ , that

$$|u(e^{i\theta})| = \sum_{k=1}^{2^\nu} |v_k(e^{i\theta})| \leq \sum_{k=1}^{2^\nu} 1/p_k$$

$$= \sum_{k=1}^{2^{\nu}} 1/(p^{4^{\nu-1}})^k \leq r_{\nu}.$$

Hence by the maximum principle,  $0 \leq u(z) \leq r_{\nu}$  in the intersection of the upper half of  $U$  and the left side of  $L_{\nu}$  in  $U$ . Hence  $|u(z)| \leq r_{\nu}$  in the left side of  $L_{\nu}$  in  $U$ . By (3) in § 2, we see that  $w(z; A'_{2^{\nu}}) \geq 1 - p_{2^{\nu}}/2^{\nu}$  on  $L_{\nu}$  and so on the right side of  $L_{\nu}$  in  $U$ . Hence if  $z$  lies between  $L_{\nu}$  and  $L_{\nu+1}$  in  $U$ ,  $b_{\nu} w(z; A'_{2^{\nu}}) \geq b_{\nu} - 2^{-\nu/2+2} \geq \Phi(|u(z)|) - 2^{-\nu/2+2}$ , or  $\Phi(|u(z)|) \leq b_{\nu} w(z; A'_{2^{\nu}}) + 2^{-\nu/2+2}$ , since  $\Phi(t) \leq b_{\nu}$  if  $t \leq r_{\nu+1}$ . On the other hand,

$$2\pi b_{\nu} w(0; A'_{2^{\nu}}) = b_{\nu}(4p_{2^{\nu}}\pi/2^{\nu}) = 8\pi 2^{-\nu/2}.$$

Hence if we set  $w(z) = \sum_{\nu=1}^{\infty} (b_{\nu} w(z; A'_{2^{\nu}}) + 2^{-\nu/2+2})$ , then  $w(0) =$

$8 \cdot \sum_{\nu=1}^{\infty} 2^{-\nu/2} < \infty$  and so  $w(z) \in \text{HP}(U)$ . Thus

$$\Phi(|u(z)|) \leq b_{\nu} w(z; A'_{2^{\nu}}) + 2^{-\nu/2+2} \leq w(z)$$

between  $L_{\nu}$  and  $L_{\nu+1}$  in  $U$ . As  $\nu$  is arbitrary, so  $\Phi(|u(z)|) \leq w(z)$  in  $U$  <sup>(5)</sup>. This shows that  $u \in H\Phi(U)$ .

*Proof of (d).* — Contrary to the assertion, assume that  $u \in \text{HP}'(U)$ . Then  $|u(z)|$  has a harmonic majorant  $h(z)$  on  $U$ . As  $u(z)$ ,  $u_n(z)$  and  $\nu_n(z)$  are positive in the upper half of  $U$  and antisymmetric with respect to the real line (i.e.  $u(z) = -u(-z)$  etc.), so  $h(z) \geq |u(z)| \geq |u_n(z)|$  in  $U$ . Clearly  $|u_n(z)| = \sum_{k=1}^n |w(z; A_k) - w(z; \tilde{A}_k)|/p_k$  and the least harmonic majorant of the subharmonic function  $|u_n(z)|$  is  $\sum_{k=1}^n w(z; A_k)/p_k$ , where  $A'_k = A_k \cup \tilde{A}_k$  as before. Hence

$$\sum_{k=1}^n w(z; A'_k)/p_k \leq h(z)$$

on  $U$  for any  $n = 1, 2, \dots$ . Thus in particular,  $\sum_{k=1}^{\infty} w(0; A'_k)/p_k \leq h(0)$ , which gives the following contradiction :

$$\infty = 2 \sum_{k=1}^{\infty} 1/k = \frac{1}{2\pi} \sum_{k=1}^{\infty} (4p_k\pi/k)/p_k \leq h(0).$$

<sup>(5)</sup> Notice that if  $z$  lies in the left of  $L_1$  in  $U$ , then  $|u(z)| \leq r_1$  and so  $0 = \Phi(|u(z)|) \leq w(z)$  there.

**4. Example 2.**

Consider functions

$$\begin{cases} \Phi(t) = \log^+ t = \max(\log t, 0) & \text{on } [0, \infty); \\ u(z) = r^{-1} \cos \theta \quad (z = r e^{i\theta}) & \text{on } U_0 = \{z; 0 < |z| < 1\}. \end{cases}$$

Then  $\Phi(t)$  is unbounded, increasing, continuous and  $\bar{d}(\Phi) = \underline{d}(\Phi) = 0$ . We can also easily see that  $u(z)$  is an  $H\Phi$ -function in  $U_0$  but not an  $HP'$ -function in  $U_0$ . But this example deeply depends on the weakness of the special boundary point 0 of  $U_0$ . However, without using such a special boundary property, we can construct such an example in the open unit disc  $U = \{z; |z| < 1\}$  by the aid of Example 1.

**EXAMPLE 2.** Let  $\Phi(t)$  and  $u(z)$  be as in Example 1. Let

$$\Phi_a(t) = \min(\Phi(t), at) \quad (0 < a < \infty).$$

Then the followings hold :

- (a)  $\Phi_a(t)$  is increasing, continuous,  $\bar{d}(\Phi_a) = a$  and  $\underline{d}(\Phi_a) = 0$ ;
- (b)  $u(z) \in H\Phi_a(U)$ ;
- (c)  $u(z) \notin HP'(U)$ .

**5. Proof of Theorem 3.**

First we prove that  $H\Phi(R) \subset HP'(R)$  for any  $R$  if  $\underline{d}(\Phi) > 0$ . Let  $u \in H\Phi(R)$  and  $\underline{d}(\Phi) = 2c > 0$ . Then there exists a point  $t_0$  in  $[0, \infty)$  such that  $\Phi(t) > ct$  ( $t \geq t_0$ ). Then for any  $t$  in  $[0, \infty)$ ,  $\Phi(t) + ct_0 \geq ct$ . As  $\Phi(|u|)$  possesses a harmonic majorant  $h$  on  $R$ , so

$$h + ct_0 \geq \Phi(|u|) + ct_0 > c|u|$$

on  $R$ . Thus  $u$  possesses a harmonic majorant  $(h + ct_0)/c$ , i.e.  $u \in HP'(R)$ .

Conversely, if  $H\Phi(R) \subset HP'(R)$  for any  $R$ , then Examples 1 and 2 show that  $\underline{d}(\Phi) > 0$  no matter whether  $\bar{d}(\Phi)$  is finite or infinite.

6. Proof of Theorem 4.

Let  $u \in H\Phi(R) \cap HP'(R)$ . We have to show that  $u \in HB'(R)$ . As  $u \in H\Phi(R)$ , so there exists an HP-function  $h$  such that  $\Phi(|u|) \leq h$  on  $R$ . Since  $u \in HP'(R)$ , we can consider  $Mu = u \vee 0 + (-u) \wedge 0 \geq |u|$  and  $Bu$ . To show that  $u \in HB'(R)$ , we have to prove that  $Bu = u$  or equivalently,  $BMu = Mu$  (see 3 in the introductory part of this paper).

By the assumption that  $\bar{d}(\Phi) = \infty$ , we can find an increasing sequence  $(r_n)_{n=1}^\infty$  of positive numbers converging to  $\infty$  such that  $\Phi(r_n) > 0$  and  $\lim_{n \rightarrow \infty} a_n = 0$ ,  $a_n = r_n / \Phi(r_n)$ . Let  $G_n = \{p \in R; |u(p)| < r_n\}$  ( $n = 1, 2, \dots$ ). Clearly

$$G_1 \subset G_2 \subset \dots \subset G_n \subset \dots, \quad R = \bigcup_{n=1}^\infty G_n.$$

Let  $(R_m)_{m=1}^\infty$  be an exhaustion of  $R$  and  $w_m$  be a harmonic function on  $R_m \cap G_n$  with the boundary value

$$w_m = \begin{cases} \min(Mu - BMu, r_n) & \text{on } (\partial R_m) \cap G_n; \\ 0 & \text{on } \partial G_n. \end{cases}$$

Since  $\min(Mu - BMu, r_n)$  is superharmonic on  $R$ ,  $w_m$  is subharmonic on  $R_m$  if we define  $w_m = 0$  in  $R_m - G_n$ , and  $w_m \geq w_{m+1}$  on  $R_m$ . Let  $w'_m$  be harmonic in  $R_m$  with the boundary value

$$w'_m = \begin{cases} \min(Mu - BMu, r_n) & \text{on } (\partial R_m) \cap G_n; \\ 0 & \text{on } \partial R_m - G_n. \end{cases}$$

Then clearly  $(w'_m)_{m=1}^\infty$  is a bounded sequence and  $0 \leq w'_m \leq Mu - BMu$ ,  $n = 1, 2, \dots$ . If  $w'$  is any limiting harmonic function of a convergent subsequence of  $(w'_m)_{m=1}^\infty$ , then  $0 \leq w' \leq Mu - BMu$ . By applying the operator  $B$ , we get

$$0 \leq Bw' \leq B(Mu - BMu) = BMu - B^2Mu = BMu - BMu = 0.$$

Since  $w'$  is bounded and positive,  $Bw' = w'$ . Hence  $w' \equiv 0$  on  $R$ . Thus  $\lim_{m \rightarrow \infty} w'_m = 0$  on  $R$ . As we have  $w'_m \geq w_m \geq 0$  on  $R_m$ , so we conclude that  $\lim_{m \rightarrow \infty} w_m = 0$  on  $R$ .

On  $(\partial R_m) \cap G_n$ ,  $|u| \leq r_n$  and  $|u| \leq Mu = BMu + (Mu - BMu)$  or  $|u| - BMu \leq Mu - BMu$ . Hence on  $(\partial R_m) \cap G_n$ ,  $|u| - BMu \leq \min(Mu - BMu, r_n)$  or  $|u| \leq BMu + w_m$ . On  $\partial G_n$ , we have  $|u| = r_n = a_n \Phi(|u|) \leq a_n h$ . Thus we conclude that  $|u| \leq a_n h +$

+  $BMu + w_m$  on  $\partial(R_m \cap G_n)$ . Since  $|u|$  is subharmonic and  $a_n h + BMu + w_m$  is harmonic on  $R_m \cap G_n$ , we can conclude that

$$|u| \leq a_n h + BMu + w_m \quad \text{on } R_m \cap G_n.$$

By letting  $m \nearrow \infty$  and then  $n \nearrow \infty$ , we conclude that  $|u| \leq BMu$  on  $R$ . By the definition of  $Mu$ , we must have  $Mu \leq BMu$  and hence  $BMu = Mu$ .

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Mitsuru Nakai,  
Mathematical Institute,  
Nagoya University,  
Nagoya (Japan).