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A RELATIONSHIP BETWEEN THE NON-ACYCLIC REIDEMEISTER TORSION AND A ZERO OF THE ACYCLIC REIDEMEISTER TORSION

by Yoshikazu YAMAGUCHI (*)

Abstract. — We show a relationship between the non-acyclic Reidemeister torsion and a zero of the acyclic Reidemeister torsion for a λ-regular SU(2) or $\mathrm{SL}(2,\mathbb{C})$-representation of a knot group. Then we give a method to calculate the non-acyclic Reidemeister torsion of a knot exterior. We calculate a new example and investigate the behavior of the non-acyclic Reidemeister torsion associated to a 2-bridge knot and SU(2)-representations of its knot group.

Résumé. — Nous montrons une relation entre la torsion de Reidemeister non-acyclique et un zéro de la torsion de Reidemeister acyclique pour une représentation λ-régulière dans SU(2) ou $\mathrm{SL}(2,\mathbb{C})$ du groupe d'un nœud. Alors nous pouvons donner une méthode pour calculer la torsion de Reidemeister non-acyclique de l'extérieur d'un nœud. Nous calculons un nouvel exemple et étudions le comportement de la torsion de Reidemeister non-acyclique associée à un nœud à deux-ponts et une SU(2)-représentations du groupe du nœud.

1. Introduction

The Reidemeister torsion is an invariant for a CW-complex and a representation of its fundamental group. In other words, this invariant associates with the local system for a representation of the fundamental group. Originally the Reidemeister torsion is defined if the local system is *acyclic*, *i.e.*, all homology groups vanish. However we can extend the definition of the Reidemeister torsion to non-acyclic cases [12, 19]. In this paper, we focus on the non-acyclic cases.

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It is known that the Fox calculus plays important roles in the study of the Reidemeister torsion [4, 9, 10, 13, 15, 19]. The many results were obtained by using the Fox calculus for the acyclic Reidemeister torsion. In particular, there are important results related to the Alexander polynomial in the knot theory [9, 10, 13, 19]. The Fox calculus is also important for non-acyclic cases [4, 15]. It is related to the cohomology theory of groups.

This paper contributes to the study of the non-acyclic Reidemeister torsion by using the Fox calculus. Our purpose is to apply the Fox calculus for the acyclic cases to the study of the non-acyclic Reidemeister torsion by using a relationship between the acyclic Reidemeister torsion and the non-acyclic one. Our main theorem says that the non-acyclic Reidemeister torsion for a knot exterior is given by the differential coefficients of the twisted Alexander invariant of the knot. The twisted Alexander invariant of a knot is the acyclic Reidemeister torsion and expressed as a one variable rational function [10]. A conjecture due to J. Dubois and R. Kashaev [6] will be solved in [22] by using our main theorem.

In the latter of this paper, we apply this relationship to study the Reidemeister torsion for the pair of a 2-bridge knot and SU(2)-representation of its knot group. We give an explicit expression of the non-acyclic Reidemeister torsion associated to 5_2 knot. This is a new example of calculation of the non-acyclic Reidemeister torsion. Furthermore, we investigate where the non-acyclic Reidemeister torsion associated to a 2-bridge knot has critical points. Note that the non-acyclic Reidemeister torsion is parametrized by the representations of a knot group. Moreover this Reidemeister torsion turns into a function on the character variety of the knot group. We will see that the critical points of the non-acyclic Reidemeister torsion associated to a 2-bridge knot are binary dihedral representations and these representations are related to the geometry of the character variety of a 2-bridge knot group.

This paper is organized as follows. In Section 2, we review the Reidemeister torsion. In particular, we give the notion of the non-acyclic Reidemeister torsion of knot exteriors [4, 15].

Section 3 includes our main theorem on a relationship between the non-acyclic Reidemeister torsion and the twisted Alexander invariant for knot exteriors. We give a formula of the non-acyclic Reidemeister torsion for a knot exterior by using a Wirtinger presentation of a knot group.

In Section 4, we apply the results of Section 3 to study the non-acyclic Reidemeister torsion for a 2-bridge knot group and SU(2)-representation of its knot group.

2. Review on the non-abelian twisted Reidemeister torsion

2.1. Notation

In this paper, we use the following notations.

- $\mathbb{F}$ is the field $\mathbb{R}$ or $\mathbb{C}$.
- G is the Lie group $\mathrm{SU}(2)$ (resp. $\mathrm{SL}(2,\mathbb{C})$) if $\mathbb{F}$ is $\mathbb{R}$ (resp. $\mathbb{C}$). The symbol $\mathfrak{g}$ denotes the Lie algebra of G.
- Ad denotes the adjoint action of G to the Lie group $\mathfrak{g}$.
- $(\ ,\ )_{\mathfrak{g}} : \mathfrak{g}\times\mathfrak{g} \to \mathbb{F}$ is a product on the $\mathfrak{g}$, which is defined by $(X,Y)_{\mathfrak{g}} = \mathrm{Tr}({}^t X\bar{Y})$.
- V denotes an n-dimensional vector space over $\mathbb{F}$.
- For two ordered bases $\mathbf{a}$ and $\mathbf{b}$ in a vector space, we denote by $(\mathbf{a}/\mathbf{b})$ the base-change matrix from $\mathbf{b}$ to $\mathbf{a}$ satisfying $\mathbf{a} = \mathbf{b}(\mathbf{a}/\mathbf{b})$. We write simply $[\mathbf{a}/\mathbf{b}]$ for the determinant $\det(\mathbf{a}/\mathbb{T}^K_\gamma\mathbf{b})$ of $(\mathbf{a}/\mathbf{b})$. We deal with ordered bases in this paper.

2.2. Torsion of a chain complex

We recall the definition of the torsion.

Let $C_* = (0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \to 0)$ be a chain complex over $\mathbb{F}$. For each i let Z_i denote the kernel of ∂_i, B_i the image of ∂_{i+1} and H_i the homology group Z_i/B_i. We say that C_* is *acyclic* if H_i vanishes for every i.

Let c^i be a basis of C_i and c be the collection $\{c^i\}_{i\geqslant 0}$. We call the pair (C_*, c) a *based chain complex*, c the preferred basis of C_* and c^i the preferred basis of C_i. Let h^i be a basis of H_i.

We construct another basis as follows. By the definitions of Z_i, B_i and H_i, the following two split exact sequences exist.

$$0 \to Z_i \to C_i \xrightarrow{\partial_i} B_{i-1} \to 0,$$
$$0 \to B_i \to Z_i \to H_i \to 0.$$

Let $\widetilde{B}_{i-1}$ be a lift of B_{i-1} to C_i and $\widetilde{H}_i$ a lift of H_i to Z_i. Then we can decompose C_i as follows.

$$\begin{aligned} C_i &= Z_i \oplus \widetilde{B}_{i-1} \\ &= B_i \oplus \widetilde{H}_i \oplus \widetilde{B}_{i-1} \\ &= \partial_{i+1}\widetilde{B}_i \oplus \widetilde{H}_i \oplus \widetilde{B}_{i-1}. \end{aligned}$$

We choose b^i a basis of B_i. We write $\widetilde{b}^i$ for a lift of b^i and $\widetilde{h}^i$ for a lift of h^i. By the construction, the set $\partial_{i+1}(\widetilde{b}^i) \cup \widetilde{h}^i \cup \widetilde{b}^{i-1}$ forms another ordered basis of C_i. We denote simply this new basis by $\partial_{i+1}(\widetilde{b}^i)\widetilde{h}^i\widetilde{b}^{i-1}$. Then the definition of $\mathrm{tor}(C_*, c, h)$ is as follows.

$$\mathrm{tor}(C_*, c, h) = \prod_i^n \left[\partial_{i+1}(\widetilde{b}^i)\widetilde{h}^i\widetilde{b}^{i-1}/c^i\right]^{(-1)^{i+1}} \in \mathbb{F}^*.$$

It is well known that $\mathrm{tor}(C_*, c, h)$ is independent of the choices of $\{b^i\}_{i\geqslant 0}$, the lifts $\{\widetilde{b}^i\}_{i\geqslant 0}$ and $\{\widetilde{h}^i\}_{i\geqslant 0}$.

We also define the torsion $\mathrm{Tor}(C_*, c, h)$ with the sign term $(-1)^{|C_*|}$ as follows [19]

$$\mathrm{Tor}(C_*, c, h) = (-1)^{|C_*|} \cdot \mathrm{tor}(C_*, c, h).$$

Here

$$|C_*| = \sum_{i\geqslant 0} \alpha_i(C_*) \cdot \beta_i(C_*),$$

where $\alpha_i(C_*) = \sum_{k=0}^i \dim C_k$ and $\beta_i(C_*) = \sum_{k=0}^i \dim H_k$.

2.3. Twisted chain complex and twisted cochain complex for CW-complex

Let W be a finite connected CW-complex and $\widetilde{W}$ its universal covering with the induced CW-structure. Since the fundamental group $\pi_1(W)$ acts on $\widetilde{W}$ by the covering transformation, the chain complex $C_*(\widetilde{W};\mathbb{Z})$ has a natural structure of a left $\mathbb{Z}[\pi_1(W)]$-module. We denote by ρ a homomorphism from $\pi_1(W)$ to G. We regard the Lie group $\mathfrak{g}$ as a right $\mathbb{Z}[\pi_1(W)]$-module by $\mathfrak{g} \times \pi_1(W) \ni (v, \gamma) \mapsto \mathrm{Ad}_{\rho(\gamma^{-1})}(v) \in \mathfrak{g}$. We use the notation $\mathfrak{g}_\rho$ for $\mathfrak{g}$ with the right $\mathbb{Z}[\pi_1(W)]$-module structure. Following [9, 15], we introduce the following notations. Set

$$C_*(W;\mathfrak{g}_\rho) = \mathfrak{g} \otimes_{\mathrm{Ad}\circ\rho} C_*(\widetilde{W};\mathbb{Z}),$$
$$C_*(W;\widetilde{\mathfrak{g}}_\rho) = \mathfrak{g}(t) \otimes_{\alpha\otimes\mathrm{Ad}\circ\rho} C_*(\widetilde{W};\mathbb{Z})$$

where $\mathfrak{g}(t)$ is $\mathbb{F}(t)\otimes\mathfrak{g}$ and α is a surjective homomorphism from $\pi_1(W)$ to the multiplicative group $\langle t\rangle$. Note that $f\otimes v\otimes(\gamma\cdot\sigma) = f\cdot t^{\alpha(\gamma)}\otimes\mathrm{Ad}_{\rho(\gamma^{-1})}(v)\otimes\sigma$. We call $C_*(W;\mathfrak{g}_\rho)$ *the $\mathfrak{g}_\rho$-twisted chain complex* and $C_*(W;\widetilde{\mathfrak{g}}_\rho)$ *the $\widetilde{\mathfrak{g}}_\rho$-twisted chain complex* of W. We also denote by $C^*(W;\mathfrak{g}_\rho)$ the $\mathbb{F}$-module consisting of the $\pi_1(W)$-equivalent homomorphisms from $C_*(\widetilde{W};\mathbb{Z})$ to $\mathfrak{g}$, *i.e.*, a homomorphism h satisfies $h(\gamma\cdot\sigma) = h(\sigma)\cdot\gamma^{-1}$ for $\gamma \in \pi_1(W)$. We call $C^*(W;\mathfrak{g}_\rho)$ *the $\mathfrak{g}_\rho$-twisted cochain complex* of W. $H_*(W;\mathfrak{g}_\rho)$ and

$H^*(W;\mathfrak{g}_\rho)$ denote the homology and cohomology groups of the $\mathfrak{g}_\rho$-twisted chain and cochain complexes.

2.4. The Reidemeister torsion for twisted chain complex

We keep the notation of the previous subsection. Let $e_1^{(i)},\dots,e_{n_i}^{(i)}$ be the set of i-dimensional cells of W. We take a lift $\widetilde{e}_j^{(i)}$ of the cell $e_j^{(i)}$ in $\widetilde{W}$. Then, for each i, $\widetilde{c}^i = \{\widetilde{e}_1^{(i)},\dots,\widetilde{e}_{n_i}^{(i)}\}$ is a basis of the $\mathbb{Z}[\pi_1(W)]$-module $C_i(\widetilde{W};\mathbb{Z})$. Let $\mathbf{B}=\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ be a basis of $\mathfrak{g}$. Then we obtain the following basis of $C_i(W;\mathfrak{g}_\rho)$:

$$\mathbf{c}_{\mathbf{B}} = \left\{\dots,\mathbf{a}\otimes\widetilde{e}_1^{(i)},\mathbf{b}\otimes\widetilde{e}_1^{(i)},\mathbf{c}\otimes\widetilde{e}_1^{(i)},\dots,\mathbf{a}\otimes\widetilde{e}_{n_i}^{(i)},\mathbf{b}\otimes\widetilde{e}_{n_i}^{(i)},\mathbf{c}\otimes\widetilde{e}_{n_i}^{(i)},\dots\right\}.$$

When $\mathbf{h}^i=\{h_1^i,\dots,h_{k_i}^i\}$ is a basis of $H_i(W;\mathfrak{g}_\rho)$, we denote by $\mathbf{h}$ the basis $\{\mathbf{h}^0,\dots,\mathbf{h}^{\dim W}\}$ of $H_*(W;\mathfrak{g}_\rho)$. Then $\mathrm{Tor}(C_*(W;\mathfrak{g}_\rho),\mathbf{c}_{\mathbf{B}},\mathbf{h})\in\mathbb{F}^*$ is well defined. Furthermore adding a sign-refinement term into $\mathrm{Tor}(C_*(W;\mathfrak{g}_\rho),$ $\mathbf{c}_{\mathbf{B}},\mathbf{h})$, we define *the Reidemeister torsion* of (W,ρ) as a vector in some 1-dimensional vector space as follows.

Definition 2.4.1 ([4, 5]). — *Let $c_{\mathbb{R}}$ be the basis over $\mathbb{R}$ of $C_*(W;\mathbb{R})$. Choose an orientation $\mathfrak{o}$ of the real vector space $\oplus_{i\geqslant 0}H_i(W;\mathbb{R})$ and provide $H_*(W;\mathbb{R})$ with a basis $h_{\mathfrak{o}}=\{h^0,\dots,h^{\dim W}\}$ such that each h^i is a basis of $H_i(W;\mathbb{R})$ and the orientation determined by $h_{\mathfrak{o}}$ agrees with $\mathfrak{o}$. Let τ_0 be either $+1$ or -1 according to the sign of $\mathrm{Tor}(C_*(W;\mathbb{R}),c_{\mathbb{R}},h_{\mathfrak{o}})$. Then we define the Reidemeister torsion $\mathcal{T}(W,\mathfrak{g}_\rho,\mathfrak{o})$ by*

$$\mathcal{T}(W,\mathfrak{g}_\rho,\mathfrak{o}) = \tau_0\cdot\mathrm{Tor}(C_*(W;\mathfrak{g}_\rho),\mathbf{c}_{\mathbf{B}},\mathbf{h})\otimes_{i\geqslant 0}\det\mathbf{h}^i\in\mathrm{Det}\,H_*(W;\mathfrak{g}_\rho),$$

where $\det\mathbf{h}^i = h_1^{(i)}\wedge\dots\wedge h_{k_i}^{(i)}$ *and*

$$\mathrm{Det}\,H_*(W;\mathfrak{g}_\rho) = \otimes_{i=0}^{\dim W}(\wedge^{\dim H_i}H_i(W;\mathfrak{g}_\rho))^{(-1)^i}.$$

Here V^{-1} means the dual space of a vector space V and the dual basis of $\det\mathbf{h}^i = h_1^{(i)}\wedge\dots\wedge h_{k_i}^{(i)}$ is $h_1^{(i)*}\wedge\dots\wedge h_{k_i}^{(i)*}$ where $h_j^{(i)*}$ is the dual element of $h_j^{(i)}$.

We made some choices in the definition of $\mathcal{T}(W,\mathfrak{g}_\rho,\mathfrak{o})$. However the following well-definedness is known [15, p. 10]:

- The sign of $\mathcal{T}(W,\mathfrak{g}_\rho,\mathfrak{o})$ is determined by the homology orientation $\mathfrak{o}$ *i.e.*, if we choose the other homology orientation, then the sign of $\mathcal{T}(W,\mathfrak{g}_\rho,\mathfrak{o})$ changes;

- $\mathcal{T}(W, \mathfrak{g}_\rho, \mathfrak{o})$ does not depend on the choice of the lift $\widetilde{e}_j^{(i)}$ for each cell $e_j^{(i)}$;
- $\mathcal{T}(W, \mathfrak{g}_\rho, \mathfrak{o})$ does not depend on the choice of the basis $\mathbf{h}$ in $\oplus_{i \geqslant 0} H_i(W; \mathfrak{g}_\rho)$.

We also have the following well-definedness.

Lemma 2.4.2. — *If the Euler characteristic of W is equal to zero, then $\mathcal{T}(W, \mathfrak{g}_\rho, \mathfrak{o})$ does not depend on the choice of the basis of $\mathfrak{g}$.*

Proof. — This follows from the definition. □

Similarly we define the Reidemeister torsion of the twisted $\widetilde{\mathfrak{g}}_\rho$-chain complex.

Definition 2.4.3. — *We define $\mathcal{T}(W, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$ by*

$$\mathcal{T}(W, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o}) = \tau_0 \cdot \mathrm{Tor}(C_*(W; \widetilde{\mathfrak{g}}), \mathbf{1} \otimes c_{\mathbf{B}}, \mathbf{h}) \otimes_{i \geqslant 0} \det \mathbf{h}^i.$$

$\mathcal{T}(W, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$ has the indeterminacy of t^m where $m \in \mathbb{Z}$. This indeterminacy is caused by the choice of the lifts $\{\widetilde{e}_j^{(i)}\}$ and the action of α.

It is also known that the sign refined torsion $\tau_0 \cdot \mathrm{Tor}(C_*(W; \mathfrak{g}_\rho), \mathbf{c}_{\mathbf{B}}, \mathbf{h})$ has the invariance under simple homotopy equivalences, and that it satisfies the following *Multiplicativity property*. Suppose we have the following exact sequence of based chain complexes:

$$0 \to (C'_*, c') \to (C_*, c' \cup \bar{c}'') \to (C''_*, c'') \to 0 \tag{1}$$

where these chain complexes are based chain complexes which consist of vector spaces with bases. Here we denote bases of C'_*, C''_* by c', c'' and a lift of c'' to C_* by $\bar{c}''$. For each i, fix the volume forms on C'_i, C_i, C''_i by using given bases and choose volume forms on $H_i(C'_*), H_i(C_*)$ and $H_i(C''_*)$. There exists the long exact sequence in homology associated to the short exact sequence (1):

$$\cdots \to H_i(C'_*) \to H_i(C_*) \to H_i(C''_*) \to H_{i-1}(C'_*) \to \cdots.$$

We denote by $\mathcal{H}_*$ this acyclic complex. Note that this acyclic complex is a based chain complex.

Proposition 2.4.4 (Multiplicativity property [12, 20]). — *We have*

$$\mathrm{Tor}(C_*) = (-1)^{\alpha(C'_*, C''_*) + \varepsilon(C'_*, C_*, C''_*)} \, \mathrm{Tor}(C'_*) \cdot \mathrm{Tor}(C''_*) \cdot \mathrm{tor}(\mathcal{H}_*),$$

where

$$\alpha(C'_*, C''_*) = \sum_{i\geqslant 0} \alpha_{i-1}(C'_*)\alpha_i(C''_*) \in \mathbb{Z}/2\mathbb{Z},$$

$$\varepsilon(C'_*, C_*, C''_*) = \sum_{i\geqslant 0} \Big[(\beta_i(C_*)+1)(\beta_i(C'_*)+\beta(C''_*)) + \beta_{i-1}(C'_*)\beta(C''_*)\Big] \in \mathbb{Z}/2\mathbb{Z}.$$

2.5. On the representation spaces

Let π be a finitely generated group and we denote by $R(\pi, G)$ the space of G-representations of π. We define the topology of this space by compact-open topology. Here we assume that π has the discrete topology and the Lie group G has the usual one. A representation $\rho : \pi \to G$ is called *central* if $\rho(\pi) \subset \{\pm\mathbf{1}\}$.

A representation ρ is called *abelian* if its image $\rho(\pi)$ is an abelian subgroup of G. A representation ρ is called *reducible* if there exists a proper non-trivial subspace U of $\mathbb{C}^2$ such that $\rho(g)(U) \subset U$ for any $g \in \pi$. A representation ρ is called *irreducible* if it is not reducible. We denote by $R^{\mathrm{red}}(\pi, G)$ the subset of reducible representations and by $R^{\mathrm{irr}}(\pi, G)$ the subset of irreducible ones. Note that all abelian representations are reducible. The Lie group G acts on $R(\pi, G)$ by conjugation. We write $[\rho]$ for the conjugacy class of $\rho \in R(\pi, G)$, and we denote by $\widehat{R}(\pi, G)$ the quotient space $R(\pi, G)/G$.

If G is $\mathrm{SU}(2)$, then one can see that the reducible representations are exactly abelian ones. Note that this does not hold for the case of $\mathrm{SL}(2,\mathbb{C})$-representations. The action by conjugation of $\mathrm{SU}(2)$ on $R(\pi, \mathrm{SU}(2))$ factors through $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm\mathbf{1}\}$. This action is free on the $R^{\mathrm{irr}}(\pi, \mathrm{SU}(2))$. We set $\widehat{R}^{\mathrm{irr}}(\pi, \mathrm{SU}(2)) = R^{\mathrm{irr}}(\pi, \mathrm{SU}(2))/\,\mathrm{SO}(3)$.

If G is $\mathrm{SL}(2,\mathbb{C})$, then the quotient space $\widehat{R}(\pi, \mathrm{SL}(2, C))$ is not Hausdorff in general. Following [14], we will focus on the *character variety* $X(\pi; \mathrm{SL}(2,\mathbb{C}))$ which is the set of *characters* of π. Associated to the representation $\rho \in R(\pi, \mathrm{SL}(2,\mathbb{C}))$, its character $\chi_\rho : \pi \to \mathbb{C}$, defined by $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$. In some sense, $X(\pi, \mathrm{SL}(2,\mathbb{C}))$ is the "algebro quotient" of $R(\pi, \mathrm{SL}(2,\mathbb{C}))$ by $\mathrm{PSL}(2,\mathbb{C})$. It is well known that $R(\pi, \mathrm{SL}(2,\mathbb{C}))$ and $X(\pi)$ have the structure of complex algebraic affine sets and two irreducible representations of π in $\mathrm{SL}(2,\mathbb{C})$ with the same character are conjugate by an element of $\mathrm{SL}(2,\mathbb{C})$. (For the details, see [14].)

2.6. The Reidemeister torsion for knot exteriors

In this subsection, we recall λ-regular representations and how to construct distinguished bases of $\mathfrak{g}_\rho$-twisted homology groups of knot exteriors for a λ-regular representation ρ. These definitions have originally been given in [15]. The original definitions are written in terms of the $\mathfrak{g}_\rho$-twisted cohomology group. We introduce the homology version by using the duality between the twisted homology and cohomology associated to *the Kronecker pairing* $C_*(W;\mathfrak{g}_\rho) \times C^*(W;\mathfrak{g}_\rho) \ni (\xi \otimes \sigma, v) \mapsto (v(\sigma), \xi)_{\mathfrak{g}} \in \mathbb{F}$ [15, p. 11].

Let K be a knot in a homology three sphere M. We give a knot exterior M_K the canonical homology orientation defined as follows. It is well known that the $\mathbb{R}$-vector space

$$H_*(M_K;\mathbb{R}) = H_0(M_K;\mathbb{R}) \oplus H_1(M_K;\mathbb{R})$$

has the basis $\{[pt],[\mu]\}$. Here $[pt]$ is the homology class of a point and $[\mu]$ is the homology class of a meridian of K. We denote by $\mathfrak{o}$ the orientation induced by $\{[pt],[\mu]\}$.

We calculate the twisted homology groups of a circle and a 2-dimensional torus before giving the definition of a natural basis of $H_*(M_K;\mathfrak{g}_\rho)$. Here S^1 consists of one 0-cell $e^{(0)}$ and one 1-cell $e^{(1)}$.

Lemma 2.6.1. — *Suppose that G is* SU(2). *If $\rho \in R(\pi_1(S^1),G)$ is central, then $H_*(S^1;\mathfrak{g}_\rho) = \mathfrak{g} \otimes H_*(S^1;\mathbb{R})$. If ρ is non-central, then we have*

$$H_1(S^1;\mathfrak{g}_\rho) = \mathbb{R}[P_\rho \otimes \widetilde{e}^{(1)}],$$

and

$$H_0(S^1;\mathfrak{g}_\rho) = \mathbb{R}[P_\rho \otimes \widetilde{e}^{(0)}]$$

where P_ρ is a vector in $\mathfrak{g}$, which satisfies that $\mathrm{Ad}(\rho(\gamma))(P_\rho) = P_\rho$ *for any $\gamma \in \pi_1(S^1)$.*

Suppose that G is $\mathrm{SL}(2,\mathbb{C})$. *If $\rho \in R(\pi_1(S^1),G)$ is central, then $H_*(S^1;\mathfrak{g}_\rho) = \mathfrak{g} \otimes H_*(S^1;\mathbb{C})$. If ρ is non-central and $\rho(\pi_1(S^1))$ has no parabolic elements, then we have*

$$H_1(S^1;\mathfrak{g}_\rho) = \mathbb{C}[P_\rho \otimes \widetilde{e}^{(1)}],$$

and

$$H_0(S^1;\mathfrak{g}_\rho) = \mathbb{C}[P_\rho \otimes \widetilde{e}^{(0)}]$$

where P_ρ is a vector in $\mathfrak{g}$, which satisfies that $\mathrm{Ad}(\rho(\gamma))(P_\rho) = P_\rho$ *for any $\gamma \in \pi_1(S^1)$. If ρ is non-central and the subgroup $\rho(\pi_1(S^1))$ is contained in a subgroup which consists of parabolic elements, then we have*

$$H_1(S^1;\mathfrak{g}_\rho) = \mathbb{C}[P_\rho \otimes \widetilde{e}^{(1)}].$$

Proof. — This is a consequence of the following fact of homology of groups. For $G = \mathbb{Z}$, it follows that $H_0(G;N) = H^1(G;N) = N_G$ and $H^0(G;N) = H_1(G;N) = N^G$ where G is a group, N is a N-module, N_G is the group of invariants of N and N^G is the group of co-invariants of N (for the details, see [1]). □

We denote by T^2 a 2-dimensional torus. Here T^2 consists of one 0-cell $e^{(0)}$, two 1-cells $e_1^{(1)}, e_2^{(1)}$ and one 2-cell $e^{(2)}$. We denote each cell $e^{(0)}, e_1^{(1)}, e_2^{(1)}$ and $e^{(2)}$ by pt, μ, λ and T^2. One can also calculate the $\mathfrak{g}_\rho$-twisted homology groups of $C_*(T^2;\mathfrak{g}_\rho)$ as follows.

Lemma 2.6.2. — *Suppose that G is* SU(2). *If $\rho \in R(\pi_1(T^2), G)$ is central, then $H_*(T^2;\mathfrak{g}_\rho) = \mathfrak{g} \otimes H_*(T^2;\mathbb{R})$. If $\rho \in R(\pi_1(T^2), G)$ is non-central, then we have*

$$\begin{aligned} H_2(T^2;\mathfrak{g}_\rho) &= \mathbb{R}[\,P_\rho \otimes \widetilde{T}^2], \\ H_1(T^2;\mathfrak{g}_\rho) &= \mathbb{R}[\,P_\rho \otimes \widetilde{\mu}\,] \oplus \mathbb{R}[\,P_\rho \otimes \widetilde{\lambda}\,], \\ H_0(T^2;\mathfrak{g}_\rho) &= \mathbb{R}[\,P_\rho \otimes \widetilde{pt}\,] \end{aligned}$$

where P_ρ is a vector of $\mathfrak{g}$ such that $\mathrm{Ad}_{\rho(\gamma)}(P_\rho) = P_\rho$ for any $\gamma \in \pi_1(T^2)$.

Suppose that G is $\mathrm{SL}(2,\mathbb{C})$. *If $\rho \in R(\pi_1(T^2), G)$ is central, then $H_*(T^2;\mathfrak{g}_\rho) = \mathfrak{g} \otimes H_*(T^2;\mathbb{C})$. If $\rho \in R(\pi_1(T^2), G)$ is non-central and $\rho(\pi_1(T^2))$ contains a non-parabolic element, then we have*

$$\begin{aligned} H_2(T^2;\mathfrak{g}_\rho) &= \mathbb{C}[\,P_\rho \otimes \widetilde{T}^2], \\ H_1(T^2;\mathfrak{g}_\rho) &= \mathbb{C}[\,P_\rho \otimes \widetilde{\mu}\,] \oplus \mathbb{C}[\,P_\rho \otimes \widetilde{\lambda}\,], \\ H_0(T^2;\mathfrak{g}_\rho) &= \mathbb{C}[\,P_\rho \otimes \widetilde{pt}\,] \end{aligned}$$

where P_ρ is a vector of $\mathfrak{g}$ such that $\mathrm{Ad}_{\rho(\gamma)}(P_\rho) = P_\rho$ for any $\gamma \in \pi_1(T^2)$.

If $\rho \in R(\pi_1(T^2), G)$ is non-central and the subgroup $\rho(\pi_1(T^2))$ is contained in a subgroup which consists of parabolic elements, then we have

$$H_2(T^2;\mathfrak{g}_\rho) = \mathbb{C}[P_\rho \otimes \widetilde{T}^2]$$

and $[P_\rho \otimes \widetilde{\lambda}]$ is a non-zero class in $H_1(M_K;\mathfrak{g}_\rho)$.

Proof. — This is a consequence of [15, Proposition 3.18]. □

Next we give the definition of regular representations for $\pi_1(M_K)$ in terms of the twisted $\mathfrak{g}_\rho$-chain complex.

Definition 2.6.3 (regular representations [15, p. 83]). — *We say that ρ is* regular *if ρ is irreducible and $\dim_{\mathbb{F}} H_1(M_K;\mathfrak{g}_\rho) = 1$.*

We let γ be a simple closed curve in ∂M_K. We say that ρ is γ-regular *if:*

(1) *ρ is regular;*

(2) *an inclusion* $\iota : \gamma \hookrightarrow M_K$ *induces the surjective homomorphism*

$$\iota_* : H_1(\gamma; \mathfrak{g}_\rho) \to H_1(M_K; \mathfrak{g}_\rho);$$

and

(3) *if* $\mathrm{Tr}(\rho(\pi_1(\partial M_K))) \subset \{\pm 2\}$, *then* $\rho(\gamma) \neq \pm \mathbf{1}$.

We fix an invariant vector $P_\rho \in \mathfrak{g}$ as above. Let γ be a simple closed curve in ∂M_K. An inclusion $\iota : \gamma \hookrightarrow M_K$ and the the Kronecker pairing between homology and cohomology induce the linear form $f^\rho_\gamma : H^1(M_K; \mathfrak{g}_\rho) \to \mathbb{F}$. By Lemma 2.6.1, it is explicitly described by

$$f^\rho_\gamma(v) = (\iota_*([\widetilde{\gamma} \otimes P_\rho]), v) = (P_\rho, v(\widetilde{\gamma}))_{\mathfrak{g}} \quad \text{for any } v \in H^1(M_K; \mathfrak{g}_\rho).$$

An alternative formulation of γ-regular representations is given in [5, 15]. Similarly, we can also give the following alternative formulation of the γ-regularity in our conventions.

Proposition 2.6.4. — *A representation* $\rho \in R^{\mathrm{irr}}(\pi_1(M_K), G)$ *is* γ*-regular if and only if the linear form* $f^\rho_\gamma : H^1(M_K; \mathfrak{g}_\rho) \to \mathbb{F}$ *is an isomorphism.*

Proof. — If f^ρ_γ is an isomorphism, then we have that $\dim_{\mathbb{F}} H^1(M_K; \mathfrak{g}_\rho) = 1$ and $\iota_*([P_\rho \otimes \widetilde{\gamma}])$ is a non-zero class in $H_1(M_K; \mathfrak{g}_\rho)$. It follows from the Kronecker pairing between the $\mathfrak{g}_\rho$-twisted homology and cohomology that $\dim_{\mathbb{F}} H_1(M_K; \mathfrak{g}_\rho)$ is also one. Hence ι_* is surjective. If ρ is γ-regular, then we have that $\dim_{\mathbb{F}} H_1(M_K; \mathfrak{g}_\rho) = 1$ and $\iota_* : H_1(\gamma; \mathfrak{g}_\rho) \to H_1(M_K; \mathfrak{g}_\rho)$ is surjective. We denote a generator of $H_1(M_K; \mathfrak{g}_\rho)$ by σ. There exists an element $[v \otimes \widetilde{\gamma}]$ of $H_1(\gamma; \mathfrak{g}_\rho)$ such that $\iota_*([v \otimes \widetilde{\gamma}]) = \sigma$.

If $\rho(\gamma)$ is central, then v satisfies that $\mathrm{Ad}(\rho(\gamma'))(v) = v$ for any $\gamma' \in \pi_1(\partial M_K)$. Therefore $\iota_*([v \otimes \widetilde{\gamma}])$ induces the isomorphism f^ρ_γ.

Suppose that $\rho(\gamma)$ is non-central, then $H_1(\gamma; \mathfrak{g}_\rho)$ is generated by $[P_\rho \otimes \widetilde{\gamma}]$. There exists an element $c \in \mathbb{F}^*$ such that $[v \otimes \widetilde{\gamma}] = c[P_\rho \otimes \widetilde{\gamma}]$. Hence $\iota_*([P_\rho \otimes \widetilde{\gamma}])$ is a non-zero class in $H_1(M_K; \mathfrak{g}_\rho)$. Therefore $\iota_*([P_\rho \otimes \widetilde{\gamma}])$ induces the isomorphism f^ρ_γ. □

We define a reference generator of $H_1(M_K; \mathfrak{g}_\rho)$ by using the above isomorphism f^ρ_γ.

Let ρ be a λ-regular representation of $\pi_1(M_K)$. By Lemma 2.6.2, the reference generator of $H_1(M_K; \mathfrak{g}_\rho)$ is defined by

$$h^{(1)}_\rho(\lambda) = \iota_* \left([P_\rho \otimes \widetilde{\lambda}] \right).$$

Moreover the reference generator of $H_2(M_K; \mathfrak{g}_\rho)$ is defined as follows.

LEMMA 2.6.5 (Cor. 3.23 [15]). — *Let $i : \partial M_K \hookrightarrow M_K$ be an inclusion map. If $\rho \in R(\pi_1(M_K), G)$ is γ-regular, then we have the isomorphism $i_* : H_2(\partial M_K; \mathfrak{g}_\rho) \to H_2(M_K; \mathfrak{g}_\rho)$.*

Using this isomorphism i_*, we define the reference generator of $H_2(M_K; \mathfrak{g}_\rho)$ by

$$h^{(2)}_\rho = i_*([P_\rho \otimes \widetilde{\partial M_K}]).$$

Remark 2.6.6. — The reference generators of $H^1(M_K; \mathfrak{g}_\rho)$ and $H^2(M_K; \mathfrak{g}_\rho)$ have been defined in [4, 5, 15] by using another metric of $\mathfrak{g}$. If we define reference generators of $H^1(M_K; \mathfrak{g}_\rho)$ and $H^2(M_K; \mathfrak{g}_\rho)$ by using our metric $(\ ,\ )_\mathfrak{g}$, then the resulting generators become the dual bases of $h^{(1)}_\rho(\lambda)$ and $h^{(2)}_\rho$ from the above propositions. (For the details, see [5, 15].)

We recall the definition of *the twisted Reidemeister torsion* for knot exteriors. Let $\rho : \pi_1(M_K) \to G$ be a λ-regular representation. We define $\mathbb{T}^K_\rho$ by the coefficient of the Reidemeister torsion $\mathcal{T}(M_K, \mathfrak{g}_\rho, \mathfrak{o})$ where we choose the reference generators $h^{(1)}_\rho(\lambda), h^{(2)}_\rho$ as a basis of $H_*(M_K; \widetilde{\mathfrak{g}})$, *i.e.*, $\mathbb{T}^K_\lambda$ is given explicitly by

$$\mathbb{T}^K_\lambda(\rho) = \tau_0 \cdot \operatorname{Tor}\left(C_*(M_K; \mathfrak{g}_\rho), \mathbf{c_B}, \left\{h^{(1)}_\rho(\lambda), h^{(2)}_\rho\right\}\right) \in \mathbb{F}^*.$$

Given the reference generator of $H_*(M_K; \mathfrak{g}_\rho)$, the basis of the determinant line $\operatorname{Det} H_*(M_K; \mathfrak{g}_\rho)$ is also given. This means that a trivialization of the line bundle $\operatorname{Det} H_*(M_K; \mathfrak{g}_\rho)$ at ρ is given. The Reidemeister torsion $\mathcal{T}(M_K, \mathfrak{g}_\rho, \mathfrak{o})$ is a section of the line bundle $\operatorname{Det} H_*(M_K; \mathfrak{g}_\rho)$. We can regard $\mathbb{T}^K_\lambda$ as a section of the line bundle $\operatorname{Det} H_*(M_K; \mathfrak{g}_\rho)$ over λ-regular representations with respect to the trivialization by $\{h^{(1)}_\rho(\lambda), h^{(2)}_\rho\}$. We also call $\mathbb{T}^K_\lambda$ *the twisted Reidemeister torsion*.

3. A relationship between acyclic Reidemeister torsion and non-acyclic Reidemeister torsion

3.1. The statement of main theorem

Our purpose is to express the twisted Reidemeister torsion by using a limit of the acyclic Reidemeister torsion.

Let K be a knot in a homology three sphere M and M_K its exterior. One of the invariants which we will investigate is the twisted Reidemeister torsion $\mathbb{T}^K_\lambda$. The other is the acyclic Reidemeister torsion $\mathcal{T}(M_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$. This invariant coincides with the twisted Alexander invariant of $\pi_1(M_K)$

[10]. The twisted Alexander invariant is computed by using the Fox calculus [9, 10]. We prove that the twisted Reidemeister torsion may be expressed as the differential coefficient of the twisted Alexander invariant of $\pi_1(M_K)$.

The invariant $\mathcal{T}(M_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$ is only defined when the local system $C_*(M_K; \widetilde{\mathfrak{g}}_\rho)$ is acyclic. On the other hand, the twisted Reidemeister torsion $\mathbb{T}^K_\lambda$ is defined on the set of λ-regular representations of $\pi_1(M_K)$. We need to check whether the local system $C_*(M_K; \widetilde{\mathfrak{g}}_\rho)$ is acyclic for a λ-regular representation ρ.

Proposition 3.1.1. — *Let ρ be an* $\mathrm{SU}(2)$ *or* $\mathrm{SL}(2,\mathbb{C})$*-representation of a knot group. If ρ is λ-regular, then the twisted chain complex $C_*(M_K; \widetilde{\mathfrak{g}}_\rho)$ is acyclic.*

Note that for a knot exterior in a homology 3-sphere, the homomorphism α satisfies $\alpha(\mu) = t$ where μ is the meridian of the knot.

Therefore $\mathbb{T}^K_\lambda$ and $\mathcal{T}(M_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$ are well defined on λ-regular representations. By the definitions, the twisted Reidemeister torsion $\mathbb{T}^K_\lambda$ is an element of $\mathbb{F}^*$ and the twisted Alexander invariant $\mathcal{T}(M_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$ is an element of $\mathbb{F}(t)^*$. Actually the following relation between $\mathbb{T}^K_\lambda \in \mathbb{F}^*$ and the rational function $\mathcal{T}(M_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o}) \in \mathbb{F}(t)^*$.

Theorem 3.1.2. — *If ρ is a λ-regular representation, then the acyclic Reidemeister torsion $\mathcal{T}(M_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$ for ρ has a simple zero at $t = 1$. Moreover the following holds:*

$$\mathbb{T}^K_\lambda(\rho) = -\lim_{t\to 1} \frac{\mathcal{T}(M_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})(t)}{t-1} = -\left.\frac{d}{dt}\mathcal{T}(M_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})\right|_{t=1}.$$

This says that we can compute the twisted Reidemeister torsion $\mathbb{T}^K_\lambda$ algebraically by using Fox calculus of the twisted Alexander invariant of K.

3.2. Proof of Proposition 3.1.1

We prove Proposition 3.1.1 by using the λ-regularity of ρ.

Proof of Proposition 3.1.1. — It is well known that any compact connected triangulated 3-manifold whose boundary is non-empty and consists of tori can be collapsed into a 2-dimensional sub-complex (see II. Cor. 11.9 in [19]). Moreover, by the simple-homotopy extension theorem, every CW-complex has the simple-homotopy type of a CW-complex which has only one vertex. We denote this 2-dimensional CW-complex by W and this deformation from M_K to W by φ. Since two $\widetilde{\mathfrak{g}}_\rho$-twisted homology groups

$H_*(M_K; \widetilde{\mathfrak{g}}_\rho)$ and $H_*(W; \widetilde{\mathfrak{g}}_\rho)$ are isomorphic, we prove that $H_*(W; \widetilde{\mathfrak{g}}_\rho)$ vanishes in the following.

The fact that $H_0(W; \widetilde{\mathfrak{g}}_\rho) = 0$ is proved in [9, Proposition 3.5]. Since the Euler characteristic of W is zero, the dimension of $H_1(W; \widetilde{\mathfrak{g}}_\rho)$ is equal to that of $H_2(W; \widetilde{\mathfrak{g}}_\rho)$. We must prove that the dimension of $H_2(W; \widetilde{\mathfrak{g}}_\rho)$ over $\mathbb{F}(t)$ is zero. It is enough to prove that the rank over $\mathbb{F}[t, t^{-1}]$ of the second homology group of the following local system is zero:

$$C_*(W; \mathfrak{g}_\rho[t, t^{-1}]) = \mathfrak{g}[t, t^{-1}] \otimes_{\alpha \otimes \mathrm{Ad} \circ \rho} C_*(\widetilde{W}; \mathbb{Z})$$

where $\mathfrak{g}[t, t^{-1}]$ is $\mathbb{F}[t, t^{-1}] \otimes \mathfrak{g}$. We denote the homology group of this chain complex by $H_*(W; \mathfrak{g}_\rho[t, t^{-1}])$. Suppose that the rank of $H_2(W; \mathfrak{g}_\rho[t, t^{-1}]) > 0$.

There exists the long exact homology sequence [18]:

$$\begin{aligned} 0 \to H_2(W; \mathfrak{g}_\rho[t, t^{-1}]) &\xrightarrow{(t-1)\cdot} H_2(W; \mathfrak{g}_\rho[t, t^{-1}]) \\ &\xrightarrow{t=1} H_2(W; \mathfrak{g}_\rho) \xrightarrow{\Delta} H_1(W; \mathfrak{g}_\rho[t, t^{-1}]) \to \cdots \end{aligned}$$

associated to the short exact sequence:

$$0 \to \mathfrak{g}[t, t^{-1}] \xrightarrow{(t-1)\cdot} \mathfrak{g}[t, t^{-1}] \xrightarrow{t=1} \mathfrak{g} \to 0.$$

Since the rank of $H_2(W; \mathfrak{g}_\rho[t, t^{-1}])$ is not zero, the multiplication with $(t-1)$ is not surjective. Hence the image of the evaluation map $(t = 1)$ is not trivial and therefore surjective since the dimension of $H_2(W; \mathfrak{g}_\rho)$ is only one. This implies that Δ is trivial. On the other hand the equation

$$\partial(1 \otimes P_\rho \otimes \widetilde{\varphi(\partial M_K)}) = (t-1) \cdot (1 \otimes P_\rho \otimes \widetilde{\varphi(\lambda)})$$

implies that $\Delta([P_\rho \otimes \widetilde{\varphi(\partial M_K)}]) = [1 \otimes P_\rho \otimes \widetilde{\varphi(\lambda)}]$. But $[1 \otimes P_\rho \otimes \widetilde{\varphi(\lambda)}]$ can not be trivial since it is mapped under the evaluation map $(t = 1)$ to $[P_\rho \otimes \widetilde{\varphi(\lambda)}]$ and the chain $P_\rho \otimes \widetilde{\varphi(\lambda)}$ represents a non-zero homology class in $H_1(W; \mathfrak{g}_\rho)$. This is a contradiction. Therefore the rank of $H_2(W; \mathfrak{g}_\rho[t, t^{-1}])$ over $\mathbb{F}[t, t^{-1}]$ is zero. Hence we have that $\dim_{\mathbb{F}(t)} H_2(W; \widetilde{\mathfrak{g}}_\rho) = 0$. Also $\dim_{\mathbb{F}(t)} H_1(W; \widetilde{\mathfrak{g}}_\rho)$ is zero. □

3.3. Proof of Theorem 3.1.2

At first, we prepare some notations and an algebraic proposition.

Let C_* is an n-dimensional chain complex which consists of left G-modules M_i $(1 \leqslant i \leqslant n)$ where G is a group. We denote by $C_*(V)$ the chain complex which consists of the vector spaces $V \otimes_\rho M_i$ where V is a right G-vector space over $\mathbb{F}$ and ρ is a homomorphism from G to $\mathrm{Aut}(V)$.

Let $H_*(V)$ be the homology groups of $C_*(V)$, $C'_*(V)$ the subchain complex which consists of a lift of $H_*(V)$ to $C_*(V)$ and $C''_*(V)$ the quotient of $C_*(V)$ by $C'_*(V)$. We denote by $h(V), c'$ and c'' the bases of $H_*(V), C'_*(V)$ and $C''_*(V)$. Note that c' is a lift of $h(V)$ to $C_*(V)$. If there exists a homomorphism α from G to the multiplicative group $\langle t\rangle$, we denote by $C_*(V(t))$ which consists of vector spaces $V(t) \otimes_{\alpha\otimes\rho} M_i$. Here we denote $\mathbb{F}(t) \otimes V$ by $V(t)$. Moreover let $C'_*(V(t))$ be the subchain complex which is given by extending the coefficients of $C'_*(V)$ to $\mathbb{F}(t)$ by using α and $C''_*(V(t))$ the quotient of $C_*(V(t))$ by $C'_*(V(t))$.

Proposition 3.3.1. — *We assume that $C_*(V(t))$ and $C'_*(V(t))$ are acyclic. The following relation holds:*

$$(1)\quad \lim_{t\to 1}(-1)^{\alpha'}\frac{\mathrm{Tor}(C_*(V(t)), 1\otimes c' \cup 1\otimes \bar{c}'')}{\mathrm{Tor}(C'_*(V(t)), 1\otimes c')} = (-1)^{\varepsilon' + |C_*(V)|}\,\mathrm{Tor}(C_*(V), c'\cup \bar{c}'', h(V))$$

where $\bar{c}''$ is a lift of c'' to $C_*(V)$, α' is $\alpha(C'_*(V(t)), C''_*(V(t)))$ in Proposition 2.4.4, and $\varepsilon' \in \mathbb{Z}/2\mathbb{Z}$ is given by $\sum_{i=0}^{n-1}\dim_{\mathbb{F}} C''_i(V)\cdot\beta_i(C_*(V))$.

Proof. — The chain complex $C''_*(V(t))$ is also acyclic from the long exact sequence of the pair $(C_*(V(t)), C'_*(V(t)))$. We can apply Proposition 2.4.4 for the short exact sequence:

$$0 \to (C'_*(V(t)), 1\otimes c') \to (C_*(V(t)), 1\otimes c' \cup 1\otimes \bar{c}'') \to (C''_*(V(t)), 1\otimes c'') \to 0.$$

Then, we obtain the following equation of the torsions.

$$(2)\quad (-1)^{\alpha'}\,\mathrm{Tor}(C_*(V(t)), 1\otimes c'\cup 1\otimes\bar{c}'') = \mathrm{Tor}(C'_*(V(t)), 1\otimes c')\cdot \mathrm{Tor}(C''_*(V(t)), 1\otimes c'').$$

Note that $\varepsilon(C'_*(V(t)), C_*(V(t)), C''_*(V(t))) = 0$ because $C_*(V(t))$, $C'_*(V(t))$ and $C''_*(V(t))$ are acyclic.

Next we consider $\mathrm{Tor}(C''_*(V(t)), c'')$. It follows from the long exact sequence of the pair $(C_*(V), C'_*(V))$ and the definition of $C'_*(V)$ that the chain complex $C''_*(V)$ is also acyclic. Since $C''_*(V)$ is acyclic, we can choose a basis $\widetilde{b}''^i$ of $\widetilde{B}''_i$ for each i. Here $\widetilde{B}''_i$ is a lift of $B''_i = \mathrm{Im}\,\partial_{i+1}(C''_{i+1}(V))$ to $C''_{i+1}(V)$.

Claim 3.3.2. — *A subset $1\otimes\widetilde{b}''^i$ in $C''_{i+1}(V(t))$ generates a subspace on which the boundary operator ∂_{i+1} is injective.*

Proof of Claim 3.3.2. — If the determinant of the boundary operator restricted on $\mathbb{F}(t)\langle 1\otimes\widetilde{b}''^i\rangle$ is zero, then substituting 1 for the parameter t

we have that the determinant of the boundary operator restricted on $\mathbb{F}\langle \widetilde{b}''^i \rangle$ is also zero. This is a contradiction to the choices of $\widetilde{b}''^i$. □

Therefore $\mathrm{Tor}(C''_*(V(t)), 1 \otimes c'')$ is represented as

$$\prod_{i=0}^{n} \left[\partial_{i+1}(1 \otimes \widetilde{b}''^i) 1 \otimes \widetilde{b}''^{i-1} / 1 \otimes c''^i \right]^{(-1)^{i+1}}.$$

We denote by $\widetilde{b}^i$ a lift $1 \otimes \widetilde{b}''^i$ to $C_*(V(t))$ simply. Note that

$$\prod_{i=0}^{n} \left[\partial_{i+1}(1 \otimes \widetilde{b}''^i)\, 1 \otimes \widetilde{b}''^{i-1} / 1 \otimes c''^i \right]^{(-1)^{i+1}} = \prod_{i=0}^{n} \left[(1 \otimes c'^i)\, \partial_{i+1}(\widetilde{b}^i)\, \widetilde{b}^{i-1} / 1 \otimes c'^i \cup 1 \otimes \bar{c}''^i \right]^{(-1)^{i+1}}.$$

We substitute these results into the equation (2) Then we have

$$\begin{aligned}
(3) \qquad & \frac{\mathrm{Tor}(C_*(V(t)), 1 \otimes c' \cup 1 \otimes \bar{c}'')}{\mathrm{Tor}(C'_*(V(t)), 1 \otimes c')} \\
&= \mathrm{Tor}(C''_*(V(t)), 1 \otimes c'') \\
&= \prod_{i=0}^{n} \left[(1 \otimes c'^i)\, \partial_{i+1}(\widetilde{b}^i)\, \widetilde{b}^{i-1} / 1 \otimes c'^i \cup 1 \otimes \bar{c}''^i \right]^{(-1)^{i+1}} \\
&= \prod_{i=0}^{n} (-1)^{\dim_{\mathbb{F}} B''_i \cdot \dim_{\mathbb{F}} H_i(V)} \Big[\partial_{i+1}(\widetilde{b}^i)\, (1 \otimes c'^i)\, \widetilde{b}^{i-1} / 1 \\
&\qquad\qquad \otimes c'^i \cup 1 \otimes \bar{c}''^i \Big]^{(-1)^{i+1}}.
\end{aligned}$$

The acyclicity of $C''_*(V)$ shows that

$$\sum_{i=0}^{n} \dim_{\mathbb{F}} B''_i \cdot \dim_{\mathbb{F}} H_i(V) \equiv \sum_{i=0}^{n-1} \dim_{\mathbb{F}} C''_i(V) \cdot \beta_i(C_*(V)) \pmod 2.$$

Substituting 1 for t, the right hand side (3) turns into

$$(-1)^{\varepsilon'} \prod_{i=0}^{n} \left[\partial_{i+1}(\widetilde{b}^i)\, \widetilde{h}^i\, \widetilde{b}^{i-1} / c'^i \cup \bar{c}''^i \right]^{(-1)^{i+1}}.$$

This is equal to $(-1)^{\epsilon' + |C_*(V)|}\, \mathrm{Tor}(C_*(V), c' \cup \bar{c}'', h(V))$.

Although the left hand side is determined up to a factor $t^m (m \in \mathbb{Z})$, the limit at $t = 1$ is determined because the factor t^m does not affect taking a limit at $t = 1$. □

We can prove Theorem 3.1.2 as an application of Proposition 3.3.1.

Proof of Theorem 3.1.2. — As in the proof of Proposition 3.1.1, let W be a 2-dimensional CW-complex with a single vertex which has the same simple-homotopy type as M_K. We denote the deformation from M_K to W by φ. The compact 3-manifold M_K is simple homotopy equivalent to W. It is enough to prove the theorem for W because of the invariance of the simple homotopy equivalence for the Reidemeister torsion. Let ρ be a λ-regular representation of $\pi_1(M_K)$. We denote by the same symbols ρ and $\mathfrak{o}$ the representation of $\pi_1(W)$ and the homology orientation of $H_*(W;\mathbb{R})$ induced from that of M_K under the map φ.

We define the subchain complex $C'_*(W;\mathfrak{g}_\rho)$ of the $\mathfrak{g}_\rho$-twisted chain complex $C_*(W;\mathfrak{g}_\rho)$ by

$$C'_2(M_K;\mathfrak{g}_\rho) = \mathbb{F}\langle P_\rho \otimes \widetilde{\varphi(\partial M_K)}\rangle, \quad C'_1(W;\mathfrak{g}_\rho) = \mathbb{F}\langle P_\rho \otimes \widetilde{\varphi(\lambda)}\rangle$$

and $C_i(W;\mathfrak{g}_\rho) = 0\,(i \neq 1,2)$ where P_ρ is an invariant vector of $\mathfrak{g}$ such that $\mathrm{Ad}_{\rho(\gamma)}(P_\rho) = P_\rho$ for any $\gamma \in \pi_1(\varphi(\partial M_K))$. The modules of this subchain complex are lifts of homology groups $H_*(W;\mathfrak{g}_\rho)$. By the definition, the boundary operators of $C'_*(W;\mathfrak{g}_\rho)$ are zero homomorphisms. Let $C''_*(W;\mathfrak{g}_\rho)$ be the quotient of $C_*(W;\mathfrak{g}_\rho)$ by $C'_*(W;\mathfrak{g}_\rho)$. Similarly, we define the subcomplex $C'_*(W;\widetilde{\mathfrak{g}}_\rho)$ of $C_*(W;\widetilde{\mathfrak{g}}_\rho)$ to be

$$C'_2(W;\widetilde{\mathfrak{g}}_\rho) = \mathbb{F}(t)\langle 1 \otimes P_\rho \otimes \widetilde{\varphi(\partial M_K)}\rangle, \quad C'_1(W;\widetilde{\mathfrak{g}}_\rho) = \mathbb{F}(t)\langle 1 \otimes P_\rho \otimes \widetilde{\varphi(\lambda)}\rangle$$

and $C'_i(W) = 0$ for $i \neq 1,2$. The boundary operators of $C'_*(W;\widetilde{\mathfrak{g}}_\rho)$ is given by

$$0 \to C'_2(W;\widetilde{\mathfrak{g}}_\rho) \xrightarrow{(t-1)\cdot} C'_1(W;\widetilde{\mathfrak{g}}_\rho) \to 0.$$

This shows that the subchain complex $C'_*(M_K;\widetilde{\mathfrak{g}}_\rho)$ is acyclic. By Proposition 3.1.1, the $\widetilde{\mathfrak{g}}_\rho$-twisted chain complex $C_*(M_K;\widetilde{\mathfrak{g}}_\rho)$ is also acyclic.

The twisted chain complex $C'_*(W;\mathfrak{g}_\rho)$ has the natural basis:

$$c' = \{P_\rho \otimes \widetilde{\varphi(\partial M_K)},\, P_\rho \otimes \widetilde{\varphi(\lambda)}\}.$$

Let c'' be a basis of $C''_*(W;\mathfrak{g}_\rho)$ and $\bar{c}''$ a lift of c'' to $C_*(W;\mathfrak{g}_\rho)$. Applying Proposition 3.3.1, we have

$$\text{(4)} \quad \lim_{t\to 1} \frac{(-1)^{\alpha'}\,\mathrm{Tor}(C_*(W;\widetilde{\mathfrak{g}}_\rho), 1\otimes c' \cup 1 \otimes \bar{c}'')}{\mathrm{Tor}(C'_*(W;\widetilde{\mathfrak{g}}_\rho), 1 \otimes c')}$$
$$= (-1)^{\varepsilon' + |C_*(W;\mathfrak{g}_\rho)|}\,\mathrm{Tor}\left(C_*(W;\mathfrak{g}_\rho), c' \cup \bar{c}'', \left\{h^{(1)}_\rho(\lambda), h^{(2)}\rho\right\}\right).$$

Claim 3.3.3.

(1) $\mathrm{Tor}(C'_*(W;\widetilde{\mathfrak{g}}_\rho), 1 \otimes c') = t - 1$.
(2) $\alpha' \equiv 0 \,(\mathrm{mod}\, 2)$.
(3) $\varepsilon' + |C_*(W;\mathfrak{g}_\rho)| \equiv 1 \,(\mathrm{mod}\, 2)$.

Proof of Claim 3.3.3.

(1) It follows by the definition.
(2) If we denote the number of 1-cells of W by k, the CW-complex W has one 0-cell, k 1-cells and $(k-1)$ 2-cells. We have $\alpha' = 0 \cdot (3k+2) + 1 \cdot (6k-2) + 2 \cdot (6k-2) \equiv 0 \pmod 2$.
(3) This follows from $\varepsilon' = (3k-4)\cdot 1 \equiv 3k-4 \pmod 2$ and $|C_*(W;\mathfrak{g}_\rho)| = 3 \cdot 0 + (3k+3) \cdot 1 + (3k+3+3k-3) \cdot 2 \equiv 3k+3 \pmod 2$.

□

The equation (4) turns into

$$\lim_{t\to 1} \frac{\mathrm{Tor}(C_*(W;\widetilde{\mathfrak{g}}_\rho), 1 \otimes c' \cup 1 \otimes \bar{c}'')}{t-1} = -\,\mathrm{Tor}\left(C_*(W;\mathfrak{g}_\rho), c' \cup \bar{c}'', \left\{h_\rho^{(1)}(\lambda), h^{(2)}\rho\right\}\right).$$

Multiplying the both sides by the alternative products of the determinants of the base-change matrices

$$\prod_{i=0}^{2} \left[c'^i \cup \bar{c}''^i / \mathbf{c_B}\right]^{(-1)^{i+1}},$$

we obtain the following equation:

$$\lim_{t\to 1} \frac{\mathrm{Tor}(C_*(W;\widetilde{\mathfrak{g}}_\rho), \mathbf{c_B})}{t-1} = -\,\mathrm{Tor}\left(C_*(W;\mathfrak{g}_\rho), \mathbf{c_B}, \left\{h_\rho^{(1)}(\lambda), h^{(2)}\rho\right\}\right).$$

Finally multiplying the both sides by the sign τ_0 gives

$$\lim_{t\to 1} \frac{\mathcal{T}(W, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})}{t-1} = -\mathbb{T}_\lambda^K(\rho).$$

Summarizing the above calculation, we have shown that the rational function $\mathcal{T}(M_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$ has a simple zero at $t=1$ and its differential coefficient at $t=1$ agrees with minus the twisted Reidemeister torsion $-\mathbb{T}_\lambda^K(\rho)$. □

3.4. A description of $\mathbb{T}_\lambda^K$ using a Wirtinger representation

Let K be a knot in S^3 and E_K its exterior. We assume that $\rho \in R(\pi_1(E_K), G)$ is λ-regular. From Theorem 3.1.2 we can describe $-\mathbb{T}_\lambda^K(\rho)$ by using the differential coefficient of $\mathcal{T}(E_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$. We will describe the differential coefficient of $\mathcal{T}(E_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$ more explicitly by using a Wirtinger representation of $\pi_1(E_K)$.

For a Wirtinger representation:

$$\pi_1(E_K) = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_{k-1} \rangle \,,$$

we obtain a 2-dimensional CW-complex W which consists of one 0-cell p, k 1-cells $x_1, \dots, x_k$ and $(k-1)$ 2-cells $D_1, \dots, D_{k-1}$ attached by the relation $r_1, \dots, r_{k-1}$. This CW-complex W is simple homotopy equivalent to E_K. Let $\alpha : \pi_1(E_K) \to \mathbb{Z} = \langle t \rangle$ such that $\alpha(\mu) = t$. Here μ is a meridian of K. Note that for all i, $\alpha(x_i)$ is equal to t in $\mathbb{Z} = \langle t \rangle$.

The following calculation is due to the result of [9, 10]. This chain complex $C_*(W; \widetilde{\mathfrak{g}}_\rho)$ is as follows:

$$0 \to \mathfrak{g}(t)^{k-1} \xrightarrow{\partial_2} \mathfrak{g}(t)^k \xrightarrow{\partial_1} \mathfrak{g}(t) \to 0$$

where

$$\partial_2 = \begin{pmatrix} \Phi(\frac{\partial r_1}{\partial x_1}) & \dots & \Phi(\frac{\partial r_{k-1}}{\partial x_1}) \\ \vdots & \ddots & \vdots \\ \Phi(\frac{\partial r_1}{\partial x_k}) & \dots & \Phi(\frac{\partial r_{k-1}}{\partial x_k}) \end{pmatrix},$$

$$\partial_1 = (\Phi(x_1 - 1),\, \Phi(x_2 - 1),\, \dots,\, \Phi(x_k - 1))\,.$$

Here we briefly denote the l-times direct sum of $\mathfrak{g}(t)$ by $\mathfrak{g}(t)^l$.

We denote by $A^1_{K,\mathrm{Ad}\circ\rho}$ $3(k-1) \times 3(k-1)$ matrix:

$$\begin{pmatrix} \Phi(\frac{\partial r_1}{\partial x_2}) & \dots & \Phi(\frac{\partial r_{k-1}}{\partial x_2}) \\ \vdots & \ddots & \vdots \\ \Phi(\frac{\partial r_1}{\partial x_k}) & \dots & \Phi(\frac{\partial r_{k-1}}{\partial x_k}) \end{pmatrix}.$$

Under this situation, the twisted Alexander invariant $\mathcal{T}(W, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$ is given by

$$\tau_0 \cdot \frac{\det A^1_{K,\mathrm{Ad}\circ\rho}}{\det(\Phi(x_1 - 1))}$$

up to a factor t^m $(m \in \mathbb{Z})$.

If $\rho(x_i)$ is conjugate to the upper triangulate matrix

$$\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix},$$

then $\mathrm{Ad}_{\rho(x_i^{-1})}$ is conjugate to the upper triangulate matrix

$$\begin{pmatrix} 1 & * & * \\ & a^2 & * \\ & & a^{-2} \end{pmatrix}.$$

Calculating $\det(\Phi(x_1-1))$, we have that

$$\det(\Phi(x_1-1)) = (t-1)\big(t^2 - \mathrm{Tr}(\rho(x_1^2))t + 1\big).$$

Since $\mathcal{T}(E_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$ has zero at $t=1$,

$$\begin{aligned}\frac{d}{dt}\mathcal{T}(E_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})\Big|_{t=1} &= \lim_{t\to 1}\frac{\mathcal{T}(E_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})}{t-1}\\ &= \lim_{t\to 1}\tau_0\cdot t^m\frac{\det A^1_{K,\mathrm{Ad}\circ\rho}(t)}{(t-1)^2(t^2-\mathrm{Tr}(\rho(x_1^2))t+1)}.\end{aligned}$$

Lemma 3.4.1. — *If* $\mathrm{Tr}\,\rho(\partial E_K)\not\subset\{\pm 2\}$, *then we have*

$$\lim_{t\to 1}\tau_0\cdot t^m\frac{\det A^1_{K,\mathrm{Ad}\circ\rho}(t)}{(t-1)^2} = \frac{\tau_0}{2}\,\frac{d^2}{dt^2}\det A^1_{K,\mathrm{Ad}\circ\rho}(t)\Big|_{t=1}.$$

Proof. — The function $\mathcal{T}(E_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$ has a simple zero at $t=1$ and the numerator $\det A^1_{K,\mathrm{Ad}\circ\rho}(t)$ is an element of $\mathbb{F}[t,t^{-1}]$. Hence $(t-1)^2$ divides $\det A^1_{K,\mathrm{Ad}\circ\rho}(t)$. We write $(t-1)^2 f(t)$ for $\det A^1_{K,\mathrm{Ad}\circ\rho}(t)$. Then the left hand side turns into $\lim_{t\to 1}\tau_0\cdot t^m f(t)$, *i.e.*, $\tau_0 f(1)$. On the other hand, the right hand side becomes as follows.

$$\begin{aligned}\frac{\tau_0}{2}\,\frac{d^2}{dt^2}\det A^1_{K,\mathrm{Ad}\circ\rho}(t)\Big|_{t=1} &= \frac{\tau_0}{2}\,\frac{d^2}{dt^2}(t-1)^2 f(t)\Big|_{t=1}\\ &= \frac{\tau_0}{2}\frac{d}{dt}\big\{2(t-1)f(t)+(t-1)^2 f'(t)\big\}\Big|_{t=1}\\ &= \frac{\tau_0}{2}\big[2f(t)+4(t-1)f'(t)+(t-1)^2 f''(t)\big]_{t=1}\\ &= \tau_0 f(1).\end{aligned}$$

□

The numerator $\det A^1_{K,\mathrm{Ad}\circ\rho}(t)$ is called *the first homology torsion* of $C_*(E_K;\widetilde{\mathfrak{g}}_\rho)$ [9]. We denote the first homology torsion by $\Delta_1(t)$. By the above calculations, we obtain the following description of $\mathbb{T}^K_\lambda(\rho)$.

Proposition 3.4.2. — *If* $\mathrm{Tr}(\rho(\partial E_K))\not\subset\{\pm 2\}$, *then we have the following expression.*

$$\mathbb{T}^K_\lambda(\rho) = -\frac{d}{dt}\mathcal{T}(E_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})\Big|_{t=1} = \frac{\tau_0\Delta_1''(1)}{2}\cdot\frac{1}{\mathrm{Tr}(\rho(x_1^2))-2}.$$

Remark 3.4.3. — If G is $\mathrm{SU}(2)$ and ρ is λ-regular, then $\mathrm{Tr}(\rho(\partial E_K))\not\subset\{\pm 2\}$.

Remark 3.4.4. — We use a Wirtinger representation of $\pi_1(E_K)$ to describe $\mathcal{T}(E_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$ in the above calculation. The twisted Alexander invariant $\mathcal{T}(E_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$ does not depend on the representation of $\pi_1(E_K)$ [21].

Since $\mathcal{T}(E_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$ is determined by the finite presentable group $\pi_1(E_K)$ and $\rho \in R(E_K, G)$, we do not necessarily need to use a Wirtinger representation on calculating $\mathcal{T}(E_K, \widetilde{\mathfrak{g}}_\rho, \mathfrak{o})$.

4. Applications.

In this section, we deal with a 2-bridge knot K in S^3 and SU(2)-representations of its knot group. In this case $\rho \in R(\pi_1(E_K), \mathrm{SU}(2))$ is irreducible if and only if $\rho(\pi_1(E_K))$ is a non-abelian subgroup of SU(2). We will show the explicit calculation of SU(2)-twisted Reidemeister torsion associated to 5_2 knot and study the critical points of the twisted Reidemeister torsion $\mathbb{T}^K_\lambda$. If K is hyperbolic and G is $\mathrm{SL}(2, \mathbb{C})$, then some features of $\mathbb{T}^K_\mu(\rho)$, given in this section, have appeared in [15, Section 4.3].

4.1. A review of a representation of a 2-bridge knot group

It is well known that $\pi_1(E_K)$ has the representation:

$$\langle x, y \,|\, wx = yw \rangle,$$

where w is a word in x and y. Here x and y represent the meridian of the knot. The method we use to describe the space of $\mathrm{SL}(2, \mathbb{C})$ and SU(2)-representations is due to R. Riley ([16]). He shows how to parametrize conjugacy classes of irreducible $\mathrm{SL}(2, \mathbb{C})$ and SU(2)-representations of any 2-bridge knot group. We review his method ([8, 16]).

Given $s, u \in \mathbb{C}$, we consider the assignment as follows:

$$x \mapsto \begin{pmatrix} s & 1 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} s & 0 \\ -su & 1 \end{pmatrix}.$$

Let W be the matrix obtained by replacing x and y by the above two matrices in the word w. This assignment defines a $\mathrm{GL}(2, \mathbb{C})$-representation if and only if $\phi(s, u) = 0$ where $\phi(s, u) = W_{11} + (1 - s)W_{12}$.

One can obtain an $\mathrm{SL}(2, \mathbb{C})$-representation from this $\mathrm{GL}(2, \mathbb{C})$-representation by dividing the above two matrices by some square root of s. If we give a path $(s(a), u(a))$ in $\mathbb{C}^2$ with $\phi(s(a), u(a)) = 0$ and some continuous branch of the square root along $s(a)$, then we obtain a path of $\mathrm{SL}(2, \mathbb{C})$-representations. Furthermore, all conjugacy classes of non-abelian $\mathrm{SL}(2, \mathbb{C})$-representations arise in this way.

According to Proposition 4 of Riley's paper [16], a pair (s, u) with $\phi(s, u) = 0$ corresponds to an SU(2)-representation if and only if $|s| = 1$, and u is real number which lies in the interval $[s+s^{-1}-2, 0] = [2\cos\theta - 2, 0]$ where $s = e^{i\theta}$. This correspondence means that the $\mathrm{SL}(2,\mathbb{C})$-representation resulting from such a pair (s, u) and some square root of s is conjugate to an $\mathrm{SU}(2)$-representation in $\mathrm{SL}(2,\mathbb{C})$.

We take the ordered basis E, H, F of $\mathfrak{sl}(2,\mathbb{C})$ as follows.

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The Lie algebra $\mathfrak{su}(2)$ is a subspace of $\mathfrak{sl}(2,\mathbb{C})$. The vectors E, H, F also form a basis of $\mathfrak{su}(2)$. Since the Euler characteristic of E_K is zero, the non-abelian Reidemeister torsion $\mathbb{T}^K_\lambda(\rho)$ does not depend on a choice of a basis of $\mathfrak{su}(2)$. We can use E, H, F as an ordered basis of $\mathfrak{su}(2)$. We denote by $\rho_{\sqrt{s},u}$ the representation corresponding to the pair $(\sqrt{s}, u)$. The representation matrices of $\mathrm{Ad}(\rho_{\sqrt{s},u}(x))$ and $\mathrm{Ad}(\rho_{\sqrt{s},u}(y))$ for this ordered basis are given as follows.

Lemma 4.1.1.

$$\mathrm{Ad}(\rho_{\sqrt{s},u}(x)) = \begin{pmatrix} s & -2 & -\frac{1}{s} \\ 0 & 1 & \frac{1}{s} \\ 0 & 0 & \frac{1}{s} \end{pmatrix}, \quad \mathrm{Ad}(\rho_{\sqrt{s},u}(y)) = \begin{pmatrix} s & 0 & 0 \\ su & 1 & 0 \\ -su^2 & -2u & \frac{1}{s} \end{pmatrix}.$$

Note that even if we choose another square root of s, we obtain the same representation matrices of the adjoint actions of $\rho_{\sqrt{s},u}(x)$ and $\rho_{\sqrt{s},u}(y)$.

4.2. SU(2)-twisted Reidemeister torsion associated to 5_2 knot

We consider 5_2 knot in the knot table of Rolfsen [17]. Note that this knot is not fibered, since its Alexander polynomial is not monic. This is the simplest example such as non-fibered in 2-bridge knots. Let K be 5_2 knot. A diagram of K is shown as in Figure 4.1.

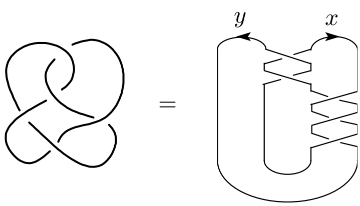

Figure 4.1. A diagram of 5_2 knot.

This knot is also called 3-twist knot. It follows from Theorem 3 of [11] that $\widehat{R}^{\mathrm{irr}}(\pi_1(E_K), \mathrm{SU}(2))$ consists of one circle and one open arc.

The knot group $\pi_1(E_K)$ has the following representation:

$$\langle x, y \mid wx = yw \rangle$$

where $w = x^{-1}y^{-1}xyx^{-1}y^{-1}$. From this representation, the Riley's polynomial of 5_2 is given by

$$W_{11} + (1-s)W_{12} = \frac{-u^3+(2(s+1/s)-3)u^2+(-(s^2+1/s^2)+3(s+1/s)-6)u+2(s+1/s)-3}{s}.$$

We may take Riley's polynomial $\phi(s,u)$ as

$$u^3 - (2(s+1/s)-3)u^2 + ((s^2+1/s^2) - 3(s+1/s)+6)u - (2(s+1/s)-3).$$

We want to know pairs (s,u) such that $s = e^{i\theta}$, u is a real number in the interval $[2\cos\theta - 2, 0]$ and $\phi(s,u) = 0$. When we regard $\phi(s,u) = 0$ as the equation of u, the relation between the number of solutions of $\phi(s,u) = 0$ and s is as follows.

(1) If $-2 \leqslant s + 1/s < (3 - \sqrt{13+16\sqrt{2}})/2$, then $\phi(s,u) = 0$ has three different simple root in $[s+1/s-2, 0]$.
(2) If $s + 1/s = (3 - \sqrt{13+16\sqrt{2}})/2$, then $\phi(s,u) = 0$ has a simple root and a multiple root in $[s+1/s-2, 0]$.
(3) If $(3 - \sqrt{13+16\sqrt{2}})/2 < s + 1/s < 3/2$, then $\phi(s,u) = 0$ has a simple root in $[s+1/s-2, 0]$.

The figure of $\widehat{R}^{\mathrm{irr}}(\pi_1(E_K), \mathrm{SU}(2))$ is given as in Figure 4.2.

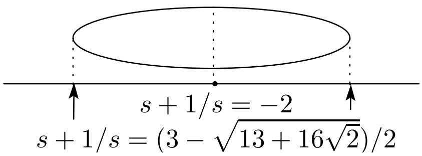

Figure 4.2. $\widehat{R}^{\mathrm{irr}}(\pi_1(E_K), \mathrm{SU}(2))$ *where* K *is* 5_2 *knot.*

We denote the SU(2)-representation corresponding to (s,u) by $\rho_{\sqrt{s},u}$. Then we can express $\mathbb{T}^K_\lambda(\rho_{\sqrt{s},u})$ from Proposition 3.4.2 as follows.

$$\mathbb{T}^K_\lambda(\rho_{\sqrt{s},u}) = \frac{\tau_0 \Delta_1''(1)}{2} \cdot \frac{1}{s+1/s-2}$$

Using a computer, we calculate a half of the differential coefficient of the second order of the numerator and simplify with the equation $\phi(s,u)=0$. Then we have

$$\frac{\tau_0\Delta_1''(1)}{2} = \tau_0(s+1/s-2)\big(-(5(s+1/s)+3)u^2+(5(s+1/s)^2 - 7(s+1/s)+1)u+1-10(s+1/s)\big).$$

Therefore we have

$$\mathbb{T}^K_\gamma(\rho_{\sqrt{s},u}) = \tau_0\big(-(5(s+1/s)+3)u^2+(5(s+1/s)^2 - 7(s+1/s)+1)u+1-10(s+1/s)\big),$$

where (u,s) satisfies $\phi(u,s)=0$.

4.3. On critical points of the SU(2)-twisted Reidemeister torsion associated to 2-bridge knots

From the example in the previous subsection, one can guess that the SU(2)-twisted Reidemeister torsion $\mathbb{T}^K_\lambda$ associated to a 2-bridge knot K is a function for the parameter $s+1/s$. Indeed the following holds.

Proposition 4.3.1. — *Let K be a 2-bridge knot and γ a simple closed curve in the boundary torus of E_K. Suppose that γ-regular* SU(2)*-representations are parametrized by $(s,u)\in \mathrm{U}(1)\times\mathbb{R}$ of Riley's method. If the trace of the meridian, $\sqrt{s}+1/\sqrt{s}$, gives a local parameter of the* SU(2)*-character variety, then the twisted Reidemeister torsion $\mathbb{T}^K_\gamma$ is a smooth function for $s+1/s$.*

Proof. — If we denote by $\rho_{\sqrt{s},u}$ a γ-regular representation corresponding to $\sqrt{s}+1/\sqrt{s}$, then there exists some homomorphism $\varepsilon:\pi_1(E_K)\to\{\pm1\}$ such that $\varepsilon\rho_{\sqrt{s},u}$ is a γ-regular representation corresponding to $-\sqrt{s}-1/\sqrt{s}$. By the construction of $\mathbb{T}^K_\gamma$, $\mathbb{T}^K_\gamma(\rho)$ is equal to $\mathbb{T}^K_\gamma(\varepsilon\rho)$. Since $\sqrt{s}+1/\sqrt{s}$ is a square root of $s+1/s+2$ and regular representations are irreducible, the twisted Reidemeister torsion $\mathbb{T}^K_\gamma$ is a smooth function for $s+1/s$. □

Corollary 4.3.2. — *If the trace of the meridian gives a local parameter of the* SU(2)*-character variety and the twisted Reidemeister torsion $\mathbb{T}^K_\lambda$ is defined, then $\mathbb{T}^K_\lambda$ is a smooth function for $s+1/s$.*

Remark 4.3.3. — All representations ρ of 2-bridge knot groups into SU(2) such that $\mathrm{Tr}(\rho(\mu))=0$ are binary dihedral representations. It follows from [7] that there exists a neighbourhood of the character of each binary dihedral representation for any 2-bridge knot, which is diffeomorphic to an open interval. From [2], the trace of the meridian gives a local parameter on a neighbourhood of the character of each dihedral representation for 2-bridge knots.

We can regard the twisted Reidemeister torsion $\mathbb{T}^K_\lambda$ as a smooth function on a neighbourhood of the character of each binary dihedral representation. Moreover these characters can be critical points of $\mathbb{T}^K_\lambda$ as follows.

Corollary 4.3.4. — *Let K be a 2-bridge knot. If a λ-regular component of the $\mathrm{SU}(2)$-character variety of $\pi_1(E_K)$ contains the characters of dihedral representations, then the function $\mathbb{T}^K_\lambda$ has a critical point at the character of each dihedral representation.*

Proof. — By Corollary 4.3.2 and Remark 4.3.3, the twisted Reidemeister torsion $\mathbb{T}^K_\lambda$ is a smooth function for $s + 1/s$. When we substitute $e^{i\theta}$ for s, we can describe $\mathbb{T}^K_\lambda(\rho)$ as

$$\frac{f(2\cos\theta)}{2\cos\theta - 2}$$

where $f(2\cos\theta)$ is a smooth function for $2\cos\theta$. This is a description of $\mathbb{T}^K_\lambda$ with respect to the local coordinate θ of $\widehat{R}^{\mathrm{irr}}(\pi_1(E_K), \mathrm{SU}(2))$. The derivation of this function for θ becomes

$$\frac{\{-2f'(2\cos\theta)(2\cos\theta - 2) + 2f(2\cos\theta)\}\sin\theta}{(2\cos\theta - 2)^2}.$$

We recall that $\mathrm{Tr}(\rho_{\sqrt{s},u}(\mu)) = \mathrm{Tr}(\rho_{\sqrt{s},u}(x)) = 2\cos(\theta/2)$. If $\mathrm{Tr}(\rho_{\sqrt{s},u}(\mu)) = 2\cos(\theta/2) = 0$, then $\sin\theta = 0$. Hence the derivation of $\mathbb{T}^K_\lambda$ vanishes if ρ satisfies $\mathrm{Tr}(\rho(\mu)) = 0$. □

Remark 4.3.5. — From [2], for 2-bridge knots, the character of a binary dihedral representation is a branch point of the two-fold branched cover from the SU(2)-character variety to the SO(3)-character variety. Moreover, every algebraic component of the SU(2)-character variety contains the character of such a representation.

Remark 4.3.6. — By [11, Theorem 10], for a knot K, the number of conjugacy class of binary dihedral representations is given by $(|\Delta_K(-1)| - 1)/2$ where $\Delta_K(t)$ is the Alexander polynomial of K. In particular, for a 2-bridge knot $b(\alpha, \beta)$ (Schubert's notation, see for example [3]), this number is given by $(\alpha - 1)/2$.

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