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CODIMENSION ONE MINIMAL FOLIATIONS AND THE FUNDAMENTAL GROUPS OF LEAVES

by Tomoo YOKOYAMA & Takashi TSUBOI (*)

Abstract. — Let $\mathcal{F}$ be a transversely orientable transversely real-analytic codimension one minimal foliation of a paracompact manifold M. We show that if the fundamental group of each leaf of $\mathcal{F}$ is isomorphic to $\boldsymbol{Z}$, then $\mathcal{F}$ is without holonomy. We also show that if $\pi_2(M) \cong 0$ and the fundamental group of each leaf of $\mathcal{F}$ is isomorphic to $\boldsymbol{Z}^k$ $(k \in \boldsymbol{Z}_{\geqslant 0})$, then $\mathcal{F}$ is without holonomy.

Résumé. — Soit $\mathcal{F}$ un feuilletage minimal de codimension un transversalement orientable, transversalement analytique réel sur une variété M paracompacte. On démontre que le feuilletage $\mathcal{F}$ est sans holonomie si le groupe fondamental de toute la feuille de $\mathcal{F}$ est isomorphe à $\boldsymbol{Z}$. On démontre aussi que le feuilletage $\mathcal{F}$ est sans holonomie si le groupe d'homotopie $\pi_2(M) \cong 0$ et que le groupe fondamental de toute la feuille de $\mathcal{F}$ est isomorphe à $\boldsymbol{Z}^k$ $(k \in \boldsymbol{Z}_{\geqslant 0})$.

1. Introduction

Let $\mathcal{F}$ be a codimension one smooth foliation of a closed manifold M. We study the relationship between the topology of the leaves of M and the topology of M. If each leaf of the foliation $\mathcal{F}$ is simply connected, then the foliation $\mathcal{F}$ is without holonomy. Then the classical result of Tischler [10] says that M fibers over the circle S^1, all leaves of $\mathcal{F}$ are diffeomorphic to a covering space of the fiber, and all leaves are dense if the leaves are not compact.

We would like to know how is the foliation if the fundamental group of each leaf is isomorphic to an elementary group. In this direction, we show the following theorems. A foliation is said minimal if all leaves are dense.

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THEOREM 1.1. — *Let $\mathcal{F}$ be a transversely orientable transversely real-analytic codimension one minimal foliation of a paracompact manifold M. Assume that the fundamental group of each leaf of $\mathcal{F}$ is isomorphic to $\boldsymbol{Z}$. Then the foliation $\mathcal{F}$ is without holonomy.*

THEOREM 1.2. — *Let $\mathcal{F}$ be a transversely orientable transversely real-analytic codimension one minimal foliation of a paracompact manifold M. Assume that $\pi_2(M) \cong 0$ and the fundamental group of each leaf of $\mathcal{F}$ is isomorphic to $\boldsymbol{Z}^k (k \in \boldsymbol{Z}_{\geqslant 0})$. Then the foliation $\mathcal{F}$ is without holonomy.*

Remark 1.3. — For foliations of closed manifolds, if we assume that the end set of each leaf is homeomorphic to neither the Cantor set nor the empty set, then the assumption of minimality in Theorems 1.1 and 1.2 are satisfied by a result of Duminy (see [2]). For, if the foliation is not minimal and there are no compact leaves, then there is an exceptional minimal set and the end set of a semi-proper leaf of the exceptional minimal set is the Cantor set.

Remark 1.4. — The assumption of transverse real-analyticity in Theorems 1.1 and 1.2 is only used to guarantee that there are no null homotopic closed transverse curves for the foliations ([5]).

Thus for example, for a codimension one real-analytic foliation $\mathcal{F}$ of a closed manifold M, if all leaves of $\mathcal{F}$ are homeomorphic to $S^1 \times \boldsymbol{R}^k$ $(k \geqslant 2)$, then the foliation is without holonomy. In this case, M is an S^1-bundle over T^{k+1}. For, by the classical result of Novikov ([8]), the universal covering $\widetilde{M}$ is homeomorphic to $\boldsymbol{R}^{k+1} \times \boldsymbol{R}$, where $\boldsymbol{R}^{k+1}$ is the universal covering of $S^1 \times \boldsymbol{R}^k$, and $\pi_1(M)/i_*(\pi_1(L))$ is free abelian, where $i : L \longrightarrow M$ is the inclusion map of a leaf L. Thus M is aspherical and homotopy equivalent to an S^1-bundle over T^{k+1}. Then by the result of Farrell-Jones [4], M is homeomorphic to the S^1-bundle over T^{k+1}.

In a similar way, for a codimension one real-analytic foliation $\mathcal{F}$ of a closed manifold M with $\pi_2(M) \cong 0$, if all leaves of $\mathcal{F}$ are homeomorphic to $T^k \times \boldsymbol{R}^\ell$ $(k + \ell \geqslant 3)$, then the foliation is without holonomy and M is a T^k-bundle over $T^{\ell+1}$.

We note that if the fundamental group of each leaf is isomorphic to a complicated group, it is no longer true that the foliation is without holonomy. In fact, we give an example (Example 3.2) of foliation with nontrivial holonomy of closed 3-manifold whose leaves are diffeomorphic. Etienne Ghys told us that he also knew of this example.

2. Proof of main theorems

To prove our main theorems, we introduce the notion of trivial fences and π_1-carriers.

DEFINITION 2.1 (Trivial fence). — *Let $\mathcal{F}$ be a transversely oriented foliation of a manifold M and $\mathcal{T}$ a foliation transverse to $\mathcal{F}$. For a compact subset K of a leaf L_0 of $\mathcal{F}$ and for $\varepsilon > 0$, an embedding $F : K \times [0,\varepsilon] \longrightarrow M$ is called a trivial positive fence if $F(K \times \{t\})$ is contained in a leaf L_t of $\mathcal{F}$, $F|K \times \{0\}$ is the inclusion $K \subset L_0$ and $F(\{x\} \times [0,\varepsilon])$ is an orientation preserving embedding to a leaf τ_x of $\mathcal{T}$. A trivial negative fence $F : K \times [-\varepsilon, 0] \longrightarrow M$ and a trivial two-sided fence $F : K \times [-\varepsilon,\varepsilon] \longrightarrow M$ are defined in a similar way.*

Remark 2.2. — For an arcwise connected compact subset K of a leaf L, a trivial positive fence on K exists, if and only if the holonomy on the positive side along any loop in K is trivial. If L is a leaf without holonomy, there is a trivial two-sided fence on any arcwise connected compact subset K of L.

DEFINITION 2.3 (π_1-carrier). — *An arcwise connected subset K of a leaf L of $\mathcal{F}$ is called a π_1-carrier for L if $\pi_1(K) \longrightarrow \pi_1(L)$ is surjective.*

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. — By a result of [3], the union of leaves without holonomy is a residual saturated subset of M. Let L be a leaf of $\mathcal{F}$ without holonomy and $K \cong S^1 \subset L_0$ be a π_1-carrier for L_0. Let $F : K \times [0,\varepsilon] \longrightarrow M$ be a trivial positive fence on K and put $F_t(x) = F(x,t)$

If the image of $(F_t)_* : \pi_1(K) \longrightarrow \pi_1(L_t)$ is of finite index, then L_t is without holonomy. For, for a loop γ in L_t, its power γ^k is in $(F_t)_*(\pi_1(K))$ and the holonomy along γ^k is trivial by the existence of the trivial fence F. Since the holonomy group of a leaf of a codimension one foliation is torsion-free, the holonomy along γ is trivial.

Assume that for a non empty open interval $(\delta_0,\delta_1) \subset [0,\varepsilon]$ and any $t \in (\delta_0,\delta_1)$, $(F_t)_*(\pi_1(K))$ is of finite index. Then $\mathcal{F}$ is without holonomy by the minimality of $\mathcal{F}$, i.e. the saturation of $F_t(K \times (\delta_0,\delta_1))$ is the whole manifold M.

Now, if $(F_t)_*(\pi_1(K))$ is of infinite index for some $t \in [0,\varepsilon]$, then that $\pi_1(L_t) \cong \boldsymbol{Z}$ implies that $(F_t)_*(\pi_1(K)) = 0$.

Thus if $\mathcal{F}$ is with nontrivial holonomy, then $\{t \in [0,\varepsilon] \,\big|\, (F_t)_*(\pi_1(K)) = 0\}$ is a dense subset of $[0,\varepsilon]$.

Assume that $\mathcal{F}$ is with nontrivial holonomy. Let $p : \widetilde{M} \longrightarrow M$ be the universal covering of M. Let $\widetilde{\mathcal{F}} = p^*\mathcal{F}$ be the pullback foliation of $\widetilde{M}$. By the above argument, for a leaf L without holonomy of $\mathcal{F}$ and any lift $\widetilde{L}$ of L, $p|\widetilde{L}$ is a diffeomorphism and $\pi_1(\widetilde{L}) \cong \boldsymbol{Z}$. For a leaf L with nontrivial holonomy, the real-analyticity of $\mathcal{F}$ implies that its lift $\widetilde{L}$ is simply connected. For, otherwise, $\widetilde{L}$ is a finite cover of L and $\widetilde{L}$ is still with nontrivial holonomy. However the argument of Haefliger [5] shows that transversely orientable real-analytic foliations on simply connected manifolds are without holonomy.

Thus the foliation $\widetilde{\mathcal{F}}$ of $\widetilde{M}$ is without holonomy and $\widetilde{\mathcal{F}}$ contains the residual subset formed by the non-simply connected lifts of leaves L of $\mathcal{F}$ without holonomy, and simply connected leaves formed by the lifts of leaves with nontrivial holonomy.

For a lift $\widetilde{L_0}$ of a leaf L_0 of $\mathcal{F}$ without holonomy, $p|\widetilde{L_0}$ is a diffeomorphism, and the positive trivial fence $F : K \times [0,\varepsilon] \longrightarrow M$ lifts to a unique positive trivial fence $\widetilde{F} : \widetilde{K} \times [0,\varepsilon] \longrightarrow \widetilde{M}$ where $\widetilde{K} = (p|\widetilde{L_0})^{-1}(K)$. Hence $\{t \in [0,\varepsilon] \mid (\widetilde{F}_t)_*(\pi_1(\widetilde{K})) = 0\}$ is a dense subset of $[0,\varepsilon]$.

The following lemma 2.4 shows that in this situation, the union of simply connected leaves is a residual subset of $\widetilde{M}$ and this completes the proof of Theorem 1.1. □

Lemma 2.4. — *Let $\widetilde{\mathcal{F}}$ be a foliation without holonomy of a simply connected manifold $\widetilde{M}$. Let $p : \widetilde{M} \longrightarrow \boldsymbol{R}$ be a continuous submersion such that a leaf of $\widetilde{\mathcal{F}}$ is a connected component of $p^{-1}(y)$ ($y \in \boldsymbol{R}$). Assume that*

(∗) *for any compact subset K of a leaf $\widetilde{L}$ and any two-sided trivial fence $F : K \times [-\varepsilon,\varepsilon] \longrightarrow \widetilde{M}$, there exists $t \in (-\varepsilon,\varepsilon)$ such that $(F_t)_*(\pi_1(K)) = 0 \subset \pi_1(\widetilde{L}_t)$.*

Then the union of simply connected leaves is a residual subset.

Proof. — First we take an increasing sequence of compact submanifolds M_i of $\widetilde{M}$ with several nice properties.

For each point $x \in \widetilde{M}$, take a closed foliated product neighbourhood $\overline{U} \cong D^{n-1} \times [-\varepsilon,\varepsilon]$ such that $D^{n-1} \times \{*\}$ is on a leaf of $\widetilde{\mathcal{F}}$ and $\{*\} \times [-\varepsilon,\varepsilon]$ is on a leaf of the transverse foliation $\widetilde{\mathcal{T}}$. Since $\widetilde{M}$ is paracompact, it is covered by countably many such distinguished neighbourhoods $\overline{U_i} \cong D_i^{n-1} \times [-\varepsilon_i,\varepsilon_i]$ ($i \in \boldsymbol{Z}_{\geqslant 0}$). Put $M_i = \bigcup_{k \leqslant i} D_k^{n-1} \times [-\varepsilon_k,\varepsilon_k]$. Then $\widetilde{M} = \bigcup_i M_i$. By modifying M_i, we may assume that M_i is a compact submanifolds of $\widetilde{M}$ with boundary and codimension two corners. ∂M_i is a union of compact submanifolds of leaves and compact submanifolds transverse to $\widetilde{\mathcal{F}}$. Moreover, we may assume that, for any leaf $\widetilde{L}_0$ of $\widetilde{\mathcal{F}}$, $\widetilde{L}_0 \cap M_i$ is a compact submanifolds of $\widetilde{L}_0$.

We observe that if $\widetilde{L_0}$ contain no components of tangential boundary of M_i, then there is a two-sided trivial fence $F : (M_i \cap \widetilde{L}_0) \times [-\varepsilon, \varepsilon] \longrightarrow \widetilde{M}$ $(\varepsilon > 0)$ such that $F((M_i \cap \widetilde{L}_0) \times \{t\})$ is contained in a leaf $\widetilde{L}$ and $p^{-1}(p(\mathrm{Im}(F))) \cap M_i = \mathrm{Im}(F)$, where $\mathrm{Im}(F)$ is the image of the fence F. For a component $(M_i \cap \widetilde{L}_0)_j$ of $M_i \cap \widetilde{L}_0$, if $\widetilde{L}_0$ contains no components of tangential boundary of M_i and the induced map $\pi_1((M_i \cap \widetilde{L}_0)_j) \longrightarrow \pi_1(\widetilde{L}_0)$ is the zero map, then for sufficiently small t in $[-\varepsilon, \varepsilon]$, $\pi_1((M_i \cap \widetilde{L}_t)_j) \longrightarrow \pi_1(\widetilde{L}_t)$ is the zero map. The reason is as follows. Take a finite generating set of $\pi_1((M_i \cap \widetilde{L}_0)_j)$. For a generator $[\gamma]$ of $\pi_1((M_i \cap \widetilde{L}_0)_j)$, there is a continuous map $h : D^2 \longrightarrow \widetilde{L}_0$ such that $\partial h = \gamma$. Then this h extends to $\widehat{h} : D^2 \times [-\varepsilon', \varepsilon'] \longrightarrow \widetilde{M}$ such that $\widehat{h}(\{*\} \times [-\varepsilon', \varepsilon']) \subset \widetilde{\tau}_{h(*)}$ and $\widehat{h}(D^2 \times \{t\}) \subset \widetilde{L}_t$. Since $\pi_1((M_i \cap \widetilde{L}_t)_j)$ is generated by the loops $[F_t(\gamma)]$ and these bound $\widehat{h}(D^2 \times \{t\})$, $\pi_1((M_i \cap \widetilde{L}_t)_j) \longrightarrow \pi_1(\widetilde{L}_t)$ is the zero map.

By the assumption $(*)$, for the trivial two-sided fence F,

$$\{t \in [-\varepsilon, \varepsilon] \mid \pi_1((M_i \cap \widetilde{L}_t)_j) \longrightarrow \pi_1(\widetilde{L}_t) \text{ is the zero map}\}$$

is a dense subset of $[-\varepsilon, \varepsilon]$, and this is an open subset by the above argument.

Since $p^{-1}(p(\mathrm{Im}(F))) \cap M_i = \mathrm{Im}(F)$, the number of connected components of $\mathrm{Im}(F)$ is finite and the number of the connected components of the tangential boundary of M_i is finite,

$$\begin{aligned} G'_i = \{s \in p(M_i) \mid\ & \pi_1((M_i \cap \widetilde{L})_j) \longrightarrow \pi_1(\widetilde{L}) \quad \text{is the zero map} \\ & \text{for any connected component } (M_i \cap \widetilde{L})_j \text{ of } M_i \cap \widetilde{L} \\ & \text{for any leaf } \widetilde{L} \text{ in } p^{-1}(s)\} \end{aligned}$$

contains an open dense subset of $p(M_i)$.

Put $G_i = (\boldsymbol{R} \setminus p(M_i)) \cup G'_i$. Then G_i contains an open dense subset of $\boldsymbol{R}$. Thus $G = \bigcap_{i=1}^{\infty} G_i$ is a residual subset of $\boldsymbol{R}$. Now, for $s \in G$, any leaf of $p^{-1}(s)$ is simply connected . Hence the union of simply connected leaves contains $p^{-1}(G)$ which is a residual subset of $\widetilde{M}$. □

Thus Theorem 1.1 is shown. The proof of Theorem 1.2 also uses the homomorphism $(F_t)_* : \pi_1(K) \longrightarrow \pi_1(L_t)$ for the trivial fence F.

Proof of Theorem 1.2. — Let L_0 be a leaf of $\mathcal{F}$ without holonomy. Let K be a π_1-carrier for L_0. K can be taken as a bouquet of k circles. Let $F : K \times [-\varepsilon, \varepsilon] \longrightarrow M$ be a two-sided trivial fence. As in the proof of Theorem 1.1, if $\mathcal{F}$ is with nontrivial holonomy, then there are no non-empty open intervals $(\delta_0, \delta_1) \subset [-\varepsilon, \varepsilon]$ such that the image of $(F_t)_* : \pi_1(K) \longrightarrow \pi_1(L_t)$ $(t \in (\delta_0, \delta_1))$ is of finite index.

Thus there are s_0 and u_0 such that $-\varepsilon < s_0 < 0 < u_0 < \varepsilon$,

$$(F_{s_0})_*(\pi_1(K)) \subset \pi_1(L_{s_0}) \cong \boldsymbol{Z}^k$$

and

$$(F_{u_0})_*(\pi_1(K)) \subset \pi_1(L_{u_0}) \cong \boldsymbol{Z}^k$$

are of infinite index. Thus $N^0_{s_0} = \mathrm{Ker}((F_{s_0})_*) \subset \pi_1(L_0) \cong \boldsymbol{Z}^k$ and $N^0_{u_0} = \mathrm{Ker}((F_{u_0})_*) \subset \pi_1(L_0) \cong \boldsymbol{Z}^k$ are nontrivial.

First we show that $N^0_{s_0} \cap N^0_{u_0} = 0$.

Lemma 2.5. — $N^0_{s_0} \cap N^0_{u_0} = 0$.

Proof. — Assume that $N^0_{s_0} \cap N^0_{u_0}$ is nontrivial. Let $\gamma : S^1 \longrightarrow L_0$ be a loop representing a nontrivial element of $N^0_{u_0} \cap N^0_{s_0}$. Let $(\widetilde{M}, \widetilde{\mathcal{F}})$ be the pull-back foliation of the universal covering $\widetilde{M}$ of M. Since $(F_{u_0})_*(\gamma) = 0$, γ lifts to a map $\widetilde{\gamma} : S^1 \longrightarrow \widetilde{L_0} \subset \widetilde{M}$ to the universal cover. Since $\pi_1(\widetilde{L_0})$ is free abelian and $[\widetilde{\gamma}] \in \pi_1(\widetilde{L_0})$ is nontrivial, $[\widetilde{\gamma}] \in H_1(\widetilde{L_0}; \boldsymbol{Z})$ is nontrivial. Since $\mathcal{F}$ is transversely real-analytic, $\widetilde{\mathcal{F}}$ is defined by a continuous submersion $\widetilde{M} \longrightarrow \boldsymbol{R}$ and $\widetilde{M} \backslash \widetilde{L_0}$ always have two connected components, $\widetilde{M}^+$ and $\widetilde{M}^-$. Since $[\gamma] \in N^0_{u_0}$, there is a continuous map $h_+ : D^2 \longrightarrow F(\gamma \times [0, u_0]) \cup L_{u_0}$ such that $\gamma = \partial h_+$. h_+ lifts to a continuous map $\widehat{h_+} : D^2 \longrightarrow \mathrm{Cl}(\widetilde{M}^+) = \widetilde{M}^+ \cup \widetilde{L_0}$ such that $\widetilde{\gamma} = \partial \widehat{h_+}$.

In a similar way, there are a continuous map $h_- : D^2 \longrightarrow F(\gamma \times [s_0, 0]) \cup L_{s_0}$ such that $\gamma = \partial h_-$ and its lift $\widehat{h_-} : D^2 \longrightarrow \mathrm{Cl}(\widetilde{M}^-) = \widetilde{M}^- \cup \widetilde{L_0}$ such that $\widetilde{\gamma} = \partial \widehat{h_-}$.

By the Mayer-Vietoris sequence for $\widetilde{M} = \mathrm{Cl}(\widetilde{M}^-) \cup \mathrm{Cl}(\widetilde{M}^+)$, we see that $\widetilde{\gamma}$ induces a nontrivial element of $H_2(\widetilde{M}; \boldsymbol{Z})$ and this contradicts the assumption that $0 = \pi_2(M) \cong \pi_2(\widetilde{M}) \cong H_2(\widetilde{M}; \boldsymbol{Z})$. Thus $N^0_{s_0} \cap N^0_{u_0} = 0$. □

We continue the proof of Theorem 1.2.

Put $t_0 = 0$, $K^{t_0} = K$ and $F^{t_0}_t = F_t : K \longrightarrow L_t$. We have, $s_0 < t_0 < u_0$ and $N^{t_0}_{s_0} \cap N^{t_0}_{u_0} = 0$. Note that for $0 = t_0 < t \leqslant u_0$, $(F^{t_0}_t)_* | N^{t_0}_{s_0}$ is injective. For otherwise, $N^{t_0}_{s_0} \cap N^{t_0}_t = 0$, contradicting Lemma 2.5

Let L_{t_1} $(t_0 < t_1 < u_0)$ be a leaf without holonomy. Take a π_1-carrier K^{t_1} for L_{t_1} containing $F^{t_0}_{t_1}(K^{t_0})$. Then we have a two-sided trivial fence $F^{t_1} : K^{t_1} \times [t_1 - \varepsilon_1, t_1 + \varepsilon_1] \longrightarrow M$. For this fence, by the same argument as before, we can find s_1 and u_1 such that $t_0 < s_1 < t_1 < u_1 < u_0$ and $N^{t_1}_{s_1} \cap N^{t_1}_{u_1} = 0$, where $F^{t_1}_t : K^{t_1} \longrightarrow L_t$ and $N^{t_1}_t = \mathrm{Ker}((F^{t_1}_t)_* : \pi_1(K^{t_1}) \longrightarrow \pi_1(L_t))$ $(t = s_1,\ u_1)$ are nontrivial.

Now note that $N^{t_1}_{s_1} \cap (F^{t_0}_{t_1})_*(N^{t_0}_{s_0}) = 0$. For otherwise, there is a loop γ in K^{t_0} such that $F^{t_0}_{t_1} \circ \gamma$ represents a nontrivial element in $N^{t_1}_{s_1} \cap (F^{t_0}_{t_1})_*(N^{t_0}_{s_0})$ and $[\gamma] \in N^{t_0}_{s_0}$. Since for $s_1 \leqslant t \leqslant u_1$, $F^{t_0}_t(K^{t_0}) \subset F^{t_1}_t(K^{t_1})$, $F^{t_1}_{s_1} \circ F^{t_0}_{t_1} \circ \gamma = F^{t_0}_{s_1} \circ \gamma$ and this represents 0. This contradicts the injectivity of $(F^{t_0}_{s_1})_*|N^{t_0}_{s_0}$.

Put $A^{t_0} = N^{t_0}_{s_0}$ and $A^{t_1} = N^{t_1}_{s_1} \oplus (F^{t_0}_{t_1})_*(A^{t_0})$. Then $\mathrm{rk}(A^{t_1}) > \mathrm{rk}(A^{t_0}) > 0$. Note that for $[\gamma] \in A^{t_1}$, we have a description of the homotopy

$$(F^{t_0}_{s_0})_*((F^{t_0}_{s_1})_*|N^{t_0}_{s_0})^{-1}(F^{t_1}_{s_1})_*[\gamma] = 0.$$

Note also that for $t_1 < t \leqslant u_1$, $(F^{t_1}_t)_*|A^{t_1}$ is injective and $A^{t_1} \cap N^{t_1}_{u_1} = 0$. For otherwise, the argument of Lemma 2.5 gives rise to a nontrivial element in $\pi_2(M)$.

For $s_1 < t_1 < u_1$, we take L_{t_2} $(t_1 < t_2 < u_1)$ without holonomy. We take a π_1-carrier K^{t_2} for L_{t_2} containing $F^{t_1}_{t_2}(K^{t_1})$. Then, using a two-sided trivial fence $F^{t_2}_t$, we find s_2 and u_2 $(t_1 < s_2 < t_2 < u_2 < u_1)$ such that $N^{t_2}_{s_2} \cap N^{t_2}_{u_2} = 0$, where $N^{t_2}_t = \ker(F^{t_2}_t)_* : \pi_1(K^{t_2}) \longrightarrow \pi_1(L_t)$ $(t = s_2, u_2)$ are nontrivial. Since for $s_2 \leqslant t \leqslant u_2$, $F^{t_1}_t(K^{t_1}) \subset F^{t_2}_t(K^{t_2})$, $N^{t_2}_{s_2} \cap (F^{t_1}_{t_2})_*(A^{t_1}) = 0$. Put $A^{t_2} = N^{t_2}_{s_2} \oplus (F^{t_1}_{t_2})_*(A^{t_1})$. Then, $\mathrm{rk}(A^{t_2}) > \mathrm{rk}(A^{t_1})$. For $t_2 < t \leqslant u_2$, $(F^{t_2}_t)_*|A^{t_2}$ is injective and $A^{t_2} \cap N^{t_2}_{u_2} = 0$. Note that, for $[\gamma] \in A^{t_2}$,

$$(F^{t_0}_{s_0})_*((F^{t_0}_{s_1})_*|A^{t_0})^{-1}(F^{t_1}_{s_1})_*((F^{t_1}_{s_2})_*|A^{t_1})^{-1}(F^{t_2}_{s_2})_*[\gamma] = 0.$$

Note also that for $t_2 < t \leqslant u_2$, $(F^{t_2}_t)_*|A^{t_2}$ is injective and $A^{t_2} \cap N^{t_2}_{u_2} = 0$. For otherwise, the argument of Lemma 2.5 again gives rise to a nontrivial element in $\pi_2(M)$.

We repeat this construction and we obtain π_1-carriers $K^{t_i} \subset L_{t_i}$ and two-sided trivial fences $F^{t_i}_t$, nontrivial subgroups $N^{t_i}_{s_i}$, $N^{t_i}_{u_i}$ of $\pi_1(K^{t_i})$, and subgroups $A^{t_i} \subset \pi_1(L_{t_i})$ $(i = 1, 2, \dots)$. Here $N^{t_i}_t = \ker(F^{t_i}_t)_* : \pi_1(K^{t_i}) \longrightarrow \pi_1(L_t)$. They satisfy

$$\begin{aligned} s_0 < t_0 < s_1 < t_1 < s_2 < t_2 < \cdots < s_i < t_i \\ < u_i < u_{i-1} < \cdots < u_2 < u_1 < u_0 \end{aligned}$$

and

$$A^{t_i} = N^{t_i}_{s_i} \oplus (F^{t_{i-1}}_{t_i})_*(A^{t_{i-1}}).$$

Since $\mathrm{rk}A^{t_i} > \mathrm{rk}A^{t_{i-1}} > \cdots > \mathrm{rk}A^{t_0} > 0$, $\mathrm{rk}A^{t_i} \geqslant i$.

Here if $A^{t_i} \cap N^{t_i}_{u_i} \neq 0$, we find a nontrivial element in $\pi_2(M)$. For, let $\gamma \in A^{t_i} \cap N^{t_i}_{u_i}$ be a loop representing nontrivial element. Since $[\gamma] \in N^{t_i}_{u_i}$, γ lifts to the universal covering, $\widetilde{\gamma} : S^1 \longrightarrow \widetilde{L}_{t_i} \subset \widetilde{M}$. By the argument similar to that in Lemma 2.5, $\widetilde{\gamma}$ bounds a disk in $\widetilde{M}^+_{t_i} \cup \widetilde{L}_{t_i}$, where $\widetilde{M}^+_{t_i}$ and

$\widetilde{M}^-_{t_i}$ are the components of $\widetilde{M} \setminus \widetilde{L}_{t_i}$. Since $[\gamma] \in A^{t_i}$,

$$(F^{t_0}_{s_0})_*((F^{t_0}_{s_1})_*|A^{t_0})^{-1}(F^{t_1}_{s_1})_*((F^{t_1}_{s_2})_*|A^{t_1})^{-1}\cdots(F^{t_{i-2}}_{s_{i-2}})_*$$
$$((F^{t_{i-2}}_{s_{i-1}})_*|A^{t_{i-2}})^{-1}(F^{t_{i-1}}_{s_{i-1}})_*((F^{t_{i-1}}_{s_i})_*|A^{t_{i-1}})^{-1}(F^{t_i}_{s_i})_*[\gamma] = 0.$$

Hence $\widetilde{\gamma}$ bounds a disk in $\widetilde{M}^-_{t_i} \cup \widetilde{L}_{t_i}$. Thus as before we see that $H_2(\widetilde{M};\boldsymbol{Z}) \neq 0$ and we find a nontrivial element in $\pi_2(M)$. This contradicts the assumption that $0 = \pi_2(M) \cong \pi_2(\widetilde{M}) \cong H_2(\widetilde{M};\boldsymbol{Z})$.

Now for $i > k$, the fact this contradicts the fact that $\pi_1(L) \cong \boldsymbol{Z}^k$. □

3. Examples

The first example shows that the assumption on the minimality of foliations in Theorems 1.1 and 1.2 are necessary.

Example 3.1. — Consider the compact manifold

$$M = S^1 \times (S^1 \times S^m \setminus \mathrm{Int}D^{1+m}),$$

where $m > 1$ and D^{1+m} is an embedded disk in $S^1 \times S^m$. It is easy to construct a foliation $\mathcal{F}$ of M tangent to the boundary $S^1 \times S^m$ and whose interior leaves are homeomorphic to $S^1 \times S^m \setminus \{*\}$. Consider the double $(DM, D\mathcal{F})$ which can be constructed as a real-analytic foliation and the compact leaf has nontrivial holonomy. It is easy to see that the fundamental group of each leaf is isomorphic to $\boldsymbol{Z}$. By taking the direct product with T^{k-1} we obtain an example with the the fundamental group of each leaf being isomorphic to $\boldsymbol{Z}^k$.

Now we construct a codimension one transversely orientable minimal foliation with nontrivial holonomy on a closed manifold whose leaves are diffeomorphic. The construction is a modification of Hirsch's example [6].

Example 3.2. — Let Σ be the surface obtained from the 2-torus by removing 3 disks. Let $\partial_1, \partial_2, \partial_3$ be the boundary components of Σ. Let $\mathcal{F}$ be the product foliation of $\Sigma \times [0,1]$ (i.e., $\mathcal{F} = \{\Sigma \times \{x\}\}_{x\in[0,1]}$). Let $f : \Sigma \longrightarrow \Sigma$ be a diffeomorphism such that $f(\partial_1) = \partial_1, f(\partial_2) = \partial_3, f(\partial_3) = \partial_2$ and let M be the manifold obtained from $\Sigma \times [0,1]$ by identifying $(x,1)$ with $(f(x),0)$. Then the boundary of M is diffeomorphic to a disjoint union of two tori T^2_1 (containing $\partial_1 \times \{0\}$) and T^2_2 (containing $\partial_2 \times \{0\}$ and $\partial_3 \times \{0\}$). Using the natural projection to $[0,1]/0 \sim 1 = \boldsymbol{R}/\boldsymbol{Z}$, T^2_1 is identified with $\partial_1 \times \boldsymbol{R}/\boldsymbol{Z}$ and T^2_2 is identified with $\partial_2 \times \boldsymbol{R}/2\boldsymbol{Z}$. We have the induced foliation $\mathcal{F}'$ of M and we see that $\mathcal{F}'|T^2_1 = \{\partial_1 \times \{t\}\}_{t\in\boldsymbol{R}/\boldsymbol{Z}}$

and $\mathcal{F}'|T_2^2 = \{\partial_2 \times \{t\}\}_{t \in \boldsymbol{R}/2\boldsymbol{Z}}$. Let $g : T_1^2 \longrightarrow T_2^2$ be a diffeomorphism sending $\partial_1 \times \{t\}$ to $\partial_2 \times \{2t\}$. Let M_g be the manifold obtained from M by the identification by g. We obtain the induced real-analytic foliation $\mathcal{F}''$ of M_g which is minimal. It is easy to see that there are countably many leaves with nontrivial holonomy. Each leaf of $\mathcal{F}''$ is an orientable surface of infinite genus and its end set is the Cantor set consisting of non planar ends. By the classifying theorem ([7], [9]) for non-compact surfaces (see also [1]), all leaves are diffeomorphic.

BIBLIOGRAPHY

[1] J. Cantwell & L. Conlon, "Leaf prescriptions for closed 3-manifolds", *Trans. Amer. Math. Soc.* **236** (1978), p. 239-261.

[2] ———, "Endsets of exceptional leaves; a theorem of G. Duminy", in *Foliations: geometry and dynamics (Warsaw, 2000)*, World Sci. Publ., River Edge, NJ, 2002, p. 225-261.

[3] D. B. A. Epstein, K. C. Millett & D. Tischler, "Leaves without holonomy", *J. London Math. Soc. (2)* **16** (1977), no. 3, p. 548-552.

[4] F. T. Farrell & L. E. Jones, "The surgery L-groups of poly-(finite or cyclic) groups", *Invent. Math.* **91** (1988), no. 3, p. 559-586.

[5] A. Haefliger, "Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes", *Comment. Math. Helv.* **32** (1958), p. 248-329.

[6] M. Hirsch, "A stable analytic foliation with only exceptional minimal sets", in *Dynamical Systems*, Lecture Notes in Math., vol. 468, Springer, Berlin, Heidelberg, New York, 1975, p. 9-10.

[7] B. Kerékjártó, *Vorlesungen uber Topologie*, vol. I, Springer, Berlin, 1923.

[8] S. P. Novikov, "Topology of foliations", *Trans. Mosc. Math. Soc.* **14** (1965), p. 268-304, translation from Tr. Mosk. Mat. Obshch. 14, 248-278 (1965).

[9] I. Richards, "On the classification of noncompact surfaces", *Trans. Amer. Math. Soc.* **106** (1963), p. 259-269.

[10] D. Tischler, "On fibering certain foliated manifolds over S^1", *Topology* **9** (1970), p. 153-154.

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