



# ANNALES

DE

# L'INSTITUT FOURIER

Giovanni MORANDO

**Tempered solutions of  $\mathcal{D}$ -modules on complex curves and formal invariants**

Tome 59, n° 4 (2009), p. 1611-1639.

[http://aif.cedram.org/item?id=AIF\\_2009\\_\\_59\\_4\\_1611\\_0](http://aif.cedram.org/item?id=AIF_2009__59_4_1611_0)

© Association des Annales de l'institut Fourier, 2009, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (<http://aif.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://aif.cedram.org/legal/>). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

*Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>*

## TEMPERED SOLUTIONS OF $\mathcal{D}$ -MODULES ON COMPLEX CURVES AND FORMAL INVARIANTS

by Giovanni MORANDO (\*)

---

ABSTRACT. — Let  $X$  be a complex analytic curve. In this paper we prove that the subanalytic sheaf of tempered holomorphic solutions of  $\mathcal{D}$ -modules on  $X$  induces a fully faithful functor on a subcategory of germs of formal holonomic  $\mathcal{D}$ -modules. Further, given a germ  $\mathcal{M}$  of holonomic  $\mathcal{D}$ -module, we obtain some results linking the subanalytic sheaf of tempered solutions of  $\mathcal{M}$  and the classical formal and analytic invariants of  $\mathcal{M}$ .

RÉSUMÉ. — Soit  $X$  une courbe analytique complexe. Dans cet article nous démontrons que le faisceau sous-analytique des solutions holomorphes tempérées des  $\mathcal{D}$ -modules sur  $X$  induit un foncteur pleinement fidèle sur une sous-catégorie des germes des  $\mathcal{D}$ -modules holonomes formels. De plus, étant donné un germe  $\mathcal{M}$  de  $\mathcal{D}$ -module holonome, nous obtenons des résultats qui lient le faisceau sous-analytique des solutions tempérées de  $\mathcal{M}$  avec les invariants formels et analytiques classiques de  $\mathcal{M}$ .

### Introduction

The search for algebraic or topological invariants of complex linear partial differential equations is classical and widely developed.

At the very first step of the study of linear differential equations, two main types of equations are distinguished: regular and irregular. To give an idea of the difference between the two kinds of equations, let us recall that, in dimension 1, the solutions of the former equations have moderate growth while the solutions of the latter have exponential-type growth.

---

*Keywords:*  $\mathcal{D}$ -modules, irregular singularities, tempered holomorphic functions, subanalytic.

*Math. classification:* 34M35, 32B20, 34Mxx.

(\*) Research supported in part by grant CPDA061823 of Padova University and in part by fellowship SFRH/BPD/29934/2006 of Fundação para Ciência e Tecnologia at Centro de Álgebra da Universidade de Lisboa.

The more general algebraic approach to the study of linear differential equations consists in considering differential equations as sheaves of modules over the ring  $\mathcal{D}_X$  of linear differential operators on a manifold  $X$ . In this framework, in [7] and [8], M. Kashiwara gives a proof of the Riemann-Hilbert correspondence which is a generalization of the 21st Hilbert's problem. For  $X$  a complex analytic manifold, M. Kashiwara defines the functor  $T\mathcal{H}om$  and he gives an explicit inverse to the functor of holomorphic solutions from the bounded derived category of complexes of  $\mathcal{D}_X$ -modules with regular holonomic cohomology to the bounded derived category of complexes of sheaves with constructible cohomology. This implies the classic result that the functor of holomorphic solutions  $\mathcal{S}(\cdot)$  is an equivalence between the category of regular meromorphic connections on  $X$  with poles along a closed submanifold  $Z$  and the category of linear representations of finite dimension of the fundamental group of  $X \setminus Z$ .

The irregular case is more complicated. In complex dimension 1, the classification of meromorphic connections obtained through the formal classification and the Stokes coefficients is nowadays well understood. Let us simply recall that, roughly speaking, the difference between a regular connection and an irregular one is based on the presence of functions of the type  $\exp \varphi$  ( $\varphi \in z^{-1/l} \mathbb{C}[z^{-1/l}]$ ,  $l \in \mathbb{Z}_{>0}$ ) in the solutions of the latter. The polynomials  $\varphi$ , appearing at exponent in the solutions of a meromorphic connection  $\mathcal{M}$ , are called determinant polynomials of  $\mathcal{M}$ . Their presence is made explicit in the Levelt-Turrittin's Formal Theorem (Theorem 1.13) and in the Hukuhara-Turrittin's Asymptotic Theorem (Theorem 1.15) which are of analytic nature. It is interesting to look for a topological description of the determinant polynomials.

In higher dimension, the study of irregular  $\mathcal{D}$ -modules is much more complicated. In [20] (see also [19]), C. Sabbah defines the notion of good model in dimension 2 and he conjectures the analogue of the Levelt-Turrittin's Formal Theorem, further he proves it for meromorphic connections of rank  $\leq 5$ . Recently, T. Mochizuki proved Sabbah's conjecture in any dimension in the algebraic case, see [15] and [16].

Given a complex analytic manifold  $X$ , in [10], M. Kashiwara and P. Schapira defined the complex of sheaves of tempered holomorphic functions  $\mathcal{O}_{X_{sa}}^t$ . The entries of the complex  $\mathcal{O}_{X_{sa}}^t$  are not sheaves on a topological space but on the subanalytic site,  $X_{sa}$ . The open sets of  $X_{sa}$  are the subanalytic open subsets of  $X$ , the coverings are locally finite coverings. If  $X$  has dimension 1, then  $\mathcal{O}_{X_{sa}}^t$  is a sheaf on  $X_{sa}$  and, for  $U$  a relatively compact subanalytic open subset of  $X$ , the sections of  $\mathcal{O}_{X_{sa}}^t(U)$

are the holomorphic functions on  $U$  which extend as distributions on  $X$  or, equivalently, which have moderate growth at the boundary of  $U$ .

Further in an example in [11], M. Kashiwara and P. Schapira explicated the sheaf of tempered holomorphic solutions of  $\mathcal{D}_{\mathbb{C}} \exp(1/z)$ . Such example suggests that tempered holomorphic functions and the subanalytic site could be useful tools in the study of ordinary differential equations. Roughly speaking, one of the ideas underlying the irregular Riemann-Hilbert correspondence in dimension 1 is to enrich, by ad hoc tools taking in account determinant polynomials and Stokes coefficients, the structure of the category of sheaves of  $\mathbb{C}$ -vector spaces where the functor of holomorphic solutions takes values (see the notion of  $\Omega$ -filtered local systems in [1] or [14]). The approach through subanalytic sheaves allows to enrich the topology of the space where the sheaf of solutions lives.

In this paper we go into the study of the subanalytic sheaf of tempered holomorphic solutions of germs of  $\mathcal{D}$ -modules. Denote by  $\mathcal{S}^t(\mathcal{M})$  the subanalytic sheaf of tempered holomorphic solutions of a holonomic  $\mathcal{D}_X$ -module  $\mathcal{M}$ . Let  $X \subset \mathbb{C}$  be an open neighborhood of 0,  $\text{Mod}(\mathbb{C}_{X_{sa}})$  the category of sheaves of  $\mathbb{C}$ -vector spaces on  $X_{sa}$ . We denote by  $\text{GM}_k$  be the category of modules over the ring of linear differential operators with formal Laurent power series “without ramification” (see Section 1.3 for a precise definition) and with Katz invariant  $< k$ . Roughly speaking, up to ramification, every meromorphic connection is formally equivalent to an element of  $\text{GM}_k$ , for  $k$  big enough. We prove that

$$\mathcal{S}^t(\cdot \otimes \mathcal{D} \exp(1/z^k)) : \text{GM}_k \longrightarrow \text{Mod}(\mathbb{C}_{X_{sa}})$$

is a fully faithful functor (**Theorem 3.5**). Further we prove that, given a germ of holonomic  $\mathcal{D}_X$ -module  $\mathcal{M}$  with Katz invariant  $< k$ , the datum of  $\mathcal{S}^t(\mathcal{M} \otimes \mathcal{D}_X \exp(1/z^{k+1}))$  is equivalent to the data of the holomorphic solutions, the determinant polynomials and their “rank” (**Theorem 3.7**).

Let us also recall that many sheaves of function spaces have been used in the study of irregular ordinary differential equations. For example, one can find in [14] the definitions of the sheaves  $\mathcal{A}^{\leq r}$  ( $r \in \mathbb{R}$ ) defined on the real blow-up of the complex plane at the origin. In [6], P. Deligne defined the sheaves  $\tilde{\mathcal{F}}^k$ , successively studied in detail in [12]. Roughly speaking, the solutions of  $\mathcal{D}_{\mathbb{C}} \exp(\varphi)$  with values in  $\mathcal{A}^{\leq r}$  (resp.  $\tilde{\mathcal{F}}^k$ ) depend only on the degree and the argument of the leading coefficient of  $\varphi$  (resp. the degree and the leading coefficient of  $\varphi$ ).

In conclusion we can say that tempered solutions on the subanalytic site give a good topological description of the determinant polynomials of a

given meromorphic connection. As further development, it would be interesting to describe precisely the image category of the functor of tempered solutions in order to give a full topological description of the space of determinant polynomials. It would also be interesting to give a good notion of Fourier transform for tempered holomorphic solutions of algebraic  $\mathcal{D}$ -modules in the same spirit of [14].

The present paper is subdivided in three sections organized as follows.

**Section 1** is devoted to the definitions, the notations and the presentation of the main results that will be needed in the rest of the paper. In particular we recall classical results on the subanalytic sets and site, on the tempered holomorphic functions and on the germs of  $\mathcal{D}$ -modules on complex curves. We recall the Levelt-Turrittin's Formal Theorem and the Hukuhara-Turrittin's Asymptotic Theorem. The latter theorem allows to endow holomorphic solutions of meromorphic connections on sufficiently small sectors with a graduation with respect to the space of determinant polynomials.

The functions of the form  $\exp(\varphi)$ ,  $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ , are the responsible for the non-tempered-growth of the solutions of an irregular  $\mathcal{D}$ -module. This motivates the study of  $\exp(\varphi)$  that we develop in **Section 2**. In particular, given  $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$  and  $U$  a relatively compact subanalytic open subset of  $\mathbb{C}$ , we give a necessary and sufficient topological condition on  $U$  so that  $\exp(\varphi) \in \mathcal{O}_{\mathbb{C}_{sa}}^t(U)$ . Further, given  $\varphi_1, \varphi_2 \in z^{-1}\mathbb{C}[z^{-1}]$ , we prove that the condition “for any  $U \subset \mathbb{C}$  relatively compact subanalytic open set,  $\exp(\varphi_1) \in \mathcal{O}_{\mathbb{C}_{sa}}^t(U)$  if and only if  $\exp(\varphi_2) \in \mathcal{O}_{\mathbb{C}_{sa}}^t(U)$ ” is equivalent to “ $\varphi_1$  and  $\varphi_2$  are proportional by a real positive constant”.

In **Section 3** we apply the results of Section 2 to the study of the functor of tempered holomorphic solutions of germs of  $\mathcal{D}_X$ -modules on a complex curve  $X$ . We prove that  $\mathcal{S}^t(\cdot \otimes_{\mathcal{D}_X} \exp(1/z^k)) : \mathbf{GM}_k \rightarrow \mathbf{Mod}(\mathbb{C}_{X_{sa}})$  is a fully faithful functor and that, given a germ of  $\mathcal{D}_X$ -module  $\mathcal{M}$  with Katz invariant  $< k$ , the datum of  $\mathcal{S}^t(\mathcal{M} \otimes_{\mathcal{D}_X} \exp(1/z^{k+1}))$  is equivalent to the data of the holomorphic solutions of  $\mathcal{M}$ , the determinant polynomials of  $\mathcal{M}$  and their rank.

*Acknowledgements.* — We thank P. Schapira for proposing this problem to our attention, C. Sabbah and N. Honda for many fruitful discussions and A. D’Agnolo for many useful remarks.

### 1. Notations and review

In this section we recall the definitions and the classical results concerning:

- (i) subanalytic sets, the subanalytic site and sheaves on it,
- (ii) the subanalytic sheaf of tempered holomorphic functions,
- (iii) germs of  $\mathcal{D}$ -modules on a complex curve.

#### 1.1. The subanalytic site

Let  $M$  be a real analytic manifold,  $\mathcal{A}$  the sheaf of real-valued real analytic functions on  $M$ .

DEFINITION 1.1.

- (i) A set  $X \subset M$  is said semi-analytic at  $x \in M$  if the following condition is satisfied. There exists an open neighborhood  $W$  of  $x$  such that  $X \cap W = \cup_{i \in I} \cap_{j \in J} X_{ij}$  where  $I$  and  $J$  are finite sets and either  $X_{ij} = \{y \in W; f_{ij}(y) > 0\}$  or  $X_{ij} = \{y \in W; f_{ij}(y) = 0\}$  for some  $f_{ij} \in \mathcal{A}(W)$ . Further,  $X$  is said semi-analytic if  $X$  is semi-analytic at any  $x \in M$ .
- (ii) A set  $X \subset M$  is said subanalytic if the following condition is satisfied. For any  $x \in M$ , there exist an open neighborhood  $W$  of  $x$ , a real analytic manifold  $N$  and a relatively compact semi-analytic set  $A \subset M \times N$  such that  $\pi(A) = X \cap W$ , where  $\pi : M \times N \rightarrow M$  is the projection.

Given  $X \subset M$ , denote by  $\overset{\circ}{X}$  (resp.  $\overline{X}$ ,  $\partial X$ ) the interior (resp. the closure, the boundary) of  $X$ .

PROPOSITION 1.2 (See [2]). — *Let  $X$  and  $Y$  be subanalytic subsets of  $M$ . Then  $X \cup Y$ ,  $X \cap Y$ ,  $\overline{X}$ ,  $\overset{\circ}{X}$  and  $X \setminus Y$  are subanalytic. Moreover the connected components of  $X$  are subanalytic, the family of connected components of  $X$  is locally finite and  $X$  is locally connected.*

Proposition 1.3 below is based on Łojasiewicz’s inequality, see [2, Corollary 6.7].

PROPOSITION 1.3. — *Let  $U \subset \mathbb{R}^n$  be an open set,  $X, Y$  closed subanalytic subsets of  $U$ . For any  $x_0 \in X \cap Y$ , there exist an open neighborhood  $W$  of  $x_0$ ,  $c, r \in \mathbb{R}_{>0}$  such that, for any  $x \in W$ ,*

$$\text{dist}(x, X) + \text{dist}(x, Y) \geq c \text{dist}(x, X \cap Y)^r.$$

DEFINITION 1.4. — Let  $\epsilon \in \mathbb{R}_{>0}$ ,  $\gamma : ] - \epsilon, \epsilon[ \rightarrow M$  an analytic map. The set  $\gamma(]0, \epsilon[)$  is said a semi-analytic arc with an endpoint at  $\gamma(0)$ .

THEOREM 1.5 (Curve Selection Lemma.). — Let  $Z \neq \emptyset$  be a subanalytic subset of  $M$  and let  $z_0 \in \overline{Z}$ . Then there exists an analytic map  $\gamma : ] - 1, 1[ \rightarrow M$ , such that  $\gamma(0) = z_0$  and  $\gamma(t) \in Z$  for  $t \neq 0$ .

For the rest of the subsection we refer to [10].

Let  $X$  be a complex analytic curve, we denote by  $\text{Op}(X)$  the family of open subsets of  $X$ . For  $k$  a commutative unital ring, we denote by  $\text{Mod}(k_X)$  the category of sheaves of  $k$ -modules on  $X$ .

Let us recall the definition of the subanalytic site  $X_{sa}$  associated to  $X$ . An element  $U \in \text{Op}(X)$  is an open set for  $X_{sa}$  if it is open, relatively compact and subanalytic in  $X$ . The family of open sets of  $X_{sa}$  is denoted  $\text{Op}^c(X_{sa})$ . For  $U \in \text{Op}^c(X_{sa})$ , a subset  $S$  of the family of open subsets of  $U$  is said an open covering of  $U$  in  $X_{sa}$  if  $S \subset \text{Op}^c(X_{sa})$  and, for any compact  $K$  of  $X$ , there exists a finite subset  $S_0 \subset S$  such that  $K \cap (\cup_{V \in S_0} V) = K \cap U$ .

We denote by  $\text{Mod}(k_{X_{sa}})$  the category of sheaves of  $k$ -modules on the subanalytic site. With the aim of defining the category  $\text{Mod}(k_{X_{sa}})$ , the adjective “relatively compact” can be omitted in the definition above. Indeed, in [10, Remark 6.3.6], it is proved that  $\text{Mod}(k_{X_{sa}})$  is equivalent to the category of sheaves on the site whose open sets are the open subanalytic subsets of  $X$  and whose coverings are the same as  $X_{sa}$ .

Given  $Y \in \text{Op}^c(X_{sa})$ , we denote by  $Y_{X_{sa}}$  the site induced by  $X_{sa}$  on  $Y$ , defined as follows. The open sets of  $Y_{X_{sa}}$  are open subanalytic subsets of  $Y$ . A covering of  $U \in \text{Op}(Y_{sa})$  for the topology  $Y_{X_{sa}}$  is a covering of  $U$  in  $X_{sa}$ .

We denote by  $\varrho : X \rightarrow X_{sa}$ , the natural morphism of sites associated to  $\text{Op}^c(X_{sa}) \rightarrow \text{Op}(X)$ . We refer to [10] for the definitions of the functors  $\varrho_* : \text{Mod}(k_X) \rightarrow \text{Mod}(k_{X_{sa}})$  and  $\varrho^{-1} : \text{Mod}(k_{X_{sa}}) \rightarrow \text{Mod}(k_X)$  and for Proposition 1.6 below.

PROPOSITION 1.6.

- (i)  $\varrho^{-1}$  is left adjoint to  $\varrho_*$ .
- (ii)  $\varrho^{-1}$  has a left adjoint denoted by  $\varrho_! : \text{Mod}(k_X) \rightarrow \text{Mod}(k_{X_{sa}})$ .
- (iii)  $\varrho^{-1}$  and  $\varrho_!$  are exact and  $\varrho_*$  is exact on  $\mathbb{R}$ -constructible sheaves.
- (iv)  $\varrho_*$  and  $\varrho_!$  are fully faithful.

Through  $\varrho_*$ , we will consider  $\text{Mod}(k_X)$  as a subcategory of  $\text{Mod}(k_{X_{sa}})$ .

The functor  $\varrho_!$  is described as follows. Let  $F \in \text{Mod}(k_X)$ , then  $\varrho_!(F)$  is the sheaf on  $X_{sa}$  associated to the presheaf  $U \mapsto F(\overline{U})$ .

Remark 1.7. — It is worth to mention that, given an analytic manifold  $X$ , there exists a topological space  $X'$  such that the category of sheaves

on  $X_{sa}$  with values in sets is equivalent to the category of sheaves on  $X'$  with values in sets. A detailed description of the semi-algebraic case and the o-minimal case are presented respectively in [3] and [5].

**1.2. Definition and main properties of  $\mathcal{O}_{X_{sa}}^t$**

For this subsection we refer to [10].

Let  $X$  be a complex analytic curve, denote by  $\overline{X}$  the complex conjugate curve and by  $X_{\mathbb{R}}$  the underlying real analytic manifold. We denote by  $X_{sa}$  the subanalytic site relative to  $X_{\mathbb{R}}$ .

Denote by  $\mathcal{O}_X$  (resp.  $\mathcal{D}_X$ ) the sheaf of holomorphic functions (resp. linear differential operators with holomorphic coefficients) on  $X$ . Denote by  $\mathcal{D}b_{X_{\mathbb{R}}}$  the sheaf of distributions on  $X_{\mathbb{R}}$  and, for a closed subset  $Z$  of  $X$ , by  $\Gamma_Z(\mathcal{D}b_{X_{\mathbb{R}}})$  the subsheaf of sections supported by  $Z$ . One denotes by  $\mathcal{D}b_{X_{sa}}^t$  the presheaf of tempered distributions on  $X_{sa}$  defined as follows,

$$\text{Op}^c(X_{sa}) \ni U \longmapsto \mathcal{D}b_{X_{sa}}^t(U) := \frac{\Gamma(X; \mathcal{D}b_{X_{\mathbb{R}}})}{\Gamma_{X \setminus U}(X; \mathcal{D}b_{X_{\mathbb{R}}})}.$$

In [10], using some results of [13], it is proved that  $\mathcal{D}b_{X_{sa}}^t$  is a sheaf on  $X_{sa}$ . This sheaf is well defined in the category  $\text{Mod}(\varrho! \mathcal{D}_X)$ . Moreover, for any  $U \in \text{Op}^c(X_{sa})$ ,  $\mathcal{D}b_{X_{sa}}^t$  is  $\Gamma(U, \cdot)$ -acyclic.

Denote by  $D^b(\varrho! \mathcal{D}_X)$  the bounded derived category of  $\varrho! \mathcal{D}_X$ -modules. The sheaf  $\mathcal{O}_{X_{sa}}^t \in D^b(\varrho! \mathcal{D}_X)$  of tempered holomorphic functions is defined as

$$\mathcal{O}_{X_{sa}}^t := R\text{Hom}_{\varrho! \mathcal{D}_{\overline{X}}}(\varrho! \mathcal{O}_{\overline{X}}, \mathcal{D}b_{X_{\mathbb{R}}}^t).$$

In [10], it is proved that, since  $\dim X = 1$ ,  $R\varrho_* \mathcal{O}_X$  and  $\mathcal{O}_{X_{sa}}^t$  are concentrated in degree 0. Hence we can write the following exact sequence of sheaves on  $X_{sa}$

$$0 \longrightarrow \mathcal{O}_{X_{sa}}^t \longrightarrow \mathcal{D}b_{X_{sa}}^t \xrightarrow{\bar{\partial}} \mathcal{D}b_{X_{sa}}^t \longrightarrow 0.$$

Let us recall that  $\mathcal{D}b_{X_{sa}}^t$  and  $\mathcal{O}_{X_{sa}}^t$  can be defined without any change on a complex analytic manifold  $X$  (see [10]).

Now we recall the definition of polynomial growth for  $\mathcal{C}^\infty$  functions on  $X_{\mathbb{R}}$  and in (1.2) we give an alternative expression for  $\mathcal{O}_{X_{sa}}^t(U)$ ,  $U \in \text{Op}^c(X_{sa})$ .

DEFINITION 1.8. — *Let  $U$  be an open subset of  $X$ ,  $f \in \mathcal{C}_{X_{\mathbb{R}}}^\infty(U)$ . One says that  $f$  has polynomial growth at  $p \in X$  if it satisfies the following condition. For a local coordinate system  $x = (x_1, x_2)$  around  $p$ , there exist a compact neighborhood  $K$  of  $p$  and  $M \in \mathbb{Z}_{>0}$  such that*

$$(1.1) \quad \sup_{x \in K \cap U} \text{dist}(x, K \setminus U)^M |f(x)| < +\infty.$$



We say that  $f \in \mathcal{C}_{X_{\mathbb{R}}}^{\infty}(U)$  has polynomial growth on  $U$  if it has polynomial growth at any  $p \in X$ . We say that  $f$  is tempered at  $p$  if all its derivatives have polynomial growth at  $p \in X$ . We say that  $f$  is tempered on  $U$  if it is tempered at any  $p \in X$ . Denote by  $\mathcal{C}_{X_{sa}}^{\infty,t}$  the presheaf on  $X_{\mathbb{R}}$  of tempered  $\mathcal{C}^{\infty}$ -functions.

It is obvious that  $f$  has polynomial growth at any point of  $U$ .

In [10] it is proved that  $\mathcal{C}_{X_{sa}}^{\infty,t}$  is a sheaf on  $X_{sa}$ . For  $U \subset \mathbb{R}^2$  a relatively compact open set, there is a simple characterization of functions with polynomial growth on  $U$ .

PROPOSITION 1.9. — *Let  $U \subset \mathbb{R}^2$  be a relatively compact open set and let  $f \in \mathcal{C}_{\mathbb{R}^2}^{\infty}(U)$ . Then  $f$  has polynomial growth if and only if there exist  $C, M \in \mathbb{R}_{>0}$  such that, for any  $x \in U$ ,*

$$|f(x)| \leq \frac{C}{\text{dist}(x, \partial U)^M}.$$

For Proposition 1.10 below, see [10].

PROPOSITION 1.10. — *One has the following isomorphism*

$$\mathcal{O}_{X_{sa}}^t \simeq R\mathcal{H}om_{\varrho! \mathcal{D}_{\overline{X}}}(\varrho! \mathcal{O}_{\overline{X}}, \mathcal{C}_{X_{sa}}^{\infty,t}).$$

Hence, for  $U \in \text{Op}^c(X_{sa})$ , we deduce the short exact sequence

$$(1.2) \quad 0 \longrightarrow \mathcal{O}_{X_{sa}}^t(U) \longrightarrow \mathcal{C}_{X_{sa}}^{\infty,t}(U) \xrightarrow{\bar{\partial}} \mathcal{C}_{X_{sa}}^{\infty,t}(U) \longrightarrow 0.$$

Now, we recall two results on the pull back of tempered holomorphic functions. We refer to [9] for the definition of  $\mathcal{D}_{X \rightarrow Y}$ , for  $f : X \rightarrow Y$  a morphism of complex manifolds. For Lemma 1.11, see [10, Lemma 7.4.7].

LEMMA 1.11. — *Let  $f : X \rightarrow Y$  be a closed embedding of complex manifolds. There is a natural isomorphism in  $D^b(\varrho! \mathcal{D}_X)$*

$$\varrho! \mathcal{D}_{X \rightarrow Y} \otimes_{\varrho! f^{-1} \mathcal{D}_Y}^L f^{-1} \mathcal{O}_Y^t \simeq \mathcal{O}_X^t.$$

For Proposition 1.12, see [17, Theorem 2.1].

PROPOSITION 1.12. — *Let  $f \in \mathcal{O}_{\mathbb{C}}(X)$ ,  $U \in \text{Op}^c(X_{sa})$  such that  $f|_{\overline{U}}$  is an injective map,  $h \in \mathcal{O}_X(f(U))$ . Then  $h \in \mathcal{O}_{\mathbb{C}_{sa}}^t(f(U))$  if and only if  $h \circ f \in \mathcal{O}_{X_{sa}}^t(U)$ .*

We conclude this subsection by recalling the definition of the sheaf of holomorphic functions with moderate growth at the origin. We follow [14]. Let  $S^1$  be the unit circle,  $S^1 \times \mathbb{R}_{\geq 0}$  the real blow-up at the origin of  $\mathbb{C}^{\times}$ .

For  $\tau \in \mathbb{R}$ ,  $r \in \mathbb{R}_{>0}$ ,  $0 < \epsilon < \pi$ , the set

$$S_{\tau \pm \epsilon, r} := \left\{ \varrho e^{i\vartheta} \in \mathbb{C}^\times; \varrho \in ]0, r[, \vartheta \in ]\tau - \epsilon, \tau + \epsilon[ \right\}$$

is called an *open sector centered at  $\tau$  of amplitude  $2\epsilon$  and radius  $r$*  or simply an *open sector*. Identifying  $S^1$  with  $[0, 2\pi[ \subset \mathbb{R}$ , we will consider sectors centered at  $\tau \in S^1$ . Further, with an abuse of language, we will say that an open sector  $S_{\tau \pm \epsilon, r}$  contains  $\vartheta \in \mathbb{R}$  or  $e^{i\vartheta} \in S^1$  if  $\vartheta \in ]\tau - \epsilon, \tau + \epsilon[ \pmod{2\pi}$ .

The sheaf on  $S^1 \times \mathbb{R}_{\geq 0}$  of holomorphic functions with moderate growth at the origin, denoted  $\mathcal{A}^{\leq 0}$ , is defined as follows. For  $U$  an open set of  $S^1 \times \mathbb{R}_{\geq 0}$ , set

$$(1.3) \quad \mathcal{A}^{\leq 0}(U) = \left\{ f \in \mathcal{O}_{\mathbb{C}}(U \setminus (S^1 \times \{0\})) \text{ satisfying the following condition: for any } (e^{i\vartheta}, 0) \in U \text{ there exist } C, M \in \mathbb{R}_{>0} \text{ and an open sector } S \subset U \text{ containing } e^{i\vartheta} \text{ such that } |f(z)| < C|z|^{-M} \text{ for any } z \in S \right\}.$$

Clearly,  $\mathcal{A}^{\leq 0}$  is a sheaf on  $S^1 \times \mathbb{R}_{\geq 0}$ .

In [18], the author defines the functor  $\nu_0^{sa}$  of specialization at 0 for the sheaves on the subanalytic site. One has that,  $\varrho^{-1}\nu_0^{sa}(\mathcal{O}_{\mathbb{C}_{sa}}^t) \simeq \mathcal{A}^{\leq 0}$ .

### 1.3. $\mathcal{D}$ -modules on complex curves and good models

In this subsection we recall some classical results on germs of  $\mathcal{D}_X$ -modules on a complex analytic curve  $X$ . For a detailed and comprehensive presentation we refer to [14], [9] and [1].

Given a complex analytic curve  $X$  and  $x_0 \in X$ , we denote by  $\mathcal{O}_X(*x_0)$  (resp.  $\mathcal{D}_X(*x_0)$ ) the sheaf on  $X$  of holomorphic functions on  $X \setminus \{x_0\}$  meromorphic at  $x_0$  (resp. the sheaf of rings of differential operators of finite order with coefficients in  $\mathcal{O}_X(*x_0)$ ). Further, we denote by  $\widehat{\mathcal{O}}(*x_0)$  (resp.  $\widehat{\mathcal{D}}(*x_0)$ ) the field of formal Laurent power series (resp. the ring of differential operators with coefficients in  $\widehat{\mathcal{O}}(*x_0)$ ). The ring  $\mathcal{O}(*x_0)$  comes equipped with a natural valuation  $v : \mathcal{O}(*x_0) \rightarrow \mathbb{Z} \cup \{+\infty\}$ .

By the choice of a local coordinate  $z$  near  $x_0$ , we can suppose that  $X \subset \mathbb{C}$  is an open neighborhood of  $x_0 = 0 \in \mathbb{C}$ .

The category of holonomic  $\mathcal{D}_X(*0)$ -modules, denoted  $\text{Mod}_h(\mathcal{D}_X(*0))$ , is equivalent to the category of local meromorphic connections.

For  $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ , set  $\mathcal{L}^\varphi := \mathcal{D}_X(*0) \exp(\varphi)$ .

For  $l \in \mathbb{Z}_{>0}$ , let  $\mu_l : \mathbb{C} \rightarrow \mathbb{C}$ ,  $z \mapsto z^l$ . We denote by  $\mu_l^*$  the inverse image functor for  $\mathcal{D}_X(*0)$ -modules.

Theorems 1.13 and 1.15 below are cornerstones in the theory of ordinary differential equations. We refer to [14], [21] and [22].

**THEOREM 1.13** (Levitt-Turrittin’s Formal Theorem). — *Let  $\mathcal{M} \in \text{Mod}_h(\mathcal{D}_X(*0))$ . There exist  $l \in \mathbb{Z}_{>0}$ , a finite set  $\Sigma \subset z^{-1}\mathbb{C}[z^{-1}]$ , a family,  $\{\mathcal{R}_\varphi\}_{\varphi \in \Sigma}$ , of regular holonomic  $\mathcal{D}_X(*0)$ -modules indexed by  $\Sigma$  and an isomorphism in  $\text{Mod}(\widehat{\mathcal{D}_X(*0)})$*

$$(1.4) \quad \mu_l^* \mathcal{M} \otimes \widehat{\mathcal{O}}_{*0} \simeq \bigoplus_{\varphi \in \Sigma} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi \otimes \widehat{\mathcal{O}}_{*0}.$$

In the literature (for example [14]) the definition of the Katz invariant of  $\mathcal{M} \in \text{Mod}_h(\mathcal{D}_X(*0))$  is given starting from the Newton polygon of  $\mathcal{M}$ . For sake of simplicity we give an equivalent definition based on the isomorphism (1.4). Clearly, the valuation  $v$  induces a map, still denoted  $v$ ,  $z^{-1}\mathbb{C}[z^{-1}] \rightarrow \mathbb{Z} \cup \{+\infty\}$ .

**DEFINITION 1.14.**

- (i) *Let  $\mathcal{M} \in \text{Mod}_h(\mathcal{D}_X(*0))$ . Suppose that (1.4) is satisfied,  $l$  is minimal and  $\Sigma \neq \{0\}$ . The Katz invariant of  $\mathcal{M}$  is  $\max_{\varphi \in \Sigma} \{-\frac{v(\varphi)}{l}\}$ . If  $\Sigma = \{0\}$  then the Katz invariant of  $\mathcal{M}$  is 0.*

*For  $k \in \mathbb{Z}_{>0}$ , we denote by  $\text{Mod}_h(\mathcal{D}_X(*0))_k$  the full abelian subcategory of  $\text{Mod}_h(\mathcal{D}_X(*0))$  whose objects have Katz invariant strictly smaller than  $k$ .*

- (ii) *Let  $\Sigma \subset z^{-1}\mathbb{C}[z^{-1}]$  be a finite set and  $\{\mathcal{R}_\varphi\}_{\varphi \in \Sigma}$  a family of regular holonomic  $\mathcal{D}_X(*0)$ -modules. A  $\mathcal{D}_X(*0)$ -module isomorphic to  $\bigoplus_{\varphi \in \Sigma} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi$ , is said a good model.*

*We denote by  $\text{GM}_k$  the full subcategory of  $\text{Mod}_h(\mathcal{D}_X(*0))_k$  whose objects are good models.*

Roughly speaking, Theorem 1.15 below says that the formal isomorphism (1.4) is analytic on sufficiently small open sectors.

Let

$$P := \sum_{j=0}^m a_j(z) \left(\frac{d}{dz}\right)^j,$$

where  $m \in \mathbb{Z}_{>0}$ ,  $a_j \in \mathcal{O}_{\mathbb{C}}(X)$  and  $a_m \neq 0$ . Denote by  $\mathcal{C}_{\mathbb{C}}^0$  the sheaf of continuous functions on  $\mathbb{C}$ . For  $l \in \mathbb{Z}_{>0}$ ,  $S$  an open sector,  $h \in \{1, \dots, l\}$ , let  $\zeta_h : S \rightarrow \mathbb{C}$  be the  $l$  different inverse functions to  $z \mapsto z^l$  defined on  $S$ .

**THEOREM 1.15** (Hukuhara-Turrittin’s Asymptotic Theorem). — *There exist a finite set  $\Sigma \subset z^{-1}\mathbb{C}[z^{-1}]$ ,  $l, r_\varphi \in \mathbb{Z}_{>0}$  ( $\varphi \in \Sigma$ ) such that for any  $\tau \in$*

$\mathbb{R}$ , there exist an open sector  $S$  containing  $\tau$ ,  $f_{\varphi kh} \in \mathcal{O}_{\mathbb{C}}(S) \cap \mathcal{C}_{\mathbb{C}}^0(\overline{S} \setminus \{0\})$  ( $\varphi \in \Sigma$ ,  $k = 1, \dots, r_{\varphi}$ ,  $h = 1, \dots, l$ ), satisfying

- (i)  $\{f_{\varphi kh}(z) \exp(\varphi \circ \zeta_h(z)); \varphi \in \Sigma, k = 1, \dots, r_{\varphi}, h = 1, \dots, l\}$  is a basis of the  $\mathbb{C}$ -vector space of holomorphic solutions of  $Pu = 0$  on  $S$ ,
- (ii) there exist  $C, M \in \mathbb{R}_{>0}$  such that, for any  $z \in S$ ,

$$C|z|^M \leq |f_{\varphi kh}(z)| \leq (C|z|^M)^{-1}$$

( $\varphi \in \Sigma$ ,  $k = 1, \dots, r_{\varphi}$ ,  $h = 1, \dots, l$ ).

It is well known that, given  $\mathcal{M} \in \text{Mod}_h(\mathcal{D}_X(*0))$ , there exists  $P \in \mathcal{D}_X(*0)$  such that  $\mathcal{M} \simeq \frac{\mathcal{D}_X(*0)}{\mathcal{D}_X(*0) \cdot P}$ . With a harmless abuse of language, we will speak without distinctions about the solutions of  $\mathcal{M}$  and the solutions of  $Pu = 0$ .

DEFINITION 1.16. — We use the notations of Theorem 1.15. Let  $\mathcal{M} \in \text{Mod}_h(\mathcal{D}_X(*0))$ .

- (i) If  $f_{\varphi kh}(z) \exp(\varphi \circ \zeta_h(z))$  is a holomorphic solution of  $\mathcal{M}$  on an open sector, then we say that  $\varphi \circ \zeta_h$  is a determinant polynomial of  $\mathcal{M}$ . We denote by  $\Omega(\mathcal{M})$  the set of all determinant polynomials of  $\mathcal{M}$ . If  $\varphi \circ \zeta_h \in \Omega(\mathcal{M})$ , we say that  $r_{\varphi} \in \mathbb{Z}_{>0}$  is the rank of  $\varphi \circ \zeta_h$ .
- (ii) Given  $\vartheta \in S^1$ , we set

$$\Omega_{\vartheta} := \left\{ \sum_{j=1}^n a_j z^{-\frac{j}{l}}; a_j \in \mathbb{C}, l, n \in \mathbb{Z}_{>0} \right\}.$$

We denote by  $\text{Mod}_{\Omega_{\vartheta}}(\mathbb{C})$  the category of finite dimensional vector spaces graded with respect to  $\Omega_{\vartheta}$ .

Theorem 1.15 implies that, given  $\vartheta \in S^1$ , the holomorphic solutions of  $\mathcal{M}$  on a sufficiently small sector containing  $\vartheta$ , can be endowed with a  $\Omega_{\vartheta}$ -graduation. Hence, there exists a functor

$$\mathcal{S}^{\Omega}(\cdot)_{\vartheta} : \text{Mod}_h(\mathcal{D}_X(*0)) \longrightarrow \text{Mod}_{\Omega_{\vartheta}}(\mathbb{C}).$$

The  $\Omega_{\vartheta}$ -graduation on the holomorphic solutions of meromorphic connections described above is the first step to have a local irregular Riemann-Hilbert correspondence in dimension 1. We refer to [1] and [14] for a complete description of  $\Omega$ -filtered local systems.

## 2. Tempered growth of exponential functions

This section is subdivided as follows. In the first part we study the family of sets where a function of the form  $\exp(\varphi)$ ,  $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ , is tempered. In the second part we use the results of the first part in order to prove that such family determines  $\varphi$  up to a multiplicative positive constant. Throughout this section  $X = \mathbb{C}$ .

### 2.1. Sets where exponential functions have tempered growth

For  $\varphi \in z^{-1}\mathbb{C}[z^{-1}] \setminus \{0\}$  and  $A \in \mathbb{R}_{>0}$ , set

$$(2.1) \quad U_{\varphi,A} := \left\{ z \in \mathbb{C}^\times; \operatorname{Re}(\varphi(z)) < A \right\},$$

further set  $U_{0,A} := \mathbb{C}$ .

First we state and prove the analogue of a result of [11].

**PROPOSITION 2.1.** — *Let  $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$  and  $U \in \operatorname{Op}^c(X_{sa})$  with  $U \neq \emptyset$ . The conditions below are equivalent.*

- (i)  $\exp(\varphi) \in \mathcal{O}_{X_{sa}}^t(U)$ .
- (ii) There exists  $A \in \mathbb{R}_{>0}$  such that  $U \subset U_{\varphi,A}$ .

Before proving Proposition 2.1, we need the following

**LEMMA 2.2** ([11]). — *Let  $W \neq \emptyset$  be an open subanalytic subset of  $\mathbb{P}^1(\mathbb{C})$ ,  $\infty \notin W$ . The following conditions are equivalent.*

- (i) There exists  $A \in \mathbb{R}_{>0}$  such that  $\operatorname{Re} z < A$ , for any  $z \in W$ .
- (ii) The function  $\exp(z)$  has polynomial growth on any semi-analytic arc  $\Gamma \subset W$  with an endpoint at  $\infty$ . That is, for any semi-analytic arc  $\Gamma \subset W$  with an endpoint at  $\infty$ , there exist  $M, C \in \mathbb{R}_{>0}$  such that, for any  $z \in \Gamma$ ,

$$(2.2) \quad |\exp(z)| \leq C(1 + |z|^2)^M.$$

*Proof.*

Clearly, (i)  $\Rightarrow$  (ii).

Let us prove (ii)  $\Rightarrow$  (i). Set  $z := x + iy$  and suppose that  $x$  is not bounded on  $W$ . There exist  $\epsilon, L \in \mathbb{R}_{>0}$  and a real analytic map

$$\begin{aligned} \gamma : [0, \epsilon[ &\longrightarrow \mathbb{P}^1(\mathbb{C}) \\ t &\longmapsto (x(t), y(t)), \end{aligned}$$

such that  $\gamma(0) = \infty$ ,  $\gamma(]0, \epsilon[) \subset W$  and  $x(]0, \epsilon[) = ]L, +\infty[$ . Since  $\gamma$  is analytic, there exist  $q \in \mathbb{Q}$ ,  $c \in \mathbb{R}$  and  $\mu \in \mathbb{R}_{>0}$  such that, for any  $t \in ]0, \epsilon[$ ,

$$\gamma(t) = \left( x(t), cx(t)^q + O(x(t)^{q-\mu}) \right).$$

Now, if (2.2) is satisfied, then  $\exp(x)$  has polynomial growth in a neighborhood of  $+\infty$ , which gives a contradiction.  $\square$

*Proof of Proposition 2.1.*

Clearly, (ii)  $\Rightarrow$  (i).

(i)  $\Rightarrow$  (ii). The result is obvious if  $\varphi = 0$ . Otherwise, we distinguish two cases.

Case 1: Suppose  $\varphi(z) = \frac{1}{z}$ .

Suppose that for any  $A \in \mathbb{R}_{>0}$ , there exists  $z_A \in U$  such that  $\operatorname{Re}\left(\frac{1}{z_A}\right) > A$ . Then, by Lemma 2.2, there exists a semi-analytic arc with an endpoint at 0,  $\Gamma \subset U$ , such that  $\exp\left(\operatorname{Re}\frac{1}{z}\right)$  has not polynomial growth on  $\Gamma$ . That is, for any  $M, C \in \mathbb{R}_{>0}$ , there exists  $z_{M,C} \in \Gamma$  satisfying

$$\exp\left(\operatorname{Re}\frac{1}{z_{M,C}}\right) \geq \frac{C}{|z_{M,C}|^M}.$$

Apply Proposition 1.3 with  $X = \bar{\Gamma}$  and  $Y = \partial U$ . There exist an open neighborhood  $V$  of 0,  $c, r \in \mathbb{R}_{>0}$  such that, for any  $z \in \Gamma \cap V$ ,

$$|z| \leq c \operatorname{dist}(z, \partial U)^r.$$

Hence,

$$\exp\left(\operatorname{Re}\frac{1}{z_{M,C}}\right) \geq \frac{c^{-1}C}{\operatorname{dist}(z_{M,C}, \partial U)^{rM}}.$$

It follows that  $\exp\left(\frac{1}{z}\right)$  is not tempered on  $U$ .

Case 2: Suppose  $\varphi(z) = \sum_{j=1}^n \frac{a_j}{z^j}$ , with  $n \in \mathbb{Z}_{>0}$  and  $a_n \neq 0$ .

Let

$$\eta(z) := \left( \sum_{j=1}^n \frac{a_j}{z^j} \right)^{-1} = \frac{z^n}{\sum_{j=1}^n a_j z^{n-j}}.$$

There exists a neighborhood  $W \subset \mathbb{C}$  of 0 such that  $\eta \in \mathcal{O}_{\mathbb{C}}(W)$ . It is well known that a non-constant holomorphic function is locally the composition of a holomorphic isomorphism and a positive integer power of  $z$ . Since it is sufficient to prove the result in a neighborhood of 0 and up to finite coverings, we can suppose that  $U \subset W$  and  $\eta|_{\bar{U}}$  is injective.

Consider the following commutative triangle,

$$\begin{array}{ccc}
 U & \xrightarrow{\eta} & \eta(U) \\
 \searrow \exp(\varphi) & & \downarrow \exp(\frac{1}{\zeta}) \\
 & & \mathbb{C}.
 \end{array}$$

Using Proposition 1.12 and Case 1, we have that

$$\begin{aligned}
 \exp(\varphi) \in \mathcal{O}^t(U) &\Leftrightarrow \exp\left(\frac{1}{\zeta}\right) \in \mathcal{O}^t(\eta(U)) \\
 &\Leftrightarrow \eta(U) \subset U_{\frac{1}{\zeta}, A} \quad \text{for some } A \in \mathbb{R}_{>0} \\
 &\Leftrightarrow U \subset U_{\varphi, A} \quad \text{for some } A \in \mathbb{R}_{>0}.
 \end{aligned}$$

□

**COROLLARY 2.3.** — *Let  $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ . Let  $S$  be an open sector of amplitude  $2\pi$ ,  $U \in \text{Op}^c(X_{sa})$ ,  $\emptyset \neq U \subset S$ ,  $\zeta : S \rightarrow \mathbb{C}$ , an inverse of  $z \mapsto z^l$ . Then  $\exp(\varphi \circ \zeta) \in \mathcal{O}^t_{X_{sa}}(U)$  if and only if there exists  $A \in \mathbb{R}_{>0}$  such that  $\zeta(U) \subset U_{\varphi, A}$ .*

*In particular, setting*

$$U_{\varphi \circ \zeta, A} := \left\{ z \in S; \text{Re}(\varphi \circ \zeta(z)) < A \right\},$$

*one has that  $\exp(\varphi \circ \zeta) \in \mathcal{O}^t_{X_{sa}}(U)$  if and only if there exists  $A \in \mathbb{R}_{>0}$  such that  $U \subset U_{\varphi \circ \zeta, A}$ .*

*Proof.* — Let  $\mu_l(z) := z^l$ . Consider the following commutative diagram,

$$\begin{array}{ccc}
 \zeta(U) & \xrightarrow{\mu_l} & U \\
 \searrow \exp(\varphi) & & \downarrow \exp(\varphi \circ \zeta) \\
 & & \mathbb{C}.
 \end{array}$$

By Proposition 1.12, we have that  $\exp(\varphi \circ \zeta) \in \mathcal{O}^t_{X_{sa}}(U)$  if and only if  $\exp(\varphi) \in \mathcal{O}^t_{X_{sa}}(\zeta(U))$ . Then the conclusion follows by Proposition 2.1.

□

Now we are going to introduce a class of subanalytic sets which plays an important role in what follows.

**DEFINITION 2.4.** — *For  $\tau \in \mathbb{R}$  we say that  $U \in \text{Op}^c(X_{sa})$  is concentrated along  $\tau$  if  $U \neq \emptyset$  is connected,  $0 \in \partial U$  and, for any open sector  $S$  containing  $\tau$ , there exists an open neighborhood  $W \subset \mathbb{C}$  of 0 such that  $U \cap W \subset S$ .*

Lemma 2.5 below follows easily from the well known fact that a non-constant holomorphic function is locally the composition of a holomorphic isomorphism and a positive integer power of  $z$ .

LEMMA 2.5. — *Let  $W \subset \mathbb{C}$  be an open neighborhood of 0,  $f \in \mathcal{O}(W)$ . Suppose that  $f$  has a zero of order  $l \in \mathbb{Z}_{>0}$  at 0. There exists  $\tau_f \in \mathbb{R}$ , depending only on the argument of  $f^{(l)}(0)$ , satisfying the following conditions.*

- (i) *For any  $\tau \in \mathbb{R}$ ,  $U \in \text{Op}^c(X_{sa})$  concentrated along  $\tau$ , there exists an open neighborhood  $W'$  of 0,  $\overline{W'} \subset W$ , such that  $f|_{\overline{U \cap W'}}$  is injective and  $f(U \cap W')$  is concentrated along  $l(\tau + \tau_f)$ .*
- (ii) *For any  $\tau \in \mathbb{R}$ ,  $V \in \text{Op}^c(X_{sa})$  concentrated along  $\tau$ , there exist an open neighborhood  $W'$  of 0,  $V_0, \dots, V_{l-1} \in \text{Op}^c(X_{sa})$ , such that  $V_j$  is concentrated along  $\frac{\tau}{l} - \tau_f + j\frac{2\pi}{l}$ , and  $f(V_j) = V \cap W'$  ( $j = 0, \dots, l - 1$ ).*

Proposition 2.6 below will play a fundamental role in the next subsection.

PROPOSITION 2.6. — *Let  $n \in \mathbb{Z}_{\geq 0}$ ,  $\tau_0 \in \mathbb{R}$ . There exists  $\tau \in \mathbb{R}$  such that, for any  $\varphi = \frac{\varrho e^{i\tau_0}}{z^n} + \tilde{\varphi} \in z^{-1}\mathbb{C}[z^{-1}]$  ( $\varrho \in \mathbb{R}_{>0}$ ,  $-v(\tilde{\varphi}) < n$ ), there exist  $U_0, \dots, U_{2n-1} \in \text{Op}^c(X_{sa})$  satisfying*

- (i)  $U_j$  is concentrated along  $\tau + j\frac{\pi}{n}$ ,
  - (ii)  $\exp(\varphi), \exp(-\varphi) \in \mathcal{O}_{X_{sa}}^t(U_j)$ ,
- ( $j = 0, \dots, 2n - 1$ ).

*Proof.* — The result is obvious if  $\varphi = 0$ . Otherwise we distinguish three cases.

Case 1: Suppose  $\varphi(z) = \frac{1}{z}$ .

Recall (2.1). For  $A \in \mathbb{R}_{>0}$ , one checks easily that the set  $U_{\frac{1}{z}, A}$  (resp.  $U_{-\frac{1}{z}, A}$ ) is the complementary of the closed disc of center  $(\frac{1}{2A}, 0)$  (resp.  $(-\frac{1}{2A}, 0)$ ) and radius  $\frac{1}{2A}$ .

Set

$$U_1 := \{(x, y) \in \mathbb{R}^2; |x| < 1, \sqrt{|x| - x^2} < y < 1\},$$

$$U_2 := \{(x, y) \in \mathbb{R}^2; |x| < 1, -1 < y < -\sqrt{|x| - x^2}\}.$$

It is easy to see that  $U_1$  (resp.  $U_2$ ) is concentrated along  $\frac{\pi}{2}$  (resp.  $\frac{3\pi}{2}$ ) and  $U_1 \cup U_2 \subset U_{\frac{1}{z}, 1} \cap U_{-\frac{1}{z}, 1}$ . Hence, by Proposition 2.1,

$$(2.3) \quad \exp(1/z), \exp(-1/z) \in \mathcal{O}_{X_{sa}}^t(U_j) \quad (j = 1, 2).$$



Case 2: Suppose that  $\varphi(z) = \frac{1}{z^m}$ , for  $m \in \mathbb{Z}_{>0}$ . Let  $\mu_m : \mathbb{C} \rightarrow \mathbb{C}$ ,  $\mu_m(z) = z^m$ . Consider the commutative triangle

$$\begin{array}{ccc} \mathbb{C}^\times & \xrightarrow{\mu_m} & \mathbb{C}^\times \\ & \searrow \exp(1/z^m) & \downarrow \exp(1/z) \\ & & \mathbb{C}. \end{array}$$

Consider  $U_1, U_2$  as in Case 1. Applying Lemma 2.5 (ii) with  $f = \mu_m$ , we obtain that there exist  $V_{1,0}, \dots, V_{1,m-1} \in \text{Op}^c(X_{sa})$  (resp.  $V_{2,0}, \dots, V_{2,m-1} \in \text{Op}^c(X_{sa})$ ) such that

- (i)  $V_{1,j}$  (resp.  $V_{2,j}$ ) is concentrated along  $\frac{\pi}{2m} + j\frac{2\pi}{m}$  (resp.  $\frac{3\pi}{2m} + j\frac{2\pi}{m}$ ),
  - (ii)  $\mu_m(V_{k,j}) = U_k$  ,
- ( $j = 0, \dots, m - 1, k = 1, 2$ ).

Clearly,  $\mu_m|_{V_{k,j}}$  is injective. By Proposition 1.12, we have that

$$\begin{aligned} \exp(1/z^m) &\in \mathcal{O}^t_{X_{sa}}(V_{k,j}) \\ (\text{resp. } \exp(-1/z^m) &\in \mathcal{O}^t_{X_{sa}}(V_{k,j})) \end{aligned}$$

if and only if

$$\begin{aligned} \exp(1/z) &\in \mathcal{O}^t_{X_{sa}}(\mu_m(V_{k,j})) = \mathcal{O}^t_{X_{sa}}(U_k) \\ (\text{resp. } \exp(-1/z) &\in \mathcal{O}^t_{X_{sa}}(\mu_m(V_{k,j})) = \mathcal{O}^t_{X_{sa}}(U_k)) \end{aligned}$$

( $j = 0, \dots, m - 1, k = 1, 2$ ). The conclusion follows.

Case 3: Suppose that

$$\varphi(z) = \sum_{j=1}^n \frac{a_j}{z^j} \in z^{-1}\mathbb{C}[z^{-1}],$$

for  $n \in \mathbb{Z}_{>0}$ ,  $a_j \in \mathbb{C}$  ( $j = 1, \dots, n$ ) and  $a_n \neq 0$ .

First, we recall the implicit function theorem for convergent power series. We denote by  $\mathbb{C}\{x\}$  (resp.  $\mathbb{C}\{x, y\}$ ) the ring of convergent power series in  $x$  (resp.  $x, y$ ). We refer to [4, Theorem 8.6.1, p. 166] for the proof.

**THEOREM 2.7.** — *Let  $F \in \mathbb{C}\{x, y\}$  be such that  $F(0, 0) = 0$ . There exists  $\eta(x) \in \cup_{l \in \mathbb{Z}_{>0}} x^{1/l}\mathbb{C}\{x^{1/l}\}$  such that  $F(x, \eta(x)) = 0$ .*

Consider

$$\begin{aligned} F(z, \eta) &:= -\eta^n + z^n \sum_{j=1}^n a_j \eta^{n-j} \\ (2.4) \quad &= -\eta^n + z^n a_1 \eta^{n-1} + \dots + z^n a_{n-1} \eta + z^n a_n. \end{aligned}$$

By Theorem 2.7, there exist  $l \in \mathbb{Z}_{>0}$ ,  $\eta(z) \in z^{1/l}\mathbb{C}\{z^{1/l}\}$  such that  $F(z, \eta(z)) = 0$ . Since  $a_n \neq 0$ , we have that  $\eta(z) \neq 0$ , for  $z \neq 0$ . It follows that  $\eta(z) \in z^{1/l}\mathbb{C}\{z^{1/l}\}$  satisfies

$$(2.5) \quad \varphi(\eta(z)) = \sum_{j=1}^n \frac{a_j}{\eta(z)^j} = \frac{1}{z^n}.$$

Further, substituting  $\eta(z)$  in (2.4), one checks that  $l = 1$  and  $\eta(z) = z\sigma(z)$ , for  $\sigma$  an invertible element of  $\mathbb{C}\{z\}$  such that  $\arg(\sigma(0)) = \frac{\arg(a_n)}{n}$ . In particular, there exists an open neighborhood  $W \subset \mathbb{C}$  of the origin such that  $\eta \in \mathcal{O}_{\mathbb{C}}(W)$ .

Now, by Case 2, there exist  $V_{k,j} \subset W$  ( $j = 0, \dots, n-1, k = 1, 2$ ) such that  $V_{1,j}$  (resp.  $V_{2,j}$ ) is concentrated along  $\frac{\pi}{2n} + j\frac{2\pi}{n}$  (resp.  $\frac{3\pi}{2n} + j\frac{2\pi}{n}$ ) and

$$(2.6) \quad \exp\left(\frac{1}{z^n}\right), \exp\left(-\frac{1}{z^n}\right) \in \mathcal{O}_{X_{sa}}^t(V_{k,j})$$

( $j = 0, \dots, n-1, k = 1, 2$ ).

As  $\eta$  has a zero of order 1 at 0, by Lemma 2.5 (i), there exists  $\tau_\eta \in \mathbb{R}$ , depending only on  $\arg(\eta(0)) = \frac{\arg(a_n)}{n}$ , such that, up to shrinking  $W$ ,

(i)  $\eta|_{\overline{V_{k,j}}}$  is injective and

(ii)  $\eta(V_{1,j})$  (resp.  $\eta(V_{2,j})$ ) is concentrated along  $\tau_\eta + \frac{\pi}{2n} + j\frac{2\pi}{n}$  (resp.  $\tau_\eta + \frac{3\pi}{2n} + j\frac{2\pi}{n}$ ),

( $j = 0, \dots, n-1$ ).

Consider the commutative triangle

$$\begin{array}{ccc} W \setminus \{0\} & \xrightarrow{\eta} & \mathbb{C}^\times \\ & \searrow \exp\left(\frac{1}{z^n}\right) & \downarrow \exp(\varphi(z)) \\ & & \mathbb{C}. \end{array}$$

By Proposition 1.12, we have that

$$\begin{aligned} &\exp(\varphi(z)) \in \mathcal{O}_{X_{sa}}^t(\eta(V_{k,j})) \\ &\left(\text{resp. } \exp(-\varphi(z)) \in \mathcal{O}_{X_{sa}}^t(\eta(V_{k,j}))\right) \end{aligned}$$

if and only if

$$\begin{aligned} &\exp(\varphi \circ \eta(z)) = \exp(1/z^n) \in \mathcal{O}_{X_{sa}}^t(V_{k,j}) \\ &\left(\text{resp. } \exp(-\varphi \circ \eta(z)) = \exp(-1/z^n) \in \mathcal{O}_{X_{sa}}^t(V_{k,j})\right) \end{aligned}$$

( $j = 0, \dots, n-1, k = 1, 2$ ). The conclusion follows from (2.6). □

*Remark 2.8.* — Recall the definition given in the end of Subsection 1.2 of the sheaf  $\mathcal{A}^{\leq 0}$  defined on  $S^1 \times \mathbb{R}_{\geq 0}$ , considered as the real blow-up at 0 of  $\mathbb{C}^\times$ . Let  $\tau \in \mathbb{R}$ ,  $U \in \text{Op}^c(X_{sa})$  concentrated along  $\tau$ , the set  $\{(\tau, 0)\} \cup U \subset S^1 \times \mathbb{R}_{\geq 0}$  is not open. Further if  $\exp(\varphi) \in \mathcal{A}_{(\tau,0)}^{\leq 0}$  then  $\exp(-\varphi) \notin \mathcal{A}_{(\tau,0)}^{\leq 0}$ .

We conclude this subsection with an easy lemma which will be useful in the next subsection. First, let us introduce some notation.

Given  $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ ,  $\varphi = \frac{\eta e^{i\tau}}{z^n} + \tilde{\varphi}$ , for  $\eta \in \mathbb{R}_{>0}$ ,  $n \in \mathbb{Z}_{>0}$ ,  $\tau \in \mathbb{R}$  and  $\tilde{\varphi} \in z^{-1}\mathbb{C}[z^{-1}]$ ,  $-v(\tilde{\varphi}) < n$ , set

$$I_\varphi := \left\{ \vartheta \in [0, 2\pi]; \cos(\tau - n\vartheta) < 0 \right\}.$$

In other words,  $I_\varphi$  is the support of  $\exp(\varphi)$  as a section of  $\mathcal{A}^{\leq 0}|_{S^1 \times \{0\}}$ .

Recall the definition of the sets  $U_{\varphi,A}$  given in (2.1).

LEMMA 2.9.

- (i) Let  $\varphi_1, \varphi_2 \in z^{-1}\mathbb{C}[z^{-1}]$ ,  $\varphi_j := \frac{\eta_j e^{i\tau_j}}{z^{n_j}} + \tilde{\varphi}_j$ , for  $\eta_j \in \mathbb{R}_{>0}$ ,  $n_j \in \mathbb{Z}_{>0}$ ,  $\tau_j \in \mathbb{R}$  and  $\tilde{\varphi}_j \in z^{-1}\mathbb{C}[z^{-1}]$ ,  $-v(\tilde{\varphi}_j) < n_j$ .  
If  $n_1 \neq n_2$  or  $\tau_1 \neq \tau_2$ , then  $I_{\varphi_1} \setminus \bar{I}_{\varphi_2} \neq \emptyset$  and  $I_{\varphi_2} \setminus \bar{I}_{\varphi_1} \neq \emptyset$ .
- (ii) Let  $\vartheta_0 \in [0, 2\pi[$  and  $\varphi \in z^{-1}\mathbb{C}[z^{-1}] \setminus \{0\}$ . If  $\vartheta_0 \in I_\varphi$ , then there exists an open sector  $S$  containing  $\vartheta_0$  such that, for any  $A \in \mathbb{R}_{>0}$ ,  $S \subset U_{\varphi,A}$ . In particular, for any  $U \in \text{Op}^c(X_{sa})$  concentrated along  $\vartheta_0$ ,  $\exp(\varphi) \in \mathcal{O}_{X_{sa}}^t(U)$ .
- (iii) Let  $\vartheta_0 \in [0, 2\pi[$  and  $\varphi \in z^{-1}\mathbb{C}[z^{-1}] \setminus \{0\}$ . If  $\vartheta_0 \notin \bar{I}_\varphi$ , then there exists an open sector  $S$  containing  $\vartheta_0$  such that, for any  $A \in \mathbb{R}_{>0}$ ,  $S \subset X \setminus U_{\varphi,A}$ . In particular, for any  $U \in \text{Op}^c(X_{sa})$  concentrated along  $\vartheta_0$ ,  $\exp(\varphi) \notin \mathcal{O}_{X_{sa}}^t(U)$ .

*Proof.* — The result follows from some easy computations. □

## 2.2. Comparison between growth of exponential functions

In this subsection we are going to use the results of the previous subsection in order to prove that if  $\varphi_1, \varphi_2 \in z^{-1}\mathbb{C}[z^{-1}]$  and, for any  $\lambda \in \mathbb{R}_{>0}$ ,  $\varphi_1 \neq \lambda\varphi_2$ , then the families  $\{U_{\varphi_1,A}\}_{A \in \mathbb{R}_{>0}}$  and  $\{U_{\varphi_2,A}\}_{A \in \mathbb{R}_{>0}}$  are not cofinal.

The main result of this subsection is Proposition 2.10 below.

PROPOSITION 2.10. — Let  $\varphi_1, \varphi_2 \in z^{-1}\mathbb{C}[z^{-1}] \setminus \{0\}$ .

- (i) Suppose that there exists  $\lambda \in \mathbb{R}_{>0}$  such that  $\varphi_1 = \lambda\varphi_2$ . Then, for any  $A \in \mathbb{R}_{>0}$ ,  $U_{\varphi_1,A} = U_{\varphi_2, \frac{A}{\lambda}}$ . In particular, for any  $U \in \text{Op}^c(X_{sa})$ ,  $\exp(\varphi_1) \in \mathcal{O}_{X_{sa}}^t(U)$  if and only if  $\exp(\varphi_2) \in \mathcal{O}_{X_{sa}}^t(U)$ .

- (ii) Suppose that, for any  $\lambda \in \mathbb{R}_{>0}$ ,  $\varphi_1 \neq \lambda\varphi_2$ . Then for any open sector  $S$  of amplitude  $> \frac{2\pi}{\max\{-v(\varphi_1), -v(\varphi_2), 2\}}$  at least one of the two conditions below is satisfied (resp. for any open neighborhood  $S$  of 0, both conditions below are satisfied).
  - (a) There exists  $U \in \text{Op}^c(X_{sa})$ ,  $U \subset S$  such that  $\exp(\varphi_1) \in \mathcal{O}_{X_{sa}}^t(U)$  and  $\exp(\varphi_2) \notin \mathcal{O}_{X_{sa}}^t(U)$ .
  - (b) There exists  $V \in \text{Op}^c(X_{sa})$ ,  $V \subset S$  such that  $\exp(\varphi_1) \notin \mathcal{O}_{X_{sa}}^t(V)$  and  $\exp(\varphi_2) \in \mathcal{O}_{X_{sa}}^t(V)$ .

*Proof.*

(i) Obvious.

(ii) For  $S$  an open sector, set  $\tilde{S} := \{\vartheta \in [0, 2\pi[; \exists r > 0 \text{ } re^{i\vartheta} \in S\}$ .

Let

$$\varphi_1(z) := \frac{\eta_1 e^{i\tau_1}}{z^{n_1}} + \tilde{\varphi}_1(z) \quad \text{and} \quad \varphi_2(z) := \frac{\eta_2 e^{i\tau_2}}{z^{n_2}} + \tilde{\varphi}_2(z),$$

for  $\eta_j \in \mathbb{R}_{>0}$ ,  $\tau_j \in [0, 2\pi[$  and  $\tilde{\varphi}_j(z) \in z^{-1}\mathbb{C}[z^{-1}]$ ,  $-v(\tilde{\varphi}_j) < n_j$  ( $j = 1, 2$ ).

Suppose that  $n_1 \neq n_2$  or  $\tau_1 \neq \tau_2$ .

By Lemma 2.9 (i),  $I_{\varphi_1} \setminus \bar{I}_{\varphi_2} \neq \emptyset$  and  $I_{\varphi_2} \setminus \bar{I}_{\varphi_1} \neq \emptyset$ .

Let  $S$  be an open neighborhood of 0,  $\vartheta_1 \in I_{\varphi_1} \setminus \bar{I}_{\varphi_2}$  and  $\vartheta_2 \in I_{\varphi_2} \setminus \bar{I}_{\varphi_1}$ . There exist  $U_k \in \text{Op}^c(X_{sa})$  concentrated along  $\vartheta_k$  such that  $U_k \subset S$  ( $k = 1, 2$ ). The result follows by Lemma 2.9 (ii),(iii).

Suppose that  $S$  is an open sector of amplitude  $> \frac{2\pi}{\max\{n_1, n_2, 2\}}$ . Then, there exists  $\vartheta \in \tilde{S}$  such that either  $\vartheta \in I_{\varphi_1} \setminus \bar{I}_{\varphi_2}$  or  $\vartheta \in I_{\varphi_2} \setminus \bar{I}_{\varphi_1}$ . Since  $\vartheta \in \tilde{S}$ , there exists  $U \in \text{Op}^c(X_{sa})$  concentrated along  $\vartheta$  such that  $U \subset S$ . The conclusion follows by Lemma 2.9 (ii),(iii).

Now suppose that  $n_1 = n_2 = n$  and  $\tau_1 = \tau_2$ . That is,

$$\varphi_1(z) = \frac{\eta_1 e^{i\tau_1}}{z^n} + \tilde{\varphi}_1(z) \quad \text{and} \quad \varphi_2(z) = \frac{\eta_2 e^{i\tau_1}}{z^n} + \tilde{\varphi}_2(z).$$

Since, for any  $\lambda \in \mathbb{R}_{>0}$ ,  $\varphi_1 \neq \lambda\varphi_2$ , we have that  $n \geq 2$ .

Set  $\psi_{21} := \varphi_2 - \frac{\eta_2}{\eta_1}\varphi_1$  and  $\psi_{12} := \varphi_1 - \frac{\eta_1}{\eta_2}\varphi_2$ . Since  $\psi_{21} \neq 0$  and  $\psi_{21} = -\frac{\eta_2}{\eta_1}(\varphi_1 - \frac{\eta_1}{\eta_2}\varphi_2) = -\frac{\eta_2}{\eta_1}\psi_{12}$ , then  $I_{\psi_{21}} = I_{-\psi_{12}}$ .

By Proposition 2.6, there exist  $\tau \in \mathbb{R}$  and  $U_0, \dots, U_{2n-1}, V_0, \dots, V_{2n-1} \in \text{Op}^c(X_{sa})$  satisfying the conditions

- (i)  $U_j, V_j$  are concentrated along  $\tau + j\frac{\pi}{n}$ ,
  - (ii)  $\exp(\varphi_1), \exp(-\varphi_1) \in \mathcal{O}_{X_{sa}}^t(U_j)$  and  $\exp(\varphi_2), \exp(-\varphi_2) \in \mathcal{O}_{X_{sa}}^t(V_j)$ ,
- ( $j = 0, \dots, 2n - 1$ ).

Since  $-v(\psi_{12}) = -v(\psi_{21}) < n$ , if  $S$  is an open sector of amplitude  $> \frac{2\pi}{n}$ , there exists  $j' \in \{0, \dots, 2n - 1\}$  such that  $\tau + j'\frac{\pi}{n} \in \tilde{S}$  and either  $\tau + j'\frac{\pi}{n} \notin \bar{I}_{\psi_{12}}$  or  $\tau + j'\frac{\pi}{n} \notin \bar{I}_{\psi_{21}}$ .

More generically,  $\{\tau + j\frac{\pi}{n}; j \in 0, \dots, 2n - 1\} \not\subset \bar{I}_{\psi_{12}}$  and  $\{\tau + j\frac{\pi}{n}; j \in 0, \dots, 2n - 1\} \not\subset \bar{I}_{\psi_{21}}$ .

Let us consider the case  $\tau + j'\frac{\pi}{n} \notin \bar{I}_{\psi_{12}}$ . Since  $V_{j'}$  is concentrated along  $\tau + j'\frac{\pi}{n}$ , Lemma 2.9 (iii) implies

$$(2.7) \quad \exp(\psi_{12}) \notin \mathcal{O}_{X_{sa}}^t(V_{j'}).$$

Suppose now that  $\exp(\varphi_1) \in \mathcal{O}_{X_{sa}}^t(V_{j'})$ . Since  $\exp(-\varphi_2), \exp(\varphi_2) \in \mathcal{O}_{X_{sa}}^t(V_{j'})$  and the product of tempered functions is tempered, we have that  $\exp(\varphi_1 - \frac{\eta_1}{\eta_2}\varphi_2) \in \mathcal{O}_{X_{sa}}^t(V_{j'})$ , which contradicts (2.7). Hence  $\exp(\varphi_1) \notin \mathcal{O}_{X_{sa}}^t(V_{j'})$ .

Let us consider the case  $\tau + j'\frac{\pi}{n} \notin \bar{I}_{\psi_{21}}$ . Since  $U_{j'}$  is concentrated along  $\tau + j'\frac{\pi}{n}$ , Lemma 2.9 (iii) implies

$$(2.8) \quad \exp(\psi_{21}) \notin \mathcal{O}_{X_{sa}}^t(U_{j'}).$$

Suppose now that  $\exp(\varphi_2) \in \mathcal{O}_{X_{sa}}^t(U_{j'})$ . Since  $\exp(\varphi_1), \exp(-\varphi_1) \in \mathcal{O}_{X_{sa}}^t(U_{j'})$  and the product of tempered functions is tempered, we have that  $\exp(\varphi_2 - \frac{\eta_2}{\eta_1}\varphi_1) \in \mathcal{O}_{X_{sa}}^t(U_{j'})$ , which contradicts (2.8). Hence  $\exp(\varphi_2) \notin \mathcal{O}_{X_{sa}}^t(U_{j'})$ . □

**COROLLARY 2.11.** — *Let  $l \in \mathbb{Z}_{>0}$   $\omega, \varphi_1, \varphi_2 \in z^{-1}\mathbb{C}[z^{-1}]$ , such that  $-v(\omega) > \max_{j=1,2} \{ \frac{-v(\varphi_j)}{l} + 1 \}$ . Let  $S$  an open sector of amplitude  $2\pi$ ,  $\zeta$  an inverse of  $z \mapsto z^l$  defined on  $S$ . The following conditions are equivalent.*

- (i)  $\varphi_1 \circ \zeta \neq \varphi_2 \circ \zeta$ .
- (ii) *At least one of the following two conditions is verified:*
  - (a) *there exists  $U \in \text{Op}^c(X_{sa}), U \subset S$  such that  $\exp(\varphi_1 \circ \zeta + \omega) \in \mathcal{O}_{X_{sa}}^t(U)$  and  $\exp(\varphi_2 \circ \zeta + \omega) \notin \mathcal{O}_{X_{sa}}^t(U)$ ;*
  - (b) *there exists  $V \in \text{Op}^c(X_{sa}), V \subset S$  such that  $\exp(\varphi_1 \circ \zeta + \omega) \notin \mathcal{O}_{X_{sa}}^t(V)$  and  $\exp(\varphi_2 \circ \zeta + \omega) \in \mathcal{O}_{X_{sa}}^t(V)$ .*

*Proof.*

(ii)  $\Rightarrow$  (i). Obvious.

(i)  $\Rightarrow$  (ii). Set  $\mu_l(z) := z^l$ .

Suppose now that  $\varphi_1 \circ \zeta \neq \varphi_2 \circ \zeta$ . It follows that, for any  $\lambda \in \mathbb{R}_{>0}$ ,  $\lambda(\varphi_1 + \omega \circ \mu_l) \neq \varphi_2 + \omega \circ \mu_l$ . Consider the sector  $\zeta(S)$  of amplitude  $\frac{2\pi}{l}$ . Since  $-v(\omega) \geq 2$ , then  $\frac{2\pi}{l} > -\frac{2\pi}{lv(\omega)}$ . Hence by Proposition 2.10 there exists  $\zeta(U) \subset \zeta(S)$  such that either  $\exp(\varphi_1 + \omega \circ \mu_l) \in \mathcal{O}_{X_{sa}}^t(\zeta(U))$  and  $\exp(\varphi_2 + \omega \circ \mu_l) \notin \mathcal{O}_{X_{sa}}^t(\zeta(U))$  or viceversa. By Proposition 1.12, it follows that either  $\exp(\varphi_1 \circ \zeta + \omega) \in \mathcal{O}_{X_{sa}}^t(U)$  and  $\exp(\varphi_2 \circ \zeta + \omega) \notin \mathcal{O}_{X_{sa}}^t(U)$  or viceversa. □

*Remark 2.12.* — There is another way to prove Proposition 2.10. Let us briefly summarize it.

Let  $\varphi_1, \varphi_2 \in z^{-1}\mathbb{C}[z^{-1}] \setminus \{0\}$ , such that, for any  $\lambda \in \mathbb{R}_{>0}$ ,  $\varphi_1 \neq \lambda\varphi_2$ . Let  $n_k = -v(\varphi_k)$  ( $k = 1, 2$ ).

Let  $C_{\varphi_k, A}$  be the boundary of the set  $U_{\varphi_k, A}$  ( $k = 1, 2$ ). One has that  $C_{\varphi_k, A}$  is the set of the zeros of a polynomial  $Q_{\varphi_k, A}(x, y) \in \mathbb{R}[x, y]$  ( $k = 1, 2$ ). Further  $C_{\varphi_k, A}$  has  $2n_k$  distinct branches at 0 determined by the Puiseux's series  $\sigma_{\varphi_k, A, 1}(x), \dots, \sigma_{\varphi_k, A, 2n_k}(x)$  obtained by solving the equation  $Q_{\varphi_k, A}(x, y) = 0$  ( $k = 1, 2$ ) with respect to  $y$ .

One checks that the first  $n_k$  terms of  $\sigma_{\varphi_k, A, j}(x)$  do not depend on  $A$  ( $k = 1, 2, j = 1, \dots, 2n_k$ ). Further, it turns out that there exists  $\vartheta_k \in [0, 2\pi[$  such that the tangent at 0 of the graph of  $\sigma_{\varphi_k, A, j}(x)$  has slope  $\tan(\vartheta_k + j\frac{\pi}{2n_k})$  ( $k = 1, 2, j = 1, \dots, 2n_k$ ).

If  $n_1 \neq n_2$  or  $\vartheta_1 \neq \vartheta_2$ , the result follows easily.

If  $n_1 = n_2$  and  $\vartheta_1 = \vartheta_2$ , one checks that there exist  $\bar{j} \in \{1, \dots, 2n_1\}$  and  $r \in \{1, \dots, n_1\}$  such that the  $r$ -th coefficients of  $\sigma_{\varphi_1, A, \bar{j}}(x)$  and  $\sigma_{\varphi_2, A, \bar{j}}(x)$  are different. Hence, there are infinitely many relatively compact subanalytic open sets concentrated along some  $\vartheta_1 + j\frac{\pi}{2n_1}$  fitting between  $\sigma_{\varphi_1, A, \bar{j}}(x)$  and  $\sigma_{\varphi_2, A, \bar{j}}(x)$ . Choosing  $U$  among these sets, one obtains that one exponential is tempered on  $U$  and the other is not.

This procedure is more intuitive than the proof we chose to expose here but it is more technical and much longer.

### 3. Tempered solutions and formal invariants of $\mathcal{D}$ -modules

In the first part of this section we are going to prove that the tempered solutions induce a fully faithful functor on good models. In the second part we will prove that the datum of tempered solutions of a meromorphic connection  $\mathcal{M}$  is equivalent to the data of determinant polynomials and holomorphic solutions of  $\mathcal{M}$ .

Let us recall that M. Kashiwara, in [8], proves that, given a complex analytic manifold  $X$  and an object  $\mathcal{M}$  of the bounded derived category of  $\mathcal{D}_X$ -modules with regular holonomic cohomology,

$$R\mathcal{H}om_{\varrho! \mathcal{D}_X}(\varrho! \mathcal{M}, \mathcal{O}_{X_{sa}}^t) \simeq R\mathcal{H}om_{\varrho! \mathcal{D}_X}(\varrho! \mathcal{M}, \mathcal{O}_X).$$

Given a complex analytic curve  $Y$ ,  $x_0 \in Y$ ,  $\mathcal{M} \in \text{Mod}_h(\mathcal{D}_{*x_0})$ , there exists a neighborhood  $X \subset Y$  of  $x_0$  such that  $\mathcal{M}$  is a holonomic  $\mathcal{D}_X$ -module. By choosing a local coordinate  $z$  near  $x_0$ , we can suppose that  $X = \mathbb{C}$  and  $x_0 = 0$ . Recall that for  $\omega \in z^{-1}\mathbb{C}[z^{-1}]$ , we set  $\mathcal{L}^\omega := \mathcal{D}_\mathbb{C} \exp(\omega)$ .

Set

$$\begin{aligned} \mathcal{S}(\mathcal{M}) &:= \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X), \\ \mathcal{S}^t(\mathcal{M}) &:= \mathcal{H}om_{\varrho_! \mathcal{D}_X}(\varrho_! \mathcal{M}, \mathcal{O}_{X_{sa}}^t), \\ \mathcal{S}_\omega^t(\mathcal{M}) &:= \mathcal{H}om_{\varrho_! \mathcal{D}_X}(\varrho_!(\mathcal{M} \otimes \mathcal{L}^\omega), \mathcal{O}_{X_{sa}}^t), \end{aligned}$$

the functors  $\varrho_*$  and  $\varrho_!$  being defined in Subsection 1.1.

### 3.1. Tempered solutions and good models

The main results of this subsection are Proposition 3.2 and Theorem 3.5 below. First, let us describe explicitly the subanalytic sheaf  $\mathcal{S}^t(\mathcal{L}^\varphi \otimes \mathcal{R})$ , for  $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$  and  $\mathcal{R}$  a regular holonomic  $\mathcal{D}_X(*0)$ -module.

Recall the definition of the sets  $U_{\varphi,A}$  given in (2.1).

LEMMA 3.1. — *Let  $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ ,  $\mathcal{R}$  a regular holonomic  $\mathcal{D}_X(*0)$ -module. Then*

$$\mathcal{S}^t(\mathcal{L}^\varphi \otimes \mathcal{R}) \simeq \varinjlim_{A>0} \varrho_* \mathcal{S}(\mathcal{R})_{U_{\varphi,A}}$$

*Proof.* — If  $\varphi = 0$ , the result follows from the fact that  $\mathcal{S}^t(\mathcal{R}) \simeq \mathcal{S}(\mathcal{R})$ .

Suppose  $\varphi \neq 0$ . Let  $V \in \text{Op}^c(X_{sa})$  be connected and simply connected. If  $0 \in V$ , then clearly  $\Gamma(V, \mathcal{S}^t(\mathcal{L}^\varphi \otimes \mathcal{R})) \simeq 0$ . Otherwise, the  $\mathbb{C}$ -vector space  $\Gamma(V, \mathcal{S}(\mathcal{L}^\varphi \otimes \mathcal{R}))$  has finite dimension  $r$  and is generated by  $h_1(z) \exp(\varphi(z)), \dots, h_r(z) \exp(\varphi(z))$ , for  $h_1, \dots, h_r \in \mathcal{O}_{\mathbb{C}}(V)$ , such that there exist  $C, M > 0$  satisfying

$$C|z|^M \leq |h_j(z)| \leq (C|z|^M)^{-1} \quad (z \in V, j = 1, \dots, r).$$

In particular, since  $\Gamma(V, \mathcal{S}^t(\mathcal{L}^\varphi \otimes \mathcal{R})) \simeq \Gamma(V, \mathcal{S}(\mathcal{L}^\varphi \otimes \mathcal{R})) \cap \mathcal{O}_{X_{sa}}^t(V)$  we have that

$$\Gamma(V, \mathcal{S}^t(\mathcal{L}^\varphi \otimes \mathcal{R})) \simeq \begin{cases} \Gamma(V, \mathcal{S}(\mathcal{L}^\varphi \otimes \mathcal{R})) & \text{if } \exp(\varphi) \in \mathcal{O}_{X_{sa}}^t(V) \\ 0 & \text{otherwise} \end{cases}.$$

The conclusion follows by Proposition 2.1. □

We can now state and proof

PROPOSITION 3.2. — *Let  $\varphi_1, \varphi_2 \in z^{-1}\mathbb{C}[z^{-1}]$ ,  $\varphi_2 \neq 0$ ,  $\mathcal{R}_1, \mathcal{R}_2$  regular holonomic  $\mathcal{D}_X(*0)$ -modules. If, for any  $\lambda \in \mathbb{R}_{>0}$ ,  $\varphi_1 \neq \lambda\varphi_2$*

$$\text{Hom}_{\mathbb{C}_{X_{sa}}}(\mathcal{S}^t(\mathcal{L}^{\varphi_1} \otimes \mathcal{R}_1), \mathcal{S}^t(\mathcal{L}^{\varphi_2} \otimes \mathcal{R}_2)) \simeq 0.$$

Otherwise,

$$(3.1) \quad \text{Hom}_{\mathbb{C}_{X_{sa}}}(\mathcal{S}^t(\mathcal{L}^{\varphi_1} \otimes \mathcal{R}_1), \mathcal{S}^t(\mathcal{L}^{\varphi_2} \otimes \mathcal{R}_2)) \simeq \text{Hom}_{\mathcal{D}_X(*0)}(\mathcal{R}_1, \mathcal{R}_2)$$

functorially in  $\mathcal{R}_1, \mathcal{R}_2$ .

*Proof.* — Suppose that, for any  $\lambda \in \mathbb{R}_{>0}$ ,  $\varphi_1 \neq \lambda\varphi_2$ . By Proposition 2.10 (ii), there exists  $W \in \text{Op}^c(X_{sa})$  such that

- (i) there exists  $A_0 > 0$  such that, for any  $A \geq A_0$ ,  $W \subset U_{\varphi_1, A}$ ,
- (ii) for any  $B > 0$ ,  $W \not\subset U_{\varphi_2, B}$ .

In particular, for any  $A > A_0$  and  $B > 0$ ,

$$(3.2) \quad U_{\varphi_1, A} \not\subset U_{\varphi_2, B}.$$

Combining (3.2) and the fact that  $\mathcal{S}(\mathcal{R}_1)$  and  $\mathcal{S}(\mathcal{R}_2)$  are locally constant sheaves on  $\mathbb{C}^\times$ , we obtain, for any  $A > A_0$  and  $B > 0$ ,

$$\text{Hom}_{\mathbb{C}_X}(\mathcal{S}(\mathcal{R}_1)_{U_{\varphi_1, A}}, \mathcal{S}(\mathcal{R}_2)_{U_{\varphi_2, B}}) = 0.$$

Now, using Lemma 3.1, we obtain

$$\begin{aligned} \text{Hom}_{\mathbb{C}_{X_{sa}}}(\mathcal{S}^t(\mathcal{L}^{\varphi_1} \otimes \mathcal{R}_1), \mathcal{S}^t(\mathcal{L}^{\varphi_2} \otimes \mathcal{R}_2)) \\ \simeq \lim_{\substack{\leftarrow \\ A > 0}} \lim_{\substack{\leftarrow \\ B > 0}} \text{Hom}_{\mathbb{C}_X}(\mathcal{S}(\mathcal{R}_1)_{U_{\varphi_1, A}}, \mathcal{S}(\mathcal{R}_2)_{U_{\varphi_2, B}}) \\ = 0. \end{aligned}$$

Suppose now there exists  $\lambda \in \mathbb{R}_{>0}$  such that  $\varphi_1 = \lambda\varphi_2$ . Then

$$(3.3) \quad U_{\varphi_1, \lambda A} = U_{\varphi_2, A}.$$

We need the following

LEMMA 3.3. — *Let  $\varphi \in z^{-1}\mathbb{C}[z^{-1}] \setminus \{0\}$ . There exists  $A_0 \in \mathbb{R}_{>0}$  such that for any  $A > A_0$ , the sets*

$$U_{\varphi, A} := \{z \in \mathbb{C}^\times; \text{Re } \varphi(z) < A\}$$

are homotopically equivalent to  $\mathbb{C}^\times$ .

*Proof.* — We prove the result in three steps:  $\varphi = 1/z$ ,  $\varphi = 1/z^n$  and  $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ .

First suppose that  $\varphi(z) = \frac{1}{z}$ . Then  $U_{\varphi, A}$  is the complementary of a closed disc and the result is obvious.

Suppose now that  $\varphi(z) = \frac{1}{z^n}$ , for some  $n \in \mathbb{Z}_{>0}$ . Let  $\mu_n : \mathbb{C} \rightarrow \mathbb{C}$ ,  $z \mapsto z^n$ . Then  $U_{\varphi, A} = \mu_n^{-1}(U_{\frac{1}{z}, A})$  and the conclusion follows.

Suppose now that  $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$  and  $-v(\varphi) = n$ . Mimicking the proof of Proposition 2.6, there exists a biholomorphism between neighborhoods of 0,  $\eta$ , such that  $\varphi(\eta(z)) = \frac{1}{z^n}$ . The conclusion follows.  $\square$



Let us conclude the proof of Proposition 3.2.

We have the following sequence of isomorphisms

$$\begin{aligned} \text{Hom}_{\mathbb{C}_{X_{sa}}}(\mathcal{S}^t(\mathcal{L}^{\varphi_1} \otimes \mathcal{R}_1), \mathcal{S}^t(\mathcal{L}^{\varphi_2} \otimes \mathcal{R}_2)) & \\ \simeq \varprojlim_{A>0} \varinjlim_{B>0} \text{Hom}_{\mathbb{C}_X}(\mathcal{S}(\mathcal{R}_1)_{U_{\varphi_1,A}}, \mathcal{S}(\mathcal{R}_2)_{U_{\varphi_2,B}}) & \\ \simeq \varprojlim_{A>0} \varinjlim_{B>0} \text{Hom}_{\mathbb{C}_X}(\mathcal{S}(\mathcal{R}_1)_{U_{\varphi_1,A}}, \mathcal{S}(\mathcal{R}_2)_{U_{\varphi_1,A}}) & \\ \simeq \varprojlim_{A>0} \varinjlim_{B>0} \text{Hom}_{\mathbb{C}_X}(\mathcal{S}(\mathcal{R}_1)_{\mathbb{C}^\times}, \mathcal{S}(\mathcal{R}_2)_{\mathbb{C}^\times}) & \\ \simeq \text{Hom}_{\mathcal{D}_X(*0)}(\mathcal{R}_1, \mathcal{R}_2), & \end{aligned}$$

where the first isomorphism follows from Lemma 3.1, the second from (3.3) and the third from Lemma 3.3.

The conclusion follows. □

We can now state the main results of this subsection.

**THEOREM 3.4.** — *Let  $\bigoplus_{\varphi \in \Sigma_1} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi$  and  $\bigoplus_{\psi \in \Sigma_2} \mathcal{L}^\psi \otimes \mathcal{P}_\psi$  be two good models. The following conditions are equivalent.*

- (i)  $\mathcal{S}^t\left(\bigoplus_{\varphi \in \Sigma_1} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi\right)_{X \setminus \{0\}} \simeq \mathcal{S}^t\left(\bigoplus_{\psi \in \Sigma_2} \mathcal{L}^\psi \otimes \mathcal{P}_\psi\right)_{X \setminus \{0\}}$ .
- (ii) *There exist  $\varphi_1, \dots, \varphi_d \in z^{-1}\mathbb{C}[z^{-1}]$  such that,*

$$\prod_{j=1}^d \mathbb{R}_{>0} \varphi_j = \mathbb{R}_{>0} \Sigma_1 = \mathbb{R}_{>0} \Sigma_2$$

and, for any  $j = 1, \dots, d$ ,

$$\bigoplus_{\varphi \in \Sigma_1 \cap \mathbb{R}_{>0} \varphi_j} \mathcal{R}_\varphi \simeq \bigoplus_{\psi \in \Sigma_2 \cap \mathbb{R}_{>0} \varphi_j} \mathcal{P}_\psi.$$

*Proof.*

(ii)  $\Rightarrow$  (i). Combining Proposition 3.2 and the fact that  $\mathcal{S}^t(\cdot)_{X \setminus \{0\}}$  is fully faithful on the category of regular holonomic  $\mathcal{D}_X(*0)$ -modules, we have that

$$\begin{aligned} \text{Hom}_{\mathbb{C}_{X_{sa}}}(\mathcal{S}^t\left(\bigoplus_{\varphi \in \Sigma_1} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi\right)_{X \setminus \{0\}}, \mathcal{S}^t\left(\bigoplus_{\psi \in \Sigma_2} \mathcal{L}^\psi \otimes \mathcal{P}_\psi\right)_{X \setminus \{0\}}) & \\ \simeq \bigoplus_{\varphi \in \Sigma_1} \bigoplus_{\psi \in \Sigma_2} \text{Hom}_{\mathbb{C}_{X_{sa}}}(\mathcal{S}^t(\mathcal{L}^\varphi \otimes \mathcal{R}_\varphi), \mathcal{S}^t(\mathcal{L}^\psi \otimes \mathcal{P}_\psi)) & \\ \simeq \bigoplus_{j=1}^d \bigoplus_{\varphi \in \Sigma_1 \cap \mathbb{R}_{>0} \varphi_j} \text{Hom}_{\mathbb{C}_{X_{sa}}}(\mathcal{S}^t(\mathcal{L}^\varphi \otimes \mathcal{R}_\varphi), \bigoplus_{\psi \in \Sigma_2 \cap \mathbb{R}_{>0} \varphi_j} \mathcal{S}^t(\mathcal{L}^\psi \otimes \mathcal{P}_\psi)) & \\ \simeq \bigoplus_{j=1}^d \text{Hom}_{\mathcal{D}_X(*0)}\left(\bigoplus_{\varphi \in \Sigma_1 \cap \mathbb{R}_{>0} \varphi_j} \mathcal{R}_\varphi, \bigoplus_{\psi \in \Sigma_2 \cap \mathbb{R}_{>0} \varphi_j} \mathcal{P}_\psi\right). & \end{aligned}$$

The functoriality of (3.1) allows to conclude.

(i)  $\Rightarrow$  (ii). First let us suppose that  $\mathbb{R}_{>0}\Sigma_1 \neq \mathbb{R}_{>0}\Sigma_2$ . Hence either  $\mathbb{R}_{>0}\Sigma_1 \not\subset \mathbb{R}_{>0}\Sigma_2$  or  $\mathbb{R}_{>0}\Sigma_2 \not\subset \mathbb{R}_{>0}\Sigma_1$ . Suppose the latter.

There exists  $\psi \in \Sigma_2$  such that for any  $\varphi \in \Sigma_1, \lambda \in \mathbb{R}_{>0}, \psi \neq \lambda\varphi$ . Suppose that  $\psi \neq 0$ . By Proposition 3.2, we have

$$\mathrm{Hom}_{\mathbb{C}_{X_{sa}}}(\mathcal{S}^t(\bigoplus_{\varphi \in \Sigma_1} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi)_{X \setminus \{0\}}, \mathcal{S}^t(\mathcal{L}^\psi \otimes \mathcal{P}_\psi)_{X \setminus \{0\}}) \simeq 0.$$

It follows that

$$\mathcal{S}^t(\bigoplus_{\varphi \in \Sigma_1} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi)_{X \setminus \{0\}} \not\cong \mathcal{S}^t(\bigoplus_{\psi \in \Sigma_2} \mathcal{L}^\psi \otimes \mathcal{P}_\psi)_{X \setminus \{0\}}.$$

Suppose that  $\psi = 0$ , then  $0 \notin \Sigma_1$  and, by Proposition 3.2,

$$\mathrm{Hom}_{\mathbb{C}_{X_{sa}}}(\mathcal{S}^t(\mathcal{L}^\psi \otimes \mathcal{P}_\psi)_{X \setminus \{0\}}, \mathcal{S}^t(\bigoplus_{\varphi \in \Sigma_1} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi)_{X \setminus \{0\}}) \simeq 0.$$

It follows that

$$\mathcal{S}^t(\bigoplus_{\varphi \in \Sigma_1} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi)_{X \setminus \{0\}} \not\cong \mathcal{S}^t(\bigoplus_{\psi \in \Sigma_2} \mathcal{L}^\psi \otimes \mathcal{P}_\psi)_{X \setminus \{0\}}.$$

The case  $\mathbb{R}_{>0}\Sigma_1 \not\subset \mathbb{R}_{>0}\Sigma_2$  is treated similarly.

Now let us suppose that  $\mathbb{R}_{>0}\Sigma_1 = \mathbb{R}_{>0}\Sigma_2 = \prod_{j=1}^d \mathbb{R}_{>0}\varphi_j$  and, there exists  $j' \in \{1, \dots, d\}$ , such that

$$\bigoplus_{\varphi \in \Sigma_1 \cap \mathbb{R}_{>0}\varphi_{j'}} \mathcal{R}_\varphi \not\cong \bigoplus_{\psi \in \Sigma_2 \cap \mathbb{R}_{>0}\varphi_{j'}} \mathcal{P}_\psi.$$

By Proposition 3.2 we have that

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}_{X_{sa}}}(\mathcal{S}^t(\bigoplus_{\varphi \in \Sigma_1} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi)_{X \setminus \{0\}}, \mathcal{S}^t(\bigoplus_{\psi \in \Sigma_2} \mathcal{L}^\psi \otimes \mathcal{P}_\psi)_{X \setminus \{0\}}) \\ \simeq \bigoplus_{j=1}^d \mathrm{Hom}_{\mathcal{D}_X(*0)}(\bigoplus_{\varphi \in \Sigma_1 \cap \mathbb{R}_{>0}\varphi_j} \mathcal{R}_\varphi, \bigoplus_{\psi \in \Sigma_2 \cap \mathbb{R}_{>0}\varphi_j} \mathcal{P}_\psi). \end{aligned}$$

The functoriality of (3.1) allows to conclude. □

We conclude the study of tempered solutions of good models with

**THEOREM 3.5.** — *Let  $\omega \in z^{-1}\mathbb{C}[z^{-1}]$ ,  $-v(\omega) \geq k$ . The functor*

$$\begin{aligned} \mathcal{S}_\omega^t(\cdot): \quad \mathrm{GM}_k &\longrightarrow \mathrm{Mod}(\mathbb{C}_{X_{sa}}) \\ \mathcal{M} &\longmapsto \mathcal{H}om_{\varrho_! \mathcal{D}_X}(\varrho_!(\mathcal{M} \otimes \mathcal{L}^\omega), \mathcal{O}_{X_{sa}}^t), \end{aligned}$$

*is fully faithful.*

*Proof.* — Clearly it is sufficient to prove that, given  $\varphi_1, \varphi_2 \in z^{-1}\mathbb{C}[z^{-1}]$ ,  $k > \max\{-v(\varphi_1), -v(\varphi_2)\}$ ,  $\mathcal{R}_1, \mathcal{R}_2$  regular holonomic  $\mathcal{D}_X(*0)$ -modules, the functor of tempered solutions induces the isomorphism

$$(3.4) \quad \text{Hom}_{\mathcal{D}_X(*0)}(\mathcal{L}^{\varphi_1} \otimes \mathcal{R}_1, \mathcal{L}^{\varphi_2} \otimes \mathcal{R}_2) \simeq \text{Hom}_{\mathbb{C}_{X_{sa}}}(\mathcal{S}_\omega^t(\mathcal{L}^{\varphi_1} \otimes \mathcal{R}_1), \mathcal{S}_\omega^t(\mathcal{L}^{\varphi_2} \otimes \mathcal{R}_2)).$$

Let us prove (3.4).

First, suppose that  $\varphi_1 \neq \varphi_2$ . Then

$$\text{Hom}_{\mathcal{D}_X(*0)}(\mathcal{L}^{\varphi_1} \otimes \mathcal{R}_1, \mathcal{L}^{\varphi_2} \otimes \mathcal{R}_2) = 0.$$

Moreover, as, for any  $\lambda \in \mathbb{R}_{>0}$ ,  $\lambda(\varphi_1 + \omega) \neq \varphi_2 + \omega$ , Proposition 3.2 implies that,

$$\text{Hom}_{\mathbb{C}_{X_{sa}}}(\mathcal{S}_\omega^t(\mathcal{L}^{\varphi_1} \otimes \mathcal{R}_1), \mathcal{S}_\omega^t(\mathcal{L}^{\varphi_2} \otimes \mathcal{R}_2)) = 0.$$

Now, suppose that  $\varphi_1 = \varphi_2$ . The result follows from Proposition 3.2 and the fact that

$$\text{Hom}_{\mathcal{D}_X(*0)}(\mathcal{R}_1, \mathcal{R}_2) \simeq \text{Hom}_{\mathcal{D}_X(*0)}(\mathcal{L}^{\varphi_1} \otimes \mathcal{R}_1, \mathcal{L}^{\varphi_2} \otimes \mathcal{R}_2).$$

□

### 3.2. Tempered solutions of ordinary differential equations

We begin this subsection by proving the analogue of Lemma 3.1 in the case of ramified determinant polynomials *i.e.* with non-integer exponents.

Recall that, for  $Y \subset X$ ,  $Y_{X_{sa}}$  is the subanalytic site on  $Y$  induced by  $X_{sa}$ , in particular the open sets of  $Y_{X_{sa}}$  are of the form  $U \cap Y$  for  $U \in \text{Op}^c(X_{sa})$ . For  $\mathcal{F} \in \text{Mod}(k_{X_{sa}})$ , we denote by  $\mathcal{F}|_Y$  the restriction of  $\mathcal{F}$  to  $Y_{X_{sa}}$ .

Recall the definitions of  $\Omega(\mathcal{M})$  and  $r_{\varphi, \mathcal{M}}$  (resp.  $U_{\varphi, \epsilon}$ ) given in Definition 1.16 (resp. Corollary 2.3).

LEMMA 3.6. — *Let  $\mathcal{M} \in \text{Mod}_h(\mathcal{D}_X(*0))$ ,  $\vartheta \in \mathbb{R}$ ,  $Y := X \setminus (\mathbb{R}_{\geq 0}e^{i\vartheta})$ . Then*

$$\mathcal{S}^t(\mathcal{M})|_Y \simeq \bigoplus_{\varphi \in \Omega(\mathcal{M})} \varinjlim_{\epsilon > 0} \varrho_* \mathbb{C}_{Y, U_{\varphi, \epsilon}}^{r_{\varphi, \mathcal{M}}}.$$

*Proof.* — As  $\mathcal{M}$  is fixed, for sake of simplicity, we drop the index  $\mathcal{M}$  in the symbol  $r_{\varphi, \mathcal{M}}$ .

Let  $V \in \text{Op}(Y_{X_{sa}})$  connected. By the Hukuhara-Turrittin’s Asymptotic Theorem 1.15, the  $\mathbb{C}$ -vector space  $\mathcal{S}(\mathcal{M})(V) \subset \mathcal{O}(V)$  is generated by  $\{h_{\varphi, j} \exp(\varphi)\}_{\substack{\varphi \in \Omega(\mathcal{M}) \\ j \in \{1, \dots, r_\varphi\}}}$ .

Hence

$$\begin{aligned} \mathcal{S}^t(\mathcal{M})(V) &\simeq \mathcal{S}(\mathcal{M})(V) \cap \mathcal{O}^t(V) \\ &\simeq \left\{ \sum_{\varphi \in \Omega(\mathcal{M})} \sum_{j=1}^{r_\varphi} c_{\varphi,j} h_{\varphi,j} \exp(\varphi) \in \mathcal{O}^t(V); c_{\varphi,j} \in \mathbb{C} \right\}. \end{aligned}$$

Since, for  $\varphi \in \Omega(\mathcal{M}), j \in \{1, \dots, r_\varphi\}, h_{\varphi,j} \exp(\varphi)$  are  $\mathbb{C}$ -linearly independent functions and  $h_{\varphi,j}, h_{\varphi,j}^{-1} \in \mathcal{O}^t(V)$ , one has that

$$\sum_{\varphi \in \Omega(\mathcal{M})} \sum_{j=1}^{r_\varphi} c_{\varphi,j} h_{\varphi,j} \exp(\varphi) \in \mathcal{O}^t(V)$$

if and only if  $\exp(\varphi) \in \mathcal{O}^t(V)$  for  $c_{\varphi,j} \neq 0$ .

The conclusion follows. □

Theorem 3.7 below states that the tempered solutions of a meromorphic connection  $\mathcal{M}$  encode the determinant polynomials, their rank and the holomorphic solutions of  $\mathcal{M}$ . Hence, from tempered holomorphic solutions of  $\mathcal{M}$ , one can't recover the Stokes coefficients (see [1] or [14] for the definition) or the formal monodromy (i.e. the monodromy of the holomorphic solutions of the good model formally isomorphic to  $\mathcal{M}$ ), which are essential invariants characterizing  $\mathcal{M}$ . In particular, tempered solutions can't give a fully faithful functor on  $\text{Mod}_h(\mathcal{D}_X(*0))_k$ .

**THEOREM 3.7.** — *Let  $k \in \mathbb{Z}_{>0}, \mathcal{M}_1, \mathcal{M}_2 \in \text{Mod}_h(\mathcal{D}_X(*0))_k$  and  $\omega \in z^{-1}\mathbb{C}[z^{-1}]$  such that  $-v(\omega) > k$ . The following conditions are equivalent.*

- (i)  $\mathcal{S}_\omega^t(\mathcal{M}_1)_{X \setminus \{0\}} \simeq \mathcal{S}_\omega^t(\mathcal{M}_2)_{X \setminus \{0\}}$ .
- (ii) (a)  $\mathcal{S}(\mathcal{M}_1)|_{X \setminus \{0\}} \simeq \mathcal{S}(\mathcal{M}_2)|_{X \setminus \{0\}}$  and  
 (b) for any  $\vartheta \in S^1, \mathcal{S}^\Omega(\mathcal{M}_1)_\vartheta \simeq \mathcal{S}^\Omega(\mathcal{M}_2)_\vartheta$  as  $\Omega_\vartheta$ -graded  $\mathbb{C}$ -vector spaces.

*Proof.*

(i)  $\Rightarrow$  (ii). Since  $\varrho^{-1}\mathcal{S}^t(\mathcal{M}_j)_{X \setminus \{0\}} \simeq \mathcal{S}(\mathcal{M}_j)_{X \setminus \{0\}}$ , the condition (a) is proved.

Suppose now that there exists  $\vartheta \in S^1$  such that  $\mathcal{S}^\Omega(\mathcal{M}_1)_\vartheta \not\simeq \mathcal{S}^\Omega(\mathcal{M}_2)_\vartheta$ . Then, either  $\Omega(\mathcal{M}_1) \neq \Omega(\mathcal{M}_2)$  or there exists  $\varphi \in \Omega(\mathcal{M}_1) \cap \Omega(\mathcal{M}_2)$  such that  $r_{\varphi, \mathcal{M}_1} \neq r_{\varphi, \mathcal{M}_2}$ . In the former case, combining the ideas of the first part of the proof of Proposition 3.2 with Lemma 3.6 and Corollaries 2.3 and 2.11, we obtain that, for any  $\vartheta \in \mathbb{R}, \mathcal{S}_\omega^t(\mathcal{M}_1)|_{X \setminus \mathbb{R}_{\geq 0}e^{i\vartheta}} \not\simeq \mathcal{S}_\omega^t(\mathcal{M}_2)|_{X \setminus \mathbb{R}_{\geq 0}e^{i\vartheta}}$ . In the latter case the result follows easily from Lemma 3.6.

(ii)  $\Rightarrow$  (i). Set  $\mathcal{S}_\omega(\cdot) := \mathcal{S}(\cdot \otimes \mathcal{L}^\omega)$ .

Let  $\vartheta_1, \vartheta_2 \in \mathbb{R}, \vartheta_1 \neq \vartheta_2 \pmod{2\pi}, Y_j := X \setminus \mathbb{R}_{\geq 0}e^{i\vartheta_j} (j = 1, 2)$ .

Since for any  $\vartheta \in S^1$ ,  $\mathcal{S}^\Omega(\mathcal{M}_1)|_{\vartheta} \simeq \mathcal{S}^\Omega(\mathcal{M}_2)|_{\vartheta}$ , then  $\Omega(\mathcal{M}_1) = \Omega(\mathcal{M}_2)$  and  $r_{\varphi, \mathcal{M}_1} = r_{\varphi, \mathcal{M}_2}$ . In particular, Lemma 3.6 implies that

$$\mathcal{S}_\omega^t(\mathcal{M}_1)|_{Y_1} \simeq \mathcal{S}_\omega^t(\mathcal{M}_2)|_{Y_1} \quad \text{and} \quad \mathcal{S}_\omega^t(\mathcal{M}_1)|_{Y_2} \simeq \mathcal{S}_\omega^t(\mathcal{M}_2)|_{Y_2}.$$

Now, we have that  $\mathcal{S}(\mathcal{M}_1)_{X \setminus \{0\}} \simeq \mathcal{S}(\mathcal{M}_2)_{X \setminus \{0\}}$  implies

$$\mathcal{S}_\omega(\mathcal{M}_1)_{X \setminus \{0\}} \simeq \mathcal{S}_\omega(\mathcal{M}_2)_{X \setminus \{0\}}.$$

We conclude thanks to the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}_\omega^t(\mathcal{M}_j)_{X \setminus \{0\}} & \longrightarrow & \mathcal{S}_\omega^t(\mathcal{M}_j)_{Y_1} \oplus \mathcal{S}_\omega^t(\mathcal{M}_j)_{Y_2} & \longrightarrow & \mathcal{S}_\omega^t(\mathcal{M}_j)_{Y_1 \cap Y_2} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{S}_\omega(\mathcal{M}_j)_{X \setminus \{0\}} & \longrightarrow & \mathcal{S}_\omega(\mathcal{M}_j)_{Y_1} \oplus \mathcal{S}_\omega(\mathcal{M}_j)_{Y_2} & \longrightarrow & \mathcal{S}_\omega(\mathcal{M}_j)_{Y_1 \cap Y_2} & \longrightarrow & 0. \end{array}$$

□

## BIBLIOGRAPHY

- [1] D. G. BABBITT & V. S. VARADARAJAN, “Local moduli for meromorphic differential equations”, *Astérisque* (1989), no. 169-170, p. 217.
- [2] E. BIERSTONE & P. D. MILMAN, “Semianalytic and subanalytic sets”, *Inst. Hautes Études Sci. Publ. Math.* (1988), no. 67, p. 5-42.
- [3] J. BOCHNAK, M. COSTE & M.-F. ROY, *Real algebraic geometry*, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)* [Results in Mathematics and Related Areas (3)], vol. 36, Springer-Verlag, Berlin, 1998, Translated from the 1987 French original, Revised by the authors, x+430 pages.
- [4] A. CHENCINER, *Courbes algébriques planes*, Publications Mathématiques de l’Université Paris VII [Mathematical Publications of the University of Paris VII], vol. 4, Université de Paris VII U.E.R. de Mathématiques, Paris, 1978, 203 pages.
- [5] M. COSTE, *An introduction to o-minimal geometry*, Istituti Ed. e Poligrafici Intern., Università di Pisa, Dipartimento di Matematica, Pisa, 2000.
- [6] P. DELIGNE, B. MALGRANGE & J.-P. RAMIS, *Singularités irrégulières*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 5, Société Mathématique de France, Paris, 2007, Correspondance et documents. [Correspondence and documents], xii+188 pages.
- [7] M. KASHIWARA, “Faisceaux constructibles et systèmes holonomes d’équations aux dérivées partielles linéaires à points singuliers réguliers”, in *Séminaire Goulaouic-Schwartz, 1979–1980 (French)*, École Polytechnique, Palaiseau, 1980, Exp. No. 19, p. 7.
- [8] M. KASHIWARA, “The Riemann-Hilbert problem for holonomic systems”, *Publ. Res. Inst. Math. Sci.* **20** (1984), no. 2, p. 319-365.
- [9] ———, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, RI, 2003, Translated from the 2000 Japanese original by Mutsumi Saito, Iwanami Series in Modern Mathematics, xvi+254 pages.
- [10] M. KASHIWARA & P. SCHAPIRA, “Ind-sheaves”, *Astérisque* (2001), no. 271, p. 136.
- [11] ———, “Microlocal study of ind-sheaves. I. Micro-support and regularity”, *Astérisque* (2003), no. 284, p. 143-164, *Autour de l’analyse microlocale*.

- [12] M. LODAY-RICHAUD & G. POURCIN, “On index theorems for linear ordinary differential operators”, *Ann. Inst. Fourier (Grenoble)* **47** (1997), no. 5, p. 1379-1424.
- [13] S. LOJASIEWICZ, “Sur le problème de la division”, *Studia Math.* **18** (1959), p. 87-136.
- [14] B. MALGRANGE, *Équations différentielles à coefficients polynomiaux*, Progress in Mathematics, vol. 96, Birkhäuser Boston Inc., Boston, MA, 1991, vi+232 pages.
- [15] T. MOCHIZUKI, “Good formal structure for meromorphic flat connections on smooth projective surfaces”, arXiv:math.AG/0803.1346.
- [16] ———, “Wild harmonic bundles and wild pure twistor  $D$ -modules”, arXiv:math.DG/0803.1344.
- [17] G. MORANDO, “An existence theorem for tempered solutions of  $\mathcal{D}$ -modules on complex curves”, *Publ. Res. Inst. Math. Sci.* **43** (2007), no. 3, p. 625-659.
- [18] L. PRELLI, “Microlocalization of subanalytic sheaves”, *C. R. Math. Acad. Sci. Paris* **345** (2007), no. 3, p. 127-132.
- [19] C. SABBAH, “Équations différentielles à points singuliers irréguliers en dimension 2”, *Ann. Inst. Fourier (Grenoble)* **43** (1993), no. 5, p. 1619-1688.
- [20] ———, “Équations différentielles à points singuliers irréguliers et phénomène de Stokes en dimension 2”, *Astérisque* (2000), no. 263, p. viii+190.
- [21] ———, *Déformations isomonodromiques et variétés de Frobenius*, Savoirs Actuels (Les Ulis). [Current Scholarship (Les Ulis)], EDP Sciences, Les Ulis, 2002, , Mathématiques (Les Ulis). [Mathematics (Les Ulis)], xvi+289 pages.
- [22] W. WASOW, *Asymptotic expansions for ordinary differential equations*, Dover Publications Inc., New York, 1987, Reprint of the 1976 edition, x+374 pages.

Manuscrit reçu le 13 décembre 2007,  
 accepté le 4 avril 2008.

Giovanni MORANDO  
 Università di Padova  
 Dipartimento di Matematica Pura ed Applicata  
 Via Trieste 63  
 35121 Padova (Italy)  
 and  
 Universidade de Lisboa  
 Centro de Álgebra  
 Av. Prof. Gama Pinto, 2  
 1649-003 Lisboa (Portugal)  
 gmorando@math.unipd.it