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FATOU-NAIM-DOOB LIMIT THEOREMS IN THE AXIOMATIC SYSTEM OF BRELOT

by Kohur GOWRISANKARAN

1. Introduction.

Let Ω be a locally compact Hausdorff space which is connected and has a countable base. Let $\mathfrak{H}$ be a class of real valued continuous functions, called harmonic functions, on open subsets of Ω such that for each open set $W \subset \Omega$, the set $\mathfrak{H}_W$, consisting of all functions in $\mathfrak{H}$ defined on W, is a real vector space. Let this class $\mathfrak{H}$ satisfy the axioms 1, 2 and 3 of M. Brelot [1]. Let, moreover, there exist a potential > 0 on Ω.

The classical Fatou-Naïm-Doob limit theorems were extended to the axiomatic system of M. Brelot in [2]. But, besides the above mentioned axioms, we had assumed the validity of axioms D and $\mathfrak{R}_u$ [2]. The object of this paper is to show that the Fatou-Naim-Doob limit theorems (cf. Theorem 8) hold good in the axiomatic set up without these supplementary axioms (viz. D and $\mathfrak{R}_u$). The method consists in proving first, the limit theorems for a special class of superharmonic functions (cf. Theorem 4), and using it systematically to prove the general result. A novel feature in our proof is the consideration of a modified Dirichlet problem. We shall mostly follow the notation of [1, 2, 3].

Notation.

S^+ : The set of all non-negative superharmonic functions on Ω.

H^+: The set of all non-negative harmonic functions on Ω.

Λ : A compact base for S^+ (compact in the T-topology [3]).

Δ_1 : The set of minimal harmonic functions contained in Λ.

For any $E \subset \Omega$ and $v \in S^+$,

$$R_v^E = \operatorname{Inf}\{w : w \in S^+ \text{ and } w \geqslant v \text{ on } E\}.$$

For any $h \in \Delta_1$, $\mathcal{F}_h = \{E \subset \Omega : R_h^{\complement E} \not\equiv h\}$ [2]. A set E is thin at $h \in \Delta_1$ if $R_h^E \not\equiv h$ (i.e. if $\mathcal{F}_h$ leaves no trace on E).

The limits of any function f following $\mathcal{F}_h$, for any $h \in \Delta_1$, are called the fine limits of $f(x)$, as x tends to h. To every harmonic function $w \in H^+$ corresponds a unique measure μ_w on Δ_1, called the canonical measure corresponding to w, such that $w = \int h\, \mu_w(dh)$. For any regular domain $\delta \subset \Omega$, and $x \in \delta$, $d\rho_x^\delta$ is the measure on $\partial\delta$ which associates to a finite continuous function f on $\partial\delta$ the integral $H_f(x)$. For the considerations below, let us fix a $u \in H^+$ with $u > 0$. Let μ_u be the canonical measure on Δ_1, corresponding to u. A function v on Ω is said to be super-u-harmonic (respectively-u-harmonic) if uv is superharmonic (resp. harmonic) on Ω.

2. Fine limits of bounded u-harmonic functions.

LEMMA 1. — *Let* $V \subset \Omega$ *be an open set. Then, for every* $x \in \Omega$, *the mapping* $h \to R_h^V(x)$ *of* $H^+ \to \mathbf{R}^+$ *is lower semi-continuous.*

Proof. — Let $h_n \in H^+$ converge to $h \in H^+$. Let $v_n = R_{h_n}^V$. Then, v_n is a non-negative superharmonic function on Ω and $v_n = h_n$ on V. Let $v = \liminf_{n\to\infty} v_n$. Let ω be a regular domain of Ω. Then,

$$v_n(y) \geqslant \int v_n(z)\rho_y^\omega(dz) \quad \text{for all } y \in \omega.$$

Hence,

$$v(y) = \liminf_{n\to\infty} v_n(y) \geqslant \liminf_{n\to\infty} \int v_n(z)\rho_y^\omega(dz)$$
$$\geqslant \int v(z)\rho_y^\omega(dz) \text{ (Fatou's Lemma).}$$

(Note here that v is a ρ_x^ω-measurable function.) Since v is also non-negative, it follows that v is an $S_{\mathcal{B}}$-function, where $\mathcal{B}$ is the class of all regular domains of Ω [1]. Hence, $\hat{v}$, the lower semi-continuous regularisation of v, is a superharmonic function. But $v(y) = h(y)$, for all $y \in V$, and hence $\hat{v} = h$ on V.

It follows that $v \geqslant \hat{v} \geqslant R_h^V$ on Ω. This gives the required lower semi-continuity.

COROLLARY. — *For any regular domain δ of Ω and all $x \in \delta$, the function $h \to \int R_h^V(z)\, \rho_x^\delta(dz)$ is lower semi-continuous on* H^+.

The corollary follows from the lemma by the use of Fatou's lemma.

LEMMA 2. — *The set $\mathcal{E}_V$ of points of Δ_1, where an open set $V \subset \Omega$ is thin, is a borel subset of Δ_1.*

Proof. — Let $\{\delta_n\}$ be a countable covering of Ω by regular domains. Let, for each n, $x_n \in \delta_n$. Define,

$$F'_n = \left\{ h \in \Lambda \cap H^+ : \int R_h^V(y)\, \rho_{x_n}^{\delta_n}(dy) < h(x_n) \right\}.$$

In view of the above lemma and its corollary, F'_n is a borel subset of Λ (in fact, a K_σ — set). Hence, $F_n = F'_n \cap \Delta_1$ is a borel subset of Δ_1. It can be proved as in [2], that $\bigcup_{n=1}^{\infty} F_n$ is precisely the set $\mathcal{E}_V$. The lemma is proved.

THEOREM 1. — *Let $V \subset \Omega$ be any open set. Then $R_u^V \equiv u$ if and only if $\mu_u(\mathcal{E}_V) = 0$.*

Proof. — Let $\mu_u(\mathcal{E}_V) = 0$. For any $x \in \Omega$, we have,

$$R_u^V(x) = \int R_h^V(x) \mu_u(dh) \quad \text{(Th. 22.3, [3])}.$$

Since $R_h^V(x) = h(x)$, for all $h \in \Delta_1 - \mathcal{E}_V$, and $\mu_u(\mathcal{E}_V) = 0$, we get,

$$R_u^V(x) = \int h(x)\, \mu_u(dh) = u(x).$$

This is true whatever be $x \in \Omega$.

Conversely, suppose that $R_u^V \equiv u$. Let $\{\delta_n\}$ be a sequence covering Ω, each δ_n being a regular domain, and consider the sets $F_n \subset \Delta_1$, as defined in the above lemma.

Let ν_k be the swept-out measure corresponding to the measure $d\rho_{x_k}^{\delta_k}$ relative to the sweeping out on V. (Th. 10. 1, [3]). (Note that $d\rho_{x_k}^{\delta_k}$ is with the compact support $\partial\delta_k$). The measure ν_k is such that, for any $v \in S^+$,

$$\int v(y)\, \nu_k(dy) = \int R_v^V(y)\, \rho_{x_k}^{\delta_k}(dy).$$

We have,

$$\int R_u^V(y)\,\rho_{x_k}^{\delta_k}(dy) = \int u(z)\,\nu_k(dz) = \int \nu_k(dz) \int h(z)\,\mu_u(dh)$$
$$= \int \mu_u(dh) \int h(z)\,\nu_k(dz) = \int \mu_u(dh) \int R_h^V(y)\,\rho_{x_k}^{\delta_k}(dy) \ldots (1)$$

(Lebesgue-Fubini Theorem).
Now,

$$\int h(x_k)\,\mu_u(dh) = u(x_k) = \int R_u^V(y)\,\rho_{x_k}^{\delta_k}(dy) \quad \text{(hypothesis)}$$
$$= \int \mu_u(dh) \int R_h^V(y)\,\rho_{x_k}^{\delta_k}(dy) \quad \text{(from (1))}.$$

It follows that,

$$\int \left[h(x_k) - \int R_h^V(y)\,\rho_{x_k}^{\delta_k}(dy)\right] \mu_u(dh) = 0 \ldots (2).$$

Since the integrand in the above equation is always $\geqslant 0$, we get, $h(x_k) = \int R_h^V(y)\,\rho_{x_k}^{\delta_k}(dy)$, for all $h \in \Delta_1$, except for a set of μ_u-measure zero. But the exceptional set where the inequality does not hold good is precisely F_k. Hence,

$$\mu_u(F_k) = 0.$$

It follows, from the above lemma, that $\mu_u(\mathcal{E}_V) = 0$. The theorem is proved.

Corollary. — *The greatest harmonic minorant of* R_u^V *is the function* $\int h\mu_u^V(dh)$ *where* μ_u^V *is the restriction of* μ_u *to* $\Delta_1 - \mathcal{E}_V$. *Hence,* R_u^V *is a potential if* V *is thin* μ_u*-almost everywhere on* Δ_1.

The proof of the corollary is exactly as in (Cor. Th. II. 2, [2]).

Theorem 2. — *Let* $w > 0$ *be a potential on* Ω. *Then* $\frac{w}{u}$ *has the fine limit zero, at* μ_u*-almost every element of* Δ_1.

Proof. — It is enough to show that, for every rational number $r > 0$, the set $V_r = \left\{x \in \Omega : \frac{w(x)}{u(x)} > r\right\}$ is thin μ_u-almost everywhere. But, since $\frac{w}{u}$ is a lower semi-continuous function, V_r is an open subset of Ω. Further, $R_u^{V_r} \leqslant \frac{w}{r}$ Hence $R_u^{V_r}$ is a potential and it follows (Cor. to Theorem 1) that V_r is thin at μ_u-almost every element of Δ_1. This is true for every $r > 0$. The proof is completed easily.

The following result is an important corollary of the above theorem.

THEOREM 3. — *Let v and w be two non-negative harmonic functions on Ω such that their canonical measures μ_v and μ_w on Δ_1 are singular relative to each other. If, $v > 0$ on Ω, then, $\frac{w}{v}$ has the fine limit zero, at μ_v-almost every element of Δ_1.*

Proof. — Let $v' = \operatorname{Inf}(v, w)$. Then it is clear that v' is a potential on Ω. By the above theorem, we can find a set $E \subset \Delta_1$ of μ_u-measure zero such that, for every $h \in \Delta_1 - E$, $\underset{x \to h}{\text{fine lim}} \frac{v'(x)}{v(x)} = 0$. From this we easily deduce that, the $\underset{x \to h}{\text{fine lim}} \frac{w(x)}{v(x)} = 0$, for every $h \in \Delta_1 - E$. This completes the proof.

THEOREM 4. — *Let w be a bounded u-harmonic function on Ω. Then, w has a fine limit at μ_u-almost every element of Δ_1.*

Proof. — Define, for a μ_u-summable function f on Δ_1,

$$\sigma_f = \int f(h) \frac{h}{u} \mu_u (dh).$$

For the characteristic function χ_E of a μ_u-measurable set $E \subset \Delta_1$, let us denote by σ_E the function σ_{χ_E} and σ'_E the function $\sigma_{\chi_{\complement E}}$. σ_f is a u-harmonic function, for every such f. Now, for a μ_u-measurable set $E \subset \Delta_1$, since $\sigma_E \leqslant 1$ on Ω,

$$\underset{x \to h}{\text{fine lim sup}}\, \sigma_E(x) \leqslant 1 \qquad \text{for all} \quad h \in \Delta_1.$$

If either $\mu_u(E)$ or $\mu_u(\complement E)$ is zero, then $\sigma'_E = 1$ (or respectively $\sigma_E = 1$), and the fine limits of σ_E and σ'_E, exist at all points of Δ_1. On the other hand, suppose $\mu_u(E) \neq 0$ and also

$$\mu_u(\complement E) \neq 0.$$

Then, $u\sigma_E$ and $u\sigma'_E$ are two harmonic functions > 0 on Ω and their canonical measures on Δ_1 (viz. μ_u restricted to E and $\complement E$) are singular relative to each other. Hence, by the Theorem 3, σ_E/σ'_E has the fine limit zero at μ_u-almost every element of $\Delta_1 - E$. It follows then that,

$$\text{fine} \limsup_{x \to h} \sigma_E(x) \leqslant \text{fine} \lim_{x \to h} \frac{\sigma_E(x)}{\sigma'_E(x)} = 0$$

for μ_u-almost every element of $\Delta_1 - E$, as $\sigma'_E(x) \leqslant 1$. Hence,

$$\text{fine} \limsup_{x \to h} \sigma_E(x) \leqslant \chi_E(h)$$

for μ_u-almost every $h \in \Delta_1 \ldots (3)$.

In particular, the inequality (3) is valid for the complement of E and we deduce that,

$$\text{fine} \liminf_{x \to h} \sigma_E(x) \geqslant \chi_E(h) \qquad \text{for} \qquad \mu_u\text{-almost every } h \in \Delta_1.$$

In any case we get, for the characteristic function χ_E of a μ_u-measurable set E contained in Δ_1,

$$\text{fine} \lim_{x \to h} \sigma_E(x) = \chi_E(h) \qquad \text{for} \qquad \mu_u\text{-almost every } h \in \Delta_1 \ldots (4).$$

Suppose, now, $f \geqslant 0$ is a μ_u-measurable function on Δ_1. Then, there exists an increasing sequence of non-negative simple functions s_n such that $\lim_{n \to \infty} s_n = f$. We deduce easily from (4) that

$$\text{fine} \lim_{x \to h} \sigma_{s_n}(x) = s_n(h) \qquad \text{for} \qquad \mu_u\text{-almost every } h \in \Delta_1.$$

Hence, σ_f satisfies,

$$\text{fine} \liminf_{x \to h} \sigma_f(x) \geqslant s_n(h) \qquad \text{for} \qquad \mu_u\text{-almost every } h \in \Delta_1.$$

Now, it is easily seen that,

$$\text{fine} \liminf_{x \to h} \sigma_f(x) \geqslant f(h) \quad \text{for} \quad \mu_u\text{-almost every } h \in \Delta_1 \ldots (5).$$

Let us now consider a bounded μ_u-measurable function g on Δ_1 (say $|g| \leqslant M$). Then, applying the inequality (5) to the two functions $\sigma_{(M\pm g)}$, and noting that, $\sigma_{M\pm g} = M \pm \sigma_g$, we get that

$$\text{fine}\lim_{x\to h} \sigma_g(x) = g(h) \qquad \text{for} \qquad \mu_u\text{-almost every } h \in \Delta_1.$$

Now, the proof of the theorem is completed by noting that any bounded u-harmonic function w is equal to $u\sigma_g$, for some bounded μ_u-measurable function g on Δ_1; this g is unique (depending on w) upto a set of μ_u-measure zero.

Remark 1. — In the course of the proof of the theorem, we have shown that, for any $f \geqslant 0$, which is μ_u-measurable,

$$\text{fine}\liminf_{x\to h_0} \int f(h)\,\frac{h(x)}{u(x)}\,\mu_u(dh) \geqslant f(h_0),$$

for μ_u-almost every $h_0 \in \Delta_1$ (viz. the inequality (5)).

Remark 2. — For any bounded u-harmonic function w on Ω, if $g(h) = \text{fine}\lim\limits_{x\to h} w(x)$, (the function g is defined upto a set of μ_u-measure zero), then g is μ_u-measurable and

$$w(x) = \int g(h)\,\frac{h(x)}{u(x)}\,\mu_u(dh).$$

In particular, if the fine limit is $\geqslant 0$ for μ_u-almost every element of Δ_1, then w is non-negative.

Remark 3. — For any bounded super-u-harmonic function v on Ω, the fine $\lim\limits_{x\to h} v(x)$ exists for μ_u-almost every $h \in \Delta_1$.

THEOREM 5. — *(The Minimum Principle). Let v be a lower bounded super-u-harmonic function on Ω. Suppose that, for every $h \in \Delta_1 - E$, fine $\limsup\limits_{x\to h} v(x) \geqslant 0$, where E is a set with $\mu_u^*(E) = 0$. Then, v is $\geqslant 0$ on Ω.*

Proof. — Let $\alpha > 0$ be such that $v \geqslant -\alpha$. Consider

$$v' = \text{Inf}\,(v, 1).$$

Then v' is a super-u-harmonic function such that $v' \geqslant -\alpha$. The theorem would be proved if we show that $v' \geqslant 0$ on Ω.

Now, it is easily seen that fine $\limsup_{x \to h} v'(x) \geqslant 0$, for all $h \in \Delta_1 - E$. But, we know, (by the Remark 3 following the Theorem 4) that, the limit of v' exists, following $\mathcal{F}_h$, for μ_u-almost every $h \in \Delta_1$; and this fine limit is precisely the fine limit of u_1, where u_1 is the greatest u-harmonic minorant of v'. Hence, we have that the fine limit of u' is $\geqslant 0$ at μ_u-almost every element of Δ_1. It follows that $u_1 \geqslant 0$ (from the Remark 2, Theorem 4). A fortiori, $v' \geqslant 0$. This completes the proof of the theorem.

3. A Dirichlet problem.

Let Σ be the set of all lower bounded super-u-harmonic functions on Ω. Corresponding to any extended real valued function f on Δ_1, define,

$$\Sigma_f = \left\{ \begin{array}{l} v \in \Sigma : \exists \text{ a set } E_v \subset \Delta_1 \text{ of } \mu_u\text{-measure zero such that for} \\ \quad \text{all } h \in \Delta_1 - E_v, \text{ fine } \liminf_{x \to h} v(x) \geqslant f(h) \end{array} \right\}$$

$$\tilde{\Sigma}_f = \left\{ \begin{array}{l} v \in \Sigma : \exists \text{ a set } F_v \subset \Delta_1 \text{ of } \mu_u\text{-measure zero such that for} \\ \quad \text{all } h \in \Delta_1 - E_v, \text{ fine } \limsup_{x \to h} v(x) \geqslant f(h) \end{array} \right\}$$

Definition. — *Corresponding to any extended real valued function f on Δ_1, define, for all $x \in \Omega$,*

$$\overline{\mathcal{H}}_{f,u}(x) = \operatorname{Inf}\{v(x) : v \in \Sigma_f\}$$
$$\underline{\mathcal{H}}_{f,u}(x) = -\overline{\mathcal{H}}_{-f,u}(x)$$

and

$$\overline{\mathfrak{D}}_{f,u}(x) = \operatorname{Inf}\{v(x) : v \in \tilde{\Sigma}_f\}.$$

It is easy to see that Σ_f is a saturated family of super-u-harmonic functions [1]. Hence $\overline{\mathcal{H}}_{f,u}$ is either identically $\pm \infty$ or it is a u-harmonic function. Moreover, from the minimum principle, we deduce that $\overline{\mathcal{H}}_{f,u} \geqslant \underline{\mathcal{H}}_{f,u}$ on Ω.

Also $\overline{\mathcal{H}}_{f,u} \geqslant \overline{\mathfrak{D}}_{f,u}$.

Definition 2. — *Let $u(\mathfrak{R})$ be the class of extended real valued functions f on Δ_1 such that, $\overline{\mathcal{H}}_{f,u} = \underline{\mathcal{H}}_{f,u}$ and this function u-harmonic on Ω. For functions $f \in u(\mathfrak{R})$, we denote $\mathcal{H}_{f,u} = \overline{\mathcal{H}}_{f,u} = \underline{\mathcal{H}}_{f,u}$.*

LEMMA 3. — *Every bounded μ_u-measurable function f on Δ_1 belongs to $u(\mathfrak{R})$ and moreover*

$$\mathcal{H}_{f,u} = \int f(h) \frac{h}{u} \mu_u(dh).$$

Proof. — The u-harmonic function $\sigma_f = \int f(h) \frac{h}{u} \mu_u(dh)$ satisfies,

$$\text{fine} \lim_{x \to h} \sigma_f(x) = f(h) \quad \text{for} \quad \mu_u\text{-almost every } h \in \Delta_1$$

(Theorem 4). Hence, $\overline{\mathcal{H}}_{f,u} \leqslant \sigma_f \leqslant \underline{\mathcal{H}}_{f,u}$. This completes the proof.

PROPOSITION 1. — *Let $\{f_n\}$ be an increasing sequence of extended real functions such that $\overline{\mathcal{H}}_{f_1,u} > -\infty$. Then,*

$$\lim \overline{\mathcal{H}}_{f_n,u} = \overline{\mathcal{H}}_{f,u}.$$

Proof. — Since $\overline{\mathcal{H}}_{f_n,u} \leqslant \overline{\mathcal{H}}_{f,u}$, for every n, it is enough to show that $\overline{\mathcal{H}}_{f,u} \leqslant \lim\limits_{n \to \infty} \overline{\mathcal{H}}_{f_n,u}$, when the limit is not $+\infty$. Let $x_0 \in \Omega$. Given $\varepsilon > 0$, choose for every n, an element $v_n \in \Sigma_{f_n}$ such that

$$\overline{\mathcal{H}}_{f_n,u}(x_0) \geqslant v_n(x_0) - \frac{\varepsilon}{2^n}.$$

Consider $w = \lim \overline{\mathcal{H}}_{f_n,u} + \sum\limits_{n=1}^{\infty} (v_n - \overline{\mathcal{H}}_{f_n,u})$. It is easily seen that w is a super-u-harmonic function. Moreover $w \geqslant v_n$, for every n. Hence w is lower bounded on Ω. Also, if E_{v_n} is the set contained in Δ_1 such that $\mu_u(E_{v_n}) = 0$ and for all $h \in \Delta_1 - E_{v_n}$, $\text{fine} \liminf\limits_{x \to h} v_n(x) \geqslant f_n(h)$, then,

$$\text{fine} \liminf_{x \to h} w(x) \geqslant f(h),$$

for all $h \in \Delta_1 - \bigcup\limits_{n=1}^{\infty} E_{v_n}$. It follows that $w \in \Sigma_f$. Hence $w \geqslant \overline{\mathcal{H}}_{f,u}$. But,

$$\overline{\mathcal{H}}_{f,u}(x_0) \leqslant w(x_0) \leqslant \lim \overline{\mathcal{H}}_{f_n,u}(x_0) + \varepsilon.$$

The proof is now completed easily.

The following proposition is proved easily.

PROPOSITION 2. — *$u(\mathfrak{R})$ is a real vector space. Moreover, for $f, g \in u(\mathfrak{R})$, $\mathcal{H}_{f,u} + \mathcal{H}_{f,u} = \mathcal{H}_{f+g,u}$.*

LEMMA 4. — *For any non-negative extended real valued function f on Δ_1, $\overline{\mathcal{H}}_{f,u} = 0$ is equivalent to the fact that $f = 0$ μ_u-almost everywhere.*

Proof. — Suppose $f = 0$ except on a set of μ_u-measure zero. Let $v \in \Sigma_f$. Then clearly $\frac{1}{n} v \in \Sigma_f$, for all positive integers n. Hence $\overline{\mathcal{H}}_{f,u} = 0$.

Conversely, suppose $\overline{\mathcal{H}}_{f,u} = 0$. Let $A_n = \left\{h : f(h) > \frac{1}{n}\right\}$. Then the characteristic function χ_n of $A_n \subset \Delta_1$ has the property that $\overline{\mathcal{H}}_{\chi_n, u} = 0$. The lemma would be proved if we show that for any set $A \subset \Delta_1$, $\overline{\mathcal{H}}_{\chi_A, u} = 0$ implies that $\mu_u^*(A) = 0$.

Let $v \in \Sigma_{\chi_A}$. That is, there exists a set E_v of μ_u-measure zero such that fine $\liminf_{x \to h} v(x) \geqslant \chi_A(h)$, for all $h \in \Delta_1 - E_v$. Given $\varepsilon > 0$, let $V_\varepsilon = \{x \in \Omega : v(x) > 1 - \varepsilon\}$. Then, V_ε is an open set and V_ε is not thin at any point of $h \in A - E_v$. Now,

$$\frac{uv}{1 - \varepsilon} \geqslant R_u^{V_\varepsilon} \geqslant \overline{\int} h\chi_{A-E_v}(h)\, \mu_u(dh) = \overline{\int} h\chi_A(h)\, \mu_u(dh).$$

This inequality is true for all $\varepsilon > 0$. Hence

$$v \geqslant \overline{\int} \frac{h}{u} \chi_A(h)\, \mu|u|(dh).$$

In turn, this inequality is true for all $v \in \Sigma\chi_A$, and we deduce,

$$\overline{\mathcal{H}}_{\chi_A, u} \geqslant \frac{1}{u} \overline{\int} h\, \chi_A(h)\, \mu_u(dh).$$

Hence, if $\overline{\mathcal{H}}\chi_{A,u} = 0$, then $\overline{\int} h\chi_A(h)\, \mu_u(dh) = 0$. Now, we deduce easily that $\mu_u^*(A) = 0$. This completes the proof.

THEOREM 6. — *Every μ_u-summable function f on Δ_1 belongs to $u(\mathfrak{R})$ and moreover, $\mathcal{H}_{f,u}(x) = \int f(h) \frac{h(x)}{u(x)} \mu_u(dh)$ on Ω.*

Proof. — Suppose f is a non-negative μ_u-summable function on Δ_1. For each positive integer n, if, $f_n = \inf(f, n)$, then

$f_n \in u(\mathcal{R})$ and $\mathcal{H}_{f_n,u} = \int f_n(h) \frac{h}{u} \lambda_u (dh)$. (Lemma 3). Hence, we have,

$$\begin{aligned}\overline{\mathcal{H}}_{f,u} &= \lim_{n\to\infty} \mathcal{H}_{f_n,u} \quad \text{(Proposition 1)}\\ &= \lim_{n\to\infty} \int f_n(h) \frac{h}{u} \mu_u(dh)\\ &= \int f(h) \frac{h}{u} \mu_u(dh).\end{aligned}$$

Also,

$$\int f(h) \frac{h}{u} \mu_u(dh) = \lim \mathcal{H}_{f_n,u} \leqslant \underline{\mathcal{H}}_{f,u}.$$

It follows that

$$f \in u(\mathcal{R}) \qquad \text{and} \qquad \mathcal{H}_{f,u} = \int f(h) \frac{h}{u} \mu_u(dh).$$

Now the proof is completed easily.

Remark. — It can be proved that any function $f \in u(\mathcal{R})$ is necessarily equal μ_u-almost everywhere to a μ_u-summable function and that $\mathcal{H}_{f,u}$ is precisely $\int f(h) \frac{h}{u} \mu_u(dh)$.

4. The Main Result.

THEOREM 7. — *Let $f \geqslant 0$ be an extended real valued function on Δ_1. Then, $\overline{\mathfrak{D}}_{f,u} = \overline{\mathcal{H}}_{f,u}$.*

Proof. — It is enough to show that $\overline{\mathfrak{D}}_{f,u} \geqslant \overline{\mathcal{H}}_{f,u}$.

First of all consider a function $f \geqslant 0$ which is bounded, say $f \leqslant M$. Consider $\tilde{\Sigma}_f^M = \{v \in \tilde{\Sigma}_f : v \leqslant M\}$. We assert that $\overline{\mathfrak{D}}_{f,u} = \operatorname{Inf}\{v : v \in \tilde{\Sigma}_f^M\}$. For, suppose $v \in \tilde{\Sigma}_f$. Then $v_M = \inf(v, M)$ is a super-u-harmonic function and satisfies

$$\text{fine} \limsup_{x \to h} v_M(x) \geqslant f(h),$$

for μ_u-almost every $h \in \Delta_1$. Hence, $v \geqslant v_M \geqslant \operatorname{Inf}\{v : v \in \tilde{\Sigma}_f^M\}$. Hence $\overline{\mathfrak{D}}_{f,u} \geqslant \operatorname{Inf}\{v : v \in \tilde{\Sigma}_f^M\}$. The opposite inequality is obvious.

Now, let $v \in \tilde{\Sigma}_f^M$. Then, by Theorem 4, Remark 3, the fine $\lim_{x \to h} v(x)$ exists for all $h \in \Delta_1 - E'_v$, where $\mu_u(E'_v) = 0$. But, by the defining property of $v \in \tilde{\Sigma}_f^M$, fine $\limsup_{x \to h} v(x) \geqslant f(h)$ for all $h \in \Delta_1 - F_v$, where $\mu_u(F_v) = 0$. It follows that,

$$\text{fine} \liminf_{x \to h} v(x) \geqslant f(h),$$

for all $h \in \Delta_1 - (E'_v \cup F_v)$. Hence, $v \geqslant \overline{\mathcal{H}}_{f,u}$. This is true for all $v \in \tilde{\Sigma}_f^M$ and we get that $\overline{\mathfrak{D}}_{f,u} \geqslant \overline{\mathcal{H}}_{f,u}$.

Let us now consider any $f \geqslant 0$. Let, for every positive integer n, $f_n = \inf(f, n)$. Then, we have,

$$\overline{\mathfrak{D}}_{f,u} \geqslant \lim \overline{\mathfrak{D}}_{f_n,u} = \lim \overline{\mathcal{H}}_{f_n,u} = \overline{\mathcal{H}}_{f,u}.$$

This completes the proof of the theorem.

THEOREM 7. — *For every μ_u-summable function f on Δ_1,*

$$\text{fine} \lim_{x \to h_0} \int f(h) \frac{h(x)}{u(x)} \mu_u(dh) = f(h_0),$$

for μ_u almost every $h_0 \in \Delta_1$.

Proof. — It is enough to prove the theorem assuming that $f \geqslant 0$. Define, for every $h_0 \in \Delta_1$,

$$\varphi'(h_0) = \text{fine} \limsup_{x \to h_0} \int f(h) \frac{h(x)}{u(x)} \mu_u(dh).$$

Let $\varphi = \sup(\varphi', f)$ and $v \in \tilde{\Sigma}_f$. Then, $v \geqslant \int f(h) \frac{h}{u} \mu_u(dh)$ and we see easily that the fine $\limsup_{x \to h} v(x) \geqslant \varphi(h)$, for μ_u-almost every $h \in \Delta_1$. It follows that $v \in \tilde{\Sigma}_\varphi$. This is true for all $v \in \tilde{\Sigma}_f$. Hence, $\overline{\mathcal{H}}_{f,u} \geqslant \overline{\mathcal{H}}_{\varphi,u}$. But $\overline{\mathcal{H}}_{f,u} = \underline{\mathcal{H}}_{f,u} \leqslant \underline{\mathcal{H}}_{\varphi,u}$. This implies that $\varphi \in u(\mathfrak{R})$ and $\mathcal{H}_{\varphi,u} = \mathcal{H}_{f,u}$. Again, $\varphi - f \geqslant 0$ and $\mathcal{H}_{\varphi - f,u} = 0$. We get, from the Lemma 4, that, $\varphi = f$, μ_u-almost everywhere. Hence,

$$\text{fine} \limsup_{x \to h_0} \int f(h) \frac{h(x)}{u(x)} \mu_u(dh) \leqslant f(h_0)$$

for μ_u-almost every $h_0 \in \Delta_1$. But we have already proved that the fine lim inf is $\geqslant f(h_0)$ for μ_u-almost every $h_0 \in \Delta_1$. Hence, we get,

$$\operatorname*{fine\,lim}_{x \to h_0} \int f(h) \frac{h(x)}{u(x)} \mu_u(dh) = f(h_0)$$

for μ_u-almost every $h_0 \in \Delta_1$, completing the proof of the theorem.

Theorem 8. (Fatou-Naïm-Doob). *For any* $v \in S^+$, $\frac{v}{u}$ *has a finite limit at* μ_u*-almost every element of* Δ_1.

Proof. — Let ν be the canonical measure on Δ_1 corresponding to the greatest harmonic minorant of v. Let ν_1 (respectively ν_2) be the absolutely continuous (resp. singular) part of ν relative to μ_u. Let f be the Radon-Nikodym derivative of ν_1 relative to μ_u (f is defined upto a set of μ_u measure zero). Then

$$v = v_1 + v_2 + v_3$$

where v_3 is a potential, $v_2 = \int h\nu_2\,(dh)$ and $v_1 = \int f(h)\, h\mu_u(dh)$. Now, $\frac{v_1}{u}$ has the fine limit f (note that f is finite μ_u almost everywhere), for μ_u-almost every element of Δ_1. Also, $\frac{v_2 + v_3}{u}$ has the fine limit zero at μ_u-almost every element of Δ_1. This completes the proof of the theorem.

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