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# JORDAN OBSTRUCTION TO THE INTEGRABILITY OF HAMILTONIAN SYSTEMS WITH HOMOGENEOUS POTENTIALS

by Guillaume DUVAL & Andrzej J. MACIEJEWSKI

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ABSTRACT. — In this paper, we consider the natural complex Hamiltonian systems with homogeneous potential  $V(q)$ ,  $q \in \mathbb{C}^n$ , of degree  $k \in \mathbb{Z}^*$ . The known results of Morales and Ramis give necessary conditions for the complete integrability of such systems. These conditions are expressed in terms of the eigenvalues of the Hessian matrix  $V''(c)$  calculated at a non-zero point  $c \in \mathbb{C}^n$ , such that  $V'(c) = c$ . The main aim of this paper is to show that there are other obstructions for the integrability which appear if the matrix  $V''(c)$  is not diagonalizable. We prove, among other things, that if  $V''(c)$  contains a Jordan block of size greater than two, then the system is not integrable in the Liouville sense. The main ingredient in the proof of this result consists in translating some ideas of Kronecker about Abelian extensions of number fields into the framework of differential Galois theory.

RÉSUMÉ. — Dans cet article, nous étudions les systèmes Hamiltoniens de potentiels homogènes  $V(q)$ ,  $q \in \mathbb{C}^n$  de degré  $k \in \mathbb{Z}^*$ . Morales et Ramis avaient donné des conditions nécessaires à l'intégrabilité de ces systèmes en termes des valeurs propres des matrices de Hessienne  $V''(c)$ , calculées aux points  $c \in \mathbb{C}^n$  tels que  $V'(c) = c$ . Le thème principal de ce travail est de montrer que d'autres obstructions à l'intégrabilité apparaissent quand  $V''(c)$  n'est pas diagonalisable. Entre autres, nous prouvons que si  $V''(c)$  possède un bloc de Jordan de taille supérieure à deux, alors le système n'est pas intégrable. Pour ce faire, nous avons adapté des idées de Kronecker sur les extensions Abéliennes de corps de nombres, dans le contexte de la théorie de Galois différentielle.

## 1. Introduction

### 1.1. Morales and Ramis results

The Galois obstruction to the integrability of Hamiltonian systems is formulated in the following theorem obtained by Morales and Ramis [11].

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*Keywords:* Hamiltonian systems, integrability, differential Galois theory.

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**THEOREM 1.1** (Morales-Ramis). — *If an Hamiltonian system is completely integrable with first integrals meromorphic in a connected neighbourhood of a phase curve  $\gamma$ , then the identity component of the differential Galois group of the variational equation along  $\gamma$  is virtually Abelian.*

In [10], Morales and Ramis applied this theorem to find obstructions to the complete integrability of Hamiltonian systems with homogeneous potentials. They considered natural systems with Hamiltonian given by

$$(1.1) \quad H(q, p) = \frac{1}{2} \sum_{i=1}^n p_i^2 + V(q_1, \dots, q_n),$$

where  $q = (q_1, \dots, q_n), p = (p_1, \dots, p_n) \in \mathbb{C}^n$ , are the canonical coordinates and momenta, respectively, and  $V(q)$  is a homogeneous potential of degree  $k \in \mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$ . To that purpose, following Yoshida [15], they studied *the variational equations* (in short: VE) associated to *a proper Darboux point* of  $V$ , (in short: PDP) which is a non-zero vector  $c \in \mathbb{C}^n$  such that

$$(1.2) \quad \text{grad } V(c) =: V'(c) = c.$$

If such a Darboux point exists, then the Hamiltonian system admits a particular solution associated with this point, namely, the rectilinear trajectory

$$t \mapsto \gamma(t) = (q(t), p(t)) := (\varphi(t)c, \dot{\varphi}(t)c) \in \mathbb{C}^{2n},$$

where  $t \mapsto \varphi(t)$  is a complex scalar function satisfying the hyper-elliptic differential equation

$$(1.3) \quad \dot{\varphi}(t)^2 = \frac{2}{k}(1 - \varphi^k(t)) \implies \ddot{\varphi}(t) = -\varphi^{k-1}(t).$$

The VE along the curve  $t \mapsto \gamma(t)$  is given by

$$(1.4) \quad \frac{d^2 \eta}{dt^2} = -\varphi^{k-2}(t)V''(c)\eta, \quad \eta \in \mathbb{C}^n.$$

The Hessian matrix  $V''(c)$  is a  $n \times n$  complex, symmetric scalar matrix. Assume that it is diagonalizable with eigenvalues  $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$  (we called them *the Yoshida coefficients*). Then, up to a linear change of unknowns, the system (1.4) splits into a direct sum of equations

$$(1.5) \quad \frac{d^2 \eta_i}{dt^2} = -\lambda_i \varphi^{k-2}(t)\eta_i, \quad i = 1, \dots, n.$$

Morales and Ramis proved the following

**THEOREM 1.2** (Morales-Ramis). — *Assume that the Hamiltonian system with Hamiltonian (1.1) and  $\deg(V) = k \in \mathbb{Z}^*$  is completely integrable by meromorphic first integrals. If  $c = V'(c)$  is a PDP of  $V$  and*

the Hessian matrix  $V''(c)$  is diagonalizable with the Yoshida coefficients  $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ , then each pair  $(k, \lambda_i)$  belongs to Table 1.1.

Table 1.1. The Morales-Ramis table.

$G(k, \lambda)^\circ$	$k$	$\lambda$	Row number
	$k = \pm 2$	$\lambda$ is an arbitrary complex number	1
$G_a$	$ k  \geq 3$	$\lambda(k, p) = p + \frac{k}{2}p(p-1)$	2
$G_a$	1	$p + \frac{1}{2}p(p-1), p \neq -1, 0$	3
$G_a$	-1	$p - \frac{1}{2}p(p-1), p \neq 1, 2$	4
{Id}	1	0	5
{Id}	-1	1	6
{Id}	$ k  \geq 3$	$\frac{1}{2} \left( \frac{k-1}{k} + p(p+1)k \right)$	7
{Id}	3	$\frac{-1}{24} + \frac{1}{6}(1+3p)^2, \frac{-1}{24} + \frac{3}{32}(1+4p)^2$	8,9
		$\frac{-1}{24} + \frac{3}{50}(1+5p)^2, \frac{-1}{24} + \frac{3}{50}(2+5p)^2$	10,11
{Id}	-3	$\frac{25}{24} - \frac{1}{6}(1+3p)^2, \frac{25}{24} - \frac{3}{32}(1+4p)^2$	12,13
		$\frac{25}{24} - \frac{3}{50}(1+5p)^2, \frac{25}{24} - \frac{3}{50}(2+5p)^2$	14,15
{Id}	4	$\frac{-1}{8} + \frac{2}{9}(1+3p)^2$	16
{Id}	-4	$\frac{9}{8} - \frac{2}{9}(1+3p)^2$	17
{Id}	5	$\frac{-9}{40} + \frac{5}{18}(1+3p)^2, \frac{-9}{40} + \frac{1}{10}(2+5p)^2$	18,19
{Id}	-5	$\frac{49}{40} - \frac{5}{18}(1+3p)^2, \frac{49}{40} - \frac{1}{10}(2+5p)^2$	20,21

The group  $G(k, \lambda)^\circ$ , appearing in the first column of Table 1.1, will be defined properly later in this sections.

### 1.2. Jordan obstruction

In order to generalise Theorem 1.2, we are going to work without the assumption that the Hessian matrix  $V''(c)$  is semi-simple. Indeed, since

the Hessian matrix  $V''(c)$  is symmetric, it is diagonalizable if it is real. But, even for a real potential coming from physics, PDP may be a complex non real vector. Therefore,  $V''(c)$  may not be diagonalizable, see Section 6 for a discussion about this point.

Our main result is the following.

**THEOREM 1.3.** — *Let  $V(q)$  be a homogeneous potential of  $n$  variables and degree  $k \in \mathbb{Z} \setminus \{-2, 0, 2\}$ , such that  $H$  is completely integrable with meromorphic first integrals. Then, at any proper Darboux point  $c = V'(c) \in \mathbb{C}^n \setminus \{0\}$ , the Hessian matrix  $V''(c)$  satisfies the following conditions:*

- (1) *For each eigenvalue  $\lambda$  of  $V''(c)$ , the pair  $(k, \lambda)$  belongs to Table 1.1.*
- (2) *The matrix  $V''(c)$  does not have any Jordan block of size  $d \geq 3$ .*
- (3) *If  $V''(c)$  has a Jordan block of size  $d = 2$  with corresponding eigenvalue  $\lambda$ , then the row number of  $(k, \lambda)$  in Table 1.1 is greater or equal to five.*

For  $k = \pm 2$ , independently of the value of  $V''(c)$ , the connected component of the Galois group of the variational equation is Abelian.

In the above statement, by **a Jordan block of size  $d$  with the eigenvalue  $\lambda$** , we mean that the Jordan form of  $V''(c)$  contains a block of the form

$$(1.6) \quad B(\lambda, d) := \begin{bmatrix} \lambda & 0 & 0 & \dots & 0 \\ 1 & \lambda & 0 & \dots & 0 \\ 0 & 1 & \lambda & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & \lambda \end{bmatrix} \in \mathbb{M}(d, \mathbb{C}),$$

where  $\mathbb{M}(d, \mathbb{C})$  denotes the set of  $d \times d$  complex matrices.

*Remark 1.4.* — Theorem 1.3 roughly states that Morales-Ramis Theorem 1.2 is optimal. Indeed, up to some exceptions, if  $H$  is completely integrable, then  $V''(c)$  must be diagonalizable with specific eigenvalues.

Our result is analogous to the Liapunov-Kowaleskaya Theorem, which states that if a given system of weight-homogeneous differential equations enjoys the Painleve property, then among other things, the linearization of the system along a certain single-valued particular solution is diagonalizable. For details, see [8]. Moreover, in the same sense, we find similarities in the classical normal form theory of vector fields, where a complicated dynamics appears in a neighbourhood of the equilibrium if the linearization of the vector field is not semi-simple.

As far as we know, except for one example given in Chapter 7 in [9], there are no explicit links between the Galois approach to the integrability and the dynamics. Nevertheless, the above analogies were our strong motivations for that study.

The proof of Theorem 1.3 is of another nature. It comes from arithmetic ideas belonging to Kronecker. He observed that in Number Theory, Abelian extensions of number fields can be characterised by simple arithmetic relations. We translate this very nice observation into the framework of the Differential Galois Theory.

### 1.3. VE, Yoshida transformations and Jordan blocks

The VE (1.4) is a system of differential equations with respect to the time variable  $t$ . First, we perform the so-called *Yoshida transformation*, in order to express the VE in terms of a new variable  $z$ . The great advantage of this transformation is that it converts our original system into a new one where the classical hypergeometric equation naturally appears. Next, we give the canonical formulae for the subsystems of VE associated to Jordan blocks.

The Yoshida transformation is a change of independent variable in equation (1.4) given by

$$(1.7) \quad t \longmapsto z = \varphi^k(t).$$

Thanks to (1.3) and the chain rule we have

$$\begin{aligned} \frac{d^2\eta}{dt^2} &= \left(\frac{dz}{dt}\right)^2 \frac{d^2\eta}{dz^2} + \frac{d^2z}{dt^2} \frac{d\eta}{dz}, \\ \left(\frac{dz}{dt}\right)^2 &= 2kz(1-z)\varphi^{k-2}(t), \quad \frac{d^2z}{dt^2} = [(2-3k)z + 2(k-1)]\varphi^{k-2}(t). \end{aligned}$$

Then, after some simplifications, (1.4), becomes

$$(1.8) \quad \frac{d^2\eta}{dz^2} + p(z)\frac{d\eta}{dz} = s(z)V''(c)\eta,$$

where

$$p(z) = \frac{2(k-1)(z-1) + kz}{2kz(z-1)} \quad \text{and} \quad s(z) = \frac{1}{2kz(z-1)}.$$

Next, after the classical Tchirnhauus change of dependent variable,

$$(1.9) \quad \eta = f(z)\zeta, \quad f(z) = \exp\left(-\frac{1}{2} \int p(z)dz\right) = z^{-(k-1)/2k}(z-1)^{-1/4},$$

equation (1.8) has the reduced form

$$(1.10) \quad \frac{d^2\zeta}{dz^2} = [r_0(z)\text{Id} + s(z)V''(c)]\zeta,$$

where

$$r_0(z) = \frac{\rho^2 - 1}{4z^2} + \frac{\sigma^2 - 1}{4(z - 1)^2} - \frac{1}{4} (1 - \rho^2 - \sigma^2 + \tau_0^2) \left( \frac{1}{z} + \frac{1}{1 - z} \right),$$

and

$$\rho = \frac{1}{k}, \quad \sigma = \frac{1}{2}, \quad \tau_0 = \frac{k - 2}{2k}.$$

Assume that  $V''(c)$  contains a Jordan block  $B(\lambda, d)$  with  $d = 3$ , for example. Then, the subsystem of (1.10) corresponding to this block can be written as

$$\frac{d^2}{dz^2} \begin{bmatrix} x \\ y \\ u \end{bmatrix} = \begin{bmatrix} x'' \\ y'' \\ u'' \end{bmatrix} = \left( \begin{bmatrix} r_0(z) & 0 & 0 \\ 0 & r_0(z) & 0 \\ 0 & 0 & r_0(z) \end{bmatrix} + s(z) \begin{bmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ u \end{bmatrix}.$$

We rewrite it in the following form

$$(1.11) \quad \begin{bmatrix} x'' \\ y'' \\ u'' \end{bmatrix} = \begin{bmatrix} r(z) & 0 & 0 \\ s(z) & r(z) & 0 \\ 0 & s(z) & r(z) \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix}.$$

where  $r(z) = r_\lambda(z)$  is given by

$$(1.12) \quad \begin{aligned} r(z) &= r_0(z) + \lambda s(z) \\ &= \frac{\rho^2 - 1}{4z^2} + \frac{\sigma^2 - 1}{4(z - 1)^2} - \frac{1}{4} (1 - \rho^2 - \sigma^2 + \tau^2) \left( \frac{1}{z} + \frac{1}{1 - z} \right), \end{aligned}$$

with

$$(1.13) \quad \rho = \frac{1}{k}, \quad \sigma = \frac{1}{2}, \quad \tau = \frac{\sqrt{(k - 2)^2 + 8k\lambda}}{2k}.$$

The above three numbers are exactly the respective exponents differences at  $z = 0$ ,  $z = 1$  and  $z = \infty$  of the reduced hypergeometric equation  $L_2 = x'' - r(z)x = 0$ . Thus, the solutions of  $L_2 = 0$  belong to the Riemann scheme

$$(1.14) \quad P \left\{ \begin{matrix} 0 & 1 & \infty \\ \frac{1}{2} - \frac{1}{2k} & \frac{1}{4} & \frac{-1 - \tau}{2} \\ \frac{1}{2} + \frac{1}{2k} & \frac{3}{4} & \frac{-1 + \tau}{2} \end{matrix} z \right\}.$$

Let us fix the following convention concerning the differential Galois groups related to system (1.11) corresponding to a block of size 3. The differential

Galois group of the first equation of this system, i.e. the differential Galois group of the equation  $L_2 = x'' - r(z)x = 0$ , with respect to the ground field  $\mathbb{C}(z)$ , is denoted  $G_1$ . This group can be determined thanks to the Kimura theorem [5], see also [3] and [4]. In fact,

$$G(k, \lambda) := G_1,$$

is the group appearing in the first column of Table 1. The differential Galois group of subsystem of (1.11) consisting of first two equations is denoted  $G_2$ . Finally, the differential Galois group of whole system (1.11) is denoted by  $G_3$ .

### 1.4. Generalities and Galois groups of the distinct VE

In this subsection, we summarise some results about differential Galois groups and classical Differential Algebra which we frequently use, see [6, 14]. Next, we compare the differential Galois groups of different forms of the VE introduced in the previous subsection.

In what follows,  $(K, \partial)$  denotes an ordinary differential field with the algebraically closed subfield of constants  $\mathbb{C}$ . We use the standard notation, e.g.,  $x' = \partial x$ ,  $x'' = \partial^2 x$ , etc., for an element  $x \in K$ .

- If a linear system  $Y' = AY$ , where  $A \in \mathbb{M}(n_1 + n_2, K)$ , splits into a direct sum

$$Y' = \begin{bmatrix} Y'_1 \\ Y'_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \quad \text{where } A_i \in \mathbb{M}(n_i, K) \quad \text{for } i = 1, 2,$$

then, with obvious notations, the identity component  $\mathcal{G}^\circ$  of its differential Galois group  $\mathcal{G}$  is a subgroup of the direct product  $\mathcal{G}_1^\circ \times \mathcal{G}_2^\circ$ . Moreover, the two projection maps  $\pi_i : \mathcal{G}^\circ \rightarrow \mathcal{G}_i^\circ$ , with  $i = 1, 2$ , are surjective. Therefore,  $\mathcal{G}^\circ$  is Abelian iff  $\mathcal{G}_1^\circ$  and  $\mathcal{G}_2^\circ$  are Abelian.

- If the system  $Y'_1 = A_1 Y_1$  is a subsystem of

$$\begin{bmatrix} Y'_1 \\ Y'_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B & A_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

then the reduction morphism  $\mathcal{G}^\circ \rightarrow \mathcal{G}_1^\circ$  is surjective. Therefore, if  $\mathcal{G}^\circ$  is Abelian, then  $\mathcal{G}_1^\circ$  is also Abelian.

LEMMA 1.5. — *Let  $E/K$  be an ordinary differential field extension with the same subfield of constants  $\mathbb{C}$ .*



- (1) Let  $f_1, \dots, f_p \in E$ , and  $f'_i \in K$ , for  $i = 1, \dots, p$ . Then the family  $\{f_1, \dots, f_p\}$  is algebraically dependent over  $K$  if and only if there exists a non trivial linear relation

$$c_1 f_1 + \dots + c_p f_p \in K \text{ with } (c_1, \dots, c_p) \in \mathbb{C}^p \setminus \{0\}.$$

- (2) Let  $T(E/K)$  be the set of elements  $f$  of  $E$  such that there exists a non-zero linear differential equation  $L \in K[\partial]$  such that  $L(f) = 0$ . Then  $T(E/K)$  is a  $K$ -algebra containing the algebraic closure of  $K$  in  $E$ . If  $E/K$  is a Picard-Vessiot extension, then  $T(E/K)$  is the Picard-Vessiot ring of  $E/K$ , and  $T(E/K) = K[Z_{i,j}][W^{-1}]$ , where  $W = \det((Z_{i,j}))$ , and  $(Z_{i,j})_{1 \leq i, j \leq n}$  is an arbitrary fundamental matrix defining the Picard-Vessiot extension.
- (3) Let  $y' = Ay$  be a differential system with  $A \in \mathbb{M}(n, K)$ , and  $K'/K$  be a finite degree extension of  $K$ . Denote by  $\mathcal{G}$  (resp. by  $\mathcal{G}'$ ) the respective Galois groups of  $y' = Ay$  when this system is considered over  $K$ , (resp. over  $K'$ ). Then  $\mathcal{G}'$  is naturally a subgroup of  $\mathcal{G}$ , and  $(\mathcal{G}')^\circ = \mathcal{G}^\circ$ .

*Proof.* — (1) is the classical Ostrowski-Kolchin theorem about the algebraic independence of integrals. Its proof may be found in [6].

(2) follows directly from Exercises 1.24 on p. 17 and Corollary 1.38 on p. 30 in [14].

(3) Let  $F'/K'$  be a Picard-Vessiot extension of  $y' = Ay$  over  $K'$ . Then  $F' = K'(Y)$ , where  $Y$  is a fundamental matrix of solutions of the system. Set  $F = K(Y)$ . Then  $F/K$  is a Picard-Vessiot extension of  $y' = Ay$  over  $K$ , for which

$$F' = K'F \quad \text{and} \quad F \subset F'.$$

Since the group  $\mathcal{G}'$  fixes  $K'$  pointwise and leaves  $F$  globally invariant, i.e.,  $\mathcal{G}' \cdot F = F$ , it may be considered as a subgroup of  $\mathcal{G}$ . Therefore, we also have the inclusion of connected components

$$(\mathcal{G}')^\circ \subset \mathcal{G}^\circ.$$

From Corollary 1.30 on p. 23 in [14] and its proof, we have

$$\dim \mathcal{G}^\circ = \dim \mathcal{G} = \text{tr. deg}(F/K).$$

But  $\text{tr. deg}(F/K) = \text{tr. deg}(K'F/K') = \text{tr. deg}(F'/K') = \dim(\mathcal{G}')^\circ$ . So, the two irreducible varieties  $(\mathcal{G}')^\circ$  and  $\mathcal{G}^\circ$  have the same dimension, hence they are equal.  $\square$

Point 3 in the above lemma is exactly Theorem 3.13 from [2] where sketch of the proof is given. This point gives the invariance of the connected component of the differential Galois group with respect to ramified coverings. See also [9] for other exposition and proofs.

Let us consider the four variational equations derived in the previous subsection, namely equations (1.4), (1.8), (1.10) and (1.11). The system (1.4) is defined over the ground fields  $\mathbb{C}(\varphi(t), \dot{\varphi}(t))$ , and the other three are defined over  $\mathbb{C}(z)$ . Let  $G(\text{VE}_t)$ ,  $G(\text{VE}_z)$ , and  $G_{\text{block}}$  be the differential Galois groups of equations (1.4), (1.8), and (1.11), respectively. Then we have the following.

PROPOSITION 1.6. — *With the notations above, we have:*

- (1) *The Galois groups of systems (1.8) and (1.10) have common connected component  $G(\text{VE}_z)^\circ$ .*
- (2) *The two connected components  $G(\text{VE}_t)^\circ$  and  $G(\text{VE}_z)^\circ$  are isomorphic.*
- (3) *The connected component  $G_{\text{block}}^\circ$  is a quotient of  $G(\text{VE}_z)^\circ$ .*

*Proof.* — (1) Set  $K = \mathbb{C}(z)$  and  $K' = \mathbb{C}(z)[f(z)]$ , where  $f(z)$  is given by (1.9). Then  $K'/K$  is a finite extension. Denote by  $Z$  a fundamental matrix of solutions of (1.10). Then  $f(z)Z$  is a fundamental matrix of solutions of (1.8). Therefore, (1.8) and (1.10) share the same Picard-Vessiot extension over  $K'$ . So they have the same Galois group  $\mathcal{G}'$  over  $K'$ . From point 3 of Lemma 1.5,

$$(\mathcal{G}')^\circ = \mathcal{G}^\circ := G(\text{VE}_z)^\circ,$$

is also the connected component of the Galois group of (1.10) when it is viewed as a system over  $K = \mathbb{C}(z)$ .

(2) Consider the Yoshida map

$$\begin{aligned} \phi : K = \mathbb{C}(z) &\rightarrow K' = \mathbb{C}(\varphi(t), \dot{\varphi}(t)), \\ z &\mapsto \varphi^k(t). \end{aligned}$$

This map is a morphism of fields which is not a differential morphism for differential fields  $(K, \frac{d}{dz})$  and  $(K', \frac{d}{dt})$ . But, since  $K'/K$  is finite, the derivation  $\frac{d}{dz}$  of  $K$  extends uniquely to a derivation of  $K'$  which is still denoted by the same symbol. Moreover,

$$(1.15) \quad \frac{d}{dt} = \frac{dz}{dt} \frac{d}{dz} = k\varphi^{k-1}\dot{\varphi} \frac{d}{dz}$$

Let  $F'/K'$  be the Picard-Vessiot extension of (1.4) over  $(K', \frac{d}{dt})$ . Then  $F'/K'$  is a Picard-Vessiot extension of (1.8) when considered over  $(K', \frac{d}{dz})$ .

From (1.15), an automorphism of  $F'/K'$  commutes with  $\frac{d}{dt}$  iff it commutes with  $\frac{d}{dz}$ . Therefore, by point 3 of Lemma 1.5, we conclude that

$$G(\text{VE}_t)^\circ = G(\text{VE}_z)^\circ.$$

(3) According to the second item given at the beginning of this subsection, the reduction map  $G(\text{VE}_z)^\circ \rightarrow G_{\text{block}}^\circ$  is surjective. Hence,  $G_{\text{block}}^\circ$  is a quotient of  $G(\text{VE}_z)^\circ$ .  $\square$

### 1.5. The plan of the paper

As shown in the above section, there are several VE, but essentially we have two connected Galois group to deal with:  $G(\text{VE}_z)^\circ$  and  $G_{\text{block}}^\circ$ , the latter being a quotient of the former.

In Section 2, we study differential equations of the form (1.11) for Jordan blocks of size  $d = 2$ . We find necessary and sufficient conditions for the connected component of the Galois group  $G_{\text{block}}^\circ = G_2^\circ$  to be Abelian, see Theorem 2.3. In this part the reader will find our interpretation of Kronecker's ideas in the framework of the Differential Galois Theory.

In Section 3, we apply this result to eliminate from Table 1.1 all the cases corresponding to  $G_1^\circ \simeq G_a$ , where  $G_a$  denotes the additive algebraic subgroup of  $\text{SL}(2, \mathbb{C})$ . Here,  $G_1 = G(k, \lambda)$  is the Galois group over  $\mathbb{C}(z)$  of the equation  $L_2 = x'' - r(z)x = 0$ . According to Theorem 2.3, we have to check if certain specific primitive integrals built from special function as Jacobi polynomials, are algebraic.

From Theorem 2.3, if  $G_1$  is finite, then  $G_2^\circ$  is Abelian. In those cases, the existence of Jordan blocks with size  $d = 2$  does not give any obstacles for the integrability. This is why we are forced to look for such obstructions considering Jordan blocks of size  $d = 3$ . This problem is investigated in Section 4, where the results of Section 2 are also used. In this part of the paper we follow the general ideas contained in Sections 2 and 3, but our considerations are much more technical.

In Section 5, we deal with the exceptional cases of potentials of degree  $k = \pm 2$ , for which we prove that  $G(\text{VE}_t)^\circ \simeq G(\text{VE}_z)^\circ$  is Abelian. The strategy employed is completely different and independent of the general frame of the paper. First, we give a direct proof of that result for  $k = 2$ . Then, we extract and discuss a general principle of symmetry contained in Table 1.1. Applying this principle, we deduce the following implication

$$G(\text{VE}_t)^\circ \text{ Abelian for } k = 2 \implies G(\text{VE}_t)^\circ \text{ is Abelian for } k = -2.$$

For the non-expert reader, we should recommend to read this section first, since for  $k = 2$ , he shall see the frame of a very simple and particular VE.

In order to justify our study, in Section 6, we prove that the Hessian matrix  $V''(c)$  for a homogeneous polynomial potential  $V$  of degree  $k$ , can be an arbitrary symmetric matrix  $A$  satisfying  $Ac = (k - 1)c$ . This is made by a dimensional arguments and study of complex symmetric matrices.

### 2. Theory for Jordan blocks of size two

Let  $(K, \partial)$  be an ordinary differential field with constant subfield  $\mathbb{C}$ . We consider the following system of two linear differential equations over  $K$ .

$$(2.1) \quad x'' = rx,$$

$$(2.2) \quad y'' = ry + sx.$$

We denote by  $F_1$  and  $F_2$  the Picard-Vessiot fields of equation (2.1), and the system (2.1)-(2.2), respectively. The differential Galois group of extension  $F_i/K$  is denoted by  $G_i$ , for  $i = 1, 2$ .

We look for the conditions under which  $G_2^\circ$  is Abelian. Since  $F_1$  may be seen as a subfield of  $F_2$ , and  $G_1$  as a quotient of  $G_2$ , we express these conditions in terms of  $G_1^\circ$ , and  $r, s \in K$ .

From now on,  $\{x_1, x_2\}$  denotes a basis of solutions of (2.1) normalised in such a way that

$$W(x_1, x_2) = \det(X) = 1, \quad \text{where } X = \begin{bmatrix} x_1 & x_2 \\ x'_1 & x'_2 \end{bmatrix}.$$

For each  $\sigma \in G_2$ , there exists matrix  $A(\sigma) \in \text{SL}(2, \mathbb{C})$ , such that  $\sigma(X) = XA(\sigma)$ . Moreover, we chose  $\{x_1, x_2\}$  such that

- if  $G_1^\circ \simeq G_a$ , then for all  $\sigma \in G_2^\circ$ , the matrix  $A(\sigma)$  is a unipotent upper triangular matrix;
- if  $G_1^\circ \simeq G_m$ , then for all  $\sigma \in G_2^\circ$ , the matrix  $A(\sigma)$  is a diagonal matrix.

We recall here that in the above statements  $G_a$  and  $G_m$  denote the additive and the multiplicative subgroups of  $\text{SL}(2, \mathbb{C})$ .

The group  $G_1^\circ$  is a connected subgroup of  $\text{SL}(2, \mathbb{C})$ . It is Abelian if and only if it is isomorphic either to  $G_a$ ,  $G_m$ , or to  $\{\text{Id}\}$ . Moreover, in [7] Kovacic proved the following

LEMMA 2.1. —

- (1)  $G_1^\circ \simeq G_a$ , iff there exists a positive integer  $m$  such that  $x_1^m \in K$ , and  $x_2$  is transcendental over  $K$ . In this case the algebraic closure of  $K$  in  $F_1$  is  $L = K[x_1]$ .

- (2)  $G_1^\circ \simeq G_m$ , iff  $x_1$  and  $x_2$  are transcendental over  $K$  but,  $(x_1x_2)^2 \in K$ . In this case the algebraic closure of  $K$  in  $F_1$  is  $L = K[x_1x_2]$ . Moreover,  $L$  is at most quadratic over  $K$ .
- (3)  $G_1$  is a finite group if and only if both  $x_1$  and  $x_2$  are algebraic over  $K$ . Moreover, if this happens then,  $G_1$  is a finite subgroup of  $SL(2, \mathbb{C})$  which is of one of the four types listed below:
- (a) Dihedral type:  $G_1$  is conjugated to a finite subgroup of

$$D^\dagger = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{bmatrix} \mid \lambda \in \mathbb{C}^* \right\} \cup \left\{ \begin{bmatrix} 0 & \lambda \\ -1/\lambda & 0 \end{bmatrix} \mid \lambda \in \mathbb{C}^* \right\}.$$

- (b) Tetrahedral type:  $G_1/\{\pm \text{Id}\} \simeq \mathfrak{A}_4$ .  
 (c) Octahedral type:  $G_1/\{\pm \text{Id}\} \simeq \mathfrak{S}_4$ .  
 (d) Icosahedral type:  $G_1/\{\pm \text{Id}\} \simeq \mathfrak{A}_5$ .

In the above,  $\mathfrak{S}_p$  and  $\mathfrak{A}_p$  denote the symmetric, and the alternating group of  $p$  elements, respectively.

DEFINITION 2.2. — Let  $\varphi = \int sx_1^2$  and  $\psi = \int x_1^{-2}$ . We define the following conditions

- ( $\alpha$ ): There exists  $c \in \mathbb{C}$  such that  $\varphi + c\psi \in L$ .  
 ( $\beta$ ):  $\varphi\psi - 2 \int \varphi \cdot \psi' = 2 \int \varphi' \psi - \varphi \cdot \psi \in L[\psi]$ .  
 ( $\gamma$ ): There exists  $\phi_1 \in L$  such that  $(\phi_1 x_1^2)' = sx_1^2$ .  
 ( $\delta$ ): There exists  $\phi_2 \in L$  such that  $(\phi_2 x_2^2)' = sx_2^2$ .

With the above notations and definitions our main result in this section is the following.

THEOREM 2.3. — The group  $G_2^\circ$  is Abelian if and only if one of the following cases occur

- (1)  $G_1$  is a finite group.  
 (2)  $G_1^\circ \simeq G_a$  and condition ( $\beta$ ) holds.  
 (3)  $G_1^\circ \simeq G_m$  and conditions ( $\gamma$ ) and ( $\delta$ ) hold.

When  $G_1^\circ \simeq G_a$ , then ( $\beta$ )  $\Rightarrow$  ( $\alpha$ ). Hence, ( $\alpha$ ) is a necessary condition for  $G_2^\circ$  to be Abelian in this case.

The proof of the theorem will be done at the end of this section. Before, for sake of clarity, we explain the main ideas of its proof.

A good illustration of the Kronecker observation in arithmetic is the following example. Let

$$f(X) = X^3 + pX + q \in \mathbb{Q}[X],$$

be an irreducible polynomial. Let  $\text{Gal}(f/\mathbb{Q})$  be its Galois group over the rationals, and  $\Delta = -4p^3 - 27q^2$  be the discriminant of  $f$ . The group  $\text{Gal}(f/\mathbb{Q})$  can be either  $\mathfrak{S}_3$ , or  $\mathfrak{A}_3$ . Moreover,

$$\text{Gal}(f/\mathbb{Q}) \simeq \mathfrak{A}_3 \iff \Delta \in (\mathbb{Q})^2.$$

In other words,  $\text{Gal}(f/\mathbb{Q})$  is Abelian, iff  $\Delta$  is a square of a rational number. This is, in the considered example, the precise ‘‘arithmetical condition’’ that governs the Abelianity of the Galois group.

In differential Galois theory, the analogue of the discriminant is the Wronskian determinant. Therefore our idea was to express the Abelianity condition for  $G_2^\circ$  in terms of certain properties of the Wronskian determinant.

We proceed in the three following steps

- (1) The very specific form of system (2.1)–(2.2), allows to express the Abelianity of  $G_2^\circ$  in terms of its subgroup  $H = \text{Gal}_\theta(F_2/F_1)$ .
- (2) Next, we translate this group conditions into properties of certain Wronskians.
- (3) In a third step, thanks to Lemma 1.5, we express these Wronskian properties in terms of the algebraicity of certain primitive integrals.

Later, in the applications, we shall not use the case of Theorem 2.3 where  $G_1^\circ \simeq G_m$ . This is because those cases only happen for potentials of degree  $k = \pm 2$  for which other kind of arguments will be applied in Section 5. Therefore, at first reading, this part of the proof of Theorem 2.3 may be avoided.

### 2.1. Group formulation of the criterion

The system (2.1) and (2.2) may be written into the matrix form:

$$(2.3) \quad \begin{bmatrix} x' \\ x'' \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ s & 0 & r & 0 \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix} = \begin{bmatrix} R & 0 \\ S & R \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix},$$

where

$$(2.4) \quad R := \begin{bmatrix} 0 & 1 \\ r & 0 \end{bmatrix}, \quad S := \begin{bmatrix} 0 & 0 \\ s & 0 \end{bmatrix}.$$

For a given basis  $\{x_1, x_2\}$  of solutions of equation (2.1), we set

$$X = \begin{bmatrix} x_1 & x_2 \\ x'_1 & x'_2 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{bmatrix},$$

where  $y_1$  and  $y_2$  are two particular solutions of (2.2), that is:

$$(2.5) \quad \begin{cases} y_1'' = ry_1 + sx_1 \\ y_2'' = ry_2 + sx_2 \end{cases}$$

Then the following  $4 \times 4$  matrix

$$\Xi_2 = \begin{bmatrix} X & 0 \\ Y & X \end{bmatrix},$$

is a fundamental matrix of solutions of (2.3). For each  $\sigma \in G_2$ , we have

$$(2.6) \quad \sigma(\Xi_2) = \begin{bmatrix} \sigma(X) & 0 \\ \sigma(Y) & \sigma(X) \end{bmatrix} = \Xi_2 M(\sigma).$$

Performing the above multiplication we can easily notice that the  $4 \times 4$  matrix  $M(\sigma)$  has the form

$$M(\sigma) = \begin{bmatrix} A(\sigma) & 0 \\ B(\sigma) & A(\sigma) \end{bmatrix}.$$

Therefore,  $G_2$  can be identified with a subgroup of  $SL(4, \mathbb{C})$ :

$$(2.7) \quad G_2 \subset G_{\max} = \left\{ M(A, B) := \begin{bmatrix} A & 0 \\ B & A \end{bmatrix} \mid A \in SL(2, \mathbb{C}), B \in M(2, \mathbb{C}) \right\}.$$

For  $M_i = M_i(A_i, B_i) \in G_{\max}$ , with  $i = 1, 2$ , we have

$$(2.8) \quad M_1 M_2 = \begin{bmatrix} A_1 A_2 & 0 \\ B_1 A_2 + A_1 B_2 & A_1 A_2 \end{bmatrix}.$$

DEFINITION 2.4. — We denote by  $H := \text{Gal}_\partial(F_2/F_1)$ . The groups  $H_{\max}$ ,  $H_a$  and  $H_m$  are the subgroups of  $G_{\max}$ , defined by,

$$\begin{aligned} H_{\max} &:= \left\{ N(B) := \begin{bmatrix} \text{Id} & 0 \\ B & \text{Id} \end{bmatrix} \mid B \in M(2, \mathbb{C}) \right\}, \\ H_a &:= \left\{ N(B) \in H_{\max} \mid B = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, a, b \in \mathbb{C} \right\}, \\ H_m &:= \left\{ N(B) \in H_{\max} \mid B = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, a, d \in \mathbb{C} \right\}. \end{aligned}$$

From (2.8), we have  $N(B_1)N(B_2) = N(B_1 + B_2)$ . So,  $H_{\max}$  is a vector group of dimension 4 isomorphic to  $(M(2, \mathbb{C}), +)$ .

Let us recall that a field extension  $E/K$  is regular iff for all  $x \in E$  we have:  $x$  is algebraic over  $K$  implies that  $x \in K$ . When  $E/K$  is a Picard-Vessiot extension, then it is regular iff its differential Galois group is connected.

PROPOSITION 2.5. — With the notations above we have the following.

- (1) The Picard Vessiot extension  $F_2/F_1$  is a regular fields extension and its Galois group  $H = \text{Gal}_\partial(F_2/F_1)$  is a vector group.
- (2) The algebraic closure of  $K$  in  $F_2$  coincides with the algebraic closure  $L$  of  $K$  in  $F_1$ .
- (3) The kernel of the restriction map  $\text{Res}^\circ : G_2^\circ \rightarrow G_1^\circ$  coincides with  $H = \text{Gal}_\partial(F_2/F_1)$ .
- (4) If  $G_1$  is finite then  $G_2^\circ = H = \text{Gal}_\partial(F_2/F_1)$  is Abelian.

*Proof.* — (1) Let  $\text{Res} : G_2 \rightarrow G_1, \sigma \mapsto \sigma|_{F_1}$ , be the restriction map. We have

$$M(\sigma) = \begin{bmatrix} A(\sigma) & 0 \\ B(\sigma) & A(\sigma) \end{bmatrix} \mapsto \text{Res}(M(\sigma)) = A(\sigma)$$

Since  $H = \text{Gal}_\partial(F_2/F_1) = \text{Ker}(\text{Res})$ , the algebraic subgroup  $H$  of  $G_2$  may be viewed as an algebraic subgroup of  $H_{\max}$ . It is therefore a vector group, hence connected.

(2) Let  $u \in F_2$  be algebraic over  $K$ . Since  $H$  is connected,  $Hu = \{u\}$ , and thus  $u \in F_1$  is algebraic over  $K$ .

(3) Since the restriction map  $\text{Res} : G_2 \rightarrow G_1$  is a surjective morphism of algebraic groups, it maps  $G_2^\circ$  onto  $G_1^\circ$ . Denoting by  $\text{Res}^\circ$  the restriction of  $\text{Res}$  to  $G_2^\circ$  and putting  $H' = \text{Ker}(\text{Res}^\circ)$ , we have the following commutative diagram of algebraic groups, whose lines are exact sequences

$$\begin{array}{ccccccccc} \{\text{Id}\} & \longrightarrow & H & \longrightarrow & G_2 & \xrightarrow{\text{Res}} & G_1 & \longrightarrow & \{\text{Id}\} \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \{\text{Id}\} & \longrightarrow & H' & \longrightarrow & G_2^\circ & \xrightarrow{\text{Res}^\circ} & G_1^\circ & \longrightarrow & \{\text{Id}\} \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \{\text{Id}\} & \longrightarrow & \{\text{Id}\} & \longrightarrow & \{\text{Id}\} & \longrightarrow & \{\text{Id}\} & \longrightarrow & \{\text{Id}\} \end{array}$$

Applying the snake lemma to the first two lines we obtain following exact sequence

$$\{\text{Id}\} \longrightarrow H/H' \longrightarrow G_2/G_2^\circ \longrightarrow G_1/G_1^\circ \longrightarrow \{\text{Id}\}$$

But  $G_2/G_2^\circ$  is finite, so  $H/H'$  is also finite. Moreover,  $H/H'$  as a quotient of vector group is also a vector group hence, it is the trivial vector group. That is  $H' = H$ , and  $G_2/G_2^\circ$  is isomorphic to  $G_1/G_1^\circ$ . Moreover, the second line of the commutative diagram reduces to the exact sequence

(2.9) 
$$\{\text{Id}\} \longrightarrow H = \text{Gal}_\partial(F_2/F_1) \longrightarrow G_2^\circ \longrightarrow G_1^\circ \longrightarrow \{\text{Id}\}.$$

(4) If  $G_1$  is finite,  $G_1^\circ = \{\text{Id}\}$  in (2.9), so  $G_2^\circ = H$  is Abelian. □



In general, from (2.9) we have

$$G_2^\circ \subset \{M(A, B) \in G_{\max} \mid A \in G_1^\circ\},$$

and, if  $G_2^\circ$  is Abelian, then  $G_1^\circ$  is an Abelian algebraic subgroup of  $SL(2, \mathbb{C})$ . If this happens, then  $G_1^\circ$  is isomorphic either to  $\{\text{Id}\}$ , or  $G_a$ , or  $G_m$ .

If  $G_1^\circ = \{\text{Id}\}$  we have seen above that  $G_2^\circ$  is an Abelian vector group. However, if  $G_1^\circ$  is isomorphic either to  $G_a$  or  $G_m$ , then we have to find conditions under which  $G_2^\circ$  is Abelian. For that purpose we need the following conjugation formula, which is obtained from (2.8) by direct computations. Namely, for all  $M(A, B) \in G_{\max}$  and all  $N(C) \in H_{\max}$ , we have

$$(2.10) \quad M(A, B)N(C)M(A, B)^{-1} = N(ACA^{-1}).$$

PROPOSITION 2.6. — *If  $G_1^\circ$  is isomorphic either to  $G_a$  or  $G_m$ , then  $G_2^\circ$  is Abelian if and only if  $H = \text{Gal}_\partial(F_2/F_1) \subset Z(G_2^\circ)$ , i.e.,  $H$  is contained in the center of  $G_2^\circ$ .*

- (1) *If  $G_1^\circ \simeq G_a$ , then  $G_2^\circ$  is Abelian iff  $H$  is a subgroup of  $H_a$ .*
- (2) *If  $G_1^\circ \simeq G_m$ , then  $G_2^\circ$  is Abelian iff  $H$  is a subgroup of  $H_m$ .*

Here  $H_a$  and  $H_m$  are the groups defined in Definition 2.4.

*Proof.* — If  $G_2^\circ$  is Abelian, then  $H \subset Z(G_2^\circ)$ . Conversely, let us assume that  $G_1^\circ$  is isomorphic either to  $G_a$ , or  $G_m$ , and  $H \subset Z(G_2^\circ)$ . For any  $M_0 \in G_2^\circ \setminus H$ , the subgroup  $\Omega$  generated by  $M_0$  and  $H$ , as well as its Zariski closure  $\bar{\Omega}$ , is an Abelian subgroup of  $G_2^\circ$ . By formula (2.9), we have  $\dim G_2^\circ = \dim H + 1$ , but

$$(\dim H + 1 \leq \dim \bar{\Omega} \leq \dim G_2^\circ) \implies \dim \bar{\Omega} = \dim G_2^\circ.$$

Since  $G_2^\circ$  is connected, we deduce that  $G_2^\circ = \bar{\Omega}$  is Abelian.

Note that  $H \subset Z(G_2^\circ)$  iff for all  $M = M(A, B) \in G_2^\circ$ , and all  $N = N(C) \in H$ , we have, thanks to (2.10),

$$MNM^{-1} = N \iff N(ACA^{-1}) = N(C) \iff ACA^{-1} = C \iff [A, C] = 0.$$

Now, we can prove the remaining points.

- (1) If  $G_1^\circ \simeq G_a$  we put

$$A = A(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We have

$$[C, A] = \begin{bmatrix} -tc & t(a-d) \\ 0 & -tc \end{bmatrix},$$

and thus

$$\begin{aligned}
 (\forall A \in G_a \ [C, A] = 0) &\iff (c = 0 \text{ and } a = d) \\
 &\iff C = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \iff H \subset H_a.
 \end{aligned}$$

(2) If  $G_1^\circ \simeq G_m$ , we proceed in a similar way and we put

$$A = A(t) = \begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix} \text{ and } C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then we obtain

$$[C, A] = \begin{bmatrix} 0 & -b(t - 1/t) \\ c(t - 1/t) & 0 \end{bmatrix},$$

and thus

$$\begin{aligned}
 (\forall A \in G_m \ [C, A] = 0) &\iff (b = 0 \text{ and } c = 0) \\
 &\iff C = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \iff H \subset H_m.
 \end{aligned}$$

□

### 2.2. From group to Wronskian relations

For two elements  $f, g \in F_2$ , we set

$$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}.$$

Observe that for  $\sigma \in G_2$  we have

$$\sigma(W(f, g)) = \begin{vmatrix} \sigma(f) & \sigma(g) \\ \sigma(f)' & \sigma(g)' \end{vmatrix} = W(\sigma(f), \sigma(g)).$$

Let  $x_1, x_2, y_1, y_2, X$  and  $Y$  be as they were defined in Section 2.1.

DEFINITION 2.7. — We define the following three conditions

- $W_1 : W(x_1, y_1) \in F_1.$
- $W_2 : W(x_2, y_2) \in F_1.$
- $W_3 : x_1W(x_1, y_2) - y_1 \in F_1.$

PROPOSITION 2.8. — Let  $H_a$  and  $H_m$  be the groups given in Definition 2.4. We have

- (1)  $H = \text{Gal}_\partial(F_2/F_1) \subset H_a$  iff condition  $W_3$  is fulfilled. Moreover,  $W_3 \Rightarrow W_1.$
- (2)  $H = \text{Gal}_\partial(F_2/F_1) \subset H_m$  iff the conditions  $W_1$  and  $W_2$  are fulfilled.

*Proof.* — Let  $\sigma \in H$ . Then  $\sigma(X) = X$ , and  $\sigma(Y) = Y + XB(\sigma)$  for a certain

$$B(\sigma) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}(2, \mathbb{C}).$$

Therefore, the action of  $\sigma$  on  $Y$  is given by the relations

$$\begin{cases} \sigma(y_1) = y_1 + ax_1 + cx_2, \\ \sigma(y_2) = y_2 + bx_1 + dx_2. \end{cases}$$

From these relations, the action of  $\sigma$  on the Wronskians is given by the following formulae

$$\begin{aligned} \sigma(W(x_1, y_1)) &= \begin{vmatrix} x_1 & y_1 + ax_1 + cx_2 \\ x'_1 & y'_1 + ax'_1 + cx'_2 \end{vmatrix} = W(x_1, y_1) + c, \\ \sigma(W(x_2, y_2)) &= \begin{vmatrix} x_2 & y_2 + bx_1 + dx_2 \\ x'_2 & y'_2 + bx'_1 + dx'_2 \end{vmatrix} = W(x_2, y_2) - b, \\ \sigma(W(x_1, y_2)) &= \begin{vmatrix} x_1 & y_2 + bx_1 + dx_2 \\ x'_1 & y'_2 + bx'_1 + dx'_2 \end{vmatrix} = W(x_1, y_2) + d, \\ \sigma(W(x_2, y_1)) &= \begin{vmatrix} x_2 & y_1 + ax_1 + cx_2 \\ x'_2 & y'_1 + ax'_1 + cx'_2 \end{vmatrix} = W(x_2, y_1) - a. \end{aligned}$$

To obtain these formulae, we used the fact that  $W(x_1, x_2) = 1$ . Moreover, we also have

$$\begin{aligned} \sigma(x_1W(x_1, y_2) - y_1) &= x_1\sigma(W(x_1, y_2)) - \sigma(y_1) \\ &= x_1W(x_1, y_2) + dx_1 - (y_1 + ax_1 + cx_2), \\ \sigma(x_1W(x_1, y_2) - y_1) &= [x_1W(x_1, y_2) - y_1] + (d - a)x_1 - cx_2. \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma(W(x_1, y_1)) = W(x_1, y_1) &\iff c = 0, \\ \sigma(W(x_2, y_2)) = W(x_2, y_2) &\iff b = 0, \\ \sigma(x_1W(x_1, y_2) - y_1) = x_1W(x_1, y_2) - y_1 &\iff [d = a \text{ and } c = 0]. \end{aligned}$$

For the last equivalence we used the fact that  $x_1$  and  $x_2$  are  $\mathbb{C}$ -linearly independent.

From the above equivalences we deduce that for  $\sigma \in H$ , we have

$$\begin{aligned} \sigma \in H_m &\iff (\sigma(W(x_1, y_1)) = W(x_1, y_1) \text{ and } \sigma(W(x_2, y_2)) = W(x_2, y_2)), \\ \sigma \in H_a &\iff \sigma(x_1W(x_1, y_2) - y_1) = x_1W(x_1, y_2) - y_1. \end{aligned}$$

and, moreover,

$$(\sigma(x_1W(x_1, y_2) - y_1) = x_1W(x_1, y_2) - y_1) \implies (\sigma(W(x_1, y_1)) = W(x_1, y_1)).$$

But for  $f \in F_2$  we have

$$f \in F_1 \iff (\forall \sigma \in H, \sigma(f) = f).$$

So,  $H \subset H_m$  if and only if conditions  $W_1$  and  $W_2$  hold. Similarly,  $H \subset H_a$  if and only if the conditions  $W_3$  is satisfied, moreover  $W_3 \Rightarrow W_1$ .  $\square$

### 2.3. From Wronskian to integral relations

#### 2.3.1. Computation of the Wronskian and resolution of equation (2.2)

From Proposition 2.5 we know that  $H = \text{Gal}_\partial(F_2/F_1)$  is a vector group, so it is solvable. Therefore, by Liouville-Kolchin solvability theorem, equation (2.2):  $y'' = ry + sx$  can be solved by finite integrations. In the following lemma we give, among others things, the explicit form of its solutions. Notice that in this lemma we do not make any assumption about the group  $G_1$ .

Let  $x_1$  be a non-zero solution of equation (2.1). According to Definition 2.2, let us set  $\varphi = \int sx_1^2$  and  $\psi = \int x_1^{-2}$ . Then  $x_2 = x_1\psi$  is another solution of (2.1), and  $W(x_1, x_2) = 1$ . Let  $y_1$  and  $y_2$  be two particular solution of (2.2) given (2.5). Then we have the following.

LEMMA 2.9. — *Up to additive constants we have*

$$\begin{aligned} W(x_1, y_1) &= \int sx_1^2, & W(x_2, y_2) &= \int sx_2^2, \\ W(x_1, y_2) &= \int sx_1x_2, & W(x_2, y_1) &= \int sx_1x_2, \\ y_1 &= x_1 \int \left( x_1^{-2} \int sx_1^2 \right), & y_2 &= x_2 \int \left( x_2^{-2} \int sx_2^2 \right), \end{aligned}$$

and

$$x_1W(x_1, y_2) - y_1 = x_1Q, \quad \text{where } Q = \varphi\psi - 2 \int \varphi\psi' = -\varphi\psi + 2 \int \psi\varphi'.$$

*Proof.* — Identities with Wronskians can be checked by a direct differentiation. Formulae for  $y_1$  and  $y_2$  are obtained by a classical variations of constants method.  $\square$

COROLLARY 2.10. — *Let  $\sigma \in H = \text{Gal}_\partial(F_2/F_1)$ , and  $N(B(\sigma)) \in H_{\max}$  be the matrix of  $\sigma$ . Then  $\text{Tr}(B(\sigma)) = 0$ .*

*Proof.* — As in the proof of Proposition 2.8 we set  $B = B(\sigma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . From this proof we know also that

$$\sigma(W(x_1, y_2)) = W(x_1, y_2) + d \quad \text{and} \quad \sigma(W(x_2, y_1)) = W(x_2, y_1) - a.$$

But from Lemma 2.9 we know that  $\Delta = W(x_1, y_2) - W(x_2, y_1)$  is a constant belonging to  $\mathbb{C}$ . Therefore, for  $\sigma \in H$ ,

$$\sigma(\Delta) = \Delta + d + a = \Delta + \text{Tr}(B) = \Delta.$$

So,  $\text{Tr}(B) = 0$ . □

### 2.3.2. Study of the conditions $W_i$

From now on, as in Lemma 2.1,  $L$  denotes the algebraic closure of  $K$  in  $F_1$ .

LEMMA 2.11. — *Let  $T(F_1/K) \subset F_1$  be the Picard Vessiot ring of  $F_1/K$ . With the notations of Lemmas 1.5 and 2.1, we have*

- (1)
  - If  $G_1^\circ \simeq G_a$ , then  $T(F_1/K) = L[x_2] = L[\psi]$ .
  - If  $G_1^\circ \simeq G_m$ , then  $T(F_1/K) = L[x_1, x_2] = L[x_1, x_1^{-1}]$ .
- (2) If  $G_1^\circ \simeq G_a$ , then the condition  $W_1$  is equivalent to  $(\alpha)$  and  $W_3$  to  $(\beta)$ .
- (3) If  $G_1^\circ \simeq G_m$ , then the condition  $W_1$  is equivalent to  $(\gamma)$  and  $W_2$  to  $(\delta)$ .

*Proof.* — (1) From the relation  $W(x_1, x_2) = 1$ , and Lemma 1.5, we have

$$T(F_1/K) = K[x_1, x_2, x'_1, x'_2] = L[x_1, x_2, x'_1, x'_2].$$

Moreover,  $F_1$  is the field of fractions of the ring  $T(F_1/K)$ . Let us compute this ring in the two particular cases.

If  $G_1^\circ \simeq G_a$ , then, by assumption,  $x_1 \in L$ , so  $x'_1 \in L$ . Since  $x_2 = x_1 \int x_1^{-2} = x_1 \psi$ , we have

$$T(F_1/K) = L[x_2] = L[x_1 \psi] = L[\psi].$$

If  $G_1^\circ \simeq G_m$ , then  $G_1^\circ$  acts on  $x_1$  by a character, so it acts on  $x'_1$  by the same character. Therefore, the logarithmic derivative  $x'_1/x_1$  is left invariant by  $G_1^\circ$ , hence belongs to  $L$ . Moreover, from Lemma 2.1,  $x_1 x_2 \in L$ , so similarly  $x'_2/x_2 \in L$ , and we have

$$T(F_1/K) = L[x_1, x_2] = L[x_1, x_1^{-1}].$$

(2) Since  $G_1^\circ \simeq G_a$ ,

$$F_1 = L(x_2) = L\left(\int x_1^{-2}\right) = L(\psi).$$

Therefore, from Lemma 2.9, the condition  $W_1$  may be written

$$W(x_1, y_1) = \int sx_1^2 \in L \left( \int x_1^{-2} \right).$$

Thus, condition  $W_1$  implies that the two primitive integrals:  $\varphi = \int sx_1^2$ , and  $\psi = \int x_1^{-2}$  are algebraically dependent over  $L$ . Hence, by the Ostrowski-Kolchin theorem (Lemma 1.5 point 1), this implies that there exists  $(c_1, c_2) \in \mathbb{C}^2 \setminus \{0, 0\}$  such that

$$c_1\varphi + c_2\psi \in L.$$

But  $c_1 = 0$  implies that  $\psi = \int x_1^{-2} \in L$ , and  $x_2 = -x_1\psi$  is algebraic over  $K$ , however it is not true. So, dividing the linear relation by  $c_1$  we get that  $W_1 \Rightarrow (\alpha)$ . Conversely, if  $(\alpha)$  holds then,  $\int sx_1^2 \in L(\int x_1^{-2}) = L(x_2) = F_1$  and  $W_1$  is satisfied.

From Lemma 2.9 we have

$$x_1Q = x_1W(x_1, y_2) - y_1,$$

where

$$Q = \varphi\psi - 2 \int \varphi\psi' = -\varphi\psi + 2 \int \psi\varphi'.$$

From Lemma 1.5, the element  $x_1Q = x_1W(x_1, y_2) - y_1 \in T(F_2/K)$ . But here,  $x_1$  is algebraic over  $K$  so from Lemma 1.5 again,  $Q = \frac{1}{x_1} \cdot x_1Q \in T(F_2/K)$ . Therefore, the condition  $Q \in F_1$ , is equivalent to  $Q \in T(F_1/K)$ , because  $T(F_1/K)$  is the algebra containing the elements of  $F_1$  which are solutions of a certain linear differential equation over  $K$ . So, we have the following equivalences

$$W_3 \iff (x_1Q \in F_1) \iff (Q \in F_1) \iff (Q \in T(F_1/K)) \iff (Q \in L[\psi]).$$

Thus, condition  $W_3$  is equivalent to condition  $(\beta)$ .

(3) Since  $G_1^\circ \simeq G_m$ , the role of  $x_1$  and  $x_2$  are symmetric. We have to prove only that the conditions  $W_1$  and  $(\gamma)$  are equivalent. As before,  $W(x_1, y_1) \in T(F_2/K)$  so,

$$W(x_1, y_1) \in F_1 \iff W(x_1, y_1) \in T(F_1/K) = L[x_1, x_1^{-1}].$$

Since  $W(x_1, y_1) = \int sx_1^2$ , condition  $W_1$  is equivalent to

$$\int sx_1^2 \in L[x_1, x_1^{-1}].$$

The above condition is fulfilled iff we have a relation of the form

$$\int sx_1^2 = \sum_{n=p}^q f_n x_1^n, \quad p \leq q; \quad p, q \in \mathbb{Z},$$

with  $f_n \in L$ . Differentiating the above equation we obtain

$$sx_1^2 = \sum_{n=p}^q (f'_n + f_n n\theta)x_1^n,$$

where  $\theta = x'_1/x_1 \in L$ .

But  $x_1$  is transcendental over  $L$ , so from the last formula we have  $f'_n + f_n n\theta = 0$  for  $n \neq 2$ , and  $s = f'_2 + 2f_2\theta$ . Thus we have

$$sx_1^2 = (\phi_1 x_1^2)',$$

with  $\phi_1 = f_2 \in L$ . This proves that condition  $W_1$  is equivalent to condition  $(\gamma)$ .  $\square$

#### 2.4. Proof of Theorem 2.3

*Proof.* — As a connected subgroup of  $\mathrm{SL}(2, \mathbb{C})$ , group  $G_1^\circ$  is isomorphic to one of the following groups:  $\{\mathrm{Id}\}$ ,  $G_a$ ,  $G_m$ , the semi-direct product  $G_a \rtimes G_m$ , or  $\mathrm{SL}(2, \mathbb{C})$ , see, e.g. [7]. If  $G_1^\circ$  is Abelian, then the last two possibilities must be excluded.

- If  $G_1^\circ = \{\mathrm{Id}\}$ , then  $G_2^\circ$  is Abelian thanks to point 5 of Proposition 2.5.
- If  $G_1^\circ \simeq G_a$ , (resp.  $G_1^\circ \simeq G_m$ ), then the proof follows from Proposition 2.8 and point 2, (resp. point 3) of Lemma 2.11.  $\square$

### 3. Elimination of the Jordan blocks with $G_1^\circ \simeq G_a$ .

We now apply the results of the previous section to the study of the connected component  $G(\mathrm{VE}_z)^\circ$  of the Galois group of the VE (1.10)

$$\frac{d^2\zeta}{dz^2} = [r_0(z)\mathrm{Id} + s(z)V''(c)]\zeta.$$

Our main result in this section is the following.

**THEOREM 3.1.** — *Assume that  $V''(c)$  has a Jordan block of size  $d \geq 2$ , and  $G_1^\circ \simeq G_a$ . Then  $G(\mathrm{VE}_z)^\circ$  is not Abelian. This corresponds to the elimination of rows 2,3, and 4 in Table 1.1.*

*Remark 3.2.* — Let  $B(\lambda, d)$  be a Jordan block of  $V''(c)$  with size  $d \geq 2$ , and eigenvalue  $\lambda$ . Since  $G_1^\circ$  is isomorphic to  $G_a$  and corresponds to the VE

$$\frac{d^2\eta}{dt^2} = -\lambda\varphi^{k-2}(t)\eta,$$

we deduce from Theorem 1.2 that necessarily, the pair  $(k, \lambda)$  must belong to row 2, 3, or 4 in Table 1.1. Now, passing to the VE in the  $z$  variable, we know that the system (1.11) with Galois group  $G_2$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} r & 0 \\ s & r \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

is a subsystem of VE (1.10). From Proposition 1.6,  $G(\text{VE}_t)^\circ \simeq G(\text{VE}_z)^\circ$  and  $G_2^\circ$  is a quotient of  $G(\text{VE}_z)^\circ$ . Therefore, it is enough to prove that  $G_2^\circ$  is not Abelian. To this aim we proceed as follows. According to Theorem 2.3, we have to prove that condition  $(\beta)$  is not fulfilled. Since  $(\beta) \Rightarrow (\alpha)$ , and  $(\alpha)$  is much easier to check than  $(\beta)$ , at first we check if  $(\alpha)$  is fulfilled. Since  $(\alpha)$  is a condition concerning the primitive integrals  $\varphi = \int sx_1^2$  and  $\psi = \int x_1^{-2}$ , where  $x_1$  is an algebraic solution of equation  $x'' = rx$ , we first have to investigate analytical properties of these integrals.

### 3.1. Assumptions and notations

We assume that  $G_1^\circ \simeq G_a$ . From Table 1.1, we must have

$$\lambda = p + \frac{k}{2}p(p - 1),$$

for a certain  $p \in \mathbb{Z}$ . In this case  $x_1$  is algebraic over  $K = \mathbb{C}(z)$  and  $x_2 = x_1 \int x_1^{-2}$  is transcendental.

**DEFINITION 3.3.** — *Let  $f(z)$  be a multivalued function of the complex variable  $z$ , and let  $z_0 \in \mathbb{P}^1$ . We say that  $e \in \mathbb{C}$  is the exponent of  $f$  at  $z_0$ , if in a neighbourhood of  $z_0$ ,  $f$  can be expressed into the following form*

$$f(z) = \zeta^e h(\zeta),$$

where  $\zeta$  is a local parameter around  $z_0$ ,  $\zeta \mapsto h(\zeta)$  is holomorphic at  $\zeta = 0$  and  $h(0) \neq 0$ .

The principal part of  $f$  at  $z_0$  is denoted  $f_{z_0}$ , i.e,  $f_{z_0} = \zeta^e h(0)$ .

We denote by  $\mathcal{M}_{z_0}$  the monodromy operator around  $z_0$ .

**LEMMA 3.4.** — *If  $G_1^\circ \simeq G_a$  then,*

- (1) *Up to a complex multiplicative constant, the algebraic solution  $x_1$  may be written in the form*

$$x_1 = z^a(z - 1)^b J(z) \quad \text{where} \quad a \in \left\{ \frac{k - 1}{2k}, \frac{k + 1}{2k} \right\}, \quad b \in \left\{ \frac{1}{4}, \frac{3}{4} \right\},$$

and  $J(z) \in \mathbb{R}[z]$  does not vanish at  $z \in \{0, 1\}$ .



(2) The function  $\psi = \int x_1^{-2}$  has the exponent  $1 - 2b$  at  $z = 1$  and  $\mathcal{M}_1(\psi) = -\psi$ .

*Proof.* — (1) For all  $\sigma \in G_1$ ,  $\sigma(x_1) = \chi(\sigma)x_1 + \mu(\sigma)x_2$  for certain  $(\chi(\sigma), \mu(\sigma)) \in \mathbb{C}^2$ . But  $\sigma(x_1)$  is still algebraic, hence  $\mu(\sigma) = 0$ , and  $\sigma(x_1) = \chi(\sigma)x_1$ . In particular  $x_1$  is an eigenvector of the monodromy operators  $\mathcal{M}_0$  and  $\mathcal{M}_1$ . For  $|k| \geq 3$ , from equation (1.13), the differences of exponents at  $z = 0$  and  $z = 1$  are not integers, hence we can deduce that  $x_1$  is a principal branch of the Riemann scheme (1.14) at  $z = 0$  and at  $z = 1$ . Therefore,  $x_1$  may be written in the form  $x_1 = z^a(z - 1)^b J(z)$  where  $a$ , (resp.  $b$ ) is an exponent at  $z = 0$  (resp. at  $z = 1$ ), and  $J(z)$  is holomorphic on  $\mathbb{C}$ . Since  $a$  and  $b$  are rational numbers,  $J(z) = x_1/z^a(z - 1)^b$  is an algebraic function which is holomorphic on  $\mathbb{C}$  hence,  $J(z)$  is a polynomial. For  $|k| = 1$ , we have  $\{\frac{k-1}{2k}, \frac{k+1}{2k}\} = \{0, 1\}$ , therefore  $x_1$  is regular at  $z = 0$ . At  $z = 1$  the difference of exponents is  $\Delta_1 = 1/2$ , so the previous arguments apply, and point 1 is still true with  $a \in \{0, 1\}$ . Moreover,  $J(0) \neq 0$  and  $J(1) \neq 0$ . Since the exponents are real,  $J$  is a solution of a second order differential equation over  $\mathbb{R}(z)$ . Thus, we can assume that  $J \in \mathbb{R}[z]$ .

(2) The function  $x_1^{-2}$  has the exponent  $-2b$  at  $z = 1$ . Thus, expanding it around  $z = 1$  and integrating, we obtain that  $\psi$  has the exponent  $1 - 2b$  at  $z = 1$ . Therefore,  $\mathcal{M}_1(\psi) = \exp[2\pi i(1 - 2b)]\psi = \exp[-4\pi i b]\psi = -\psi$ , because  $b \in \{\frac{1}{4}, \frac{3}{4}\}$ . □

Now, thanks to Remark 3.2, at first we have to test conditions  $(\alpha)$  for  $\varphi$  and  $\psi$ . If we set

$$\theta := z^a(z - 1)^b,$$

then, using Lemma 3.4, we have the explicit formulae

$$\varphi = \int s x_1^2 = \int s \theta^2 J^2 = \frac{1}{2k} \int z^{2a-1} (z - 1)^{2b-1} J^2(z) dz,$$

$$\psi = \int x_1^{-2} = \int \frac{1}{\theta^2 J^2}.$$

### 3.2. Algebraicity of $\psi$ and $\varphi$

Since  $G_1^\circ \simeq G_a$ , we know that  $\psi$  is not algebraic and we have the following.

LEMMA 3.5. — *Let  $|k| \geq 3$ . If condition  $(\alpha)$  holds then  $\varphi$  is algebraic.*

*Proof.* — Let  $\tilde{L} = \mathbb{C}(z)[\theta^2]$  where  $\theta = z^a(z - 1)^b$ , and

$$a \in \left\{ \frac{k-1}{2k}, \frac{k+1}{2k} \right\}, \quad \text{and} \quad b \in \left\{ \frac{1}{4}, \frac{3}{4} \right\}.$$

This is an algebraic extension of  $K = \mathbb{C}(z)$  of degree

$$N = \begin{cases} |k| & \text{when } k \in 2\mathbb{N}, \\ 2|k| & \text{when } k \notin 2\mathbb{N}. \end{cases}$$

Indeed, the minimal equation for  $\theta^2$  is  $(\theta^2)^N = z^{2Na}(z - 1)^{2Nb} \in \mathbb{C}[z]$ . Therefore, a basis of  $\tilde{L}/K$  is  $\{\theta^{-2}, 1, \theta^2, \dots, (\theta^2)^{N-2}\}$ , and  $N - 2 \geq 2$  since  $|k| \geq 3$ . As  $\varphi' \in \tilde{L}$ , and  $\psi' \in \tilde{L}$ , from the Ostrowski-Kolchin theorem (see point 1 of Lemma 1.5), we deduce that condition  $(\alpha)$  holds iff there exists  $c \in \mathbb{C}$  such that,  $\varphi + c\psi \in \tilde{L}$ . But  $\varphi + c\psi \in \tilde{L}$  iff there exists a family  $(f_{-1}, \dots, f_{N-2}) \in \mathbb{C}(z)^N$  such that

$$\varphi + c\psi = \sum_{i=-1}^{N-2} f_i(\theta^2)^i.$$

Differentiating the above equality, we obtain

$$\begin{aligned} \varphi' + c\psi' &= \sum_{i=-1}^{N-2} \left( f'_i + 2i \frac{\theta'}{\theta} f_i \right) \theta^{2i}, \\ \frac{1}{2kz(z-1)} \theta^2 J^2(z) + \frac{c}{\theta^2 J^2(z)} &= \sum_{i=-1}^{N-2} \left( f'_i + 2i \frac{\theta'}{\theta} f_i \right) \theta^{2i}. \end{aligned}$$

From this equation, we necessarily have

$$\begin{aligned} \frac{c}{\theta^2 J^2} &= \left( f'_{-1} - 2 \frac{\theta'}{\theta} f_{-1} \right) \frac{1}{\theta^2} \iff c\psi = \frac{f_{-1}}{\theta^2}, \\ \frac{1}{2kz(z-1)} \theta^2 J^2 &= \left( f'_1 + 2 \frac{\theta'}{\theta} f_1 \right) \theta^2 \iff \varphi = f_2 \theta^2. \end{aligned}$$

The first equation implies that  $c = 0$  because  $\psi$  is not algebraic. The second equation implies that  $\varphi$  is algebraic. Moreover,  $\varphi$  is algebraic iff there exists  $f \in \mathbb{C}(z)$  such that

$$(3.1) \quad \frac{J^2(z)}{z(z-1)} = f' + 2 \frac{\theta'}{\theta} f.$$

□

Since  $\theta = z^a(z-1)^b$ , equation (3.1) is equivalent to  $J^2 = T(f) := z(z-1)f' + 2((a+b)z-a)f$ . Therefore we have the equivalence

$$(3.2) \quad \varphi = \int sx_1^2 = f\theta^2 \iff J^2 = T(f) = z(z-1)f' + 2((a+b)z-a)f.$$

### 3.3. Algebraicity of $\varphi$ and condition $(\alpha)$

At the end of the previous subsection we showed that  $\varphi$  is algebraic iff the equation  $J^2 = T(f)$  defined by (3.2), has a rational solution  $f$ . The next Lemma gives an answer to this problem.

LEMMA 3.6. — *Let  $J \in \mathbb{R}[z]$  such that  $J(0)J(1) \neq 0$ . Then,*

- (1) *If  $a \neq 1$ , then the equation  $J^2 = T(f)$  does not have rational solutions and  $\varphi$  is not algebraic.*
- (2) *If  $a = 1$ , and the equation  $J^2 = T(f)$  has a solution  $f \in \mathbb{C}(z)$ , then  $f(z) = c(z^{-2} + 2bz^{-1}) + g(z)$  where  $c \neq 0$  is a constant, and  $g(z)$  is a polynomial.*

*Proof.* — Let  $f \in \mathbb{C}(z)$  be such that  $J^2 = T(f)$ , in particular  $T(f)$  is a polynomial. We separate into three steps our further reasoning.

*First step.* We prove that  $f$  has only few poles, precisely we claim that

- (1) if  $a \neq 1$ , then  $f \in \mathbb{R}[z]$ ;
- (2) if  $a = 1$ , then  $f(z) = c(z^{-2} + 2bz^{-1}) + g(z)$  with  $c \in \mathbb{C}$ , and  $g(z) \in \mathbb{C}[z]$ .

Indeed, if  $f$  has a pole of order  $n$  at  $t$ , setting  $f_t = c(z-t)^{-n}$ , we have the following possibilities for the principal part of  $T(f)$ :

$$\begin{aligned} T(f)_t &= \frac{-cnt(t-1)}{(z-t)^{n+1}} \quad \text{for } t \notin \{0, 1\}, \\ T(f)_0 &= \frac{c(n-2a)}{z^n} \quad \text{for } t = 0, \\ T(f)_1 &= \frac{c(2b-n)}{(z-1)^n} \quad \text{for } t = 1. \end{aligned}$$

If  $t \notin \{0, 1\}$ , then  $T(f)_t \neq 0$ , so  $t$  is not a pole of  $f$ . Similarly, since  $2b-n \neq 0$ ,  $T(f)_1 \neq 0$ , and  $t = 1$  cannot be a pole of  $f$ . Now, the formula  $T(f)_0 = c(n-2a)z^{-n}$  is valid iff  $n-2a \neq 0$ . But  $n-2a = n-1 \pm \frac{1}{k}$ , thus

$$(n-2a=0) \iff \left(n=1 \mp \frac{1}{k}\right) \iff (n=2 \quad \text{and} \quad a=1 \quad \text{and} \quad k=\mp 1).$$

Therefore, if  $a \neq 1$ , then  $f$  does not have pole at  $z = 0$ , and  $f$  must be a polynomial. Now if  $f$  has a pole at  $z = 0$ , according to the previous

equivalence, we must have  $a = 1$  and  $n = 2$ . But, if  $a = 1$ , then  $T(f) = z(z - 1)f' + 2((1 + b)z - 1)f$ , and we have

$$T\left(\frac{1}{z^2}\right) = \frac{2b}{z} \quad \text{and} \quad T\left(\frac{1}{z}\right) = -\frac{1}{z} + 2b + 1.$$

If  $f$  is a solution which is not a polynomial, it must have a pole of order two at zero, and, for the compensation, we must have  $f(z) = c(z^{-2} + 2bz^{-1}) + g(z)$ , where  $c \in \mathbb{C}$  and  $g(z) \in \mathbb{C}[z]$ .

*Second step.* We now treat the particular case  $a = 0$ . If  $f$  is a rational solution of the equation  $J^2 = T(f)$ , then, by the first step,  $f$  is a polynomial. Evaluating this equation at  $z = 0$  we get

$$J^2(0) = -2af(0).$$

Therefore, if  $a = 0$ , then  $J(0) = 0$ , but this is not true. Thus, in this case, the equation does not have rational solutions.

*Third step.* Under the assumption that  $a \neq 0$  we claim that the equation  $T(f) = J^2$  does not have polynomial solutions. Since  $\theta = z^a(z - 1)^b$ , equivalence (3.2) can be written in the following form

$$z^{2a-1}(z - 1)^{2b-1}J^2(z) = \frac{d}{dz}(f(z)z^{2a}(z - 1)^{2b}),$$

where

$$a \in \left\{ \frac{k - 1}{2k}, \frac{k + 1}{2k} \right\}, \quad b \in \left\{ \frac{1}{4}, \frac{3}{4} \right\}.$$

Hence, since  $a \neq 0$ , we have  $2a = 1 \pm \frac{1}{k} > 0$ , and moreover,  $2b \geq \frac{1}{2}$ . Therefore integrating between 0 and 1 we get

$$f(z)z^{2a}(z - 1)^{2b}\Big|_0^1 = 0 = \int z^{2a-1}(z - 1)^{2b-1}J^2(z)dz > 0,$$

since the integrand is positive. The above contradiction proves the claim. As a conclusion, if  $a \neq 1$ , the equation  $J^2 = T(f)$  does not have rational solution. This proves Point 1. When  $a = 1$ , and  $J^2 = T(f)$  possesses a rational solution, the latter cannot be a polynomial. and by the first step, point 2 follows. □

In the case  $a = 1$ , which happens only for  $k = \pm 1$ ,  $\varphi$  can be algebraic, so condition  $(\alpha)$  can be satisfied. For example, computations with Riemann schemes show that for row 4 in Table 1.1, when  $(k, \lambda) = (-1, -2)$ , we have  $x_1 = z(z - 1)^{3/4}$ , and

$$\varphi = \int z(1 - z)^{1/2} dz = \frac{6z^2 - 2z - 4}{15} \sqrt{1 - z} \in L = \mathbb{C}(z)[x_1] = \mathbb{C}(z)[(1 - z)^{1/4}],$$

is therefore algebraic and condition  $(\alpha)$  is satisfied. Nevertheless, for those cases we have the following.

LEMMA 3.7. — *Let us assume that  $a = 1$ . If condition  $(\alpha)$  holds, then condition  $(\beta)$  is not satisfied.*

*Proof.* — By Definition 2.2 and Theorem 2.3, we have to check if the condition

$$Q = \varphi\psi - 2 \int \varphi\psi' \in L[\psi],$$

is satisfied. By assumption,  $\varphi + c\psi \in L$ , for a certain  $c \in \mathbb{C}$ . Thus, we have

$$(Q \in L[\psi]) \iff \left( I(z) = \int \psi' \cdot \varphi \in L[\psi] \right),$$

where

$$I(z) = \int \psi' \cdot \varphi = \int \frac{f\theta^2}{J^2\theta^2} = \int \frac{f(z)}{J^2(z)} dz.$$

As  $a = 1$ , by point 2 of Lemma 3.6,  $f(z) = c(z^{-2} + 2bz^{-1}) + g(z)$  with  $c \neq 0$ . Therefore  $I(z)$  may be expressed by a formula of the form

$$I(z) = \gamma_0 \text{Log}(z) + \sum \gamma_i \text{Log}(z - z_i) + h(z),$$

where,  $h(z) \in \mathbb{C}(z)$ ,  $\gamma_0 = -2bc/J^2(0) \neq 0$ ,  $\gamma_i \in \mathbb{C}$ , and  $z_i$  are roots of  $J(z)$ . In particular  $z_i \notin \{0, 1\}$ . Hence,  $I(z) \in L[\psi]$  if and only if  $I(z)$  and  $\psi(z)$  are algebraically dependent. But, by the Ostrowski-Kolchin theorem, this happens if and only if we have a non trivial linear relation with complex coefficients

$$\mu I(z) + \nu \psi(z) = \omega(z) \in L.$$

However,  $\mathcal{M}_1(I(z)) = I(z)$  and, from Lemma 3.4,  $\mathcal{M}_1(\psi) = \exp[-\pi i]\psi = -\psi$ . Applying the monodromy operator to the previous equation yields

$$\mu I(z) - \nu \psi(z) = \mathcal{M}_1(\omega(z)).$$

So,  $2\mu I(z) = \omega(z) + \mathcal{M}_1(\omega(z))$  is algebraic. As  $I(z)$  is not algebraic, because  $\gamma_0 \neq 0$ , we deduce that  $\mu = \nu = 0$  and condition  $(\beta)$  is not satisfied.  $\square$

*Proof of Theorem 3.1.* — By Remark 3.2, it is enough to show that  $G_2^0$  is not Abelian. Since here  $G_1^0 \simeq G_a$ , from Theorem 2.3, it remains to show that conditions  $(\alpha)$  and  $(\beta)$  are not simultaneously satisfied.

From Remark 3.2 again, the pair  $(k, \lambda)$  must belong to rows 2, 3, or 4 of Table 1.1. In particular, either  $|k| \geq 3$ , or  $k = \pm 1$ .

- For  $|k| \geq 3$ , condition  $(\alpha)$  is not satisfied. Indeed, from Lemma 3.5, condition  $(\alpha)$  implies that  $\varphi$  is algebraic and, from point 1 of Lemma 3.6, we know that in this case  $\varphi$  is not algebraic.

- For  $k = \pm 1$ , condition  $(\alpha)$  may be satisfied but if this happens, by Lemma 3.7, condition  $(\beta)$  is not satisfied.

The above finishes the proof. □

#### 4. Elimination of the Jordan blocks with $G_1^\circ \simeq \{\text{Id}\}$

Our main result in this section is the following.

**THEOREM 4.1.** — *Assume that  $V''(c)$  has a Jordan block of size  $d \geq 3$ , and  $G_1$  is a finite subgroup of  $\text{SL}(2, \mathbb{C})$ . Then  $G(\text{VE}_z)^\circ$  is not Abelian. This eliminates the rows with numbers from 5 to 21 in Table 1.1.*

*Remark 4.2.* — Let  $B(\lambda, d)$  be a Jordan block of  $V''(c)$  with size  $d \geq 3$  and eigenvalue  $\lambda$ . Since  $G_1$  is finite and correspond to the VE

$$\frac{d^2\eta}{dt^2} = -\lambda\varphi^{k-2}(t)\eta,$$

we deduce from Theorem 1.2, that necessarily, the pair  $(k, \lambda)$  must belong to rows 5 to 21 of Table 1.1. Now, passing to the VE in the  $z$  variable, we know that the system

$$\begin{bmatrix} x'' \\ y'' \\ u'' \end{bmatrix} = \begin{bmatrix} r & 0 & 0 \\ s & r & 0 \\ 0 & s & r \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix},$$

with Galois group  $G_3$  is a subsystem of VE (1.10). From Proposition 1.6,  $G(\text{VE}_t)^\circ \simeq G(\text{VE}_z)^\circ$  and  $G_3^\circ$  is a quotient of  $G(\text{VE}_z)^\circ$ . Therefore it is enough to prove that  $G_3^\circ$  is not Abelian.

Recall from Lemma 2.1 that if  $G_1$  is finite, then it is one of the following types

- (1) Dihedral type:  $G_1$  is conjugated to a finite subgroup of

$$D^\dagger = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{bmatrix} \mid \lambda \in \mathbb{C}^* \right\} \cup \left\{ \begin{bmatrix} 0 & \lambda \\ -1/\lambda & 0 \end{bmatrix} \mid \lambda \in \mathbb{C}^* \right\}$$

- (2) Tetrahedral type:  $G_1/\{\pm\text{Id}\} \simeq \mathfrak{A}_4$
- (3) Octahedral type:  $G_1/\{\pm\text{Id}\} \simeq \mathfrak{S}_4$
- (4) Icosahedral type:  $G_1/\{\pm\text{Id}\} \simeq \mathfrak{A}_5$

From Theorem 2.3, we know that if  $G_1$  is finite,  $G_2^\circ$  is Abelian, where  $G_2$  is the Galois group of the two first equations of the above system. This why we have to consider Jordan blocks of size  $d \geq 3$ , in order to find obstructions to the integrability. At first, we build some theoretical results in the spirit of Section 2.

#### 4.1. Theory for Jordan blocks of size three

Now we assume that the size of the Jordan block is three. With the notations of Section 2, the subsystem of the variational equations corresponding to the block, can be written in the following two equivalent forms

$$(4.1) \quad \begin{cases} x'' = rx \\ y'' = ry + sx \\ u'' = ru + sy \end{cases} \iff \begin{bmatrix} x' \\ x'' \\ y' \\ y'' \\ u' \\ u'' \end{bmatrix} = \begin{bmatrix} R & 0 & 0 \\ S & R & 0 \\ 0 & S & R \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \\ u \\ u' \end{bmatrix},$$

where  $R$  and  $S$  are  $2 \times 2$  matrices given by (2.4).

Let us fix more notations.

- $F_1/K$  is the Picard-Vessiot extension associated to the equation  $L_2(x) = x'' - rx = 0$ . Its Galois group is still denoted by  $G_1$ .
- $F_2/K$  is the Picard-Vessiot extension associated to the first two equations of (4.1). Its Galois group, is still denoted by  $G_2$ .
- $F_3/K$  is the Picard-Vessiot extension over  $K$  associated to (4.1). Its Galois group is denoted by  $G_3$ .

*Remark 4.3.* — We have the following inclusions of differential fields

$$K \subset F_1 \subset F_2 \subset F_3.$$

All the results of Section 2 can be applied to the extension  $F_2/K$ . In particular, since  $G_1$  is finite, from Theorem 2.3,  $G_2^\circ$  is Abelian. Therefore,  $G_2^\circ$  is an Abelian quotient of  $G_3^\circ$ .

We fix a basis  $\{x_1, x_2\}$  of the solution space  $V$  of  $L_2 = 0$ . Let  $(y_1, y_2, u_1, u_2)$  be an element of  $F_3^4$  such that

$$\begin{cases} y_1'' = ry_1 + sx_1 \\ y_2'' = ry_2 + sx_2 \end{cases} \quad \text{and} \quad \begin{cases} u_1'' = ru_1 + sy_1 \\ u_2'' = ru_2 + sy_2 \end{cases}.$$

Then, we set

$$X = \begin{bmatrix} x_1 & x_2 \\ x_1' & x_2' \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}, \quad U = \begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix}, \quad \Xi_3 = \begin{bmatrix} X & 0 & 0 \\ Y & X & 0 \\ U & Y & X \end{bmatrix}.$$

Similarly as in Section 2,  $\Xi_3$  is a fundamental matrix of solutions of (4.1).

For all  $\sigma \in G_3$ , the equation  $\sigma(\Xi_3) = \Xi_3 M(\sigma)$ , forces  $\sigma$  to be represented by a  $6 \times 6$  matrix  $M(\sigma)$  of the form

$$M(\sigma) = \begin{bmatrix} A(\sigma) & 0 & 0 \\ B(\sigma) & A(\sigma) & 0 \\ C(\sigma) & B(\sigma) & A(\sigma) \end{bmatrix}.$$

PROPOSITION 4.4. — Assume that  $G_1$  is finite. Then  $G_3^\circ$  is Abelian iff there exists a basis  $\{x_1, x_2\}$  of  $V = \text{Sol}(L_2)$  such that one of the following condition is satisfied

- $\varphi_1 = \int sx_1^2 \in F_1$  and  $\int \varphi_1' \psi_1 \in F_1$  where  $\psi_1 = \int x_1^{-2}$ .
- $\int sx_1^2 \in F_1$  and  $\int sx_2^2 \in F_1$ .

If  $G_3^\circ$  is Abelian, then there exists at least one non-zero  $x \in V = \text{Sol}(L_2)$  such that  $\int sx^2 \in F_1$ .

Proof. — We consider  $G_3^\circ$  as a subgroup of  $\text{SL}(6, \mathbb{C})$ . The elements of  $G_3^\circ$  are matrices the form

$$P(B, C) := \begin{bmatrix} \text{Id} & 0 & 0 \\ B & \text{Id} & 0 \\ C & B & \text{Id} \end{bmatrix}.$$

The product and the commutators of two such matrices are given by

$$\begin{aligned} P(B_1, C_1)P(B_2, C_2) &= P(B_1 + B_2, C_1 + C_2 + B_1B_2), \\ [P(B_1, C_1)P(B_2, C_2)] &= P(0, [B_1, B_2]). \end{aligned}$$

Set

$$\mathbb{B} := \{B \in \mathbb{M}(2, \mathbb{C}) \mid \exists C \in \mathbb{M}(2, \mathbb{C}) \text{ and } P(B, C) \in G_3^\circ\}.$$

Then, thanks to the above formulae,  $G_3^\circ$  is Abelian iff any two matrices belonging to  $\mathbb{B}$  commute. This is the case iff, up to conjugation,  $\mathbb{B}$  is contained either in the set of upper triangular matrices with diagonal of the form  $a\text{Id}$ , or,  $\mathbb{B}$  is contained in the set of diagonal matrices. For any of this two cases, thanks to a conjugation formula similar to (2.10), we can find a basis  $\{x_1, x_2\}$  of  $V$  such that the representation of the elements of  $\mathbb{B}$  in this basis are either upper triangular or diagonal.

From point 4 of Proposition 2.5, we have  $G_2^\circ = H = \text{Gal}_\partial(F_2/F_1)$ . Let

$$\pi_2 : G_3^\circ \rightarrow G_2^\circ, \quad P(B, C) \mapsto \begin{bmatrix} \text{Id} & 0 \\ B & \text{Id} \end{bmatrix} = N(B),$$

be the projection. With the notations of Proposition 2.6, the two above conditions for  $\mathbb{B}$  are respectively equivalent to  $\pi_2(G_3^\circ) = G_2^\circ \subset H_a$ , and  $\pi_2(G_3^\circ) = G_2^\circ \subset H_m$ .



But, from Proposition 2.8, we have

$$\begin{cases} G_2^\circ \subset H_a & \iff W_1 \text{ and } W_3 \text{ hold,} \\ G_2^\circ \subset H_m & \iff W_1 \text{ and } W_2 \text{ hold.} \end{cases}$$

Now, from Definition 2.7 and Lemma 2.9, condition  $W_1$  holds iff  $W(x_1, y_1) = \varphi_1 = \int sx_1^2 \in F_1$ , and the same result holds for condition  $W_2$ . From the same definition and lemma, condition  $W_3$  holds iff

$$Q = \varphi_1\psi_1 - 2 \int \varphi_1\psi_1' = -\varphi_1\psi_1 + 2 \int \varphi_1'\psi_1 \in F_1.$$

But  $\psi_1 = x_2/x_1 \in F_1$ , and  $\varphi_1$  also belongs to  $F_1$  if  $W_1$  is assumed to be satisfied. Therefore,  $W_3$  holds iff  $\int \varphi_1'\psi_1 \in F_1$  □

PROPOSITION 4.5. — *Let  $V = \text{Sol}(L_2)$ . Assume that  $G_1$  is finite. Then we have the following properties*

- (1) *Let  $x_1$  be a non-zero element of  $V$ . If  $\int sx_1^2 \in F_1$ , then for all  $\sigma \in G_1$ ,  $\int s\sigma(x_1)^2 \in F_1$ .*
- (2) *For all  $x \in V$ ,  $\int sx^2 \in F_1$ , iff there exists a basis  $\{x_1, x_2\}$  of  $V$  such that*

$$\int sx_1^2 \in F_1 \text{ and } \int sx_1x_2 \in F_1 \text{ and } \int sx_2^2 \in F_1.$$

- (3) *Assume that  $\int sx_1^2 \in F_1$ , and  $\int sx^2 \notin F_1$ , then  $G_1$  is of dihedral type.*

*Proof.* — (1) Let  $x_1$  be any non zero element of  $V$ . Since  $\int sx_1^2 = W(x_1, y_1) \in F_2$ , for all  $\sigma \in G_2$  we have

$$\sigma\left(\int sx_1^2\right) = \sigma(W(x_1, y_1)) = W(\sigma(x_1), \sigma(y_1)) = \int s\sigma(x_1)^2.$$

Therefore, if  $\int sx_1^2 \in F_1$ , then  $\sigma(\int sx_1^2) = \int s\sigma(x_1)^2 \in F_1$ . Since  $F_1/K$  is a Picard-Vessiot extension contained in  $F_2/K$ , the restriction morphism  $\text{Res} : G_2 \rightarrow G_1$  is surjective, therefore the integrals  $\int s\sigma(x_1)^2 \in F_1$  for all  $\sigma \in G_1$ .

(2) Assume that for all  $x \in V$ ,  $\int sx^2 \in F_1$ , and let  $\{x_1, x_2\}$  be a basis of  $V$ . Then the three particular integrals

$$\int s(x_1 + x_2)^2 \text{ and } \int sx_1^2 \text{ and } \int sx_2^2,$$

belong to  $F_1$ . Taking the difference of those integrals we deduce that  $\int sx_1x_2 \in F_1$ . Conversely, each  $x \in V$  can be written in the form  $x = \lambda x_1 + \mu x_2$ . Therefore,

$$\int sx^2 = \lambda^2 \int sx_1^2 + 2\lambda\mu \int sx_1x_2 + \mu^2 \int sx_2^2 \in F_1.$$

(3) For the action of  $G_1$  on  $\mathbb{P}(V) \simeq \mathbb{P}^1$ , when we look at the orbit  $\Omega$  of  $[x_1]$ , three cases may a priori happen:

- a)  $\text{Card}(\Omega) = 1$ .
- b)  $\text{Card}(\Omega) = 2$ .
- c)  $\text{Card}(\Omega) \geq 3$ .

Let us first prove that with the assumption of point 3, case c) cannot happen. Indeed case c) implies that there exists  $x_2 = \sigma_1(x_1)$  which is not collinear to  $x_1$ , and also there exists  $x_3 = \sigma_2(x_1) = \lambda x_1 + \mu x_2$  with  $\lambda\mu \neq 0$ . From point 1, this implies that the three integrals

$$sx_1^2 \quad \text{and} \quad \int sx_2^2 \quad \text{and} \quad \int s(\lambda x_1 + \mu x_2)^2,$$

belong to  $F_1$ . So,  $\int sx_1x_2$  belongs to  $F_1$ . Thus, from point 2, for all  $x \in V$ ,  $\int sx^2 \in F_1$  which is not true. There remains to show that in cases a) and b),  $G_1$  is of dihedral type.

In case b), let  $\Omega = \{[x_1], [x_2]\}$ . This means that  $\{x_1, x_2\}$  is a basis of  $V$ . Moreover, any conjugate of  $x_1$  or  $x_2$  is either collinear to  $x_1$ , or to  $x_2$ . Hence, in the basis  $\{x_1, x_2\}$ , the representation of  $G_1$  is of dihedral type.

In Cases a), since  $\Omega = \{[x_1]\}$ ,  $x_1$  is an eigenvector of any  $\sigma \in G_1$ . We find a second common eigenvector for any  $\sigma \in G_1$ , using the following classical averaging argument due to Hermann Weyl in the representation theory. Let  $\langle \cdot, \cdot \rangle$  be an arbitrary Hermitian product on  $V \simeq \mathbb{C}^2$  for which  $x_1$  is not an isotropic vector (i.e.  $\langle x_1, x_1 \rangle \neq 0$ ). Consider the average

$$(X, Y) = \sum_{\sigma \in G_1} \langle \sigma(X), \sigma(Y) \rangle.$$

The pairing  $(\cdot, \cdot)$  is a new Hermitian product on  $V$  for which  $G_1$  is unitarian. Therefore the orthogonal of the line  $\mathbb{C}x_1$  is another line of the form  $\mathbb{C}x_2$  which is also globally  $G_1$ -invariant. Therefore  $G_1$  is diagonalizable in the basis  $\{x_1, x_2\}$ . This proves that  $G_1$  is of dihedral type. □

PROPOSITION 4.6. — Assume that  $G_1$  is finite,  $K = \mathbb{C}(z)$ ,  $s = \frac{1}{2kz(z-1)}$ , and consider the following properties

- (1) For all  $x \in V$ ,  $\int sx^2 \in F_1$ .
- (2)  $F_2 = F_1$  and  $G_2 \simeq G_1$ .
- (3) There exists  $M \in \text{GL}(2, K)$  such that  $S = M' + [M, R]$ , where  $R$  and  $S$  are given by (2.4).
- (4) There exists a non-zero rational solution  $v \in \mathbb{C}(z)$  to the equation

$$L_4(v) = [z(z-1)L_2^{\otimes 2}(v)]' = 0,$$

where  $L_2^{\otimes 2}$  denotes the second symmetric power of  $L_2$ .

Then we have  $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4)$ .

*Proof.* —  $(1) \Leftrightarrow (2)$  Let  $\{x_1, x_2\}$  be a basis of  $V$ . From point 2 of Proposition 4.5, property 1 is equivalent to

$$W(x_1, y_1) \in F_1 \quad \text{and} \quad W(x_1, y_2) \in F_1 \quad \text{and} \quad W(x_2, y_2) \in F_1.$$

By Proposition 2.8 and Corollary 2.10, these three Wronskians are fixed by the elements

$$\sigma = \begin{bmatrix} \text{Id} & 0 \\ B(\sigma) & \text{Id} \end{bmatrix} \in \text{Gal}_\partial(F_2/F_1)$$

iff  $B(\sigma) = 0$ . So, property 1 is equivalent to  $\text{Gal}_\partial(F_2/F_1) = \{\text{Id}\}$ , that is to property 2.

$(2) \Leftrightarrow (3)$  From the exact sequence

$$\begin{array}{ccccccc} \{\text{Id}\} & \longrightarrow & \text{Gal}_\partial(F_2/F_1) & \longrightarrow & G_2 & \longrightarrow & G_1 & \longrightarrow & \{\text{Id}\} \\ & & & & \begin{bmatrix} A(\sigma) & 0 \\ B(\sigma) & A(\sigma) \end{bmatrix} & \longrightarrow & A(\sigma) & & \end{array}$$

we have

$$(\text{Gal}_\partial(F_2/F_1) = \{\text{Id}\}) \iff (\forall \sigma \in G_2, B(\sigma) = 0).$$

But the general formulae for the action of  $G_2$  are  $\sigma(X) = XA(\sigma)$  and  $\sigma(Y) = YA(\sigma) + XB(\sigma)$ . This implies that

$$\sigma(YX^{-1}) = YX^{-1} + XB(\sigma)A^{-1}(\sigma)X^{-1}.$$

So,  $\sigma(YX^{-1}) = YX^{-1}$  iff  $B(\sigma) = 0$ . Therefore  $\text{Gal}_\partial(F_2/F_1) = \{\text{Id}\}$  iff  $YX^{-1} \in \text{GL}(2, K)$ .

Now we are looking for the differential equation satisfied by  $M = YX^{-1}$ . From

$$\Xi_2 = \begin{bmatrix} X & 0 \\ MX & X \end{bmatrix}, \quad \Xi'_2 = \begin{bmatrix} X' & 0 \\ M'X + MX' & X' \end{bmatrix} = \begin{bmatrix} R & 0 \\ S & R \end{bmatrix} \Xi_2,$$

we obtain

$$\begin{aligned} X' &= RX, \\ M'X + MX' &= SX + RMX, \\ M'X + MRX &= SX + RMX, \\ M' + MR - RM &= S. \end{aligned}$$

This proves  $(2) \Leftrightarrow (3)$ .

$(3) \Rightarrow (4)$ . We write  $M = \begin{bmatrix} u & v \\ f & g \end{bmatrix}$ , and we insert this expression into the above differential equation. This gives a system of four equations. By

expressing  $f$  and  $g$  in terms of  $u$  and  $v$ , the original equation is equivalent to the system

$$\begin{cases} f = u' + rv, \\ g = u + v', \\ u' = -v''/2, \\ s = u'' + r'v + 2rv'. \end{cases}$$

From the above,  $v$  satisfies  $L_3(v) := v''' - 4rv' - 2r'v = -2s$ . But  $L_3(v) = L_2^{\otimes 2}(v)$  is the second symmetric power of  $L_2(v) = v'' - rv$ . Now, for  $K = \mathbb{C}(z)$  and  $s = \frac{1}{2kz(z-1)}$ ,

$$L_3(v) = -2s = \frac{-1}{kz(z-1)} \implies L_4(v) := [z(z-1)L_2^{\otimes 2}(v)]' = 0.$$

Hence, if  $M \in GL(2, \mathbb{C}(z))$ , then  $v \in \mathbb{C}(z)$ , and this implies that the equation  $L_4(v) = 0$  has a non-zero rational solution. □

Surprisingly, the differential equation  $S = M' + [M, R]$  has the form the classical Euler equation for the angular momentum of a rigid body, see [1] pp.142-143.

### 4.2. Type of $G_1$ when it is finite

In order to apply the previous theory, we need to compute  $G_1 = G(k, \lambda)$  when  $G_1^\circ = \{\text{Id}\}$  in Table 1.1. Table 4.1 below gives this information.

To determine the last column of Table 4.1, we used the following facts.

- The exponents of  $L_2$  at  $\{0, 1, \infty\}$  are

$$\varepsilon_0 = \left\{ \frac{k-1}{2k}, \frac{k+1}{2k} \right\}, \quad \varepsilon_1 = \left\{ \frac{1}{4}, \frac{3}{4} \right\}, \quad \varepsilon_\infty = \left\{ \frac{\tau-1}{2}, -\frac{\tau+1}{2} \right\},$$

with

$$\tau = \frac{1}{2k} \sqrt{(k-2)^2 + 8k\lambda}.$$

- Therefore, the reduced differences of exponents are

$$\Delta_0 = \left| \frac{1}{k} \right|, \quad \Delta_1 = \frac{1}{2}, \quad \Delta_\infty = |\tau| \pmod{\mathbb{Z}}.$$

Thanks to the Schwarz Table, see p.128 in [12], we can compute  $G_1/\{\pm \text{Id}\} = G(k, \lambda)/\{\pm \text{Id}\}$  which is the image of  $G_1$  in  $\text{PSL}(2, \mathbb{C})$ , and completely determines the type of  $G_1$ .

Table 4.1. Type of  $G(k, \lambda)$  with  $G(k, \lambda)^\circ = \{\text{Id}\}$ .

row	$k$	$\lambda$	Exponents of $L_2$ at $\{0, \infty\}$	$G(k, \lambda)$
5	1	0	$(0, 1), (-1/4, -3/4)$	cyclic-dihedral
6	-1	1	$(0, 1), (-1/4, -3/4)$	cyclic-dihedral
7	$ k  \geq 3$	$\frac{1}{2} \left( \frac{k-1}{k} + p(p+1)k \right)$	$\varepsilon_0, \left( \frac{1}{4}(2p-1), -\frac{1}{4}(2p+3) \right)$	dihedral
8	3	$\frac{-1}{24} + \frac{1}{6}(1+3p)^2$	$\left( \frac{1}{3}, \frac{2}{3} \right), \left( -\frac{2}{3} - \frac{1}{2}p, -\frac{1}{3} + \frac{1}{2}p \right)$	tetrahedral
9	3	$\frac{-1}{24} + \frac{3}{32}(1+4p)^2$	$\left( \frac{1}{3}, \frac{2}{3} \right), \left( -\frac{5}{8} - \frac{1}{2}p, -\frac{3}{8} + \frac{1}{2}p \right)$	octahedral
10	3	$\frac{-1}{24} + \frac{3}{50}(1+5p)^2$	$\left( \frac{1}{3}, \frac{2}{3} \right), \left( -\frac{3}{5} - \frac{1}{2}p, -\frac{2}{5} + \frac{1}{2}p \right)$	icosahedral
11	3	$\frac{-1}{24} + \frac{3}{50}(2+5p)^2$	$\left( \frac{1}{3}, \frac{2}{3} \right), \left( -\frac{7}{10} - \frac{1}{2}p, -\frac{3}{10} + \frac{1}{2}p \right)$	icosahedral
12	-3	$\frac{25}{24} - \frac{1}{6}(1+3p)^2$	$\left( \frac{1}{3}, \frac{2}{3} \right), \left( -\frac{2}{3} - \frac{1}{2}p, -\frac{1}{3} + \frac{1}{2}p \right)$	tetrahedral
13	-3	$\frac{25}{24} - \frac{3}{32}(1+4p)^2$	$\left( \frac{1}{3}, \frac{2}{3} \right), \left( -\frac{5}{8} - \frac{1}{2}p, -\frac{3}{8} + \frac{1}{2}p \right)$	octahedral
14	-3	$\frac{25}{24} - \frac{3}{50}(1+5p)^2$	$\left( \frac{1}{3}, \frac{2}{3} \right), \left( -\frac{3}{5} - \frac{1}{2}p, -\frac{2}{5} + \frac{1}{2}p \right)$	icosahedral
15	-3	$\frac{25}{24} - \frac{3}{50}(2+5p)^2$	$\left( \frac{1}{3}, \frac{2}{3} \right), \left( -\frac{7}{10} - \frac{1}{2}p, -\frac{3}{10} + \frac{1}{2}p \right)$	icosahedral
16	4	$\frac{-1}{8} + \frac{2}{9}(1+3p)^2$	$\left( \frac{3}{8}, \frac{5}{8} \right), \left( -\frac{2}{3} - \frac{1}{2}p, -\frac{1}{3} + \frac{1}{2}p \right)$	octahedral
17	-4	$\frac{9}{8} - \frac{2}{9}(1+3p)^2$	$\left( \frac{3}{8}, \frac{5}{8} \right), \left( -\frac{2}{3} - \frac{1}{2}p, -\frac{1}{3} + \frac{1}{2}p \right)$	octahedral
18	5	$\frac{-9}{40} + \frac{5}{18}(1+3p)^2$	$\left( \frac{2}{5}, \frac{3}{5} \right), \left( -\frac{2}{3} - \frac{1}{2}p, -\frac{1}{3} + \frac{1}{2}p \right)$	icosahedral
19	5	$\frac{-9}{40} + \frac{1}{10}(2+5p)^2$	$\left( \frac{2}{5}, \frac{3}{5} \right), \left( -\frac{7}{10} - \frac{1}{2}p, -\frac{3}{10} + \frac{1}{2}p \right)$	icosahedral
20	-5	$\frac{49}{40} - \frac{5}{18}(1+3p)^2$	$\left( \frac{2}{5}, \frac{3}{5} \right), \left( -\frac{2}{3} - \frac{1}{2}p, -\frac{1}{3} + \frac{1}{2}p \right)$	icosahedral
21	-5	$\frac{49}{40} - \frac{1}{10}(2+5p)^2$	$\left( \frac{2}{5}, \frac{3}{5} \right), \left( -\frac{7}{10} - \frac{1}{2}p, -\frac{3}{10} + \frac{1}{2}p \right)$	icosahedral

### 4.3. Application of the theory when $G_1$ is not of dihedral type

If  $G_1$  is finite but not of dihedral type, then the main point in our proof of Theorem 4.1 will be to show that equation  $L_4(v) = 0$  does not have rational solutions. This is why we need to compute the exponents of  $L_4$  at the singularities.

LEMMA 4.7. — *With the notation  $2\varepsilon_i = \{2a, 2b\}$ , for  $\varepsilon_i = \{a, b\}$ , the respective exponents of  $L_4$  at  $z \in \{0, 1, \infty\}$  are the following*

$$\{1, 2, 2\varepsilon_0\}, \{1, 2, 2\varepsilon_1\}, \{-1, -1, 2\varepsilon_\infty\}.$$

*Proof.* — If  $\varepsilon_i = \{a, b\}$  are the exponents of  $L_2$  at the singularity  $i \in \{0, 1, \infty\}$ , then the exponents of  $L_3 = L_2^{\otimes 2}$  at the same singularity are  $\{a + b, 2a, 2b\}$ . Since at  $z = 0$  and  $z = 1$ ,  $a + b = 1$ , and at  $z = \infty$ ,  $a + b = -1$ , this gives the exponents of  $L_3$ .

Let  $\chi_3$  and  $\chi_4$  be the characteristic polynomials of the equations  $L_3 = 0$  and  $L_4 = 0$ , respectively.

In a neighbourhood of  $z = 0$  we have the following. If

$$L_3(z^\rho) = \chi_3(\rho)z^{\rho-3} + \dots,$$

then

$$L_4(z^\rho) = (\chi_3(\rho)z^{\rho-2} + \dots)' = (\rho - 2)\chi_3(\rho)z^{\rho-3} + \dots.$$

So  $\chi_4(\rho) = (\rho - 2)\chi_3(\rho)$ .

In a neighbourhood of  $z = 1$ , we obtain a similar result thanks to the formula  $z(z - 1) = (z - 1)^2 + (z - 1)$ .

In a neighbourhood  $z = \infty$  we have the following. If

$$v = x^\rho + \dots = z^{-\rho} + \dots,$$

then the first term of  $v'''$  is proportional to  $x^{\rho+3}$ . So, we have

$$\begin{aligned} L_4(x^\rho) &= \left(\frac{1}{x^2}\chi_3(\rho)x^{\rho+3} + \dots\right)' \\ &= (\chi_3(\rho)x^{\rho+1} + \dots)' \\ &= \left(\chi_3(\rho)\frac{1}{z^{\rho+1}} + \dots\right)' \\ &= -(\rho + 1)(\chi_3(\rho)x^{\rho+2} + \dots) \end{aligned}$$

Hence, up to the sign,  $\chi_4(\rho) = (\rho + 1)\chi_3(\rho)$ .

Therefore, at  $z = 0$  and  $z = 1$ , the exponents of  $L_4$  are those of  $L_3$  together with  $\rho = 2$ . At  $z = \infty$ , the exponents of  $L_4$  are those of  $L_3$  together with  $\rho = -1$ . □

COROLLARY 4.8. — *For all the rows in Table 4.1, except maybe for rows 5 and 6, the equation  $L_4 = 0$  does not have non-zero rational solutions. In particular, when  $G_1 = G(k, \lambda)$  is finite but not dihedral,  $L_4$  does not have non-zero rational solutions.*

*Proof.* — From Table 4.1 and Lemma 4.7, we see that for each possible case, the exponents of  $L_4$  at  $z = 0$  and  $z = 1$  are greater or equal to zero. So, if we look for a rational solution  $v$  of  $L_4 = 0$ ,  $v$  must be a polynomial

of degree equal to the opposite of one exponent at the infinity. Therefore,  $\deg(v) \in \{1, -2\varepsilon_\infty\}$ . Hence,  $\deg(v)$  must be equal to 1, unless maybe,  $-2\varepsilon_\infty$  contains an integral number  $\geq 2$ . But for all the rows of Table 4.1, the set  $2\varepsilon_\infty$  does not contain any integral number, so the possible polynomial to check are of the form  $v = z + d$ . We have

$$L_4(z + d) = [z(z - 1)L_2^{\otimes 2}(z + b)]' = -2[z(z - 1)F(z)]',$$

where

$$F(z) = -\frac{1}{2}L_2^{\otimes 2}(z + d) = 2r + r'(z + d).$$

Thus,

$$L_4(z + d) = 0 \iff F(z) = \frac{c}{z(z - 1)}$$

for a certain  $c \in \mathbb{C}$ . Let us study the behaviour of  $F(z)$  around  $z = 0$  and  $z = 1$ . From now, we assume that we are not in the cases of rows 5 and 6, in particular  $|k| \geq 3$ .

Around  $z = 0$ ,  $r(z)_0 = \frac{a}{z^2}$  with  $a = \frac{(1/k)^2 - 1}{4} \neq 0$ . Therefore,  $r(z)'_0 = \frac{-2a}{z^3}$ . So, if  $d \neq 0$ ,  $F(z)_0 = \frac{-2ad}{z^3}$ . This is incompatible with  $F(z) = \frac{c}{z(z-1)}$ . Hence, we must check this equation with  $d = 0$  (i.e., with  $F(z) = 2r + zr'$ ).

Around  $z = 1$ ,  $r(z)_1 = \frac{-3}{16(z-1)^2}$  therefore,  $F(z)_1 = \frac{3}{8(z-1)^3}$ , and this is still incompatible with  $F(z) = \frac{c}{z(z-1)}$ .

Thus, for all the rows except maybe for rows 5 and 6,  $L_4(z + d) \neq 0$ , and  $L_4 = 0$  does not have a non-zero rational solution. □

*Proof of Theorem 4.1 for  $G_1$  finite but not dihedral.* — Let us assume that  $G_1$  is finite and is not of dihedral type. This corresponds to cases of Table 4.1, whose row numbers are greater than 7. From Corollary 4.8,  $L_4 = 0$  does not have non-zero rational solutions. Therefore, from Proposition 4.6, there exists a non-zero  $x \in V = \text{Sol}(L_2)$ , such that  $\int sx^2 \notin F_1$ . So, from Proposition 4.5,  $\forall x \in V \setminus \{0\}, \int sx^2 \notin F_1$  since  $G_1$  is not of dihedral type. As a consequence, from Proposition 4.4,  $G_3^\circ$  is not Abelian and we can conclude thanks to Remark 4.2. □

#### 4.4. Application when $G_1$ is of dihedral type

We have to investigate the cases appearing in row 5, 6 and 7 in Table 4.2, which for the convenience of the reader, we give in Table 4.2.

We follow the strategy applied above. That is, we prove that  $G_3^\circ$  is not Abelian because all the integrals  $\int sx^2$  are not algebraic. What is more difficult here is that we cannot deduce this fact from the existence of one

particular non-algebraic integral. We begin with the simple cases of rows 5 and 6. Next we consider the case of row 7 which is more technical.

Table 4.2. Cases when  $G(k, \lambda)$  is of dihedral type.

row	$k$	$\lambda$	Exponents of $L_2$ at $\{0, \infty\}$	$G(k, \lambda)$
5	1	0	$(0, 1), (-1/4, -3/4)$	cyclic-dihedral
6	-1	1	$(0, 1), (-1/4, -3/4)$	cyclic-dihedral
7	$ k  \geq 3$	$\frac{1}{2} \left( \frac{k-1}{k} + p(p+1)k \right)$	$\varepsilon_0, ((2p-1)/4, -(2p+3)/4)$	dihedral

4.4.1. The case of rows 5 and 6

From Table 4.2, the common Riemann scheme of  $L_2$  is

$$P \left\{ \begin{matrix} 0 & 1 & \infty \\ 0 & 1/4 & -1/4 \\ 1 & 3/4 & -3/4 \end{matrix} z \right\}.$$

A basis of solutions is therefore

$$x_1 = (z - 1)^{1/4}, \quad x_2 = (z - 1)^{3/4}.$$

Since  $x_1x_2 = z - 1 \in \mathbb{C}(z)$ , here  $G_1$  is cyclic and isomorphic to  $\mathbb{Z}/4\mathbb{Z}$ .

PROPOSITION 4.9. — *In the cases of rows 5 and 6, we have*

- (1) *For any non-zero solution  $x$  of  $L_2 = 0$ , the integral  $\varphi = \int sx^2$  is not algebraic.*
- (2) *The group  $G^\circ = G_3^\circ$  is not Abelian.*

*Proof.* — Thanks to Proposition 4.4, the second point is a consequence of the first one.

Since arbitrary solution of  $L_2 = 0$  can be written as  $x = \alpha x_1 + \beta x_2$ , the general form of  $\varphi$  is

$$\begin{aligned} \varphi &= \int \frac{1}{z(z-1)} (\alpha(z-1)^{1/4} + \beta(z-1)^{3/4})^2, \\ \varphi &= \alpha^2 \int \frac{dz}{z\sqrt{z-1}} + \beta^2 \int \frac{\sqrt{z-1}}{z} dz + 2\alpha\beta \text{Log}(z), \\ \varphi &= \alpha^2 \varphi_1 + \beta^2 \varphi_2 + 2\alpha\beta \text{Log}(z) \end{aligned}$$

Since  $G_1 \simeq \mathbb{Z}/4\mathbb{Z}$ , there exists  $\sigma \in G_2$ , such that  $\sigma(x_1) = ix_1$  and  $\sigma(x_2) = -ix_2$ . As  $\sigma(\varphi) = \int s\sigma(x^2)$ , we have

$$\sigma(\varphi) = \int s(i\alpha x_1 - i\beta x_2)^2 = -\alpha^2 \varphi_1 - \beta^2 \varphi_2 + 2\alpha\beta \text{Log}(z).$$



If  $\varphi \in F_1$ , then  $\sigma(\varphi) \in F_1$ , and so  $4\alpha\beta\text{Log}(z) = \varphi + \sigma(\varphi) \in F_1$  is algebraic. Therefore  $\alpha\beta = 0$ , and  $\varphi$  is proportional either to  $\varphi_1$ , or to  $\varphi_2$ . But in those two remaining cases, the Taylor expansion of the integrand around  $z = 0$  shows that each  $\varphi_j$  for  $j \in \{1, 2\}$  can be written in the form  $\varphi_j = \pm i\text{Log}(z) + f_j(z)$ , where  $f_j(z)$  is holomorphic around  $z = 0$ . Therefore,  $\mathcal{M}_0(\varphi_j) = \mp 2\pi + \varphi_j$ , and  $\varphi_j$  cannot be algebraic since it has an infinite number of conjugates by the iteration of  $\mathcal{M}_0$ . □

4.4.2. The case of row 7

Here, from Table 4.2,  $k$  and  $p$  are relative integers with  $|k| \geq 3$ , and the Riemann scheme of  $L_2$  is

$$P_1 \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \frac{1}{2} - \frac{1}{2k} & \frac{1}{4} & \frac{2p-1}{4} \\ \frac{1}{2} + \frac{1}{2k} & \frac{3}{4} & \frac{-2p-3}{4} \end{array} \right\} z.$$

PROPOSITION 4.10. — *In the case of row 7 we have*

- (1) *For any non-zero solution  $x$  of  $L_2 = 0$ , the integral  $\varphi = \int sx^2$  is not algebraic.*
- (2) *The group  $G^\circ = G_3^\circ$  is not Abelian.*

As in Proposition 4.9 above, the second point is a consequence of the first one. But the proof of the first point is going to be divided into several steps since it is more technical.

Notice that if we change  $k$  to  $k' = -k$ , or  $p$  is to  $p' = -p - 1$ , then the Riemann scheme of  $L_2$  is not changed. Therefore, to prove Proposition 4.10 it is enough to consider the cases with  $k \geq 3$  and  $p \geq 0$ .

**The group  $D_{2N}^\dagger$ .** The differences of exponents of  $L_2$  are  $\Delta_0 = 1/k$ ,  $\Delta_1 = 1/2$ , and  $\Delta_\infty = p+1/2$ . So, the reduced exponents differences are  $1/k$ ,  $1/2$  and  $1/2$ . Therefore, from [12] p.128-129, the projective Galois group of  $L_2$ , i.e., the image of  $G_1$  in  $\text{PSL}(2, \mathbb{C})$ , is isomorphic to the dihedral group  $D_{2k}$ , which is of order  $2k$ . From Lemma 2.1,  $G_1$  is necessarily conjugated to a finite subgroup of  $D^\dagger$  which is not cyclic. That is,  $G_1$  is not a subgroup of the diagonal group  $D_{\text{iag}} = \left\{ \begin{bmatrix} \zeta & 0 \\ 0 & 1/\zeta \end{bmatrix}, \zeta \in \mathbb{C}^* \right\}$ . Let  $W = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \text{SL}(2, \mathbb{C})$  be the Weyl matrix. We have the following properties.

- (1)  $D^\dagger = D_{\text{iag}} \cup WD_{\text{iag}}$ ,
- (2) For all  $R \in \text{SL}(2, \mathbb{C})$ ,  $WRW^{-1} = R^{-1}$ ,
- (3) For all  $D \in D_{\text{iag}}$ ,  $(WD)^2 = W^2 = -\text{Id}$ , and  $WD$  is conjugated to  $W$  by an element of  $WD_{\text{iag}}$ .

By Property 3, we can assume that  $W \in G_1$ . As  $W^2 = -\text{Id}$ , the diagonal subgroup of  $G_1$ , i.e.,  $G_1 \cap D_{\text{diag}}$ , contains  $-\text{Id}$ . Since it is a finite cyclic group, it is of even order  $N$ , for a certain  $N \in 2\mathbb{N}^*$ . Therefore, as a subgroup of  $D^\dagger$ , the group  $G_1$  is generated by  $W$  and by a matrix  $R_\zeta = \begin{bmatrix} \zeta & 0 \\ 0 & 1/\zeta \end{bmatrix}$  where,  $\zeta$  is a primitive  $N$ -th root of unity. This is the group  $D_{2N}^\dagger$  of order  $2N$ , whose presentation is

$$D_{2N}^\dagger = \langle W, R_\zeta \mid W^2 = -\text{Id}, R_\zeta^N = \text{Id}, WR_\zeta W^{-1} = R_\zeta^{-1} \rangle.$$

The image of  $D_{2N}^\dagger$  in  $\text{PSL}_2(\mathbb{C})$  is the dihedral group  $D_N = D_{2k}$ , in the considered situation.

If  $\{x_1, x_2\}$  is a basis of  $V$  in which the representation of  $G_1$  is  $D_{2N}^\dagger$ , then the actions of  $W$  and  $R_\zeta$  on this basis are given by the formulae

$$\begin{cases} W(x_1) &= x_2 \\ W(x_2) &= -x_1 \end{cases} \quad \text{and} \quad \begin{cases} R_\zeta(x_1) &= \zeta x_1 \\ R_\zeta(x_2) &= \frac{1}{\zeta} x_2 \end{cases}.$$

Therefore,  $W(x_1x_2) = -x_1x_2$ , and  $R_\zeta(x_1x_2) = x_1x_2$ . So,  $(x_1x_2)^2 \in K = \mathbb{C}(z)$ , and  $L = K[x_1x_2]$  is quadratic over  $K$ . The group  $\text{Gal}(F_1/L) = \langle R_\zeta \rangle$  is cyclic of order  $N = 2k$ . Since  $x_1$  has  $N$  distinct conjugates under  $\text{Gal}(F_1/L)$ , we have  $F_1 = L[x_1]$ . Moreover,  $x_1^N \in L = \mathbb{C}(z)[x_1x_2]$ .

**Algebraicity of the general integral  $\varphi = \int sx^2$ .**

LEMMA 4.11. — *Let  $\{x_1, x_2\}$  be a basis of  $V$  in which the representation of  $G_1$  is  $D_{2N}^\dagger$ . Then the following statements hold true.*

- (1) *If there exists  $x_0 = \alpha x_1 + \beta x_2 \in V$  with  $\alpha\beta \neq 0$  such that  $\varphi_0 = \int sx_0^2 \in F_1$ , then for all  $x \in V$  the general integral  $\varphi = \int sx^2 \in F_1$ .*
- (2)  *$\varphi_1 = \int sx_1^2 \in F_1$  iff  $\varphi_2 = \int sx_2^2 \in F_1$ .*
- (3)  *$\varphi_1 = \int sx_1^2 \in F_1$  iff there exist  $\phi \in \mathbb{C}(z)[x_1x_2]$  such that  $\int sx_1^2 = \phi x_1^2$ .*
- (4) *If  $\{y_1, y_2\}$  is a basis of  $V$  such that  $y_1y_2$  is at most quadratic over  $\mathbb{C}(z)$ , then up to a permutation of the indices,  $y_1$  is proportional to  $x_1$  and  $y_2$  is proportional to  $x_2$ .*

*Proof.* — (1) Since

$$R_\zeta(x_0) = \zeta\alpha x_1 + \frac{\beta}{\zeta}x_2, \quad \text{and} \quad \int sx_0^2 = \alpha^2\varphi_1 + 2\alpha\beta \int sx_1x_2 + \beta^2\varphi_2 \in F_1,$$

we deduce that

$$\zeta^2 \int sR_\zeta(x_0^2) = \zeta^4\alpha^2\varphi_1 + \zeta^2 2\alpha\beta \int sx_1x_2 + \beta^2\varphi_2 \in F_1.$$

Since  $N = 2k \geq 6$ , we can find two primitive  $N$ -th roots of unity  $\zeta$  and  $\zeta'$ , such that  $\text{card}\{1, \zeta^2, \zeta'^2\} = 3$ . Therefore, we obtain an identity of the form

$$\begin{bmatrix} 1 & 1 & 1 \\ \zeta^4 & \zeta^2 & 1 \\ \zeta'^4 & \zeta'^2 & 1 \end{bmatrix} \begin{bmatrix} \alpha^2 \varphi_1 \\ 2\alpha\beta \int sx_1x_2 \\ \beta^2 \varphi_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \in F_1^3,$$

where the  $3 \times 3$  Vandermonde matrix on the left hand side is invertible. It implies that  $\varphi_1, \int sx_1x_2, \varphi_2 \in F_1$ , because  $\alpha\beta \neq 0$ . Therefore, by Proposition 4.5, any general integral  $\varphi = \int sx^2 \in F_1$ .

(2) From Proposition 4.5 again,

$$\left( \int sx_1^2 \in F_1 \right) \implies \left( \int sW(x_1^2) = \int sx_2^2 \in F_1 \right).$$

(3) If  $\int sx_1^2 \in F_1$ , then, as  $F_1 = L[x_1]$ , we have

$$\int sx_1^2 = \sum_{i=0}^{N-1} \phi_i x_1^i,$$

and this equality implies that

$$sx_1^2 = \sum_{i=0}^{N-1} \left( \phi'_i + i \frac{x'_1}{x_1} \phi_i \right) x_1^i.$$

But,  $x_1^N \in L$  implies that  $x'_1/x_1 \in L$ , and the above formula gives an expansion of  $sx_1^2$  in the  $L$ -basis  $\{1, x_1, \dots, x_1^{N-1}\}$ . Therefore  $\phi'_i + i \frac{x'_1}{x_1} \phi_i = 0$  for  $i \neq 2$  and

$$\phi_2 + 2 \frac{x'_1}{x_1} \phi_2 = s,$$

that is,  $\int sx_1^2 = \phi_2 x_1^2$  with  $\phi_2 \in L = \mathbb{C}(z)[x_1x_2]$ .

(4) If  $y_1y_2$  is at most quadratic over  $\mathbb{C}(z)$ , its orbit under  $G_1$  contains at most two elements. Looking at the orbit under the subgroup generated by  $R_\zeta$ , we deduce that  $y_1y_2$  must be fixed by the subgroup of the rotations  $R_\lambda$  where  $\lambda$  ranges over the  $k = N/2$  roots of unity. Now, let us write

$$y_1 = ax_1 + bx_2 \text{ and } y_2 = cx_1 + dx_2 \text{ with } ad - bc \neq 0.$$

We get the following two expressions

$$\begin{cases} y_1y_2 = acx_1^2 + (bc + ad)x_1x_2 + bdx_2^2, \\ R_\lambda(y_1y_2) = \lambda^2 acx_1^2 + (bc + ad)x_1x_2 + \frac{bd}{\lambda^2} x_2^2. \end{cases}$$

But from the proof of point 1, it follows that the family  $\{x_1^2, x_1x_2, x_2^2\}$  is  $\mathbb{C}$ -linearly independent. Therefore, when  $\lambda$  is a  $k$ -th roots of unity, the

equality  $R_\lambda(y_1y_2) = y_1y_2$  implies that

$$(\lambda^2 - 1)ac = 0 = (1 - \frac{1}{\lambda^2})bd.$$

As  $k \geq 3$ , we deduce that  $ac = bd = 0$ , and, up to a permutation,  $y_1$  is proportional to  $x_1$  and  $y_2$  is proportional to  $x_2$ . □

From the above lemma,  $\{x_1, x_2\}$  is a distinguished basis of  $V$ , and we have to compute it in order to study the algebraicity of  $\varphi_1$ .

**Solutions of  $L_2 = 0$  and the Jacobi polynomials.** The Jacobi polynomials  $J_n^{(\alpha, \beta)}(t)$  with parameters  $(\alpha, \beta)$ , and  $n \in \mathbb{N}$  are defined by the following formulae

$$J_n^{(\alpha, \beta)}(t) = \frac{(t - 1)^{-\alpha}(t + 1)^{-\beta}}{2^n n!} \frac{d^n}{dt^n} ((t - 1)^{\alpha+n}(t + 1)^{\beta+n}),$$

see p. 95 in [12]. The polynomial  $J_n^{(\alpha, \beta)}(t)$  is of degree  $n$ , and belongs to the Riemann scheme

$$P_J \left\{ \begin{matrix} -1 & \infty & 1 \\ 0 & -n & 0 \\ -\beta & \alpha + \beta + n + 1 & -\alpha \end{matrix} \right. t \left. \right\},$$

thus it is a solution of the following equation

$$(4.2) \quad (1 - t^2) \frac{d^2w}{dt^2} + [(\beta - \alpha) - (\alpha + \beta + 2)t] \frac{dw}{dt} + n(\alpha + \beta + n + 1)w = 0$$

If  $\alpha$  and  $\beta$  are real and greater than  $-1$ , then the pairing

$$\langle P, Q \rangle = \int_{-1}^1 (1 - t)^\alpha (1 + t)^\beta P(t)Q(t)dt,$$

defines a scalar product on  $\mathbb{R}[t]$ . It can be shown, see, e.g., page 97 in [12], that the family  $\{J_n^{(\alpha, \beta)}(t)\}_{n \in \mathbb{N}}$  is an orthogonal basis for this scalar product. From this it can be proved, see Ex. 2.39 on p. 94 in [13], that the roots of  $J_n^{(\alpha, \beta)}(t)$  are simple and contained in the real interval  $] - 1, 1[$ .

In what follows, we use the Jacobi polynomials with parameters

$$(\alpha, \beta) = (-1/k, 1/k),$$

and we denote them by  $J_n(t)$ .

Using the following change of variable

$$(4.3) \quad t = \sqrt{1 - z} \iff z = 1 - t^2,$$

the solutions of  $L_2 = 0$  can be expressed in the terms of variable  $t$ . We have the following.

LEMMA 4.12. — *Let  $k$  and  $p$  be natural integers with  $k \geq 3$ , and*

$$\begin{cases} x_1 = (1 - t^2)^{1/2}(-t^2)^{1/4} \left(\frac{t + 1}{t - 1}\right)^{1/2k} J_p(t), \\ x_2 = i(1 - t^2)^{1/2}(-t^2)^{1/4} \left(\frac{t + 1}{t - 1}\right)^{-1/2k} J_p(-t). \end{cases}$$

*Then,  $\{x_1, x_2\}$  is a basis of  $V$  in which the representation of  $G_1$  is  $D_{2N}^\dagger$ , and  $L = \mathbb{C}(z)[x_1x_2] = \mathbb{C}(z)(\sqrt{1 - z}) = \mathbb{C}(t)$ .*

Before proving the above lemma, we use it to finish the proof of Proposition 4.10.

**The integral  $\int sx_1^2 \neq \phi x_1^2$ .** From Lemmas 4.11 and 4.12,  $\int sx_1^2 \in F_1$  iff there exists  $\phi \in \mathbb{C}(z)(\sqrt{1 - z}) = \mathbb{C}(t)$  such that  $\int sx_1^2 = \phi x_1^2$ , or equivalently

$$(4.4) \quad \frac{d}{dz} (\phi x_1^2) = sx_1^2.$$

But here all the quantities may be expressed in term of  $t = \sqrt{1 - z}$ . Applying the chain rule we obtain

$$(4.5) \quad \frac{d}{dt} (\phi x_1^2) = -2tsx_1^2.$$

From Lemma 4.12,

$$x_1^2 = (1 - t^2)(-t^2)^{1/2} \left(\frac{t + 1}{t - 1}\right)^{1/k} J_p^2(t) = it(1 - t^2) \left(\frac{t + 1}{t - 1}\right)^{1/k} J_p^2(t).$$

Since

$$s = \frac{1}{2kz(1 - z)} = \frac{1}{2kt^2(t^2 - 1)},$$

equation (4.5) reads

$$(4.6) \quad \frac{d}{dt} \left( \phi(t)t(1 - t^2) \left(\frac{t + 1}{t - 1}\right)^{1/k} J_p^2(t) \right) = \frac{1}{k} \left(\frac{t + 1}{t - 1}\right)^{1/k} J_p^2(t).$$

If we set  $\psi(t) = \phi(t)t(1 - t^2)J_p^2(t)$ , then  $\phi \in \mathbb{C}(t)$  iff  $\psi \in \mathbb{C}(t)$ . In terms of  $\psi(t)$ , equation (4.4) has the form

$$(4.7) \quad \frac{d}{dt} \left( \psi(t) \left(\frac{1 + t}{1 - t}\right)^{1/k} \right) = \frac{1}{k} \left(\frac{1 + t}{1 - t}\right)^{1/k} J_p^2(t),$$

$$(4.8) \quad \frac{d\psi}{dt} + \frac{1}{k} \left( \frac{1}{1 + t} + \frac{1}{1 - t} \right) \psi = \frac{1}{k} J_p^2(t).$$

We can use the above equation to study the local behaviour of the function  $\psi(t)$ . A simple analysis shows that if  $\psi(t)$  is rational, then it has a simple

zero at  $t = \pm 1$ . Moreover, by the Cauchy theorem,  $\psi$  has no pole in  $] - 1, 1[$ . Since  $0 < 1/k \leq 1/3$ , if  $\psi$  is rational, then the real function

$$t \mapsto \psi(t) \left( \frac{1+t}{1-t} \right)^{1/k},$$

vanishes at  $t = \pm 1$ . Therefore, integrating (4.7), we get

$$0 = \int_{-1}^1 \frac{1}{k} \left( \frac{1+t}{1-t} \right)^{1/k} J_p^2(t) dt,$$

but it is impossible since the integrand is positive on  $] - 1, 1[$ .

Therefore,  $\varphi_1 = \int sx_1^2$  does not belong to  $F_1$ .

*Proof of Proposition 4.10.* — Since  $\varphi_1 = \int sx_1^2 \notin F_1$ , by Lemma 4.11,  $\varphi_2 = \int sx_2^2 \notin F_1$ , and for all non-zero  $x \in V$  the general integral  $\varphi = \int sx^2$  does not belong to  $F_1$ . Therefore, by Proposition 4.4,  $G_3^0$  is not Abelian.  $\square$

*Proof of Lemma 4.12.* — We can prove the first part of the lemma directly by making a change of dependent and independent variables in the equation  $x'' = rx$ . Namely, if we put

$$y = x(z(t)) = (1 - t^2)^{1/2} (-t^2)^{1/4} \left( \frac{t+1}{t-1} \right)^{1/2k} w(t),$$

where

$$z = z(t) = 1 - t^2,$$

then  $w(t)$  satisfies equation (4.2) with  $\beta = -\alpha = 1/k$  and  $n = p$ . Also, we can prove this part of the lemma applying successive transformations of Riemann schemes, see Chapter VI in [12].

This implies that the function

$$(4.9) \quad y_1 = (1 - t^2)^{1/2} (-t^2)^{1/4} \left( \frac{t+1}{t-1} \right)^{1/2k} J_p(t).$$

is a solution of  $L_2 = 0$  expressed in  $t$  variable. Moreover, it can be easily shown that

$$(4.10) \quad y_2 = \mathcal{M}_1(y_1) = i(1 - t^2)^{1/2} (-t^2)^{1/4} \left( \frac{t+1}{t-1} \right)^{-1/2k} J_p(-t).$$

Hence,

$$(4.11) \quad y_1 y_2 = t(1 - t^2) J_p(t) J_p(-t) \in \mathbb{C}[t].$$

Since  $\mathbb{C}(t)/\mathbb{C}(z)$  is quadratic,  $y_1 y_2$  is at most quadratic over  $\mathbb{C}(z)$ . Therefore, from point 4 of Lemma 4.11 we deduce that, up to a permutation,  $y_1$  is proportional to  $x_1$  and  $y_2$  is proportional to  $x_2$ . Therefore,  $\{y_1, y_2\}$  is a basis of  $V$  in which the representation of  $G_1$  is  $D_{2N}^\dagger$ , and we can call it  $\{x_1, x_2\}$ .

Since  $\mathbb{C}(z)[y_1 y_2] = L = \mathbb{C}(z)[x_1 x_2] \subset \mathbb{C}(t)$ , and  $W(x_1 x_2) = -x_1 x_2$ , the element  $x_1 x_2$  is quadratic over  $\mathbb{C}(z)$ . Thus, we deduce that

$$L = \mathbb{C}(z)[x_1 x_2] = \mathbb{C}(t),$$

and this finishes the proof.  $\square$

**Conclusion.** From this study, it follows that the first three points of Theorem 1.3 are proved.

## 5. Symmetries in Table 1.2 and potentials of degree $k = \pm 2$

In this section, we notice an important symmetry contained in Table 1.2. We use it to prove Theorem 1.3, for the exceptional cases when  $\deg(V) = k = \pm 2$ .

### 5.1. Symmetries in Table 1.2

Let us recall that the reduced VE (1.11) depends on two rational functions  $r, s \in \mathbb{C}(z)$ . The function  $r$  is defined by the equations (1.12) and (1.13); the function  $s$  is the following

$$s = \frac{1}{2kz(z-1)}.$$

In Table 1.1 there are symmetries between the rows for which  $k$  is changed into  $\tilde{k} = -k$ . In fact we have the following.

**PROPOSITION 5.1.** — *If the pair  $(k, \lambda)$  is changed into the pair  $(\tilde{k}, \tilde{\lambda}) = (-k, 1 - \lambda)$ , then*

- (1) *The pair of function  $(r, s)$  is changed into  $(\tilde{r}, \tilde{s}) = (r, -s)$ .*
- (2) *For all  $d \geq 1$  the differential Galois groups  $G_d$  and  $\tilde{G}_d$  of the subsystems of the VE associated to the Jordan blocks  $B(\lambda, d)$  and  $B(\tilde{\lambda}, d)$  are isomorphic.*

*Proof.* — If  $k$  is changed into  $\tilde{k} = -k$ , then from (1.14) the Riemann schemes  $P$  (resp.  $\tilde{P}$ ) of the equations  $x'' = rx$  (resp.  $x'' = \tilde{r}x$ ) have the same exponents at  $z = 0$  and at  $z = 1$ . Now,  $P = \tilde{P}$  iff  $\tilde{r} = \pm r$ . From (1.13) this happens iff  $\tilde{\lambda} = 1 - \lambda$ . Therefore, if  $(\tilde{k}, \tilde{\lambda}) = (-k, 1 - \lambda)$ , then  $P = \tilde{P}$ , and  $\tilde{G}_1 = G_1$ . Moreover, from (1.14) again, we have  $(\tilde{r}, \tilde{s}) = (r, -s)$ .

Now, let us make the following change of variables

$$\tilde{x} = -x, \quad \tilde{y} = y, \quad \tilde{u} = -u,$$

in the system

$$x'' = rx, \quad y'' = ry + sx, \quad u'' = ru + sy.$$

We can easily obtain

$$\tilde{x}'' = \tilde{r}\tilde{x}, \quad \tilde{y}'' = \tilde{r}\tilde{y} + \tilde{s}\tilde{x}, \quad \tilde{u}'' = \tilde{r}\tilde{u} + \tilde{s}\tilde{y}$$

By considering the first two equations of both systems we see that the two Picard-Vessiot extensions  $F_2/\mathbb{C}(z)$  and  $\tilde{F}_2/\mathbb{C}(z)$  are equal. So their differential Galois group  $G_2$  and  $\tilde{G}_2$  coincide. Similarly, by considering the three equation of both systems we have

$$F_3 = \tilde{F}_3 \text{ and } G_3 = \tilde{G}_3.$$

This arguments are can be obviously generalised for any Jordan block of size  $d \geq 3$ . □

As a consequence, Table 1.1 remains stable for the involutive pairing  $(k, \lambda) \leftrightarrow (\tilde{k}, \tilde{\lambda})$ . For example, for rows 2, 3 and 4, we have

$$\lambda(k, p) + \lambda(-k, 1 - p) = 1.$$

So, if  $\lambda = \lambda(k, p)$  then  $\tilde{\lambda} = \lambda(-k, 1 - p)$ .

### 5.2. The case $k = \pm 2$

**PROPOSITION 5.2.** — *Let  $V(q)$  be a homogeneous potential of degree  $k = \pm 2$ . Then at an arbitrary PDP, the connected component  $G(\text{VE}_t)^\circ \simeq G(\text{VE}_z)^\circ$  is Abelian.*

*Proof.* — Let us assume that  $k = 2$ . The VE (1.4)  $\dot{\eta} = -\varphi^{k-2}V''(c)\eta$  reduces to the following linear differential system with constant coefficients

$$\dot{\eta} = -V''(c)\eta.$$

Let  $F/\mathbb{C}(\varphi(t), \dot{\varphi}(t))$  be the Picard-Vessiot extension associated to this system. It is generated over  $\mathbb{C}(\varphi(t), \dot{\varphi}(t))$  by the entries of a  $n \times n$  matrix  $\Xi(t) = \exp(St)$ , where,  $S$  is a constant matrix such that

$$(5.1) \quad S^2 = -V''(c).$$

Since it is always possible to extract a square root of a complex matrix, (5.1) has a solution whose spectrum consists of numbers  $\mu_i$  with  $\mu_i^2 = -\lambda_i$ , where the  $\lambda_i$  belong to the spectrum of  $V''(c)$ . By considering the Jordan decomposition  $S = D + N$  of  $S$  with  $D$  conjugated to  $\text{diag}(\mu_1, \dots, \mu_n)$ , the entries of  $\Xi(t)$  are polynomial in  $t$  combinations of the exponential  $\exp(\mu_i t)$ .

Since the hyperelliptic equation (1.3) is now

$$\dot{\varphi}(t)^2 + \varphi(t)^2 = 1 \Rightarrow \ddot{\varphi}(t) = -\varphi(t),$$

the associated ground field is  $\mathbb{C}(\varphi(t), \dot{\varphi}(t)) = \mathbb{C}(\exp(it))$ . Therefore, the connected component  $G(\text{VE}_t)^\circ$  is either a torus, or the direct product of a



torus and  $G_a$ . The latter case happens only if some of the above mentioned polynomials appearing inside  $\Xi(t)$  are not constant. In both cases  $G(\text{VE}_t)^\circ$  is Abelian, and the same happens for the connected component  $G(\text{VE}_z)^\circ$ , by Proposition 1.6.

As a consequence for any system of the form (1.11), corresponding to a Jordan block of size  $d \geq 1$  with  $k = 2$ , the connected component  $G_d^\circ$  is Abelian. Moreover, this result is independent of the value of the eigenvalue  $\lambda$ .

Now, let  $\tilde{k} = -2$ . Over the ground field  $\mathbb{C}(z)$ , the VE (1.10), can be written as a direct sum of  $m$  systems of the form (1.11), corresponding to Jordan blocks of sizes  $d_i$  and eigenvalue  $\tilde{\lambda}_i$ . If we denote by  $\tilde{G}_{d_i}$  their respective Galois groups for  $1 \leq i \leq m$ , then from Section 1.4 we know that  $G(\text{VE}_z)^\circ$  is an algebraic subgroup of the direct product

$$\tilde{G}_{d_1}^\circ \times \cdots \times \tilde{G}_{d_m}^\circ.$$

But from the above principle of symmetries, each  $\tilde{G}_{d_i} \simeq G_{d_i}$ , where  $G_{d_i}$  is the Galois group of system (1.11) corresponding to Jordan blocks of size  $d_i$  and eigenvalue  $\lambda_i = 1 - \tilde{\lambda}_i$ , with  $k = 2$ . Since each  $G_{d_i}^\circ$  is Abelian, so does  $G(\text{VE}_z)^\circ$  and  $G(\text{VE}_t)^\circ$ . □

### 6. About the applications of Theorem 1.3

From now,  $n$  and  $k$  are fixed integers with  $n \geq 2$  and  $k \in \mathbb{Z}^*$ ,  $c \in \mathbb{C}^n \setminus \{0\}$  is a fixed non-zero complex vector. In  $\mathbb{C}^n$  we define the following pairing

$$\langle x, y \rangle := \sum_{i=1}^n x_i y_i, \quad \text{where } x = (x_1, \dots, x_n) \in \mathbb{C}^n, \quad y = (y_1, \dots, y_n) \in \mathbb{C}^n.$$

Our aim in Section 6.1 is to show the existence of a great amount of homogeneous polynomial potentials of degree  $k$  such that  $c$  is a PDP of  $V$  and  $V''(c) = A$  is a  $n \times n$  symmetric matrix as general as possible. As a consequence, there are a lot of potentials such that  $V''(c)$  is not diagonalizable. Next, in Section 6.3, we find an explicit condition for the integrability which does not involve the eigenvalues of the Hessian.

#### 6.1. From polynomial potential to symmetric matrices

Here we assume that  $k \geq 3$ , and we consider the following sets

$$R_{n,k} = \{V(q) \in \mathbb{C}[q_1, \dots, q_n] \mid V \text{ is homogeneous, and } \deg(V) = k\},$$

$$R_{n,k}(c) = \{V(q) \in R_{n,k} \mid V'(c) = c\},$$

$$\text{Sym}_n = \{A \in M_n(\mathbb{C}) \mid A^T = A\},$$

$$\text{Sym}_{n,k}(c) = \{A \in \text{Sym}_n \mid Ac = (k - 1)c\}.$$

All these sets are affine spaces of respective complex dimensions

$$\dim R_{n,k} = \binom{n+k-1}{n-1}, \quad \dim R_{n,k}(c) = \binom{n+k-1}{n-1} - n,$$

$$\dim \text{Sym}_n = \binom{n+1}{2}, \quad \dim \text{Sym}_{n,k}(c) = \binom{n+1}{2} - n.$$

Now, the Hessian map

$$h : R_{n,k}(c) \rightarrow \text{Sym}_n, \quad V \mapsto V''(c)$$

is an affine morphism whose image is contained in  $\text{Sym}_{n,k}(c)$ . Indeed, from the Euler identity,

$$V'(c) = c \implies V''(c)c = (k - 1)c.$$

More precisely we have the following property whose proof follows from computations of dimensions.

**PROPOSITION 6.1.** — *The image of the Hessian map coincides with  $\text{Sym}_{n,k}(c)$ . In other words, if  $n \geq 2$  and  $k \geq 3$ , then for any complex symmetric matrix satisfying  $Ac = (k - 1)c$ , there exists a homogeneous polynomial potential  $V$  of  $n$  variables and degree  $k$  such that  $c$  is PDP of  $V$  and  $V''(c) = A$ .*

### 6.2. Non diagonalizable complex symmetric matrices

Let us assume that  $k \in \mathbb{Z}^*$ , we show that there are a lot of non-diagonalizable symmetric matrices belonging to the space  $\text{Sym}_{n,k}(c)$ . The most reachable ones belong to

$$\text{Spec}_{k-1} := \{A \in \text{Sym}_{n,k}(c) \mid \text{Spec}(A) = \{k - 1\}\},$$

which is the set of matrices such that  $\lambda = k - 1$  is the only eigenvalue of  $A$ . Indeed, any such  $A$  is either equal to  $(k - 1) \cdot \text{Id}$ , or non-diagonalizable.

**PROPOSITION 6.2.** — *With the notations above we have:*

- (1) *If  $n \geq 3$ , then the space  $\text{Sym}_{n,k}(c)$  contains non-diagonalizable matrices.*
- (2) *For  $n = 2$ , the space  $\text{Sym}_{2,k}(c)$  contains non diagonalizable matrices iff  $c$  is isotropic, i.e.,  $\langle c, c \rangle = 0$ .*
- (3) *Moreover, when  $n = 2$  and  $c$  is isotropic,  $\text{Sym}_{2,k}(c) = \text{Spec}_{k-1}$ .*

*Proof.* — By triangularizing a  $n \times n$  matrix  $A$ , we see that  $\lambda = k - 1$  is its only eigenvalue iff  $A$  satisfies the following  $n$  algebraic equations

$$(6.1) \quad \frac{1}{n} \operatorname{Tr} A^p = (k - 1)^p \quad \text{for} \quad 1 \leq p \leq n.$$

If  $A \in \operatorname{Sym}_{n,k}(c)$ , then  $\lambda = k - 1$  is one of the eigenvalues of  $A$ , so we only need the  $(n - 1)$  first equations of (6.1) to ensure that its  $(n - 1)$  remaining eigenvalues coincide with  $k - 1$ . This proves that  $\operatorname{Spec}_{k-1}$  is an algebraic affine subset of  $\operatorname{Sym}_{n,k}(c)$ , whose dimension satisfies

$$\dim \operatorname{Spec}_{k-1} \geq \dim \operatorname{Sym}_{n,k}(c) - (n - 1) = \binom{n + 1}{2} - 2n + 1.$$

Hence  $\dim \operatorname{Spec}_{k-1} \geq 1$ , as soon as  $n \geq 3$ . This proves point 1.

Let  $n = 2$ , and  $A \in \operatorname{Sym}_{2,k}(c)$ . The line  $(\mathbb{C}c)^\perp$  of vectors of  $\mathbb{C}^2$  which are orthogonal to  $c$  is globally left invariant by  $A$ . Therefore, if  $(\mathbb{C}c)^\perp \neq \mathbb{C}c$ , then  $A$  is diagonalizable in a basis  $(c, v)$ , where  $v \in (\mathbb{C}c)^\perp$ . So, if  $\operatorname{Sym}_{2,k}(c)$  contains a matrix which is not diagonalizable, then we have  $(\mathbb{C}c)^\perp = \mathbb{C}c$ . That is,  $c$  is isotropic. Conversely, let us assume that  $c$  is isotropic. Let  $A \in \operatorname{Sym}_{2,k}(c)$ . If  $\lambda \neq k - 1$  is another eigenvalue of  $A$ , then  $\ker(A - \lambda \operatorname{Id})$  is a line which is orthogonal to  $\mathbb{C}c$  and different from it. This is impossible since  $(\mathbb{C}c)^\perp = \mathbb{C}c$ . Therefore  $k - 1$  is the only possible eigenvalue of each matrix belonging to  $\operatorname{Sym}_{2,k}(c)$ . Hence,  $\operatorname{Sym}_{2,k}(c) = \operatorname{Spec}_{k-1}$ , and point 3 is proved. Since

$$\dim \operatorname{Sym}_{2,k}(c) = 1 = \dim \operatorname{Spec}_{k-1},$$

except for the matrix  $(k - 1) \cdot \operatorname{Id}$ , any other matrix of  $\operatorname{Sym}_{2,k}(c)$  is not diagonalizable. This proves point 2.  $\square$

### 6.3. New necessary condition for integrability

Here we focus our attention on some potentials admitting isotropic PDP. Although the eigenvalues of the Hessian  $V''(c)$  does not give any obstacle to the integrability, we exhibit a new one. In the following we set  $c_0 = (1, i)$ .

**PROPOSITION 6.3.** — *Let  $V(q) = V(q_1, q_2)$  be a two degrees of freedom homogeneous potential of the following form*

$$V(q) := (q_1^2 + q_2^2)W(q)$$

where  $W(q)$  is a homogeneous function with

$$\deg W \in \mathbb{Z} \setminus \{-4, -2, -1, 0\}, \quad W(c_0) \in \mathbb{P}^1 \setminus \{0, \infty\}.$$

If the Hamiltonian system associated with this potential is completely integrable, then

$$i \frac{\partial W}{\partial q_1}(c_0) + \frac{\partial W}{\partial q_2}(c_0) = 0.$$

*Proof.* — If  $k = \deg V = 2 + \deg W$ , then  $k \in \mathbb{Z} \setminus \{-2, 0, 1, 2\}$ , i.e., we have either  $|k| \geq 3$ , or  $k = -1$ . From  $V(q) = (q_1^2 + q_2^2)W(q)$  we get

$$V'(c_0) = 2W(c_0) \cdot c_0.$$

So,  $V'(c) = c$  for  $c = \mu c_0$ , where  $2\mu^{k-2}W(c_0) = 1$ . Hence, according to point 3 of Proposition 6.2, it follows that  $V''(c) \in \text{Spec}_{k-1}$ , i.e.,  $\lambda = k - 1$  is the only eigenvalue of  $V''(c)$ . Hence,  $G_1^\circ \simeq G_a$ , and the potential satisfies the conditions appearing in the row 2 or 3 of Table 1.1. Thus Theorem 1.2 does not give any obstacles for the integrability of  $V$ . Now, Theorem 1.3 gives an obstacle iff  $V''(c)$  is not diagonalizable. This happens iff  $V''(c) \neq (k - 1) \cdot \text{Id}$ , i.e., iff

$$\frac{\partial^2 V}{\partial q_1 \partial q_2}(c) \neq 0 \iff \frac{\partial^2 V}{\partial q_1 \partial q_2}(c_0) \neq 0$$

But a direct computation shows that the last condition is equivalent to the following one

$$i \frac{\partial W}{\partial q_1}(c_0) + \frac{\partial W}{\partial q_2}(c_0) \neq 0. \quad \square$$

Let  $V(q) = (q_1^2 + q_2^2)W(q)$ , with  $\deg W \in \mathbb{Z} \setminus \{-4, -2, -1, 0\}$ , and  $W(c_0) \in \mathbb{P}^1 \setminus \{0, \infty\}$ . The condition

$$i \frac{\partial W}{\partial q_1}(c_0) + \frac{\partial W}{\partial q_2}(c_0) = 0,$$

is therefore a non-trivial condition for the integrability of  $V$ . For example, let us take  $W(q) = aq_1 + bq_2$ . By the above proposition, if  $V$  is integrable, then  $W(q) = a(q_1 - iq_2)$ , for a certain  $a \in \mathbb{C}$ . Moreover, in this case  $V = a(q_1^2 + q_2^2)(q_1 - iq_2)$  is indeed integrable with the additional first integral

$$(6.2) \quad F = ip_1^2 + 6p_1p_2 - 5ip_2^2 + 8aq_2(q_1 - iq_2)^2.$$

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