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LONG TIME ASYMPTOTICS OF THE
CAMASSA–HOLM EQUATION ON THE HALF-LINE

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ABSTRACT. — We study the long-time behavior of solutions of the initial-boundary value (IBV) problem for the Camassa–Holm (CH) equation $u_t - u_{txx} + 2u_x + 3uu_x = 2u_xu_{xx} + uu_{xxx}$ on the half-line $x \geqslant 0$. The paper continues our study of IBV problems for the CH equation, the key tool of which is the formulation and analysis of associated Riemann–Hilbert factorization problems. We specify the regions in the quarter space-time plane $x > 0$, $t > 0$ having qualitatively different asymptotic pictures, and give the main terms of the asymptotics in terms of spectral data associated with the initial and boundary values.

RÉSUMÉ. — Nous étudions le comportement asymptotique, pour de grandes valeurs du temps t, de solutions de problèmes aux limites pour l'équation de Camassa–Holm (CH) $u_t - u_{txx} + 2u_x + 3uu_x = 2u_xu_{xx} + uu_{xxx}$ sur la demi-droite $x \geqslant 0$. Cet article prolonge nos travaux antérieurs sur les problèmes aux limites pour l'équation de Camassa–Holm, travaux dont la clef est la formulation et l'analyse de problèmes de Riemann–Hilbert associés. Dans le quart de plan espace-temps $x > 0$, $t > 0$, nous distinguons des régions où les solutions ont un comportement asymptotique qualitativement différent, et nous calculons pour chacune d'elles le terme principal de l'asymptotique en termes de données spectrales associées aux valeurs initiales et au bord.

1. Introduction

The Camassa–Holm (CH) equation [12]

(1.1) $$u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_xu_{xx} + uu_{xxx},$$

where $\omega \geqslant 0$ is a parameter, and $u \equiv u(x,t)$, has attracted considerable attention recently, due to a number of interesting, distinguished features. Particularly, being considered as a model of unidirectional wave propagation [12, 13, 18, 15, 31], it accounts for both waves existing globally for all

t and waves blowing up at a certain finite critical time $t = t_{\mathrm{cr}}$, depending on the form of the initial data. For $\omega > 0$, it possesses analytic solitary waves — solitons [16, 32, 35]; moreover, any initial data from a large class of decreasing, as $|x| \to \infty$, functions evolve, for large time, into a train of solitons [9]. For $\omega = 0$, the equation supports weak, non-analytic solutions, called "peakons", which are stable patterns interacting like solitons [12, 20, 1, 2].

Although there are plenty of results concerning the initial value problem for (1.1), where the equation is supplemented by the initial conditions $u(x, 0) = u_0(x)$, $x \in (-\infty, \infty)$, much less is known for initial boundary value (IBV) problems, where, for example, in the case of the half-line setting, the data are

– the initial conditions $u_0(x)$, $x \geqslant 0$, and
– a set of boundary data prescribed at $x = 0$, consisting of (a subset of) the boundary values $u(0, t) = v_0(t)$, $u_x(0, t) = v_1(t)$, and $u_{xx}(0, t) = v_2(t)$.

It is not obvious how many boundary values can be prescribed without overdetermining the IBV problem. In [34], the existence and uniqueness of strong solutions to the IBV problem with boundary condition $u(0, t) \equiv 0$ and decaying initial data from certain functional classes has been studied for the case $\omega = 0$. In [14] the authors demonstrated that the IBV problem with boundary data $u(0, t) = v_0(t)$ is well-posed in the case $v_0(t) \leqslant 0$, in the sense that a strong solution, if it exists up to a certain time, is unique.

The CH equation is known to be formally integrable: it has a Lax pair representation as compatibility condition for the system of linear equations

$$(1.2) \qquad \begin{aligned} \psi_{xx} &= \frac{1}{4}\psi + \lambda(m + \omega)\psi, \\ \psi_t &= \left(\frac{1}{2\lambda} - u\right)\psi_x + \frac{1}{2}u_x\psi, \end{aligned}$$

where $\psi = \psi(x, t, \lambda)$, $\lambda \in \mathbb{C}$ and $m \equiv m(x, t) := u - u_{xx}$. This has been exploited in a number of papers dealing with initial value problems, see, e.g., [16, 17, 19, 33]. In [8], we have developed an inverse scattering approach to the IBV problems for the CH equation with $\omega \neq 0$, which is based on the *simultaneous* spectral analysis of the two eigenvalue equations of the Lax pair (1.2) in the domain $0 \leqslant x < +\infty$, $0 \leqslant t \leqslant T < \infty$.

A general approach, allowing the inverse scattering technique to study the IBV problems, was initiated by Fokas [25, 26] and further developed by several authors; see [3, 6, 28]. On the boundary of the (x, t) domain, the analysis leads to spectral problems for both equations of the Lax pair, the

x-equation (for $t = 0$) and the t-equation (for $x = 0$). In [8] we provide a characterization of the solution of the initial boundary value problem for the CH equation (1.1) in terms of the solution of a matrix Riemann–Hilbert (RH) factorization problem in the complex plane of the spectral parameter, for which the data are determined in terms of spectral functions associated with the initial and boundary values of the solution.

In the present paper, we continue our study of the IBV problem for the CH equation started in [8] and present the results of the analysis of the long-time behavior of a solution of this problem announced in [8]. In what follows, we assume that $\omega = 1$, the analysis being valid, by simple rescaling, for all $\omega \neq 0$. Our approach is based on the long-time analysis of the associated RH problem [8], and is, in fact, an adaptation of the nonlinear steepest descent method for studying asymptotics of solutions of integrable nonlinear equations introduced by Deift and Zhou [22]. Recently [9] we have applied such an approach to the study of the long-time behavior of the solution of the initial value problem for the CH equation (on the whole x line). We have shown that the asymptotic picture in the case of the CH equation is richer than that for the (closely related) KdV equation, containing more qualitatively different phenomena. Moreover, the asymptotic picture in the case of the CH equation, being restricted to the quarter plane $x > 0$, $t > 0$, shows all the distinctive regions, with qualitatively different behavior, presented on the whole half-plane $-\infty < x < \infty$, $t > 0$ for the KdV case [21]: there are

(i) a soliton region,

(ii) a similarity region (region of decaying modulated oscillations),

(iii) a self-similar region (described in terms of Painlevé transcendents),

and possibly

(iv) a collisionless shock region.

We show that the solution of the IBV problem for the CH equation with decaying boundary values in the case $v_0(t) \leqslant 0$, assuming that it exists for all t and satisfies the inequality $m(x,t) + 1 > 0$ for all $x \geqslant 0$ and $t \geqslant 0$, exhibits all these regions (we label them adopting the terminology as above), and that the precise values of the parameters involved in the leading asymptotic terms are determined in terms of spectral functions associated with the initial data $u_0(x)$ and the boundary values $v_j(t)$, $j = 0, 1, 2$.

The paper is organized as follows. In Section 2 we present an appropriate Riemann–Hilbert formalism for the IBV problem. Section 3 deals with the "soliton" region. In Sections 4 and 5, we present the decaying asymptotics in the "similarity" region and the "self-similar" region, respectively. Section 6
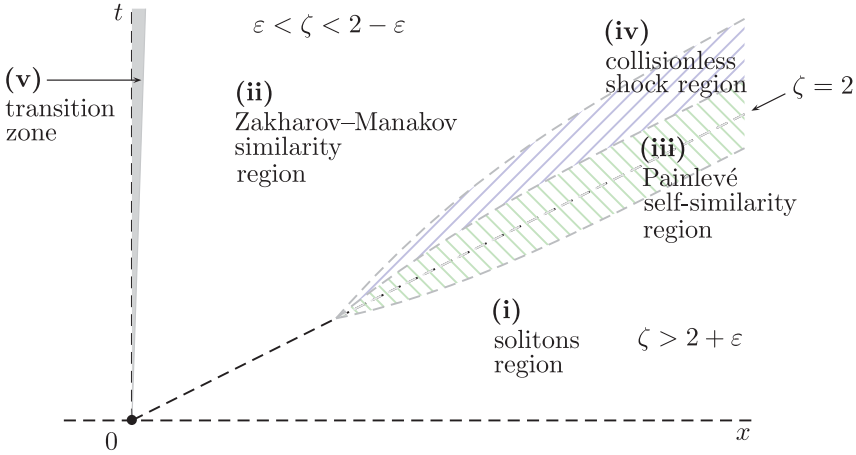
Figure 1.1. *The different regions in the (x, t)-quarter-plane, $\zeta := \frac{x}{t}$.*

discusses the "collisionless shock" region. The asymptotics in these regions are given in Theorems 3.2, 4.2, 5.1, and 6.1, respectively.

2. The RH formalism for the IBV problem

The "building blocks" for the construction of a RH problem, whose solution is then used to give a representation of the solution to the IBV problem in question, are dedicated solutions of the associated Lax pair ("eigenfunctions") with good control on the Riemann sphere of the spectral parameter [3, 4, 8]. Here we present a brief account of the needed solutions and their properties (more details are given in [8]) and give a Riemann–Hilbert formalism, different from that used in [8]: we will see that the new formalism is better adapted for the long time study that follows.

2.1. Assumptions

Before presenting the formalism, we comment on assumptions we made in Section 1:

- $v_0(t) \leqslant 0$,
- $m(x, t) + 1 > 0$ for all $x \geqslant 0$ and $t \geqslant 0$.

Both assumptions ensure that the dedicated solutions of the Lax pair, see (2.9) below, have a well-controlled behavior, as functions of the spectral parameter, simultaneously for all $x \geqslant 0$ and $t \geqslant 0$ (see the expressions in the exponentials in (2.9)).

While $v_0(t) \leqslant 0$ is a condition on the boundary data, the assumption $m(x, t) + 1 > 0$ is about the solution of the IBV problem. With this respect, we notice that for the problem on the whole line $x \in (-\infty, \infty)$, if the initial condition is such that $m(x, 0) + 1 > 0$ for all x (and that $m(x, 0)$ vanishes to 0 sufficiently fast as $x \to \pm\infty$), then there is a unique global solution for this problem such that $m(x, t) + 1 > 0$ for all $x \geqslant 0$ and $t \geqslant 0$, see [16]. To the best of our knowledge, there are no such results for IBV problems with general boundary data. On the other hand, the global existence of a solution satisfying $m(x, t) + 1 > 0$ can be established under the assumption of global solvability (for all $x \geqslant 0$ and $t \geqslant 0$) of the Riemann–Hilbert problem constructed below, see Proposition 2.6.

With this respect we notice that methods other than (any variant of) the inverse scattering method may provide better characterization of the global existence and uniqueness of solutions of initial boundary value problems. On the other hand, the main concern in the present paper is the asymptotic behavior of a solution (provided it exists globally in time).

Thus, assuming that there exists a smooth solution $u(x, t)$ of the CH equation, $x > 0$, $t > 0$, decaying sufficiently fast as $x \to +\infty$ (for all t) and satisfying the assumptions above, we are looking for a representation of this solution in terms of the solution of an associated RH problem, in the complex plane of an appropriate "spectral parameter", the data for which (the jump matrix and possibly residue conditions, in the case of a singular RH problem) are determined by the initial and boundary values of $u(x, t)$ (for $t = 0$ and for $x = 0$). Then, this representation is used, in the spirit of the nonlinear steepest descent method, for deriving various asymptotic results.

2.2. Spectral functions

Since the Lax pair equations have apparently different structures near the singular points, $\lambda = 0$ and $\lambda = \infty$, it turns to be useful to work with *two* variants of the Lax pair, the first having an appropriate form near $\lambda = \infty$ and the second, near $\lambda = 0$. These Lax pairs are found to be in the

form of 2×2 systems of first order differential equations [7, 10, 9, 8, 11],

(2.1)
$$\tilde{\Phi}_{\infty x} + ikp_x\sigma_3\tilde{\Phi}_\infty = U_\infty\tilde{\Phi}_\infty,$$
$$\tilde{\Phi}_{\infty t} + ikp_t\sigma_3\tilde{\Phi}_\infty = V_\infty\tilde{\Phi}_\infty,$$

and

(2.2)
$$\tilde{\Phi}_{0x} + ik\sigma_3\tilde{\Phi}_0 = U_0\tilde{\Phi}_0,$$
$$\tilde{\Phi}_{0t} + \frac{ik}{2\lambda}\sigma_3\tilde{\Phi}_0 = V_0\tilde{\Phi}_0,$$

where

(2.3)
$$k^2 = -\lambda - \frac{1}{4},$$

(2.4)
$$p_x = \sqrt{m+1}, \qquad p_t = \frac{1}{2\lambda} - u\sqrt{m+1},$$

(2.5a) $\quad U_\infty(x,t,k) = \dfrac{1}{4}\dfrac{m_x}{m+1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \dfrac{1}{8ik}\dfrac{m}{\sqrt{m+1}}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix},$

$$V_\infty(x,t,k) = \left[\frac{m_t}{4(m+1)} + \frac{u_x}{2}\right]\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$+ \frac{1}{8ik}\frac{u(m+2)}{\sqrt{m+1}}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$+ \frac{ik}{4\lambda}\left\{\sqrt{m+1}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \frac{1}{\sqrt{m+1}}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\right\}$$

(2.5b)
$$+ \frac{ik}{2\lambda}\sigma_3,$$

(2.6a)
$$U_0(x,t,k) = \frac{\lambda}{2ik}m(x,t)\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix},$$

$$V_0(x,t,k) = \frac{u_x}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{u}{4ik}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(2.6b)
$$+ \frac{\lambda u}{2ik}\left\{2\sigma_3 - m\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\right\}.$$

These forms of the Lax pair follow from (1.2) if we define
$$\tilde{\Phi}_\infty := G_\infty\left(\begin{smallmatrix} \psi \\ \psi_x \end{smallmatrix}\right), \quad \tilde{\Phi}_0 := G_0\left(\begin{smallmatrix} \psi \\ \psi_x \end{smallmatrix}\right)$$

with
$$G_0(k) = \frac{1}{2}\begin{pmatrix} 1 & -\frac{1}{ik} \\ 1 & \frac{1}{ik} \end{pmatrix}, \; G_\infty(x,t,k) = G_0(k)\begin{pmatrix} (m+1)^{1/4} & 0 \\ 0 & (m+1)^{-1/4} \end{pmatrix}.$$

Remarks 2.1. — (a) At $\lambda = 0$, *i.e.*, at $k = \pm\frac{i}{2}$, we have

$$U_0(x,t,k)\big|_{k=\pm i/2} = 0 \text{ for all } (x,t).$$

(b) Both expressions in (2.4) are consistent because of

$$(2.7) \qquad \left(\sqrt{m+1}\right)_t = -\left(u\sqrt{m+1}\right)_x,$$

which is an equivalent form of (1.1). This allows us to define p satisfying (2.4) by

$$(2.8) \quad p(x,t,k) = \int_0^x \sqrt{m(\xi,t)+1}\,d\xi - \int_0^t u(0,\zeta)\sqrt{m(0,\zeta)+1}\,d\zeta + \frac{t}{2\lambda},$$

thus specifying p by the condition $p(0,0,k) = 0$.

(c) The assumption $u(0,t) \leqslant 0$ implies that $p(x,t,k) - \frac{t}{2\lambda} \geqslant 0$ for all x,t.

The fact that U_∞, V_∞ in (2.5) are such that $U_\infty, V_\infty = O(1)$ as $k \to \infty$ and the terms of order k^0 in U_∞ and V_∞ are off-diagonal and that $U_\infty, V_\infty = o(1)$ as $x \to +\infty$, allows us to define solutions to (2.5), $\tilde{\Phi}_{\infty j}(x,t,k)$, $j = 1,2,3$, analytic in the corresponding domains in the k-plane (relative to the contour $\{\operatorname{Im} k = 0\} \cup \{|k| = \frac{1}{2}\}$) and having well-controlled properties as $k \to \infty$, via:

$$\tilde{\Phi}_{\infty j}(x,t,k) := \Phi_{\infty j}e^{-ikp(x,t,\lambda)\sigma_3},$$

where the $\Phi_{\infty j}$ are solutions of associated Volterra integral equations

$$\Phi_{\infty j}(x,t,k) = I + \int_{(x_j,t_j)}^{(x,t)} e^{-ik(p(x,t,k)-p(y,\tau,k))\hat{\sigma}_3} \times$$

$$(2.9) \qquad\qquad \times \{U_\infty(y,\tau,k)\Phi_{\infty j}(y,\tau,k)dy + V_\infty(y,\tau,k)\Phi_{\infty j}(y,\tau,k)d\tau\}$$

with $(x_1,t_1) = (0,+\infty)$, $(x_2,t_2) = (0,0)$, and $(x_3,t_3) = (+\infty,t)$ (for more details see [8]).

Notation. — We use the following standard notations:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and } e^{\hat{\sigma}_3}A := e^{\sigma_3}Ae^{-\sigma_3}$$

for any 2×2 matrix A.

Similarly, define

$$\tilde{\Phi}_{0j} := \Phi_{0j}e^{-i(kx+\frac{k}{2\lambda}t)},$$

where the Φ_{0j} are solutions of the integral equations

$$\Phi_{0j}(x,t,k) = I + \int_{(x_j,t_j)}^{(x,t)} e^{-ik[(x-y)+\frac{ik}{2\lambda}(t-\tau)]\hat{\sigma}_3} \times$$

$$(2.10) \qquad\qquad \times \{U_{0j}(y,\tau,k)\Phi_{0j}(y,\tau,k)dy + V_0(y,\tau,k)\Phi_{0j}(y,\tau,k)d\tau\}.$$

The spectral functions appear naturally when relating the eigenfunctions introduced above. Indeed, since the eigenfunctions $\Phi_{\infty j}$ and Φ_{0j} are solutions of systems of differential equations originating from the same system (1.2), they are related (in the domain where they are defined) by matrices independent of (x, t). Introducing the spectral matrices

$$(2.11) \qquad \begin{aligned} S(k) &= \Phi_{01}(0, 0, k), & \tilde{S}(k) &= \Phi_{\infty 1}(0, 0, k), \\ s(k) &= \Phi_{\infty 3}(0, 0, k), & \tilde{s}(k) &= \Phi_{03}(0, 0, k) \end{aligned}$$

and taking into account the normalization conditions at $(x, t) = (x_j, t_j)$, we have:

$$(2.12) \qquad \begin{aligned} \Phi_{01}(x, t, k) &= \Phi_{02}(x, t, k) e^{-i(kx + \frac{k}{2\lambda} t)\hat{\sigma}_3} S(k), \\ \Phi_{03}(x, t, k) &= \Phi_{02}(x, t, k) e^{-i(kx + \frac{k}{2\lambda} t)\hat{\sigma}_3} \tilde{s}(k), \\ \Phi_{\infty 1}(x, t, k) &= \Phi_{\infty 2}(x, t, k) e^{-ikp(x,t,\lambda)\hat{\sigma}_3} \tilde{S}(k), \\ \Phi_{\infty 3}(x, t, k) &= \Phi_{\infty 2}(x, t, k) e^{-ikp(x,t,\lambda)\hat{\sigma}_3} s(k). \end{aligned}$$

Obviously, $\det S = \det \tilde{S} = \det s = \det \tilde{s} \equiv 1$. Since all matrices $W = U_0, V_0, U_\infty, U_\infty$ satisfy the symmetry relations

$$(2.13) \qquad \overline{W(\,\cdot\,, \cdot\,, \bar{k})} = W(\,\cdot\,, \cdot\,, -k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W(\,\cdot\,, \cdot\,, k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the spectral matrices $S(k)$, $s(k)$, $\tilde{S}(k)$, and $\tilde{s}(k)$ can be written as

$$(2.14) \qquad \begin{aligned} s(k) &= \begin{pmatrix} \overline{a(\bar{k})} & b(k) \\ \overline{b(\bar{k})} & a(k) \end{pmatrix}, & S(k) &= \begin{pmatrix} \overline{A(\bar{k})} & B(k) \\ \overline{B(\bar{k})} & A(k) \end{pmatrix}, \\ \tilde{s}(k) &= \begin{pmatrix} \overline{\tilde{a}(\bar{k})} & \tilde{b}(k) \\ \overline{\tilde{b}(\bar{k})} & \tilde{a}(k) \end{pmatrix}, & \tilde{S}(k) &= \begin{pmatrix} \overline{\tilde{A}(\bar{k})} & \tilde{B}(k) \\ \overline{\tilde{B}(\bar{k})} & \tilde{A}(k) \end{pmatrix}, \end{aligned}$$

where $a(-k) = \overline{a(\bar{k})}$, $A(-k) = \overline{A(\bar{k})}$, $b(-k) = \overline{b(\bar{k})}$, $B(-k) = \overline{B(\bar{k})}$, and similarly for the entries of $\tilde{S}$ and $\tilde{s}$.

The direct spectral mapping

$$\{u(x, 0)\} \longmapsto \{a(k), b(k)\}$$

is determined via the solution $\Phi_{\infty 3}(x, 0, k)$ of the integral equation

$$(2.15) \quad \Phi_{\infty 3}(x, 0, k) = I - \int_x^\infty e^{ik \int_x^y \sqrt{m(\xi, 0) + 1}\, d\xi \hat{\sigma}_3} (U_\infty \Phi_{\infty 3})(y, 0, k) dy$$

taken at $x = 0$.

In a similar way, the direct spectral mapping

$$\{u^{(j)}(0, t)\}_{j=0}^2 \longmapsto \{A(k), B(k)\}$$

is defined via the solution of the integral equation

$$(2.16) \qquad \Phi_{01}(0,t,k) = I - \int_t^\infty e^{-\frac{ik}{2\lambda}(t-\tau)\hat{\sigma}_3}(V_0\Phi_{01})(0,\tau,k)\mathrm{d}\tau$$

taken at $t = 0$.

The spectral matrices introduced above are obviously not independent. Indeed, by the definitions, Φ_{0j} and $\Phi_{\infty j}$ are related to the same system of ODEs (1.2). Hence they are related by certain matrices $C_j(k)$ independent of (x,t) (they are defined by the normalizations in (2.9) and (2.10)):

$$\Phi_{\infty j}(x,t,k) = Q(x,t)\Phi_{0j}(x,t,k)e^{-i(kx+\frac{k}{2\lambda}t)}C_j(k)e^{ikp(x,t,k)}, \quad j = 1,2,3,$$

where

$$(2.17) \qquad Q(x,t) := G_\infty(x,t,k)G_0^{-1}(k) = \frac{1}{2}\begin{pmatrix} \varkappa + \varkappa^{-1} & \varkappa - \varkappa^{-1} \\ \varkappa - \varkappa^{-1} & \varkappa + \varkappa^{-1} \end{pmatrix}$$

with

$$\varkappa = \varkappa(x,t) = [m(x,t)+1]^{1/4}.$$

Consequently, the spectral matrices are related by

$$(2.18\mathrm{a}) \qquad \tilde{S}(k) = Q(0,0)S(k)e^{-ik\nu(0)\sigma_3},$$

$$(2.18\mathrm{b}) \qquad s(k) = Q(0,0)\tilde{s}(k)e^{-ik\nu(0)\sigma_3},$$

where we have introduced

$$(2.19) \qquad \nu(t) := \int_0^\infty \left(\sqrt{m(\xi,t)+1}-1\right)\mathrm{d}\xi.$$

Introducing also

$$(2.20) \qquad \tilde{p}(t) := -\int_0^t u(0,\zeta)\sqrt{m(0,\zeta)+1}\,\mathrm{d}\zeta,$$

notice the conservation law

$$(2.21) \qquad \nu(t) + \tilde{p}(t) = \nu(0),$$

that follows from (2.7).

2.3. Direct analysis

Analytic properties, in the complex k-plane, of the eigenfunctions, see [8], allow us, given the spectral functions $\{a(k),b(k),A(k),B(k)\}$, to construct a family of piece-wise (relative to a certain contour Σ) meromorphic, 2×2 matrix-valued functions parametrized by (x,t) such that:

(i) its limiting values on the contour are related by the spectral functions defined above;

(ii) the residue conditions at the poles are also expressed in terms of the spectral functions;

(iii) being evaluated at a certain point of the k-plane, it gives a solution of the Camassa-Holm equation, $u(x,t)$;

(iv) the limiting values of $u(x,t)$ at $t = 0$ and at $x = 0$ generate, via the direct scattering mappings, the spectral functions used in the construction of the jump conditions and the residue conditions; more specifically, $u(x,0)$ generates $\{a(k),b(k)\}$ and $u(0,t),u_x(0,t),u_{xx}(0,t)$ generate $\{A(k), B(k)\}$.

Properties (i)–(iii) allow interpreting this construction as a Riemann–Hilbert problem, with data given on the contour and at the poles. Property (iv) then allows determining the solution of the Camassa–Holm equation with given initial data $u(x,0)$ and boundary data $u(0,t), u_x(0,t), u_{xx}(0,t)$, and thus allows reducing the long time analysis of $u(x,t)$ to the long time analysis of the associated RH problem.

Thus, assuming that $u(x,t)$ satisfies the CH equation, let us define a matrix-valued function $M(x,t,k)$ by using appropriate solutions of the Lax pair equations (eigenfunctions) as follows. Let

$$D_1 = \{k \mid |k| < \tfrac{1}{2}, \operatorname{Im} k > 0\},$$
$$D_2 = \{k \mid |k - \tfrac{i}{2}| < \varepsilon, \ |k| > \tfrac{1}{2}\}.$$

• For $\operatorname{Im} k > 0$,

(2.22)

$$M(x,t,k) = \begin{cases} \left(\dfrac{\Phi^{(1)}_{\infty 1}(x,t,k)}{\hat{a}(k)} \quad \Phi^{(2)}_{\infty 3}(x,t,k) \right), & k \in D_1, \\[3mm] Q(x,t) \left(\dfrac{\Phi^{(1)}_{02}(x,t,k)}{\tilde{a}(k)} \quad \Phi^{(2)}_{03}(x,t,k) \right) E(x,t,k), & k \in D_2, \\[3mm] \left(\dfrac{\Phi^{(1)}_{\infty 2}(x,t,k)}{a(k)} \quad \Phi^{(2)}_{\infty 3}(x,t,k) \right) & k \notin D_1 \cup D_2, \end{cases}$$

where $Q(x,t)$ is given by (2.17), and

$$E(x,t,k) = \mathrm{e}^{ik(p(x,t,k) - x - \frac{t}{2\lambda} - \nu(0))\sigma_3},$$
$$\hat{a} = (\tilde{S}^{-1}s)_{22} = a\bar{\tilde{A}} - b\bar{\tilde{B}}.$$

• For $\operatorname{Im} k < 0$,

$$M(x,t,k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{M(x,t,\bar{k})} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let Σ be the following contour, see Figure 2.1:

(2.23)
$$\Sigma = \mathbb{R} \cup \left\{ k \mid |k| = \frac{1}{2} \right\} \cup \left\{ k \mid |k| \geqslant \frac{1}{2}, \left| k - \frac{i}{2} \right| = \varepsilon \right\} \cup \left\{ k \mid |k| \geqslant \frac{1}{2}, \left| k + \frac{i}{2} \right| = \varepsilon \right\}.$$
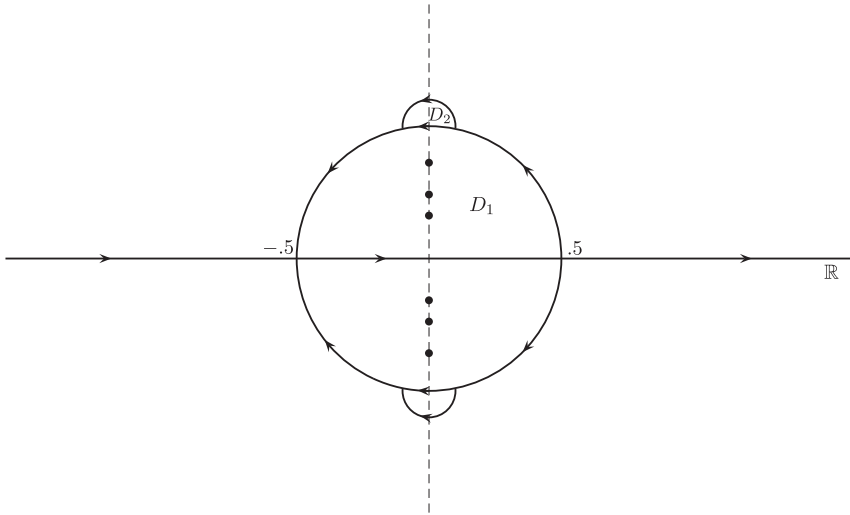

Figure 2.1. Contour Σ.

Then $M(x, t, k)$ has the following properties (*cf.* [8]):

(a) *Symmetry properties*:

(2.24)
$$\overline{M(x, t, \bar{k})} = M(x, t, -k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M(x, t, k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(b) *Analyticity*: $M(x, t, k)$ is meromorphic in $\mathbb{C} \setminus \Sigma$ and continuous up to the boundary of each connected component of $\mathbb{C} \setminus \Sigma$ except possibly at $k = 0$.

(c) *Residue conditions*: We assume that $\hat{a}(k)$ has at most a finite number of purely imaginary zeros in D_1, at $k = i\nu_j$, $0 < \nu_j < \frac{1}{2}$, $j = 1, \ldots, N$. Then

$$\text{Res}_{k=i\nu_j} M^{(1)}(x, t, k) = i\gamma_j e^{-2\nu_j p(x, t, i\nu_j)} M^{(2)}(x, t, i\nu_j),$$

$$\text{Res}_{k=-i\nu_j} M^{(2)}(x, t, k) = -i\gamma_j e^{-2\nu_j p(x, t, i\nu_j)} M^{(1)}(x, t, -i\nu_j),$$

where

$$\gamma_j = -i \frac{\bar{\bar{B}}(i\nu_j)}{\dot{\hat{a}}(i\nu_j) a(i\nu_j)}.$$

We assume $\gamma_j > 0$ for all j. Concerning these new assumptions, see Remark 2.2 below.

(d) *Behavior at $k = 0$:*

$$(2.25) \qquad M(x,t,k) = \frac{\delta(x,t)}{ik} \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} + \mathrm{O}(1), \quad k \to 0, \ \mathrm{Im}\, k > 0,$$

for some $\delta(x,t) \in \mathbb{R}$.

(e) *Behavior at $k = \infty$:*

$$(2.26) \qquad M(x,t,k) = I + \mathrm{O}\!\left(\tfrac{1}{k}\right), \quad k \to \infty.$$

(f) *Jump relation across Σ:*

$$(2.27) \qquad M_-(x,t,k) = M_+(x,t,k)J(x,t,k), \qquad k \in \Sigma \setminus \{0\},$$

where $M_\pm(x,t,k)$ are the limits of $M(x,t,k')$ when $k' \in \mathbb{C} \setminus \Sigma$ approaches $k \in \Sigma$ from the $\pm$ side of the contour, and

$$J(x,t,k) = \mathrm{e}^{-ikp(x,t,k)\sigma_3} J_0(k) \mathrm{e}^{ikp(x,t,k)\sigma_3}$$

where $J_0(k)$ is defined as follows.

• For $\mathrm{Im}\, k \geqslant 0$,

$$(2.28)$$

$$J_0(k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ R(k) & 1 \end{pmatrix} & |k - \tfrac{i}{2}| > \varepsilon, \ |k| = \tfrac{1}{2}, \\[2mm] \begin{pmatrix} 1 & 0 \\ -H(k) & 1 \end{pmatrix} & |k - \tfrac{i}{2}| = \varepsilon, \ |k| > \tfrac{1}{2}, \\[2mm] \begin{pmatrix} 1 & 0 \\ R(k) + H(k) & 1 \end{pmatrix} & |k - \tfrac{i}{2}| < \varepsilon, \ |k| = \tfrac{1}{2}, \\[2mm] \begin{pmatrix} 1 & \bar{r}(k) \\ -r(k) & 1 - |r(k)|^2 \end{pmatrix} & \mathrm{Im}\, k = 0, \ |k| > \tfrac{1}{2}, \\[2mm] \begin{pmatrix} 1 & \bar{r}(k) - \bar{R}(k) \\ -r(k) + R(k) & 1 - |r(k) - R(k)|^2 \end{pmatrix} & \mathrm{Im}\, k = 0, \ |k| < \tfrac{1}{2}, \end{cases}$$

where

$$(2.29) \qquad H(k) = \frac{\varkappa_0 - \varkappa_0^{-1}}{2a(k)\tilde{a}(k)} \mathrm{e}^{-ik\nu(0)}, \qquad R(k) = -\frac{\bar{\bar{B}}(k)}{a(k)\hat{a}(k)}.$$

• For $\mathrm{Im}\, k < 0$,

$$(2.30) \qquad J_0(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\overline{J_0(\bar{k})}\right)^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(g) *Behavior at $k = \frac{i}{2}$:*

(2.31) $$M\left(x, t, \tfrac{i}{2}\right) = Q(x, t)e^{\frac{1}{2}\int_x^\infty(\sqrt{m(\xi,t)+1}-1)\mathrm{d}\xi\,\sigma_3}.$$

Remark 2.2. —   The contour we use here is well adapted for the long time study. Namely, the trigonal structure of the jump matrix on $\Sigma \setminus \mathbb{R}$ provides that as $t \to +\infty$, these jumps decay to the identity matrix, which is of primary importance for the long time analysis. The price for this, however, is that we deal now with a piecewise meromorphic function (instead of a piecewise holomorphic one, as in [8]), with prescribed residue conditions. The particular structure of the poles and residue conditions assumed above is also related to the long time analysis: violating these assumptions would lead to singularities for $u(x, t)$ (the situation here is similar to that for the KdV equation, see, *e.g.*, , [27]).

2.4. Formulation of the RH-problem

The construction above suggests introducing a new scale,

$$(x, t) \mapsto (z(x, t), t),$$

as follows:

$$z(x, t) = \int_0^x \sqrt{m(\xi, t) + 1}\,\mathrm{d}\xi.$$

Indeed, in this new scale, the jump matrix J becomes explicitly given in terms of the spectral functions and the boundary values:

$$J(x, t, k) = J(z, t, k)\big|_{z=z(x,t)},$$

if one defines

(2.32) $$J(z, t, k) := e^{-ik(z+\tilde{p}(t)+\frac{t}{2\lambda})\sigma_3}J_0(k)e^{ik(z+\tilde{p}(t)+\frac{t}{2\lambda})\sigma_3}.$$

Now the properties of M stated above suggest the representation of $u(x, t)$, the solution to the CH equation, in terms of the solution of a Riemann–Hilbert problem, for which the data (jump matrices and residue relations) are determined by the initial values $u(x, 0)$ and the boundary values $u(0, t)$, $u_x(0, t)$, $u_{xx}(0, t)$, via the associated spectral functions.

The Riemann–Hilbert problem is as follows.

RH-PROBLEM 1. —   *Given $\{a(k), b(k), A(k), B(k)\}$, for all $z \geqslant 0$ and $t \geqslant 0$, find a 2-vector function $\mu(z, t, k) \equiv \big(\mu_1(z, t, k)\ \ \mu_2(z, t, k)\big)$ such that*

- *$\mu(z, t, k)$ is meromorphic in $k \in \mathbb{C} \setminus \Sigma$ and continuous up to the boundary in each connected component of $\mathbb{C} \setminus \Sigma$.*

- $\mu_-(z,t,k) = \mu_+(z,t,k)J(z,t,k)$ for $k \in \Sigma$, where $\mu_\pm(z,t,k)$ are the limits of $\mu(z,t,k')$ as $k' \in \mathbb{C} \setminus \Sigma$ approaches $k \in \Sigma$ from the corresponding side.
- $\mu(z,t,k)$ has poles at zeros of $\hat{a}(k)$, $k = i\nu_j$, $j = 1, \ldots, N$, and at their conjugates, with the residue conditions

(2.33)
$$\operatorname{Res}_{k=i\nu_j} \mu_1(z,t,k) = i\gamma_j e^{-2\nu_j(z+\tilde{p}(t)+\frac{2t}{-1+4\nu_j^2})} \mu_2(z,t,i\nu_j),$$
$$\operatorname{Res}_{k=-i\nu_j} \mu_2(z,t,k) = -i\gamma_j e^{-2\nu_j(z+\tilde{p}(t)+\frac{2t}{-1+4\nu_j^2})} \mu_1(z,t,-i\nu_j),$$

where
$$\gamma_j = -i\frac{\bar{\bar{B}}(i\nu_j)}{\hat{a}(i\nu_j)a(i\nu_j)} > 0.$$

- $\mu_1(z,t,-k) = \mu_2(z,t,k)$.
- $\mu(z,t,k) = (1 \quad 1) + O(\frac{1}{k})$ as $k \to \infty$.

2.5. Solution of the IBV problem

PROPOSITION 2.3. — *Let $u(x,t)$, $x > 0$, $t > 0$ be a solution of the CH equation satisfying the above assumptions. Let $\{a(k), b(k), A(k), B(k)\}$ be the spectral functions associated with the initial and boundary values of u, see (2.15) and (2.16).*

Then u can be represented in terms of the unique solution $\mu(z,t,k) \equiv (\mu_1(z,t,k) \quad \mu_2(z,t,k))$ of the above Riemann–Hilbert problem as follows:

(2.34)
$$x(z,t) = z + \log\frac{\mu_1(z,t,\frac{i}{2})}{\mu_2(z,t,\frac{i}{2})} - \int_0^t u(0,\tau)\sqrt{u(0,\tau) - u_{xx}(0,\tau) + 1}\,d\tau - \nu(0),$$

$$u(z,t) = \partial_t \log\frac{\mu_1(z,t,\frac{i}{2})}{\mu_2(z,t,\frac{i}{2})}$$
$$+ \left(\frac{1}{\mu_1^2(z,t,\frac{i}{2})\mu_2^2(z,t,\frac{i}{2})} - 1\right)u(0,t)\sqrt{u(0,t) - u_{xx}(0,t) + 1}.$$

Remark 2.4. — The spectral functions $\tilde{A}$, $\tilde{B}$, $\tilde{a}$ involved in the construction of the jump matrix and the residue conditions above are determined by the given spectral functions $\{a(k), b(k)\}$ and $\{A(k), B(k)\}$ via (2.18), taking into account that

$$a\left(\frac{i}{2}\right) = \frac{\varkappa_0 + \varkappa_0^{-1}}{2}e^{-\frac{\nu(0)}{2}}, \qquad b\left(\frac{i}{2}\right) = \frac{\varkappa_0 - \varkappa_0^{-1}}{2}e^{-\frac{\nu(0)}{2}},$$

where $\varkappa_0 = (m(0,0)+1)^{1/4}$, see [8].

Remark 2.5. —   The representation formulas (2.34) follow from (2.31) and the relationship between the scales, see [8].

The advantage of the RH problem above for the long time study is that for all z, t with $\frac{z}{t} > \varepsilon$ and for all points $k \in \Sigma \backslash \mathbb{R}$, the jump matrix $J(z, t, k)$ decays to the identity matrix as $t \to +\infty$, which suggests that this part of the contour does not contribute to the leading asymptotic terms. However, since this decay is not uniform as k approaches the real axis, one expects that the contribution of (small) neighborhoods of the intersection points $k = \pm\frac{1}{2}$ may be significant [22] in the regions where the asymptotics is generally decaying to 0. Actually, we will see in the next sections that in particular regions of the (x, t) quadrant, contour deformations (specific to each region) allow us to determine precisely which points do contribute to the leading asymptotic terms.

Notice that the "hats" in Figure 2.1, *i.e.*, the domain D_2 and its complex conjugate, are relevant in the general case when $\varkappa_0 \neq 1$ (that corresponds to $m(0, 0) \neq 0$). Introducing the "hats" regularizes the RH problem in the sense that $J_0(k)$ is continuous near $k = \pm\frac{i}{2}$ (actually, $J_0(k) \to I$ as $k \to \pm\frac{i}{2}$). In the particular case $m(0, 0) = 0$, one does not need "hats": $M(x, t, k)$ can be defined by the same formula for all k such that $\operatorname{Im} k > 0$ and $|k| > \frac{1}{2}$ (similarly for k such that $\operatorname{Im} k < 0$ and $|k| < \frac{1}{2}$), that simplifies the contour reducing it to the union of the circle $|k| = \frac{1}{2}$ and the real axis.

The analysis of the construction above leads (the details are similar to those in [8]) to the following proposition characterizing the existence of the global solution of the CH equation in terms of the existence of a solution of the Riemann–Hilbert problem.

PROPOSITION 2.6. — *Let* $\{u_0(x), x \geqslant 0; \{v_j(t)\}_{j=0}^2, t \geqslant 0\}$ *be a set of smooth, decaying functions satisfying*

   (i) $(\partial_x^j u_0)(0) = v_j(0)$, $j = 0, 1, 2$.
   (ii) $m_0(x) + 1 \equiv u_0(x) - u_{0xx}(x) + 1 > 0$ *for all* x.
   (iii) $v_0(t) - v_2(t) + 1 > 0$ *for all* t.
   (iv) $v_0(t) \leqslant 0$ *for all* t.
   (v) *The associated spectral functions satisfy the "global relation":*

$$a(k)\tilde{B}(k) - \tilde{A}(k)b(k) = 0 \;\; \text{if } \operatorname{Im} k > 0 \text{ and } |k| > \frac{1}{2}.$$

Assume that the above Riemann–Hilbert problem has a solution

$$\mu(z, t, k) \equiv \big(\mu_1(z, t, k) \quad \mu_2(z, t, k)\big)$$

for all $z \geqslant 0$ and $t \geqslant 0$ such that $\mu_1(z, t, \frac{i}{2}) > 0$ and $\mu_2(z, t, \frac{i}{2}) > 0$ are related by

$$(2.35) \qquad \left(1 + 2\partial_z \log f(z,t)\right)q^2(z,t) = 1 \text{ for all } (z,t),$$

where

$$f(z,t) := \left(\frac{\mu_1(z, t, \frac{i}{2})}{\mu_2(z, t, \frac{i}{2})}\right)^{\frac{1}{2}}, \qquad q(z,t) := \left(\mu_1\left(z, t, \frac{i}{2}\right)\mu_2\left(z, t, \frac{i}{2}\right)\right)^{\frac{1}{2}}.$$

Then the initial boundary value problem

$$(2.36) \quad u_t - u_{txx} + 2u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad \text{for } x > 0,\ 0 < t < T,$$

$$(2.37) \quad u(x, 0) = u_0(x), \qquad\qquad\qquad\qquad\qquad \text{for } x > 0,$$

$$(2.38) \quad (\partial_x)^j u(0, t) = v_j(t),\ \ j = 0, 1, 2, \qquad\qquad \text{for } 0 < t < T,$$

has a unique solution, $u(x,t)$, that vanishes as $x \to +\infty$ for all t and such that $m(x,t) + 1 \equiv u(x,t) - u_{xx}(x,t) + 1 > 0$ for all (x,t). This solution can be represented in terms of the solution of the associated Riemann–Hilbert problem in the parametric form (2.34).

Remark 2.7. — Assuming that $u(x,t)$ exists, condition (2.35) is necessarily satisfied by the solution of the associated RH problem, as it follows from (2.31). On the other hand, this condition is an important part of the characterization of the functions $\{a(k), b(k), A(k), B(k)\}$ as "admissible" spectral functions (*i.e.*, associated with functions $\{u_0(x)\}, x \geqslant 0$ and $\{v_j(t)\}_{j=0}^2, t \geqslant 0$ which are indeed the boundary values of a certain solution of the CH equation). Another form of this condition (again as a restriction on the behavior of the solution of a RH problem at $k = \frac{i}{2}$) is given in [8].

3. The soliton region: $\frac{x}{t} > 2 + \varepsilon$

Following the nonlinear steepest descent method [22], the study of the long-time behavior of $u(x,t)$ is reduced to the study of the long-time behavior of the solution of the (oscillatory) Riemann–Hilbert problem. In turn, the latter is treated via a series of "deformations" of the RH problem

$$\mu \rightsquigarrow \mu^{(1)} \rightsquigarrow \mu^{(2)} \rightsquigarrow \cdots \rightsquigarrow \mu^{(\text{mod})},$$

which involves deformations of the contour, approximations of the jump matrix with well-controlled error, and absorption of a part of the jump matrix into the solution, aimed at reducing to an exactly solvable RH problem for $\mu^{(\text{mod})}$ [23, 22, 21]. In this section and in the next one, we will

closely follow [9] mainly emphasizing the impact of the boundary conditions on the scheme applied in the whole line case.

The primary deformation $\mu \rightsquigarrow \mu^{(1)}$, for all asymptotic regions, is related to the deformation of the contour dictated by the "signature table", which is the decomposition of the complex k-plane into the regions where the exponents involved in the jump matrix decay to 0 as $t \to +\infty$. The entries of the jump matrix $J(z,t,k)$, see (2.32), for the RH problem for the Camassa–Holm equation involve the exponentials $e^{\pm 2it\theta}$ in the off-diagonal matrix entries, where

$$(3.1) \qquad \theta(z,t,k) = \hat{\zeta}k - \frac{2k}{1+4k^2} + \frac{\tilde{p}(t)}{t}k \quad \text{with} \ \ \hat{\zeta} = \frac{z}{t}.$$

Since $\tilde{p}(t) \to \tilde{p}(\infty) = \nu(0)$, the last term in (3.1) vanishes as $t \to +\infty$ and thus the "signature table" indicating the regions of the complex k-plane where $\operatorname{Im}\theta(k)$ keeps its sign is (asymptotically) the same as in the case of the initial value problem [9].

In the case $\hat{\zeta} > 2 + \varepsilon$ for any $\varepsilon > 0$, the signature table allows deforming the contour of the RH problem in such a way that the part $\Sigma \backslash \mathbb{R}$ is separated, on one hand, from the real axis, and on the other hand, from the domain with $\operatorname{Im}\theta(k) \cdot \operatorname{Im} k < 0$ (where the exponentials in J are growing with t), see Figure 3.1. Particularly, if $\{i\nu_j\}_{j=1}^{N}$ are the zeros of $\hat{a}(k)$, with $0 < \nu_1 < \cdots < \nu_N < \frac{1}{2}$, then for $0 < \varepsilon < \varepsilon_0$, where $\nu_1 = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{2+\varepsilon_0}}$, the separating distance is $\delta = \frac{\sqrt{\varepsilon}}{4\sqrt{2}}$.

Let μ be the solution of the RH problem 1, § 2.4. Define $\mu^{(1)}$ as follows:

$$(3.2) \quad \mu^{(1)} = \begin{cases} \mu \begin{pmatrix} 1 & 0 \\ -R(k)e^{2it\theta} & 1 \end{pmatrix}, & \operatorname{Im} k > 0, \ |k| < \frac{1}{2} \text{ and } k \notin \tilde{D}_1, \\[2mm] \mu \begin{pmatrix} 1 & -\bar{R}(\bar{k})e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & \operatorname{Im} k < 0, \ |k| < \frac{1}{2} \text{ and } k \notin \bar{\tilde{D}}_1, \\[2mm] \mu, & \text{otherwise.} \end{cases}$$

Then $\mu^{(1)}$ satisfies the following RH problem:
- $\mu_-^{(1)}(k) = \mu_+^{(1)}(k)J^{(1)}(k)$, for $k \in \tilde{\Sigma}$,
- residue conditions (2.33),
- $\mu^{(1)}(k) \to (1 \ \ 1)$ as $k \to \infty$,

where

$$J^{(1)} = \begin{cases} J, & k \in \tilde{\Sigma} \setminus \mathbb{R}, \\[2mm] \begin{pmatrix} 1 & \bar{r}e^{-2it\theta} \\ -re^{2it\theta} & 1 - |r|^2 \end{pmatrix}, & k \in \mathbb{R}. \end{cases}$$
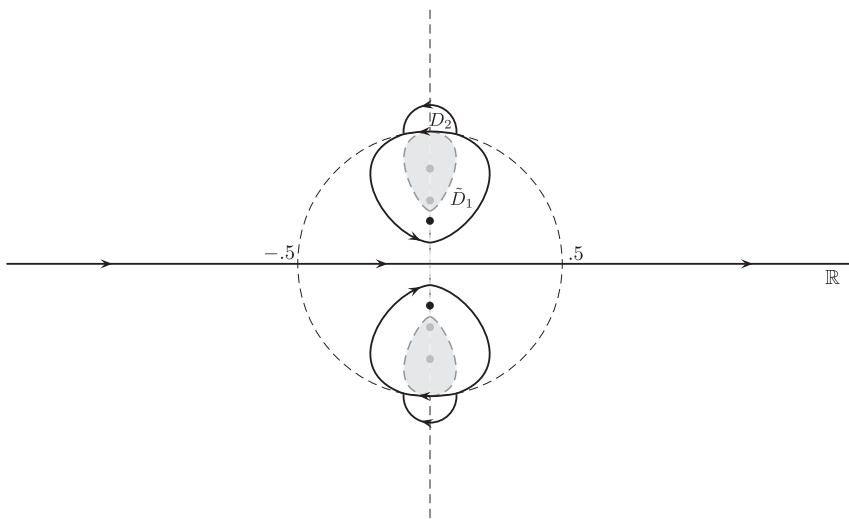
Figure 3.1. Deformed contour of the RH problem for the soliton region
$\hat{\zeta} > 2 + \varepsilon$. The filled domains are those with "bad" sign of $\operatorname{Im} \theta$, where
the exponentials in $J(z, t, k)$ are growing: $\operatorname{Im} \theta(k) < 0$ for $\operatorname{Im} k > 0$
and $\operatorname{Im} \theta(k) > 0$ for $\operatorname{Im} k < 0$.

Here $\tilde{\Sigma}$ is shown in Figure 3.1: it consists of $\mathbb{R}$ and the boundaries of $\tilde{D}_1$,
$\bar{\tilde{D}}_1$, D_2, and $\bar{D}_2$, where $\bar{D}$ denotes the domain symmetric to D with respect
to the real axis.

Now the jump matrix on $\Sigma \setminus \mathbb{R}$ decays uniformly, exponentially fast (as
$\mathrm{e}^{-2\delta t}$), to the identity matrix and thus does not contribute to the leading
term of the asymptotics, only the error term depending on ε. Consequently,
the solution of the RH problem for $\mu^{(1)}$ coincides, up to exponentially small
(in time) error, with the solution $\mu^{(2)}$ of the RH problem:

- $\mu_-^{(2)}(k) = \mu_+^{(2)}(k) J^{(1)}(k)$, for $k \in \mathbb{R}$,
- residue conditions (2.33),
- $\mu^{(2)} \to (1 \ \ 1)$ as $k \to \infty$,

which is exactly the RH problem for the initial value problem, with the re-
flection coefficient $r(k)$ and the residue conditions (2.33) at $\{\nu_j\}$. It follows
that one can proceed exactly as in the case of the initial value problem [9].
Specifically, the solution of the RH problem with poles can be written as

(*cf.* [27])
(3.3)

$$\left(\mu_1^{(2)}(z,t,k) \quad \mu_2^{(2)}(z,t,k)\right) = \begin{pmatrix} 1 & 1 \end{pmatrix} \prod_{j=1}^{N} (kI - B_j(z,t)) M_{\text{reg}}(z,t,k) D(k),$$

where

$$D(k) \equiv \text{diag}\{\tilde{d}_1 \quad \tilde{d}_2\} = \text{diag}\left\{\prod_{j=1}^{N} \frac{1}{k - i\nu_j}, \prod_{j=1}^{N} \frac{1}{k + i\nu_j}\right\}$$

and $M_{\text{reg}}(z,t,k)$ is the solution of the 2×2 matrix-valued regular (piecewise holomorphic relative to $\mathbb{R}$) RH problem with jump matrix $J_{\text{reg}} = DJ^{(1)}D^{-1}$, $k \in \mathbb{R}$. The 2×2 matrices B_j are determined recursively, by solving linear algebraic equations, in the same way as in the case of the KdV equation, see [27]:

(3.4)
$$(i\nu_j I + B_j)\tilde{M}_{j-1}(z,t,i\nu_j) \begin{pmatrix} 1 \\ -d_j(z,t) \end{pmatrix} = 0,$$

$$(-i\nu_j I + B_j)\tilde{M}_{j-1}(z,t,-i\nu_j) \begin{pmatrix} -d_j(z,t) \\ 1 \end{pmatrix} = 0, \qquad j = 1,\ldots,N,$$

where

$$\tilde{M}_0 = M_{\text{reg}},$$

$$\tilde{M}_j(z,t,k) = (kI - B_j(z,t))\,\tilde{M}_{j-1}(z,t,k), \quad j = 1,\ldots,N-1,$$

and

$$d_j(z,t) = \gamma_j \frac{\prod_{l=1,l\neq j}^{N}(\nu_j - \nu_l)}{\prod_{l=1}^{N}(\nu_j + \nu_l)} e^{-2\nu_j(z - \frac{2}{1-4\nu_j^2}t + \tilde{p}(t))}.$$

Remark 3.1. — Notice that it is only when the poles are purely imaginary and $\gamma_j > 0$ in the residue conditions that the solutions of these algebraic equations lead to $(\mu_1 \quad \mu_2)$ which is non-singular for all (z,t) (the solutions itselves are singular, but the singularities cancel when multiplying from the left by the vector $(1 \quad 1)$). Hence this justifies the assumptions on the poles and residue conditions made above in order that the spectral functions correspond to a globally non-singular solution of the CH equation (this is in complete analogy with the KdV equation [27]).

Now the asymptotics $M_{\text{reg}} \to I$ as $t \to +\infty$ follows from the whole line problem analysis in the case without poles. Namely, the factorization of the jump matrix

$$J_{\text{reg}} = \begin{pmatrix} 1 & 0 \\ -\frac{\tilde{d}_2 r}{\tilde{d}_1} e^{2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\tilde{d}_1 \bar{r}}{\tilde{d}_2} e^{-2it\theta} \\ 0 & 1 \end{pmatrix}$$

and appropriate rational approximations of $r(k)$ and $\bar{r}(k)$ are used in order to deform the contour $\mathbb{R}$ into $\Sigma_+ \cup \Sigma_-$, where $\Sigma_\pm = \{k \mid \operatorname{Im} k = \pm\varepsilon\}$ for some small ε and to absorb the trigonal factors into the solution of a new RH problem:

$$(3.5) \qquad M_{\mathrm{reg}}^{(1)} = \begin{cases} M_{\mathrm{reg}}, & |\operatorname{Im} k| > \varepsilon, \\[2mm] M_{\mathrm{reg}} \begin{pmatrix} 1 & 0 \\ -\frac{\tilde{d}_2 r}{\tilde{d}_1} e^{2it\theta} & 1 \end{pmatrix}, & 0 < \operatorname{Im} k < \varepsilon, \\[3mm] M_{\mathrm{reg}} \begin{pmatrix} 1 & \frac{\tilde{d}_1 \bar{r}}{\tilde{d}_2} e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & 0 > \operatorname{Im} k > -\varepsilon. \end{cases}$$

Since $M_{\mathrm{reg}}^{(1)}$ satisfies a RH problem with a jump matrix on $\Sigma_+ \cup \Sigma_-$, which approaches I, as $t \to +\infty$, exponentially fast, it follows that $M_{\mathrm{reg}}^{(1)} \to I \equiv M^{(\mathrm{mod})}$ as $t \to +\infty$. Hence $\mu^{(\mathrm{mod})} = (1 \ 1)M^{(\mathrm{mod})}$ is trivial in this region.

Now, (3.3) implies that the solution of the original RH problem approaches that for the reflectionless ($r(k) \equiv 0$) case, and thus

$$u(x,t) = u_{\mathrm{sol}}(x,t) + o(1), \quad t \to +\infty,$$

where the error term depends on ε and where $u_{\mathrm{sol}}(x,t)$ is a pure N-soliton solution of the CH equation, which corresponds to the Riemann–Hilbert problem with $r(k) \equiv 0$ and with the residue parameters $\{\nu_j\}_{j=1}^N$ and $\{\gamma_j\}_{j=1}^N$.

If $N = 1$, then (*cf.* (3.3))

$$(3.6) \qquad \begin{pmatrix} \mu_1(z,t,k) & \mu_2(z,t,k) \end{pmatrix} = \begin{pmatrix} \frac{k-B_1(z,t)}{k-i\nu_1} & \frac{k+B_1(z,t)}{k+i\nu_1} \end{pmatrix} + o(1),$$

where

$$B_1(z,t) = i\nu_1 \frac{1 - g_1(z,t)}{1 + g_1(z,t)}$$

with

$$g_1(z,t) = \exp\left\{-2\nu_1\left(z - \frac{2}{1 - 4\nu_1^2}t - z_0\right)\right\} \text{ and } z_0 = \frac{1}{2\nu_1}\log\frac{\gamma_1}{2\nu_1} - \nu(0).$$

Consequently, the leading term of the long-time asymptotics in this case is given by the 1-soliton solution with parameters determined by ν_1, γ_1, and

$\nu(0)$:

$$u(z,t) = \partial_t \log \frac{\mu_1(z,t,\frac{i}{2})}{\mu_2(z,t,\frac{i}{2})}$$

(3.7)
$$= \frac{16\nu_1^2}{1 - 4\nu_1^2} \frac{1}{1 + 4\nu_1^2 + (1 - 4\nu_1^2)\cosh\phi_1(z,t)} + o(1),$$

$$x(z,t) = z + \log \frac{1 + g_1 \frac{1+2\nu_1}{1-2\nu_1}}{1 + \frac{1-2\nu_1}{1+2\nu_1}g_1(z,t)} + o(1),$$

where

$$\phi_1 = -2\nu_1(z - v_1 t - z_0) \text{ and } v_1 = \frac{2}{1 - 4\nu_1^2}.$$

Notice that since $0 < \nu_1 < 1/2$, the soliton velocity v_1 is always greater than 2 and thus the soliton asymptotics resides only in the region $\zeta > 2$. Since $x = z + O(1)$ in the region $\hat{\zeta} > 2 + \varepsilon$, the soliton region can be equivalently described by the relation $\frac{x}{t} > 2 + \varepsilon$ with $0 < \varepsilon < \varepsilon_0$.

For arbitrary N, one can proceed in a similar way, taking in (3.3) $M_{\text{reg}} \equiv I$ and constructing B_j by solving linear systems of algebraic equations. Matsuno [35] has proposed a convenient representation for the purely N-soliton solutions in determinant form:

(3.8)
$$u(z,t) = \partial_t \log \frac{f_2(z,t)}{f_1(z,t)},$$

$$x(z,t) = z + \log \frac{f_2(z,t)}{f_1(z,t)} + d,$$

where $f_{1,2}$ have determinantal expressions $f_{1,2} = \det G_{1,2}$, for some $N \times N$ matrices $G_{1,2}$, and d is a constant. This form turns out to be particularly useful for analyzing the long time behavior of the multisoliton solutions, giving immediately that (as it is supposed to be!), as $t \to +\infty$, the multisoliton solution develops into a sum of 1-soliton solutions. Comparing (3.8) with our representation formulas (2.34) allows relating the soliton parameters with the spectral data in the case of the IBV problem, which finally leads to a superposition formula for the long time asymptotics in the soliton region:

THEOREM 3.2 (soliton region (i)). — *Let $\varepsilon > 0$ be sufficiently small. If (x,t) is in the region $\zeta := \frac{x}{t} > 2 + \varepsilon$, then, for $t \to +\infty$,*

(3.9)
$$u(x,t) = \sum_{j=1}^{N} F_j(Z_j(X))\Big|_{X=x-v_jt-\tilde{x}_{0j}} + o(1),$$

where

$$(3.10) \qquad v_j = \frac{2}{1 - 4\nu_j^2} \,,$$

$$(3.11) \qquad F_j(Z) = \frac{16\nu_j^2}{1 - 4\nu_j^2} \frac{1}{1 + 4\nu_j^2 + (1 - 4\nu_j^2)\cosh(2\nu_j Z)} \,,$$

$$(3.12) \qquad X_j(Z) = Z + \log \frac{1 - 2\nu_j + (1 + 2\nu_j)\mathrm{e}^{-2\nu_j Z}}{1 + 2\nu_j + (1 - 2\nu_j)\mathrm{e}^{-2\nu_j Z}}$$

and

$$(3.13) \qquad \tilde{x}_{0j} = \frac{1}{2\nu_j} \log \frac{\gamma_j}{2\nu_j} + \frac{1}{2\nu_j} \sum_{l=1}^{j-1} \log \left(\frac{\nu_j - \nu_l}{\nu_j + \nu_l} \right)^2$$

$$+ \log \frac{1 + 2\nu_j}{1 - 2\nu_j} + \sum_{l=1}^{j-1} \log \left(\frac{1 + 2\nu_l}{1 - 2\nu_l} \right)^2 - \nu(0).$$

Remark 3.3. — Alternatively, the asymptotic development of a multi-soliton solution can be done by deforming the singular RH problem into a regular one, by introducing additional parts of the contour consisting of small circles around the zeros of $\hat{a}(k)$ and rewriting the residue conditions as associated jump conditions on this parts, see [5].

Remark 3.4. — In the phase shift formula (3.13), the first two terms are similar to those for the case of the KdV equation [21] whereas the other terms are specific for the CH equation. More precisely, they originate from the relationship between the scales (x,t) and (z,t).

Remark 3.5. — The soliton asymptotics for the solution of the IBV problem has the same form as for the solution of the whole line (Cauchy) problem [9], the difference being that now the soliton parameters are determined by zeros and residues of functions which are combinations of the spectral functions associated with both the initial and boundary values. Thus solitons may occur in situations where the initial data on their own do not generate solitons (being considered as initial data for an initial value problem): the soliton velocities are determined by zeros of $\hat{a}(k)$ but not by zeros of $a(k)$ (as in the case of the whole line problem).

4. The similarity region: $\varepsilon < \frac{x}{t} < 2 - \varepsilon$

In the case $\varepsilon < \frac{z}{t} < 2 - \varepsilon$, the signature table (see Figure 4.1) suggests deforming the contour for the RH problem in such a way that it passes

through the stationary phase points $\pm\kappa_0$, where

$$(4.1) \qquad \kappa_0^2 = \kappa_0^2(\hat{\zeta}) = \frac{\sqrt{1 + 4\hat{\zeta}} - 1 - \hat{\zeta}}{4\hat{\zeta}},$$

so that

$$\hat{\zeta} = 2\frac{1 - 4\kappa_0^2}{(1 + 4\kappa_0^2)^2}.$$

Thus κ_0 varies from 0 (for $\hat{\zeta} = 2$) to $1/2$ (for $\hat{\zeta} = 0$).

ASSUMPTION. — *For simplicity, we will present the asymptotic analysis in this and the other regions where the leading terms decay to 0, in the "solitonless" case, i.e., assuming that $\hat{a}(k) \neq 0$ for any $k \in D_1$.*

Remark 4.1. — The presence of solitons affects the asymptotics in these regions in the form of an additional phase shift. It can be worked out again by using the representation (3.3) in order to calculate, recursively, the behavior of B_j's in the corresponding region. Alternatively, as in the soliton region, this can be done by working with the regular version of the RH problem on the augmented contour [5].
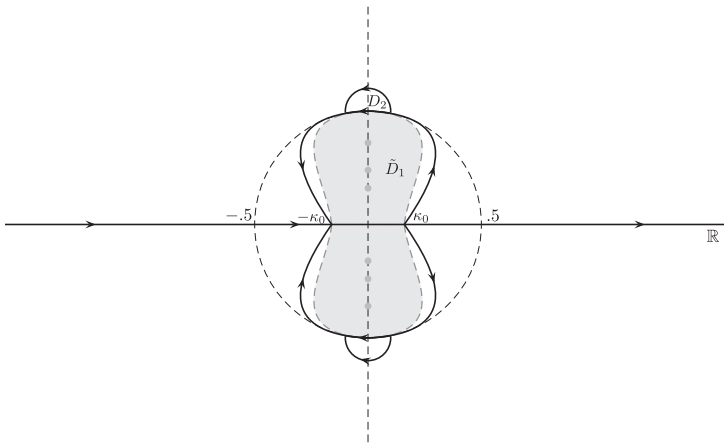

Figure 4.1. Contour transformation for the similarity region.

The initial transformation of the RH problem, $\mu \rightsquigarrow \mu^{(1)}$, is described by the same formulas (3.2) as in the soliton region, but now the jump matrix

$J^{(1)} \equiv J^{(1)}(x,t,k)$ is as follows:

$$J^{(1)} = \begin{cases} J, & k \in \tilde{\Sigma} \setminus \mathbb{R}, \\[2mm] \begin{pmatrix} 1 & \bar{r}e^{-2it\theta} \\ -re^{2it\theta} & 1-|r|^2 \end{pmatrix}, & k \in \mathbb{R} \setminus [-\kappa_0, \kappa_0], \\[4mm] \begin{pmatrix} 1 & (\bar{r}-\bar{R})e^{-2it\theta} \\ (R-r)e^{2it\theta} & 1-|r-R|^2 \end{pmatrix}, & k \in (-\kappa_0, \kappa_0), \end{cases}$$

where $\tilde{\Sigma}$ is shown in Figure 4.1.

Since $J^{(1)}$ decays exponentially to I for all $k \in \tilde{\Sigma} \setminus \mathbb{R}$ outside small neighborhoods of the points $\pm\kappa_0$, we can follow the scheme of the asymptotic analysis in the case of the whole line problem [9, 22], which involves the following steps:

(i) The jump matrices $J^{(1)}$ on $(-\infty, -\kappa_0)$, $(-\kappa_0, \kappa_0)$, (κ_0, ∞) are factorized into products of trigonal matrices:

$$(4.2) \qquad J^{(1)} = \begin{pmatrix} 1 & 0 \\ -re^{2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \bar{r}e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } k \in \mathbb{R} \setminus [-\kappa_0, \kappa_0],$$

and, for $k \in (-\kappa_0, \kappa_0)$,

$$(4.3) \quad \begin{pmatrix} 1 & \frac{\bar{r}-\bar{R}}{1-|r|^2}e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1-|r-R|^2} & 0 \\ 0 & 1-|r-R|^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-r+R}{1-|r-R|^2}e^{2it\theta} & 1 \end{pmatrix}.$$

(ii) The diagonal factor in (4.3) for $k \in (-\kappa_0, \kappa_0)$ is removed by using the diagonal matrix $D = \text{diag}\{\delta^{-1}, \delta\}$: $\mu^{(2)} = \mu^{(1)}D$, where δ is the solution of the scalar RH problem:
- δ is analytic in $\mathbb{C} \setminus [-\kappa_0, \kappa_0]$,
- $\delta_+ = \delta_-(1 - |r-R|^2)$ for $k \in (-\kappa_0, \kappa_0)$,
- $\delta \to 1$ as $k \to \infty$.

(iii) The trigonal factors are absorbed into the RH problem for $\mu^{(3)}$:

$$(4.4) \ \ \mu^{(3)} = \begin{cases} \mu^{(2)} \begin{pmatrix} 1 & 0 \\ -r\delta^{-2}e^{2it\theta} & 1 \end{pmatrix}, & \text{Im } k > 0, \ k \text{ near } \mathbb{R} \setminus [-\kappa_0, \kappa_0], \\[4mm] \mu^{(2)} \begin{pmatrix} 1 & \frac{\bar{r}-\bar{R}}{1-|r-R|^2}\delta_+^2 e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & \text{Im } k > 0, \ k \text{ near } (-\kappa_0, \kappa_0), \\[4mm] \mu^{(2)} \begin{pmatrix} 1 & 0 \\ \frac{r-R}{1-|r-R|^2}\delta_-^{-2}e^{2it\theta} & 1 \end{pmatrix}, & \text{Im } k < 0, \ k \text{ near } (-\kappa_0, \kappa_0), \\[4mm] \mu^{(2)} \begin{pmatrix} 1 & -\bar{r}\delta^2 e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & \text{Im } k < 0, \ k \text{ near } \mathbb{R} \setminus [-\kappa_0, \kappa_0]. \end{cases}$$

(iv) The contributions from the parts of the contour separated from the real axis can be neglected, since the jump matrices on them decay rapidly to the identity matrix.

(v) After appropriate scaling, the resulting "model" problem is formulated on two crosses centered at $\pm\kappa_0$, with constant jump matrices, which allows solving them explicitly.

Notice that the specific feature of the IBV problem is that on $(-\kappa_0, \kappa_0)$, where the signature table dictates the use of a lower/diagonal/upper factorization, the construction of the jump matrix involves $r(k) - R(k)$ instead of $r(k)$ as in the case of the whole line problem. The presence of R is consistent with the absorption of the trigonal factors when transforming the contour to the crosses at $k = \pm\kappa_0$, since the jump matrices for μ on the parts of the contour intersecting with $\mathbb{R}$ at $k = \pm\kappa_0$, involve $R(k)$ (see the first line of (2.28)).

The δ-function removing the diagonal factor is given by (*cf.* [9])

$$\delta(k) = \left(\frac{k - \kappa_0}{k + \kappa_0}\right)^{ih_0} e^{\chi(k)}$$

with

(4.5)
$$h_0 = -\frac{1}{2\pi} \log\left(1 - |r(\kappa_0) - R(\kappa_0)|^2\right),$$

$$\chi(k) = \frac{1}{2\pi i} \int_{-\kappa_0}^{\kappa_0} \log\left(\frac{1 - |r(s) - R(s)|^2}{1 - |r(\kappa_0) - R(\kappa_0)|^2}\right) \frac{ds}{s - k}.$$

Near $k = \kappa_0$, the part of the phase

$$\tilde{\theta}(k) = \hat{\zeta}k - \frac{2k}{1 + 4k^2}$$

can be written as

(4.6) $\tilde{\theta}(k) = \theta(\kappa_0) + 8f_0(k - \kappa_0)^2\big(1 + (k - \kappa_0)g(k)\big),$

where

(4.7)
$$g(k) = \frac{16\kappa_0^3 k - 12\kappa_0 k - 12\kappa_0^2 + 1}{\kappa_0(3 - 4\kappa_0^2)(1 + 4k^2)},$$

$$\theta(\kappa_0) = -\frac{16\kappa_0^3}{(1 + 4\kappa_0^2)^2}, \quad f_0 = \frac{\kappa_0(3 - 4\kappa_0^2)}{(1 + 4\kappa_0^2)^3} > 0.$$

Thus the factor $\delta(k)e^{-it\theta(k)}$ near $k = \kappa_0$ can be approximated as

$$\delta(k)e^{-it\theta(k)} \approx \delta_{\kappa_0}\hat{k}^{ih_0}e^{-i\hat{k}^2/4},$$

where the scaled spectral variable $\hat{k}$ is defined by

$$k - \kappa_0 = \frac{\hat{k}}{\sqrt{32 f_0 t}}$$

and

(4.8) $\delta_{\kappa_0} = \left(128 \frac{\kappa_0^3 (3 - 4\kappa_0^2)}{(1 + 4\kappa_0^2)^3} t\right)^{-\frac{i h_0}{2}} e^{i \frac{16\kappa_0^3}{(1+4\kappa_0^2)^2} t} e^{\chi(\kappa_0)} e^{-i\kappa_0 \nu(0)}$

(the last factor comes from $e^{it(\theta - \tilde{\theta})}$ as $t \to +\infty$) with

(4.9) $\chi(\kappa_0) = -\frac{1}{2\pi i} \int_{-\kappa_0}^{\kappa_0} \log|\kappa_0 - s| \, d \log(1 - |r(s) - R(s)|^2).$

Similarly for k near $-\kappa_0$.

Conjugating out the constant diagonal factors $\delta_{\pm\kappa_0}^{\sigma_3}$, we arrive at two 2×2 problems, for $\hat{M}^{\pm\kappa_0}(\hat{k})$, relative to the crosses centered at $\hat{\kappa} = 0$, with constant jump matrices $\hat{J}(\kappa_0)$ and $\hat{J}(-\kappa_0)$, where

$$\hat{J}(k) = \begin{pmatrix} 1 & \bar{r}(k) - \bar{R}(k) \\ R(k) - r(k) & 1 - |r(k) - R(k)|^2 \end{pmatrix}.$$

Their solutions are given explicitly, in terms of parabolic cylinder functions, see [22]. Particularly, the large-$\hat{k}$ behavior of $\hat{M}^{\pm\kappa_0}(\hat{k})$ is given by

(4.10) $$\hat{M}^{\pm\kappa_0}(\hat{k}) = I - \frac{\hat{M}_1^{\pm\kappa_0}}{\hat{k}} + O\left(\frac{1}{\hat{k}^2}\right),$$

where

(4.11) $\hat{M}_1^{\kappa_0} = \begin{pmatrix} 0 & i\bar{\beta}^{\kappa_0} \\ -i\beta^{\kappa_0} & 0 \end{pmatrix}$ with $\beta^{\kappa_0} = \frac{(r(\kappa_0) - R(\kappa_0))\Gamma(-ih_0)h_0}{\sqrt{2\pi} \, e^{i\pi/4} \, e^{-\pi h_0/2}}$

and $\hat{M}_1^{-\kappa_0} = -\overline{\hat{M}_1^{\kappa_0}}$.

Now recall that in order to determine the solution of the Camassa–Holm equation, we need $\mu_1(\frac{i}{2})$ and $\mu_2(\frac{i}{2})$. Tracing back the deformations of the RH problem gives

(4.12)
$\begin{pmatrix} \mu_1 & \mu_2 \end{pmatrix} \left(\frac{i}{2}\right) \sim \begin{pmatrix} 1 & 1 \end{pmatrix} \left(\delta_{\kappa_0}^{\sigma_3} M^{\kappa_0}\left(\frac{i}{2}\right)\delta_{\kappa_0}^{-\sigma_3} + \delta_{-\kappa_0}^{\sigma_3} M^{-\kappa_0}\left(\frac{i}{2}\right)\delta_{-\kappa_0}^{-\sigma_3} - I\right)\delta^{\sigma_3}\left(\frac{i}{2}\right),$

where $M^{\pm\kappa_0}(k)$ solves the RH problem on the corresponding cross in the k-plane. Since $M^{\pm\kappa_0}(k) \approx \hat{M}^{\pm\kappa_0}(\hat{k})$, taking into account the relations between k and $\hat{k}$ gives

(4.13) $$M^{\pm\kappa_0}\left(\frac{i}{2}\right) \sim I - \frac{\hat{M}_1^{\pm\kappa_0}}{\left(\frac{i}{2} \pm \kappa_0\right)\sqrt{32 f_0 t}}.$$

Collecting (4.7), (4.8), (4.9), (4.11), (4.12), and (4.13), we obtain

(4.14)

$$\mu_1\left(\tfrac{i}{2}\right)\delta^{-1}\left(\tfrac{i}{2}\right) \sim 1 + \frac{1}{\sqrt{8f_0 t}} \operatorname{Re} \frac{i\delta_{\kappa_0}^{-2}\beta^{\kappa_0}}{\tfrac{i}{2} - \kappa_0} = 1 + \left(\frac{h_0}{2f_0 t(1 + 4\kappa_0^2)}\right)^{\frac{1}{2}} \cos\phi_1,$$

$$\mu_2\left(\tfrac{i}{2}\right)\delta\left(\tfrac{i}{2}\right) \sim 1 + \frac{1}{\sqrt{8f_0 t}} \operatorname{Re} \frac{-i\delta_{-\kappa_0}^2\beta^{\kappa_0}}{\tfrac{i}{2} + \kappa_0} = 1 - \left(\frac{h_0}{2f_0 t(1 + 4\kappa_0^2)}\right)^{\frac{1}{2}} \cos\phi_2,$$

where

(4.15)

$$\phi_1(t) = \frac{32\kappa_0^3}{(1 + 4\kappa_0^2)^2} t - h_0 \log\left(\frac{128\kappa_0^3(3 - 4\kappa_0^2)}{(1 + 4\kappa_0^2)^3} t\right) + \phi_0 + \arctan(2\kappa_0)$$

$$\phi_2 = \phi_1 - 2\arctan(2\kappa_0),$$

with

$$\phi_0 = \frac{\pi}{4} - \arg(r(\kappa_0) - R(\kappa_0)) - 2\kappa_0\nu_0$$

(4.16)

$$+ \arg\Gamma(ih_0) + \frac{1}{\pi}\int_{-\kappa_0}^{\kappa_0} \log|\kappa_0 - s|\, d\log(1 - |r(s) - R(s)|^2).$$

Therefore,

(4.17) $\log\dfrac{\mu_1}{\mu_2}\left(\dfrac{i}{2}\right) - 2\log\delta\left(\dfrac{i}{2}\right)$

$$= \left(\frac{2h_0(1 + 4\kappa_0^2)}{\kappa_0(3 - 4\kappa_0^2)t}\right)^{\frac{1}{2}} \times$$

$$\times \cos\left(\frac{32\kappa_0^3}{(1 + 4\kappa_0^2)^2} t - h_0 \log\left(\frac{128\kappa_0^3(3 - 4\kappa_0^2)}{(1 + 4\kappa_0^2)^3} t\right) + \phi_0\right) + o\left(\frac{1}{\sqrt{t}}\right).$$

Finally, one arrives at the asymptotic formula in the similarity region, in the "solitonless" case:

THEOREM 4.2 (similarity region (ii)). — *Let $\varepsilon > 0$ be sufficiently small. If (x, t) is in the region $\varepsilon < \zeta := \frac{x}{t} < 2 - \varepsilon$, then, for $t \to +\infty$,*

(4.18) $$u(x, t) = \frac{c_1}{\sqrt{t}} \sin(c_2 t + c_3 \log t + c_4)\bigl(1 + o(1)\bigr),$$

where

(4.19)

$$c_1 = -\left(\frac{32h_0\kappa_0}{(3 - 4\kappa_0^2)(1 + 4\kappa_0^2)}\right)^{\frac{1}{2}}, \qquad c_2^{(0)} = \frac{32\kappa_0^3}{(1 + 4\kappa_0^2)^2}, \qquad c_3^{(0)} = -h_0,$$

$$c_4^{(0)} = \phi_0 - h_0 \log\frac{128\kappa_0^3(3 - 4\kappa_0^2)}{(1 + 4\kappa_0^2)^3} + \frac{4}{\pi}\kappa_0 \int_{-\kappa_0}^{\kappa_0} \frac{\log(1 - |r(s) - R(s)|^2)}{1 + 4s^2}\, ds,$$

and κ_0 is the function of x/t given by (4.1) with $\hat{\zeta}$ replaced by x/t.

Remark 4.3. — As in the soliton region, the form of the asymptotics is the same as in the whole line case, the effect of the boundary conditions being seen in the dependence of the parameters on the boundary values via the spectral functions involved in $R(k)$.

5. The self-similar region: $\left|\frac{x}{t} - 2\right| t^{2/3} < C$

The approximations adopted in the similarity region require the stationary phase points, κ_0 and $-\kappa_0$, to be well separated, in order that their contributions to the asymptotics could be considered separately. This is not the case when ζ becomes close to 2 and thus κ_0 and $-\kappa_0$ become close to each other (and to 0). This case requires another scaling, which, on one hand, has to be consistent with $\zeta \to 2$ as $t \to +\infty$ and, on the other hand, would approximate the phase factor $e^{it\theta}$ in a form that does not involve any large parameters. Indeed, developing $t\theta(k)$ for k close to 0 and $\frac{z}{t}$ close to 2, one obtains

$$(5.1) \qquad t\theta(k) \approx \frac{4}{3}\hat{k}^3 + s\hat{k},$$

where $\hat{k}$ is the scaled spectral parameter

$$(5.2) \qquad \hat{k} = (6t)^{1/3}k$$

and

$$(5.3) \qquad s = 6^{-1/3}\left(\frac{x}{t} - 2\right)t^{2/3}.$$

The numerical coefficients in these formulas are chosen in such a way that the form of the scaled phase factor coincide with that of a familiar RH problem. Since the scaled phase involves $\hat{k}^3$ and a linear term (with a coefficient s), one can recognize the RH problem for the Painlevé II equation, whose exponential phase factors have exactly the same numeric coefficients as in (5.1), see, *e.g.*, , [30].

This suggests that there exists an asymptotic region parametrized by a parameter s taking finite values from $-C$ to C, for any $C > 0$. Thus this region can be described as follows:

$$\left|\frac{x}{t} - 2\right| t^{2/3} < C.$$

The relevant contour transformation for the RH problem is shown in Figure 5.1.

More precisely, one should consider a family of contours of the type shown in Figure 4.1. But, since the intersection points $\pm\kappa_0$ collide to 0 as
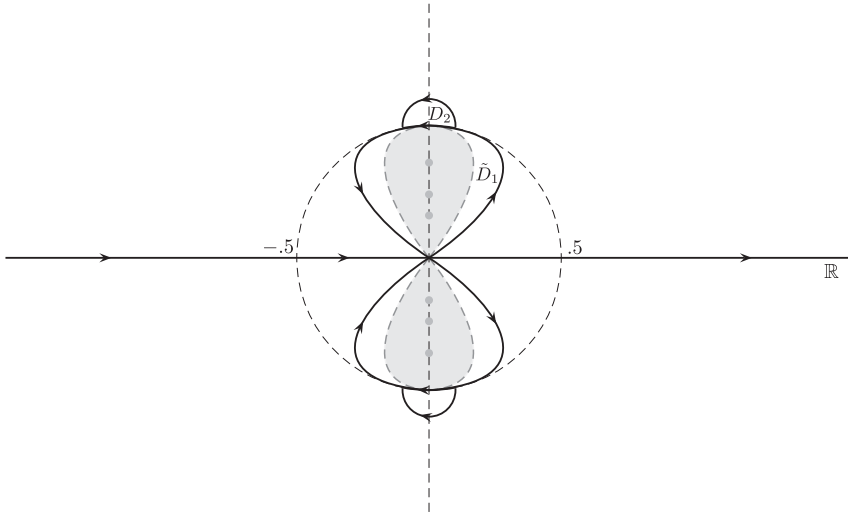
Figure 5.1. Contour transformation for the self-similar region.

$t \to +\infty$, the factorization of the jump matrix on $\mathbb{R} \setminus [-\kappa_0, \kappa_0]$ is the same as in the similarity case, whereas we did not need a different factorization (involving a diagonal factor) on $(-\kappa_0, \kappa_0)$: the constant matrix

(5.4) $$\hat{J}_0 := \begin{pmatrix} 1 & \bar{r}(0) - \bar{R}(0) \\ -r(0) + R(0) & 1 - |r(0) - R(0)|^2 \end{pmatrix}$$

provides an appropriate approximation for all $k \in (-\kappa_0, \kappa_0)$. Notice that, as for the similarity region, the presence of $R(0)$ in this formula is consistent with the absorption of the trigonal factors on $\mathbb{R} \setminus [-\kappa_0, \kappa_0]$ when transforming the contour to the cross at $k = 0$, since the jump matrices of the initially deformed RH problem, see Figure 5.1, on the parts of the contour intersecting with $\mathbb{R}$ at $k = 0$, involve $R(k)$.

Thus the "model" RHP, asymptotically equivalent to the original one, is formulated, in the $\hat{k}$-plane, on the cross at $\hat{k} = 0$,

(5.5) $$\hat{M}_-(\hat{k}, s) = \hat{M}_+(\hat{k}, s)\hat{J}(\hat{k}, s)$$

with

(5.6) $$\hat{J}(\hat{k}, s) = e^{-i(\frac{4}{3}\hat{k}^3 + s\hat{k})}\hat{J}_0 e^{i(\frac{4}{3}\hat{k}^3 + s\hat{k})},$$

where $\hat{J}_0$ is defined by (5.4).

The solution $(\mu_1\ \mu_2)$ of this RH problem can be written as $(\mu_1\ \mu_2)(i/2) = (1\ 1)M(i/2)$, where $M(k)$ is the solution of the corresponding 2×2 matrix problem. The scaling relation (5.2) indicates that in order to evaluate $M(k)$ at $k = \frac{i}{2}$ as $t \to +\infty$, we have to evaluate $\hat{M}(\hat{k})$ as $\hat{k} \to \infty$. Writing $\hat{M}(\hat{k}, s) = I + \hat{M}_1(s)/\hat{k} + \ldots$ we have

$$M\left(\frac{i}{2}\right) \approx \hat{M}\left(\frac{i}{2}(6t)^{1/3}\right) \approx I - i\left(\frac{4}{3}\right)^{1/3}\frac{1}{t^{1/3}}\hat{M}_1(s).$$

We observe now that the RHP for $\hat{M}$ is precisely that for a Painlevé II-type system, see, e.g., , [30] (more details can be found in [29]). The symmetry $r(0) = \overline{r(0)}$, $R(0) = \overline{R(0)}$ implies that the large-$\hat{k}$ term, $\hat{M}_1(s)$, can be expressed as

$$\hat{M}_1(s) = \frac{i}{2}\begin{pmatrix} \int^s u^2 & -u \\ u & -\int^s u^2 \end{pmatrix},$$

where $u(s)$ is a real-valued solution of the Painlevé II equation

$$u''(s) = su(s) + 2u^3(s).$$

In order to specify this solution, one can calculate its large-s asymptotics using again the nonlinear steepest descent method, this time for the RHP for $\hat{M}$. It is shown in [29] that, as $s \to +\infty$, $u(s) \sim (r(0)-R(0))\frac{1}{2\sqrt{\pi}}s^{-\frac{1}{4}}e^{-\frac{2}{3}s^{3/2}}$, or, in terms of the classical Airy function,

$$u(s) \sim (r(0) - R(0))\,\mathrm{Ai}(s).$$

Now, going back from $\hat{M}_1(s)$ to u and using the fact that $\partial_t s \sim -\left(\frac{4}{3}\right)^{1/3}t^{-1/3}$, see (5.3), we arrive at the asymptotic formula in the self-similar region, again for the "solitonless" case:

THEOREM 5.1 (self-similar region (iii)). — *Let $C > 0$ be arbitrary. If (x,t) is in the region*

$$\left|\frac{x}{t} - 2\right|t^{2/3} < C,$$

then, for $t \to +\infty$,

(5.7) $$u(x,t) = -\left(\frac{4}{3}\right)^{2/3}\frac{1}{t^{2/3}}\left(u^2(s) + u'(s)\right)\left(1 + o(1)\right),$$

where

$$s = 6^{-1/3}\left(\frac{x}{t} - 2\right)t^{2/3}.$$

6. The shock wave region

The region between the similarity region and the self-similar region can be described by a pair of relations amongst x and t:

$$(6.1) \qquad \frac{x}{t} - 2 \to 0, \qquad \left(\frac{x}{t} - 2\right) t^{2/3} \to \infty.$$

Thus $s \to +\infty$ (where s is defined in (5.3)), which suggests introducing such a scaling for the RH problem that the phase factor $e^{it\theta}$ takes the form $e^{is\hat{\theta}}$, where the properties of $\hat{\theta}(\hat{k})$ allows deforming the RH problem in the same spirit as above: through appropriate factorizations, where certain factors decay to the identity matrix as $s \to +\infty$ and thus can be neglected.

Introducing the scaled spectral variable

$$\hat{k} := \frac{k}{\kappa_0},$$

the RH problem for $M(k)$, i.e., $M_-(k) = M_+(k)J(k)$ reduces to a problem for $M^{(1)}(\hat{k}) = M(k)$:

$$M_-^{(1)}(\hat{k}) = M_+^{(1)}(\hat{k})J^{(1)}(\hat{k})$$

on a contour similar to that in Figure 4.1, but with $\pm\kappa_0$ replaced by ± 1. As in the previous cases, the main contribution to the leading terms of the asymptotics for u comes from the vicinities of the stationary phase points. Due to the chosen scaling, they are $\hat{k} = \pm 1$. Indeed, the jump matrix on $(-\infty, -1)$ and $(1, \infty)$ is factorized, as above, into factors exponentially decaying to the identity matrix. On the other hand, we have

$$t\theta(k) \approx \tau\hat{\theta}(\hat{k})$$

as $\kappa_0 \to 0$ and $s \to +\infty$, where τ differs from s only by a constant factor:

$$(6.2) \qquad \tau = \frac{1}{24\sqrt{6}} \left(2 - \frac{z}{t}\right)^{\frac{3}{2}} t$$

and $\hat{\theta}(\hat{k}) = 4\hat{k}^3 - 12\hat{k}$. Thus, as $\kappa_0 \to 0$ and $s \to +\infty$, the jump matrix $J^{(1)}(\hat{k}) = J(k)$ is well approximated, for $\hat{k} \in (-1, 1)$, by

(6.3) $J^{(2)}(\hat{k}) =$

$$e^{-i\tau\hat{\theta}(\hat{k})\sigma_3} \begin{pmatrix} 1 & \bar{r}(\kappa_0\hat{k}) - \bar{R}(\kappa_0\hat{k}) \\ -r(\kappa_0\hat{k}) + R(\kappa_0\hat{k}) & 1 - |r(\kappa_0\hat{k}) - R(\kappa_0\hat{k})|^2 \end{pmatrix} e^{i\tau\hat{\theta}(\hat{k})\sigma_3}.$$

Notice that the constant factor in (6.2) has been chosen in such a way that the phase factor $\hat{\theta}$ has exactly the same form as in the case of the KdV equation [21]. Therefore, similarly to the KdV case, one of two scenarios may occur:

i) $|r(0) - R(0)| \neq 1$. In this case, the central factor in (6.3) admits a factorization as in (4.3), where the diagonal factor is bounded near $\hat{k} = \pm 1$. Then the signature table for $\hat{\theta}(\hat{k})$ allows proceeding as in the similarity region, which leads to the fact that the similarity region asymptotics matches the self-similar region asymptotics and thus new asymptotic forms do not appear in the long time behavior of u.

ii) $|r(0) - R(0)| = 1$. In this case, the diagonal entry $(1 - |r(\kappa_0\hat{k}) - R(\kappa_0\hat{k})|^2)^{-1}$ in the factorization (4.3) becomes unbounded as $\kappa_0 \to 0$, which prevents from using the same arguments as in the previous case. We will show below, that in this case, a new asymptotic form indeed occurs.

Notice that, generically, $r(0) = -1$ and $R(0) = 0$. Hence, the second case is generic, and the following analysis will be devoted to this case. First, we notice that, similarly to the analysis for the self-similar region, the solution of the vector RH problem $(\mu_1 \quad \mu_2)$ can be written as $(\mu_1 \quad \mu_2)(\mathrm{i}/2) = (1\ 1)M(\mathrm{i}/2)$, where $M(k)$ is the solution of the corresponding 2×2 matrix problem normalized by the condition $M(k) \to I$ as $k \to \infty$. Due to the chosen scaling of the spectral parameter, in order to evaluate $M(k)$ at $k = \mathrm{i}/2$ as $t \to +\infty$, we have to evaluate $M^{(1)}(\hat{k})$ as $\hat{k} \to \infty$. Writing, for $j = 1, 2$,

$$M^{(j)}(\hat{k}, \kappa_0, \tau) = I + M_1^{(j)}(\kappa_0, \tau)/\hat{k} + \dots,$$

we have

(6.4) $M\left(\dfrac{\mathrm{i}}{2}\right) \approx M^{(1)}\left(\dfrac{\mathrm{i}}{2}\dfrac{1}{\kappa_0}\right) \approx I - 2\mathrm{i}\kappa_0 M_1^{(1)}(\kappa_0, \tau) \approx I - 2\mathrm{i}\kappa_0 M_1^{(2)}(\kappa_0, \tau).$

Recall that $M^{(2)}(\hat{k}, \kappa_0, \tau)$ is the solution of the RH problem in the $\hat{k}$ plane (relative to the real axis as a contour), parametrized by the parameters (κ_0, τ), and satisfying

- the jump condition (across the real axis):

$$M_-^{(2)}(\hat{k}) = M_+^{(2)}(\hat{k})J^{(2)}(\hat{k}),$$

- the normalization condition $M^{(2)}(\hat{k}) \to I$ as $\hat{k} \to \infty$.

The next step in the series of deformations of the RH problem, stipulated by the discussion above, consists in modifying the jump conditions near $\hat{k} = \pm 1$ in such a way that:

(i) There is no need to use a factorization of type of the second case in (4.3) in order to provide the exponential decay (to I) of the trigonal factors after "pushing" them out from the real axis, into the corresponding half-planes.

(ii) The signature table for the phase factor, away from certain vicinities of $\hat{k} = \pm 1$, is as for $\hat{\theta}(\hat{k})$.

An appropriate tool for such a deformation is the "g-function mechanism": one is looking for replacing the phase $4\hat{k}^3 - 12\hat{k}$ by a scalar function $g(\hat{k})$ satisfying certain analytic properties. Introduce

$$M^{(3)}(\hat{k}) = M^{(2)}(\hat{k})e^{(-i\tau 4(\hat{k}^3 - 3\hat{k}) + i\tau g(\hat{k}))\sigma_3},$$

where $g(\hat{k})$, analytic in $\mathbb{C} \setminus \mathbb{R}$, is still to be defined. Then

$$M_-^{(3)} = M_+^{(3)} J^{(3)},$$

where $J^{(3)}$ can be written in three different ways:

(6.5) $\qquad J^{(3)} = \begin{pmatrix} e^{-i\tau(g_+ - g_-)} & (\bar{r} - \bar{R})e^{-i\tau(g_+ + g_-)} \\ (-r + R)e^{i\tau(g_+ + g_-)} & (1 - |r - R|^2)e^{i\tau(g_+ - g_-)} \end{pmatrix},$

(6.6)

$\begin{pmatrix} 1 & \frac{(\bar{r} - \bar{R})e^{-2i\tau g_+}}{1 - |r - R|^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{e^{-i\tau(g_+ - g_-)}}{1 - |r - R|^2} & 0 \\ 0 & (1 - |r - R|^2)e^{i\tau(g_+ - g_-)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{(-r + R)e^{2i\tau g_-}}{1 - |r - R|^2} & 1 \end{pmatrix},$

(6.7) $\begin{pmatrix} 1 & 0 \\ (-r + R)e^{2i\tau g_+} & 1 \end{pmatrix} \begin{pmatrix} e^{-i\tau(g_+ - g_-)} & 0 \\ 0 & e^{i\tau(g_+ - g_-)} \end{pmatrix} \begin{pmatrix} 1 & (\bar{r} - \bar{R})e^{-2i\tau g_-} \\ 0 & 1 \end{pmatrix}.$

We are looking for such g that the expression (6.5) is appropriate (in the sense that it gives a useful representation in the limit $\tau \to +\infty$) for $\hat{k}$ near the critical points ± 1, i.e., for certain intervals $\hat{k} \in (-b, -a) \cup (a, b)$, where $0 < a < 1$ and $1 < b < \infty$, whereas (6.6) is appropriate for $\hat{k} \in (-a, a)$ and (6.7) is appropriate for $\hat{k} \in (-\infty, -b) \cup (b, \infty)$.

1. $\hat{k} \in (-b, -a) \cup (a, b)$.

   Notice that it is of importance here that $1 - |r(\kappa_0 \hat{k}) - R(\kappa_0 \hat{k})|^2 \to 0$ as $\kappa_0 \to 0$. More precisely, $1 - |r(\kappa_0 \hat{k}) - R(\kappa_0 \hat{k})|^2 \approx 2\gamma\kappa_0^2\hat{k}^2$ for some $\gamma > 0$. This suggest choosing g such that the diagonal entries in (6.5) both vanish as $\kappa_0 \to 0$ and $\tau \to +\infty$: the $(1,1)$ entry decays if we have $\text{Im}(g_+ - g_-) < 0$ whereas the $(2,2)$ entry should decay due to the factor $\kappa_0^2 \to 0$, which has to compensate the growth of $e^{i\tau(g_+ - g_-)}$. Then, to have the off-diagonal terms to be (asymptotically) independent of $\hat{k}$, we require $g_+ + g_- = \text{const}$ on these intervals.

2. $\hat{k} \in (-a, a)$.

   First, to get rid of the trigonal factors in (6.6) (by absorbing them into $M^{(3)}$), we need that for $\hat{k}$ outside $(-a, a)$ but near $(-a, a)$, the following property holds: $\text{Im}\, g(\hat{k}) \cdot \text{Im}\, \hat{k} > 0$. Second, for the diagonal factor in (6.6), it is useful to have $g_+ - g_- = \text{const}$ such that

$\mathrm{Im}(g_+ - g_-) < 0$ and the growth of $\mathrm{e}^{\mathrm{i}\tau(g_+ - g_-)}$ as $\tau \to +\infty$ is exactly compensated by the decay of κ_0. Thus we require

$$(6.8) \qquad \kappa_0^2 \mathrm{e}^{\mathrm{i}\tau(g_+ - g_-)} \equiv 1.$$

3. $\hat{k} \in (-\infty, -b) \cup (b, \infty)$.
  Here we require $g_+ - g_- = 0$ and $\mathrm{Im}\, g(\hat{k}) \cdot \mathrm{Im}\, \hat{k} < 0$ for $\hat{k}$ just outside these intervals (thus outside $(-b, -a) \cup (a, b)$, the sign of $\mathrm{Im}\, g$ behaves as that of $\mathrm{Im}(4\hat{k}^3 - 12\hat{k})$).

4. In order to preserve the normalization condition, we require $g(\hat{k}) - (4\hat{k}^3 - 12\hat{k}) = \mathrm{O}(1/\hat{k})$ as $\hat{k} \to \infty$.

Now we notice that all these properties are fulfilled by

$$(6.9)$$
$$g(z) = 12 \int_b^z \sqrt{(p^2 - a^2)(p^2 - b^2)}\, \mathrm{d}p + 12 \int_0^a \sqrt{(p^2 - a^2)(p^2 - b^2)}\, \mathrm{d}p,$$

for any $0 < a < 1 < b < \infty$ such that $a^2 + b^2 = 2$, see [21], where the branch for the square root is chosen such that $g(z)$ is analytic in the cut plane with the branch cuts $(-b, -a)$ and (a, b), and $\sqrt{(p^2 - a^2)(p^2 - b^2)} > 0$ for $p > b$.

Actually, the parameter a specifies a region of applicability of the construction above: the relation (6.8) becomes

$$(6.10) \qquad \kappa_0^2 \mathrm{e}^{24\tau \int_a^b \sqrt{(p^2 - a^2)(b^2 - p^2)}\, \mathrm{d}p} = 1.$$

In order to express this region in terms of the original variables (x, t), from the definitions (4.1) and (6.2) of κ_0 and τ we have (asymptotically):

$$(6.11) \quad \kappa_0^2 = \frac{1}{24}\left(2 - \frac{z}{t}\right)\left(1 + \mathrm{O}\left(\left(2 - \frac{z}{t}\right)^2\right)\right) = \left(\frac{\tau}{2t}\right)^{\frac{2}{3}}\left(1 + \mathrm{O}\left(\left(\frac{\tau}{t}\right)^{\frac{2}{3}}\right)\right).$$

On the other hand, from (6.10) it follows that

$$(6.12) \qquad 0 < -\frac{\log \kappa_0}{\tau} \leqslant 8^{\frac{3}{2}}$$

(the left bound corresponds to $a = 1$, $b = 1$ and the right bound corresponds to $a = 0$, $b = \sqrt{2}$). Combining (6.11) with (6.12) we obtain that $\tau \sim \log t$ and that a region, where this asymptotic analysis is applicable, is given by the bounds

$$(6.13) \qquad \sqrt{3} < \frac{\left(2 - \frac{z}{t}\right)^{\frac{3}{2}} t}{\log t} < C$$

for any $C > 0$. Finally, we notice that in the region in question, x and z are growing with t whereas $x - z$, see (2.34), is bounded (this follows from

the analysis below), which finally gives the description of this region in the form

$$(6.14) \qquad \sqrt{3} < \frac{\left(2 - \frac{x}{t}\right)^{\frac{3}{2}} t}{\log t} < C.$$

Now let us return to the asymptotic analysis of the RH problem for $M^{(3)}$ as $t \to +\infty$ in the region (6.14). This problem is exactly the same as in the case of the KdV equation [21]. Let's provide some details. Due to the properties of g, $M^{(3)}$ is asymptotically equivalent to $M^{(4)}$, which solves the new RH problem:

- $M^{(4)}_-(\hat{k}) = M^{(4)}_+(\hat{k}) J^{(4)}(\hat{k})$ for $\hat{k} \in (-b, b)$, where

$$(6.15) \qquad J^{(4)}(\hat{k}) = \begin{cases} \begin{pmatrix} 0 & -e^{-2i\tau C_a} \\ e^{2i\tau C_a} & 0 \end{pmatrix}, & \hat{k} \in (a, b), \\[2ex] \begin{pmatrix} \frac{1}{2\gamma \hat{k}^2} & 0 \\ 0 & 2\gamma \hat{k}^2 \end{pmatrix}, & \hat{k} \in (-a, a), \\[2ex] \begin{pmatrix} 0 & -e^{2i\tau C_a} \\ e^{-2i\tau C_a} & 0 \end{pmatrix}, & \hat{k} \in (-b, -a), \end{cases}$$

with

$$(6.16) \qquad C_a = 12 \int_0^a \sqrt{(p^2 - a^2)(b^2 - p^2)}\,dp > 0.$$

- $M^{(4)}(\hat{k}) \to I$ as $\hat{k} \to \infty$.

Indeed, the trigonal factors in the jump matrices (6.6) and (6.7) can be absorbed into the solution of an appropriately modified RH problem, the diagonal entries in (6.5) vanish asymptotically whereas the diagonal factor in (6.7) equals I and the off-diagonal entries in (6.5) are independent of $\hat{k}$.

A remarkable feature of the RH problem for $M^{(4)}$ is that its solution can be written explicitly. To achieve this, it is convenient to proceed in two steps:

1. First, to get rid of the diagonal jump matrix on $(-a, a)$.
2. Then, to solve the resulting RH problem on two slits, $(-b, -a)$ and (a, b), explicitly in terms of theta-functions of the associated Riemann surface.

When performing the first step, it is important to not "damage" the property of $J^{(4)}$ on $(-b, -a)$ and (a, b) to be independent of $\hat{k}$. This can be done by applying the "g-function" mechanism.

We look for a function $G(\hat{k})$, analytic in $\mathbb{C} \setminus [-b, b]$, such that:

(i) $G_+ - G_- = -i \log 2\gamma \hat{k}^2$ for $\hat{k} \in (-a, a)$;

(ii) $G_+ + G_- = \Delta$ for $\hat{k} \in (a, b)$, where Δ is a constant;

(iii) $G_+ + G_- = -\Delta$ for $\hat{k} \in (-b, -a)$;

(iv) $G(\hat{k}) = O(1/\hat{k})$ as $\hat{k} \to \infty$.

Then

$$M^{(5)}(\hat{k}) := M^{(4)}(\hat{k}) e^{-iG(\hat{k})\sigma_3}$$

will solve the following RH problem:

- $M_-^{(5)}(\hat{k}) = M_+^{(5)}(\hat{k}) J^{(5)}(\hat{k})$ for $\hat{k} \in (-b, -a) \cup (a, b)$ where

$$
\text{(6.17)} \qquad J^{(5)}(\hat{k}) =
\begin{cases}
\begin{pmatrix} 0 & -e^{-2i\tau C_a + \Delta} \\ e^{2i\tau C_a - \Delta} & 0 \end{pmatrix}, & \hat{k} \in (a, b), \\
\begin{pmatrix} 0 & -e^{2i\tau C_a - \Delta} \\ e^{-2i\tau C_a + \Delta} & 0 \end{pmatrix}, & \hat{k} \in (-b, -a).
\end{cases}
$$

- $M^{(5)}(\hat{k}) \to I$ as $\hat{k} \to \infty$.

In order to determine $G(\hat{k})$, it is sufficient to apply the integral Cauchy formula for the function

$$h := \frac{G}{f} \quad \text{where} \quad f(\hat{k}) = \sqrt{(\hat{k}^2 - a^2)(\hat{k}^2 - b^2)},$$

the branch of the square root being fixed as in (6.9). Noting that the sum of the limiting values of G in (ii) and (iii) is transformed into the difference for h, we have:

(6.18)

$$G(\hat{k}) = \frac{f(k)}{2\pi i} \left[\int_{-b}^{-a} \frac{-\Delta \, dp}{(p - \hat{k}) f_+(p)} + \int_{-a}^{a} \frac{-i \log 2\gamma p^2 \, dp}{(p - \hat{k}) f(p)} + \int_{a}^{b} \frac{\Delta \, dp}{(p - \hat{k}) f_+(p)} \right].$$

Then the constant Δ is determined from the requirement that the r.h.s. of (6.18) is $O(1/k)$: equating the term of order k to zero gives

$$2 \int_{a}^{b} \frac{\Delta \, dp}{f_+(p)} = \int_{-a}^{a} \frac{-i \log 2\gamma p^2 \, dp}{f(p)}$$

and thus

$$\text{(6.19)} \qquad \Delta = i \int_{0}^{a} \frac{\log 2\gamma p^2 \, dp}{f(p)} \left(\int_{a}^{b} \frac{dp}{f_+(p)} \right)^{-1} \in \mathbb{R}.$$

Finally, the explicit solution of the RH problem for $M^{(5)}$, which is a RH problem with piecewise constant (depending only on external parameters), off-diagonal jump matrix on two slits, comes from the theory of compact Riemann surfaces and associated Riemann theta functions. Particularly, relevant to the problem under consideration is the two-sheeted Riemann surface X of genus 1 associated to $f(z)$, the first sheet of which is fixed by

the condition $f(z) > 0$ for $z > b$ on this sheet. We can choose a homology basis $\{\mathfrak{a}, \mathfrak{b}\}$ on X as follows: the cycle $\mathfrak{a}$ goes counterclockwise around the interval $(-b, -a)$ on the first sheet, and the cycle $\mathfrak{b}$ goes counterclockwise around $(-a, a)$ in such a way that its part in the lower half-plane is on the first sheet whereas the part in the upper half-plane is on the second sheet. The basic holomorphic differential ω on X is fixed by the condition $\int_{\mathfrak{a}} \omega = 1$ and thus has the form

$$\omega = \frac{e}{f(z)} \mathrm{d}z \quad \text{with}$$

$$e = \frac{1}{2} \left(\int_a^b \frac{\mathrm{d}p}{f_+(p)} \right)^{-1}.$$

Let E be the associated $\mathfrak{b}$-period:

$$E = 4e \int_0^a \frac{\mathrm{d}p}{f(p)}.$$

Let θ be the associated Riemann theta function:

$$\theta(s) = \sum_{m \in \mathbb{Z}} \mathrm{e}^{2\pi \mathrm{i} m s + E\pi \mathrm{i} m^2}.$$

This function is even and has the following periodicity properties:

$$\theta(s + 1) = \theta(s),$$

$$\theta(s + E) = \mathrm{e}^{-2\pi \mathrm{i} s - \pi \mathrm{i} E} \theta(s).$$

The latter properties allow determining the main "building block" for the solution of the RH problem for $M^{(5)}$: defining

$$h(s) = \frac{\theta(s + d + c)}{\theta(s + c)},$$

where c and d are arbitrary constants, we have that

$$h(s + 1) = h(s),$$

$$h(s + E) = \mathrm{e}^{-2\pi \mathrm{i} d} h(s).$$

Now introduce

$$u(z) = \int_b^z \omega, \quad z \in \mathbb{C} \setminus (-b, b),$$

where the contour of integration is on the first sheet of X, and the values of u are in $\mathbb{C}/\mathbb{Z}$. We have:

(i) $u_+(z) + u_-(z) = 0$ for $z \in (a, b)$;
(ii) $u_+(z) + u_-(z) = -E$ for $z \in (-b, -a)$;
(iii) $u_+(z) - u_-(z) = 1$ for $z \in (-a, a)$.

Then direct calculations show that the matrix-valued function
(6.20)
$$\tilde{M}(\hat{k}) := \begin{pmatrix} \frac{\nu(\hat{k})+\nu^{-1}(\hat{k})}{2} \frac{\theta(u(\hat{k})+\frac{E}{4}-\frac{\phi}{\pi})}{\theta(u(\hat{k})+\frac{E}{4})} & \frac{\nu(\hat{k})-\nu^{-1}(\hat{k})}{-2i} \frac{\theta(-u(\hat{k})+\frac{E}{4}-\frac{\phi}{\pi})}{\theta(-u(\hat{k})+\frac{E}{4})} e^{-i\phi} \\ \frac{\nu(\hat{k})-\nu^{-1}(\hat{k})}{2i} \frac{\theta(u(\hat{k})-\frac{E}{4}-\frac{\phi}{\pi})}{\theta(u(\hat{k})-\frac{E}{4})} e^{i\phi} & \frac{\nu(\hat{k})+\nu^{-1}(\hat{k})}{2} \frac{\theta(-u(\hat{k})-\frac{E}{4}-\frac{\phi}{\pi})}{\theta(-u(\hat{k})-\frac{E}{4})} \end{pmatrix},$$

where

(6.21)
$$\phi = 2\tau C_a - \Delta,$$

with $\Delta \in \mathbb{R}$ given by (6.19) and $C_a > 0$ by (6.16), and where

$$\nu(k) = \left[\frac{(k-a)(k+b)}{(k+a)(k-b)} \right]^{\frac{1}{4}}$$

with branch cuts $(-b, -a)$ and (a, b) and $\nu(\infty) = 1$, satisfies the jump conditions $\tilde{M}_- = \tilde{M}_+ J^{(5)}$. Notice that the function $\theta(-u(\hat{k}) + \frac{E}{4})$ has a unique zero at $\hat{k} = 0$ (on the first sheet of X), which is also a zero of $\nu(\hat{k}) - \nu^{-1}(\hat{k})$. On the other hand, the function $\theta(u(\hat{k}) + \frac{E}{4})$ has no zeros on the first sheet of X (actually, the unique zero of $\theta(u(\hat{k}) + \frac{E}{4})$ considered on the whole Riemann surface X is at $\hat{k} = 0$ on the second sheet). Hence, $\tilde{M}$ is holomorphic on $\mathbb{C} \setminus ((-b, -a) \cup (a, b))$.

Thus the solution to the RH problem for $M^{(5)}$ is given by

$$M^{(5)}(\hat{k}) = \tilde{M}(\infty)^{-1} \tilde{M}(\hat{k}),$$

where

$$\tilde{M}(\infty) = \begin{pmatrix} \frac{\theta(u(\infty)+\frac{E}{4}-\frac{\phi}{\pi})}{\theta(u(\infty)+\frac{E}{4})} & 0 \\ 0 & \frac{\theta(-u(\infty)-\frac{E}{4}-\frac{\phi}{\pi})}{\theta(-u(\infty)-\frac{E}{4})} \end{pmatrix}.$$

The final step consists in relating u, which is originally defined in terms of $M(k)$, to the constructed solution $M^{(5)}(\hat{k})$. The asymptotic relationship between the transformations of the RH problem

$$M^{(2)}(\hat{k}) \approx M^{(5)}(\hat{k}) e^{iG(\hat{k})\sigma_3} e^{-i\tau(g(\hat{k})-(4\hat{k}^3-12\hat{k}))\sigma_3}$$

yields

(6.22)
$$M_1^{(2)} \approx M_1^{(5)} + iG_1\sigma_3 - i\tau g_1\sigma_3,$$

where

(6.23)
$$G(\hat{k}) = \frac{G_1}{\hat{k}} + \dots, \quad \text{as } \hat{k} \to \infty,$$

(6.24)
$$g(\hat{k}) - (4\hat{k}^3 - 12\hat{k}) = \frac{g_1}{\hat{k}} + \dots \quad \text{as } \hat{k} \to \infty,$$

and

$$M^{(5)}(\hat{k}) = I + \frac{M_1^{(5)}}{\hat{k}} + \dots.$$

From (6.20) it follows that $M_1^{(5)}$ is given by

(6.25) $$M_1^{(5)} = \begin{pmatrix} 0 & m^{12} \\ m^{21} & 0 \end{pmatrix}$$

with

$$m^{12} = \left(M_1^{(5)}\right)_{12} = -\frac{\mathrm{i}(a-b)}{2} \frac{\theta(-u(\infty) + \frac{E}{4} - \frac{\phi}{\pi})\theta(u(\infty) + \frac{E}{4})}{\theta(u(\infty) + \frac{E}{4} - \frac{\phi}{\pi})\theta(-u(\infty) + \frac{E}{4})} e^{-\mathrm{i}\phi},$$

$$m^{21} = \left(M_1^{(5)}\right)_{21} = \frac{\mathrm{i}(a-b)}{2} \frac{\theta(u(\infty) - \frac{E}{4} - \frac{\phi}{\pi})\theta(-u(\infty) - \frac{E}{4})}{\theta(-u(\infty) - \frac{E}{4} - \frac{\phi}{\pi})\theta(u(\infty) - \frac{E}{4})} e^{\mathrm{i}\phi}.$$

Now, recalling (6.4), we have:

(6.26) $$\log \frac{\mu_1}{\mu_2} = -2\mathrm{i}\kappa_0\left[\left(M_1^{(5)}\right)_{21} - \left(M_1^{(5)}\right)_{12} + 2\mathrm{i}G_1 - 2\mathrm{i}\tau g_1\right]\left(1 + o(1)\right).$$

Recalling (2.34), we have to estimate the derivative of the above expression with respect to t keeping z fixed. Recall that $\hat{\zeta} = z/t$. From the definitions of κ_0 and τ, see (4.1) and (6.2), and the estimate (6.11) we have

$$\kappa_0 = \left(\frac{\tau}{2t}\right)^{\frac{1}{3}}\left[1 + \mathrm{O}\left(\left(\frac{\tau}{t}\right)^{\frac{2}{3}}\right)\right],$$

$$\partial_t\kappa_0 = \left[(\partial_{\hat{\zeta}}\kappa_0)(\hat{\zeta})\right]\partial_t\hat{\zeta}$$

$$= \frac{1}{2\kappa_0(\hat{\zeta})} \times \left[(\partial_{\hat{\zeta}}\kappa_0^2)(\hat{\zeta})\right] \times \left(-\frac{z}{t^2}\right) = \frac{2^{\frac{1}{3}}}{24}\tau^{-\frac{1}{3}}t^{-\frac{2}{3}}\left[1 + \mathrm{O}\left(\left(\frac{\tau}{t}\right)^{\frac{2}{3}}\right)\right],$$

$$\partial_t\tau = 6^{\frac{3}{2}}\left(\frac{\tau}{2t}\right)^{\frac{1}{3}}\left[1 + \mathrm{O}\left(\left(\frac{\tau}{t}\right)^{\frac{2}{3}}\right)\right].$$

Finally, from (6.26) and (6.4) we obtain, always in the "solitonless" case:

THEOREM 6.1 (shock-wave region (iv)). — *For (x,t) in the region*

$$\sqrt{3} < \frac{\left(2 - \frac{x}{t}\right)^{\frac{3}{2}} t}{\log t} < C$$

for any $C > \sqrt{3}$, we have, as $t \to +\infty$:

(6.27) $$u(x,t) = -2^{\frac{1}{3}}6^{\frac{3}{2}}\left(\frac{\tau}{t}\right)^{\frac{2}{3}}\left(g_1(6^{-\frac{5}{2}} + 2) + \partial_\tau\Psi\right)\left(1 + o(1)\right),$$

where τ is given by (6.2) and

(6.28) $$\Psi = (b-a)\,\mathrm{Re}\left\{\frac{\theta(u(\infty) - \frac{E}{4} - \frac{\phi}{\pi})\theta(-u(\infty) - \frac{E}{4})}{\theta(-u(\infty) - \frac{E}{4} - \frac{\phi}{\pi})\theta(u(\infty) - \frac{E}{4})} e^{\mathrm{i}\phi}\right\}.$$

Here g_1 is defined by (6.24), $\Delta \in \mathbb{R}$ is given by (6.19), and $\phi = 2\tau C_a - \Delta$ is defined in (6.21) with $C_a > 0$ given in (6.16).

Remark 6.2. — Recall that in this region, $\tau \sim \log t$, and Ψ oscillates with τ.

Remark 6.3. — Differentiability of the asymptotics of solutions of RH problems is studied in [24]. Alternatively, one can avoid differentiating the solution of the RH problem (with respect to a parameter t) by using the expression for u in terms of the term of order $\mathrm{O}(k - \frac{i}{2})$ in the expansion of the solution of the RH problem as $k \to \frac{i}{2}$, see [5].

Dedicated to Bernard Malgrange, mentor and friend

BIBLIOGRAPHY

[1] R. BEALS, D. H. SATTINGER & J. SZMIGIELSKI, "Multipeakons and the classical moment problem", *Adv. Math.* **154** (2000), no. 2, p. 229-257.

[2] ———, "The string density problem and the Camassa-Holm equation", *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **365** (2007), no. 1858, p. 2299-2312.

[3] A. BOUTET DE MONVEL, A. S. FOKAS & D. SHEPELSKY, "The mKdV equation on the half-line", *J. Inst. Math. Jussieu* **3** (2004), no. 2, p. 139-164.

[4] ———, "Integrable nonlinear evolution equations on a finite interval", *Comm. Math. Phys.* **263** (2006), no. 1, p. 133-172.

[5] A. BOUTET DE MONVEL, A. KOSTENKO, D. SHEPELSKY & G. TESCHL, "Long-Time Asymptotics for the Camassa–Holm Equation", *SIAM J. Math. Anal.* **41** (2009), no. 4, p. 1559-1588.

[6] A. BOUTET DE MONVEL & D. SHEPELSKY, "Initial boundary value problem for the mKdV equation on a finite interval", *Ann. Inst. Fourier (Grenoble)* **54** (2004), no. 5, p. 1477-1495, xv, xxi.

[7] ———, "Riemann-Hilbert approach for the Camassa-Holm equation on the line", *C. R. Math. Acad. Sci. Paris* **343** (2006), no. 10, p. 627-632.

[8] ———, "The Camassa-Holm equation on the half-line: a Riemann-Hilbert approach", *J. Geom. Anal.* **18** (2008), no. 2, p. 285-323.

[9] ———, "Long-time asymptotics of the Camassa–Holm equation on the line", in *Proceedings of the Conference on Integrable Systems, Random Matrices, and Applications: A conference in honor of Percy Deift's 60th birthday* (Providence, RI) (J. Baik, L.-C. Li, T. Kriecherbauer, K. McLaughlin & C. Tomei, eds.), Contemporary Mathematics, vol. 458, Amer. Math. Soc., 2008, p. 99-116.

[10] ———, "Riemann-Hilbert problem in the inverse scattering for the Camassa-Holm equation on the line", in *Probability, geometry and integrable systems*, Math. Sci. Res. Inst. Publ., vol. 55, Cambridge Univ. Press, Cambridge, 2008, p. 53-75.

[11] ———, "A class of linearizable problems for the Camassa–Holm equation on the half-line", In preparation, 2009.

[12] R. CAMASSA & D. D. HOLM, "An integrable shallow water equation with peaked solitons", *Phys. Rev. Lett.* **71** (1993), no. 11, p. 1661-1664.

[13] R. CAMASSA, D. D. HOLM & J. M. HYMAN, "A new integrable shallow water equation", Hutchinson, John W. et al. (eds.), Advances in Applied Mechanics. Vol. 31, Boston, MA: Academic Press, p. 1-33, 1994.

[14] R. CAMASSA, J. HUANG & L. LEE, "Integral and integrable algorithms for a nonlinear shallow-water wave equation", *J. Comput. Phys.* **216** (2006), no. 2, p. 547-572.

[15] A. CONSTANTIN & H. P. MCKEAN, "A shallow water equation on the circle", *Comm. Pure Appl. Math.* **52** (1999), no. 8, p. 949-982.

[16] A. CONSTANTIN, "On the scattering problem for the Camassa-Holm equation", *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **457** (2001), no. 2008, p. 953-970.

[17] A. CONSTANTIN, V. S. GERDJIKOV & R. I. IVANOV, "Inverse scattering transform for the Camassa-Holm equation", *Inverse Problems* **22** (2006), no. 6, p. 2197-2207.

[18] A. CONSTANTIN & D. LANNES, "The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations", *Arch. Ration. Mech. Anal.* **192** (2009), no. 1, p. 165-186.

[19] A. CONSTANTIN & J. LENELLS, "On the inverse scattering approach to the Camassa-Holm equation", *J. Nonlinear Math. Phys.* **10** (2003), no. 3, p. 252-255.

[20] A. CONSTANTIN & W. A. STRAUSS, "Stability of a class of solitary waves in compressible elastic rods", *Phys. Lett. A* **270** (2000), no. 3-4, p. 140-148.

[21] P. DEIFT, S. VENAKIDES & X. ZHOU, "The collisionless shock region for the long-time behavior of solutions of the KdV equation", *Comm. Pure Appl. Math.* **47** (1994), no. 2, p. 199-206.

[22] P. DEIFT & X. ZHOU, "A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation", *Ann. of Math. (2)* **137** (1993), no. 2, p. 295-368.

[23] P. A. DEIFT, A. R. ITS & X. ZHOU, "Long-time asymptotics for integrable nonlinear wave equations", in *Important developments in soliton theory*, Springer Ser. Nonlinear Dynam., Springer, Berlin, 1993, p. 181-204.

[24] P. A. DEIFT & X. ZHOU, "Long-time asymptotics for integrable systems. Higher order theory", *Comm. Math. Phys.* **165** (1994), no. 1, p. 175-191.

[25] A. S. FOKAS, "A unified transform method for solving linear and certain nonlinear PDEs", *Proc. Roy. Soc. London Ser. A* **453** (1997), no. 1962, p. 1411-1443.

[26] ———, "Integrable nonlinear evolution equations on the half-line", *Comm. Math. Phys.* **230** (2002), no. 1, p. 1-39.

[27] A. S. FOKAS & A. R. ITS, "An initial-boundary value problem for the Korteweg-de Vries equation", *Math. Comput. Simulation* **37** (1994), no. 4-5, p. 293-321, Solitons, nonlinear wave equations and computation (New Brunswick, NJ, 1992).

[28] A. S. FOKAS, A. R. ITS & L.-Y. SUNG, "The nonlinear Schrödinger equation on the half-line", *Nonlinearity* **18** (2005), no. 4, p. 1771-1822.

[29] A. S. FOKAS, A. R. ITS, A. A. KAPAEV & V. Y. NOVOKSHENOV, *Painlevé transcendents*, Mathematical Surveys and Monographs, vol. 128, American Mathematical Society, Providence, RI, 2006, The Riemann-Hilbert approach, xii+553 pages.

[30] A. R. ITS, "The Riemann-Hilbert problem and integrable systems", *Notices Amer. Math. Soc.* **50** (2003), no. 11, p. 1389-1400.

[31] R. S. JOHNSON, "Camassa-Holm, Korteweg-de Vries and related models for water waves", *J. Fluid Mech.* **455** (2002), p. 63-82.

[32] ———, "On solutions of the Camassa-Holm equation", *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **459** (2003), no. 2035, p. 1687-1708.

[33] J. Lenells, "The scattering approach for the Camassa-Holm equation", *J. Nonlinear Math. Phys.* **9** (2002), no. 4, p. 389-393.

[34] S. Ma & S. Ding, "On the initial boundary value problem for a shallow water equation", *J. Math. Phys.* **45** (2004), no. 9, p. 3479-3497.

[35] Y. Matsuno, "Parametric representation for the multisoliton solution of the Camassa-Holm equation", *J. Phys. Soc. Japan* **74** (2005), no. 7, p. 1983-1987.

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