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Trond Stølen GUSTAVSEN & Runar ILE

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REPRESENTATION THEORY FOR LOG-CANONICAL SURFACE SINGULARITIES

by Trond Stølen GUSTAVSEN & Runar ILE (*)

ABSTRACT. — We consider the representation theory for a class of log-canonical surface singularities in the sense of reflexive (or equivalently maximal Cohen-Macaulay) modules and in the sense of finite dimensional representations of the local fundamental group. A detailed classification and enumeration of the indecomposable reflexive modules is given, and we prove that any reflexive module admits an integrable connection and hence is induced from a finite dimensional representation of the local fundamental group.

RÉSUMÉ. — Nous considérons la théorie des représentations pour une classe des singularités des surfaces log-canoniques dans le sens de modules réflexifs (ou d'une manière équivalente, modules maximals de Cohen-Macaulay) et dans le sens de représentations de dimension finie du groupe fondamental local. Une classification et une énumération détaillées des modules réflexifs indécomposables sont données, et nous montrons que n'importe quel module réflexif admet une connexion intégrable, et par conséquent est induit par une représentation de dimension finie du groupe fondamental local.

1. Introduction

A normal surface singularity (X, x) may be studied through its representations, and although a comprehensive understanding seems to be out of reach, results have been obtained for classes of normal surfaces singularities both in terms of reflexive (or equivalently maximal Cohen-Macualay) modules and in terms of finite dimensional representations of the local

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fundamental group $\pi_1^{\text{loc}}(X,x)$. The case of quotient singularities is particularly transparent. For such a singularity there is a one-to-one correspondence between the reflexive modules and the finite dimensional representations of the group, see [11, 2, 6]. The situation is much more complicated for other classes of singularities, and we believe that the thesis of Constantin P. M. Kahn is among the most comprehensive studies in this direction, see [12]. He develops a technique involving a reduction cycle on the minimal resolution, and applies it to the case of a simple elliptic surface singularity where he uses Atiyah's classification of vector bundles on an elliptic curve to give a detailed classification of the reflexive modules. Notably, he is able to show that any reflexive module is induced from a representation of the local fundamental group. He also shows that a simple elliptic surface singularity is of tame representation type with respect to reflexive modules while the local fundamental group is of wild representation type.

It is known that a normal surface singularity (X,x) is log-canonical if and only if $\pi_1^{\text{loc}}(X,x)$ is finite or solvable, and that the class of log-canonical surface singularities constitutes exactly of the quotient singularities, the simple elliptic singularities, the cusp singularities, and finite quotients of these, see [14] and [27]. Thus we find it natural to consider the class of log-canonical surface singularities, and in the present article we consider a quotient (X,x) of a simple elliptic surface singularity (Y,y). The elliptic quotients contains a large class of rational surface singularities in addition to the simple elliptic, see Theorem 4.1.

Among three main theorems, we regard the following as the most important.

Theorem A. — *Every reflexive module M on an elliptic quotient (X,x) admits an integrable connection $\nabla : \text{Der}_{\mathbb{C}}(\mathcal{O}_{X,x}) \to \text{End}_{\mathbb{C}}(M)$.*

By a Riemann-Hilbert correspondence proved in [10], this theorem implies that M is induced from a finite dimensional representation of $\pi_1^{\text{loc}}(X,x)$ and that it admits a $\mathcal{D}_{X,x}$-module structure where $\mathcal{D}_{X,x}$ is the ring of differential operators on (X,x). It should be noted that the ring $\mathcal{D}_{X,x}$ is not noetherian for elliptic quotients and is then of course far from being generated in degree one, see [4, 18]. Nevertheless, the first order structure given by the integrable connection, induces a $\mathcal{D}_{X,x}$-module structure on M.

Kurt Behnke proved in [3] that the fundamental module on a cusp singularity admits an integrable connection and conjectured that all reflexive modules on a cusp singularity admits a connection. We offer the following extension.

Conjecture. — *Every reflexive module M on a log-canonical surface singularity (X,x) admits an integrable connection $\nabla : \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x}) \to \mathrm{End}_{\mathbb{C}}(M)$.*

By Theorem A the conjecture is open for cusp singularities and their quotients.

Theorem B. — *Let (X,x) be an elliptic quotient that is not simple elliptic. If the log-index m divides r, there is a finite number of one-parameter families of reflexive modules of rank r on (X,x). If m does not divide r there are finitely many reflexive modules. In particular; if $(r,m)=1$, there are exactly $r\,|\det(C_iC_j)|$ indecomposable reflexive modules of rank r where (C_iC_j) is the intersection matrix of the minimal resolution.*

The formulation in Theorem B is a simplified version of the classification in Theorem 4.7. A reflexive module M is an invariant module if there exists a finite covering $\pi : (Y,y) \to (X,x)$ such that the double dual of the pull back, $(\pi^*M)^{\vee\vee}$, is a free module, or, equivalently, if it corresponds to a profinite representation of the local fundamental group. As a consequence of the full classification, we also obtain an identification of the invariant modules. When the rank divides the log-index, we enumerate the families containing invariant modules. For each such family there is a map $E \to C \cong \mathbb{P}^1$ from an elliptic curve to the parameter space C, and the dense image of the torsion points parameterizes the invariant modules. In the case r does not divide m we enumerate the finitely many invariant modules.

All elliptic quotients are quasi-homogenous, so we may consider graded reflexive modules on an elliptic quotient. In fact, we are only able to prove Theorem A and Theorem B for graded modules. Fortunately, we also obtain:

Theorem C. — *Every reflexive module M on (X,x) is gradable.*

The paper is organized as follows: In Section 2 we give preliminaries. In particular we review important results from the thesis of Kahn, see [12]. In order to obtain our detailed classification, we need to extend Atiyah's classification of vector bundles on an elliptic curve to vector bundles with action of a finite group G acting without translations. This is done in Section 3. In Section 4 we prove that any reflexive module on (X,x) is gradable, and carry through the classification of the reflexive modules. In Section 5 we prove that every reflexive module M on (X,x) admits an integrable connection. For this we need to extend a part of Weil's theorem to show that indecomposable vector bundles on an elliptic curve E of degree 0 with compatible G-action admits a G-equivariant integrable connection. In Section 6 we give details in an example with the two dimensional representations of the local fundamental group worked out in an appendix.

We remark that [5] also has a classification of reflexive modules on elliptic quotients. Our classification in Theorem 4.7 is however more explicit, and in particular our formula for the number of indecomposable reflexive modules in ranks not divisible by the log index, is new.

2. Notation and preliminaries

In this section we fix notation that will be used throughout the paper, and review important results from the thesis of Kahn, see [12].

We work over the field $\mathbb{C}$ of complex numbers, and denote by (X,x) the germ of a normal complex analytic space of dimension two. Choosing a representative X of $(X,x) \subset (\mathbb{C}^n, 0)$, then for a small ball B_ε in $\mathbb{C}^n$ of radius ε, the link of (X,x), defined as $L := X \cap \partial B_\varepsilon$, is a smooth, compact, connected and oriented real 3-manifold and is independent of ε up to diffeomorphism, see [21]. *The local fundamental group* is defined as $\pi_1^{\mathrm{loc}}(X,x) := \pi_1(L)$, and since (X,x) is homeomorphic to the cone over L after interchanging X with $X \cap B_\varepsilon$, one has $\pi_1^{\mathrm{loc}}(X,x) = \pi_1(X \setminus \{x\})$. We will always assume that X is such a representative and that $X \setminus \{x\}$ is smooth.

The local ring of germs of holomorphic functions will be denoted by $\mathcal{O}_{X,x}$, and $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on the complex space X.

Let M be a coherent $\mathcal{O}_{X,x}$-module. Recall that a coherent $\mathcal{O}_{X,x}$-module M is *reflexive* if the canonical map $M \to M^{\vee\vee}$ of the module into its double dual is an isomorphism. Since (X,x) is a normal surface singularity, M is reflexive if and only if M is maximal Cohen-Macaulay, *i.e.*, $\operatorname{depth} M = 2$. We will denote by $\mathsf{Ref}_{X,x}$, the category of reflexive $\mathcal{O}_{X,x}$-modules.

The $\mathbb{C}$-vector space $\mathrm{End}_{\mathbb{C}}(M)$ of $\mathbb{C}$-linear maps is an $\mathcal{O}_{X,x}$-bi-module, and a *connection* on M is an $\mathcal{O}_{X,x}$-linear map $\nabla : \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x}) \to \mathrm{End}_{\mathbb{C}}(M)$ which for all $f \in \mathcal{O}_{X,x}$, $m \in M$ and $D \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x})$ satisfy the *Leibniz rule*

$$\nabla(D)(fm) = D(f)m + f\nabla(D)(m). \tag{2.1}$$

A morphism $\varphi : (M_1, \nabla_1) \to (M_2, \nabla_2)$ is an $\mathcal{O}_{X,x}$-module homomorphism $\varphi : M_1 \to M_2$ such that $\varphi\nabla_1(D) = \nabla_2(D)\varphi$ for all $D \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x})$ (φ is a *horizontal* map). A connection $\nabla : \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x}) \to \mathrm{End}_{\mathbb{C}}(M)$ is said to be *integrable* if it is a $\mathbb{C}$-Lie-algebra homomorphism.

The thesis [12] of Constantin P. M. Kahn (see also his paper [13]) contains several results of importance for the present work. To fix notation and for the convenience of the reader, we briefly summarize some of these results.

Kahn develops a method to classify reflexive modules on a normal surface singularity using full sheaves and a reduction cycle which he applies to a class of normal surface singularities: (Y, y) is said to *be simple elliptic* of type $\mathrm{El}(b)$ if the exceptional divisor in the minimal resolution is a smooth elliptic curve E with $E^2 = -b$ and $b \geqslant 1$, see [25] and [15]. Kahn's classification of reflexive modules on a simple elliptic surface singularity relies on Atiyah's classification of vector bundles on an elliptic curve, see [1]. There are bijective maps $\alpha_{r,d} : \mathrm{Pic}^0 E \to \mathcal{B}_{\mathrm{ind}}(r, d)$ where $\mathcal{B}_{\mathrm{ind}}(r, d)$ is the set of indecomposable vector bundles of rank r and degree d, and for the classification of reflexive modules on a simple elliptic surface singularity, Kahn proves the following theorem, where $1 \in \mathrm{Pic}^0 E$ denotes the neutral element.

Theorem 2.1 ([12, 5.16]). — *Let $p : \widetilde{Y} \to Y$ be the minimal resolution of a simple elliptic surface singularity (Y, y) of type $\mathrm{El}(b)$ and let $E = p^{-1}(y)$. Every reflexive module N on Y gives a vector bundle $R(N) := (p^*N)^{\vee\vee} \otimes \mathcal{O}_E$. The images of indecomposable reflexive modules under R are up to isomorphism the following*

$$\alpha'_{r,d}(\lambda) := \begin{cases} \alpha_{r,d}(\lambda) & \text{if } d < br \text{ or } d = br \text{ and } \lambda \neq 1; \\ \mathcal{O}_E \oplus \alpha_{r-1,b(r-1)}(1) & \text{if } d = br \text{ and } \lambda = 1; \\ \mathcal{O}_E^n \oplus \alpha_{r-n,b(r-n)+n}(\lambda) & \text{if } d = br + n,\ 0 < n < r. \end{cases}$$

By a theorem of Grauert the minimal resolution is isomorphic to the total space of any line bundle of degree $-b$ on E, see [8] and [25]. The elliptic curve E is given as the zero section and $\widetilde{Y} \setminus E$ is a principal $\mathbb{C}^*$-bundle $s : \widetilde{Y} \setminus E \to E$. Since $\widetilde{Y} \setminus E \cong Y \setminus \{y\}$ we may identify s with a map $s : Y \setminus \{y\} \to E$.

The reflexive modules corresponding to the vector bundles $\alpha'_{r,d}(\lambda)$ in Theorem 2.1 are related to the $\alpha_{r,d}(\lambda)$ in the following way.

Proposition 2.2 ([12, 5.18, 5.19]). — *Let N be an indecomposable reflexive module on a simple elliptic surface singularity (Y, y). If $(p^*N)^{\vee\vee} \otimes \mathcal{O}_E \cong \alpha'_{r,d}(\lambda)$, then*

$$(p^*N)^{\vee\vee}|_{\widetilde{Y}\setminus E} \cong N|_{Y\setminus\{0\}} \cong s^*\alpha_{r,d}(\lambda).$$

Moreover, the set

$$\{(r, d, \lambda) \in \mathbb{N} \times \mathbb{N} \times \mathrm{Pic}^0 E \mid r \leqslant d < (b + 1)r\}$$

is in one-to-one correspondence with the set of isomorphism classes of indecomposable reflexive modules on (Y,y) *through the map* $(r,d,\lambda)\mapsto (i_*(s^*\alpha_{r,d}(\lambda)))$ *where* $i : Y\setminus\{y\}\to Y$ *is the inclusion.*

For a simple elliptic surface singularity of type $\mathrm{El}(b)$ Kahn finds that the local fundamental group is given as $\pi_1^{\mathrm{loc}}(Y,y)=H_b$ where H_b is the *discrete Heisenberg group* generated by three elements α, β and γ subject to the following relations:

$$[\alpha,\gamma]=1,\ [\beta,\gamma]=1 \text{ and } [\alpha,\beta]=\gamma^b.$$

He gives the following classification of the finite dimensional representations of $\pi_1^{\mathrm{loc}}(X,x)$. Assume $1\leqslant r\leqslant d<(b+1)r$ and that $(r,d)=1$. Define $\tau_{r,d}:H_b\to\mathrm{GL}(r,\mathbb{C})$ by

$$\tau_{r,d}(\alpha)=\begin{pmatrix}1&0&0&\cdots&0\\0&\zeta^b&0&\cdots&0\\0&0&\zeta^{2b}&&\vdots\\\vdots&\vdots&&\ddots&0\\0&0&\cdots&0&\zeta^{(r-1)b}\end{pmatrix},\ \tau_{r,d}(\beta)=\begin{pmatrix}0&0&\cdots&0&1\\1&0&\cdots&0&0\\0&1&0&&\vdots\\\vdots&&\ddots&\ddots&\vdots\\0&\cdots&0&1&0\end{pmatrix}$$

and $\tau_{r,d}(\gamma)=\zeta I_r$ where $\zeta=e^{-2\pi id/br}$. We denote by $\mathrm{Unip}(r,\mathbb{C})\subset\mathrm{GL}(r,\mathbb{C})$ the subgroup of unipotent upper triangular matrices of rank r.

Theorem 2.3 ([12, 6.13]). — *We have that*

(1) *To every indecomposable representation* $\rho : H_b\to\mathrm{GL}(n,\mathbb{C})$ *there is a character* $\chi : H_b\to\mathbb{C}^*$ *with* $\chi(\gamma)=1$, *a representation* $\tau_{r,d}$ *as above where* r *is a divisor of* n *and a representation* $\sigma : H_b\to\mathrm{Unip}(h,\mathbb{C})$ *where* $h=n/r$ *such that* ρ *is conjugate to* $\chi\otimes\tau_{r,d}\otimes\sigma$.

(2) *Two indecomposable representations* $\chi\otimes\tau_{r,d}\otimes\sigma$ *and* $\chi'\otimes\tau'_{r',d'}\otimes\sigma'$ *are conjugate if and only if* $r=r'$, $d=d'$, $\chi^r=\chi'^r$ *and* σ *and* σ' *are conjugate.*

(3) *A representation* $\chi\otimes\tau_{r,d}\otimes\sigma$ *is indecomposable if and only if* σ *is indecomposable.*

To each representation ρ there is by Proposition 2.2 a vector bundle F_ρ on E which corresponds to the local system $\mathbb{V}_\rho$ on $\widetilde{Y}\backslash E$ corresponding to ρ, i.e., $\mathcal{F}_\rho:=s^*F_\rho=\mathbb{V}_\rho\otimes_{\mathbb{C}}\mathcal{O}_{\widetilde{Y}\backslash E}$. If $\rho=\chi\otimes\tau_{r,d}\otimes\sigma$ then $\mathcal{F}_\rho=\mathcal{F}_\chi\otimes\mathcal{F}_{\tau_{r,d}}\otimes\mathcal{F}_\sigma$. Kahn shows that every $\alpha'_{r,d}(\lambda)$ is obtained as $F_\chi\otimes F_{\tau_{r,d}}\otimes F_\sigma$, and uses this to show that every vector bundle on $\widetilde{Y}\setminus E$ up to isomorphism is given as $\mathcal{F}_\rho$ for some representation ρ, [12, 6.16].

For any normal surface singularity (Y,y) there is an equivalence

$$\mathsf{Rep}_{\pi_1^{\mathrm{loc}}(Y,y)} \to \mathsf{Ref}_{Y,y}^{\nabla}$$

between the category of finite dimensional representations of the local fundamental group and the category of pairs (M,∇) where M is a reflexive $\mathcal{O}_{X,x}$-module and ∇ is an integrable connection. The correspondence is given by $\rho \mapsto ((i_*\mathcal{F}_\rho)^{\vee\vee}, \nabla_\rho)$ where $i : Y \setminus \{y\} \to Y$ is the inclusion and ∇_ρ is naturally constructed, see [10, 3.1] for details. Thus we have:

Theorem 2.4. — *Let (Y,y) be a simple elliptic surface singularity, and let N be a reflexive module on (Y,y). Then N admits an* integrable *connection*

$$\nabla : \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{Y,y}) \to \mathrm{End}_{\mathbb{C}}(N).$$

A surjective map of germs of normal surface singularities $\pi : (Y,y) \to (X,x)$ is a *Galois covering* if it is finite, étale on the complement $Y \setminus \{y\} \to X \setminus \{x\}$ (for some choice of representatives) and $G = G(Y/X) := \mathrm{Aut}((Y,x)/(X,y))$ acts freely on $Y \setminus \{y\}$. If π is a Galois covering, $\mathcal{O}_{Y,y}^G = \mathcal{O}_{X,x}$ (see also [10]). An $S = \mathcal{O}_{Y,y}$-module N has an action of $G = G(Y/X)$ *compatible* with the action of G on S, if $\sigma(sn) = \sigma(s)\sigma(n)$ for $\sigma \in G$.

The skew group ring $S[G]$ is the free S-module with the elements of G as basis and with multiplication given by $(s_1\sigma_1)(s_2\sigma_2) = s_1\sigma_1(s_2)\sigma_1\sigma_2$, where the s_i are in S and the σ_i are in G. It will be convenient to consider modules with compatible G-action as modules over the skew group ring, and we will denote the category of $S[G]$-modules that are reflexive as S-modules by $\mathsf{Ref}_{Y,y}^G$.

Proposition 2.5 ([10, 4.4]). — *Let $\pi : (Y,y) \to (X,x)$ be a Galois covering, with $G = G(Y/X)$. Let $R = \mathcal{O}_{X,x}$ and $S = \mathcal{O}_{Y,y}$. The rank preserving functor $F : \mathsf{Ref}_{Y,y}^G \to \mathsf{Ref}_{X,x}$ given by $F(N) = \pi_* N^G$ is an additive equivalence of categories, and an inverse equivalence is given by $F^{(-1)}(M) = (S \otimes_R M)^{\vee\vee}$.*

By Theorem 2.4, $M = F(N)$ has a connection ∇^N if (Y,y) is simple elliptic. A connection ∇^M on $M = N^G$ is given by

$$\nabla^N(D)(m) = \sum_{g \in G} g\nabla^N(D)(g^{-1}m)$$

for $D \in \mathrm{Der}_{\mathbb{C}}(S)^G = \mathrm{Der}_{\mathbb{C}}(R)$, see [26]. It is however not clear in general that this is an *integrable* connection, *i.e.*, it is not clear that it is a Lie-algebra homomorphism since it is not true in general that the quotient of a flat bundle by a finite group is flat, see [16]. That there exists an integrable connection on M is the content of Theorem 5.1.

3. Vector bundles on elliptic curves with group action

In order to give a detailed classification of the reflexive modules on elliptic quotients, it turns out that we need to classify vector bundles on an elliptic curve with group action, and in this section we do this by extending Atiyah's classification.

If G is a subgroup of the group of automorphisms of a ringed space $(Z, \mathcal{O}_Z)$, a compatible G-action on a sheaf of $\mathcal{O}_Z$-modules $\mathcal{F}$ is; for each $\sigma \in G$ a map $\sigma^{\#}_{\mathcal{F}} : \mathcal{F} \to \sigma_* \mathcal{F}$ which is $\mathcal{O}_Z$-linear via the morphism of sheaves of rings $\sigma^{\#} : \mathcal{O}_Z \to \sigma_* \mathcal{O}_Z$, *i.e.*, $\sigma^{\#}_{\mathcal{F}}(sf) = \sigma^{\#}(s)\sigma^{\#}_{\mathcal{F}}(f)$ for $s \in \mathcal{O}_Z(U)$ and $f \in \mathcal{F}(U)$ with the following two properties: The action has to be associative, *i.e.*, for all σ, $\tau \in G$, $(\tau\sigma)^{\#}_{\mathcal{F}} = (\tau_* \sigma^{\#}_{\mathcal{F}})\tau^{\#}_{\mathcal{F}}$, and the neutral element $e \in G$ has to act as the identity on $\mathcal{F}$. The adjoint isomorphism maps $\sigma^{\#}_{\mathcal{F}}$ to $\phi_\sigma \in \operatorname{Hom}_{\mathcal{O}_Z}(\sigma^* \mathcal{F}, \mathcal{F})$ and the associativity condition becomes $\phi_\sigma(\sigma^* \phi_\tau) = \phi_{\tau\sigma}$, *i.e.*, the diagram

$$\begin{array}{ccc} \sigma^*\tau^*\mathcal{F} & \xrightarrow{\sigma^*\phi_\tau} & \sigma^*\mathcal{F} \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\phi_\sigma} \\ (\tau\sigma)^*\mathcal{F} & \xrightarrow{\phi_{\tau\sigma}} & \mathcal{F} \end{array} \tag{3.1}$$

commutes, and $\phi_e = \mathrm{id}$. We will use this equivalent description of a group action on $\mathcal{F}$.

A map $f : (\mathcal{F}, \phi) \to (\mathcal{F}', \phi')$ of sheaves of $\mathcal{O}_Z$-modules with compatible G-action is a map $f : \mathcal{F} \to \mathcal{F}'$ of sheaves of $\mathcal{O}_Z$-modules commuting with the G-action; $f\phi_\sigma = \phi'_\sigma(\sigma^* f) : \sigma^*\mathcal{F} \to \mathcal{F}'$ for all $\sigma \in G$. Let $\mathsf{mod}^G_{\mathcal{O}_Z}$ denote the resulting category. Even though $\mathcal{O}_Z[G]$ in general is not even a presheaf of rings (only well defined for G-invariant open sets), we will for convenience call objects $\mathcal{F}$ of $\mathsf{mod}^G_{\mathcal{O}_Z}$ for $\mathcal{O}_Z[G]$-modules. The homological algebra in $\mathsf{mod}^G_{\mathcal{O}_Z}$ was first considered by Grothendieck in [9, Chap. V].

If H is a subgroup of G, there is a functor $I^G_H : \mathsf{mod}^H_{\mathcal{O}_Z} \to \mathsf{mod}^G_{\mathcal{O}_Z}$ defined by $I^G_H(\mathcal{F}) = \bigoplus_{[\sigma_i] \in H\backslash G} \sigma_i^* \mathcal{F}$ where the G-action is defined as follows: Given σ and i, there is a unique j and a unique $\tau \in H$ such that $\sigma_i\sigma = \tau\sigma_j$. Let ϕ_σ restricted to $\sigma^*(\sigma_i^*\mathcal{F})$ be the composition $\sigma^*(\sigma_i^*\mathcal{F}) \cong \sigma_j^*(\tau^*\mathcal{F}) \to \sigma_j^*\mathcal{F}$ where the last map is $\sigma_j^*(\phi^H_\tau)$.

Lemma 3.1. — *Suppose G is a subgroup of the group of automorphisms of a ringed space $(Z, \mathcal{O}_Z)$, H is a subgroup of G and $\mathcal{F}$ is a sheaf of $\mathcal{O}_Z$-modules. Let $a(H, \mathcal{F})$ denote the set of isomorphism classes of H-actions on $\mathcal{F}$.*

(a) *Let* $\{\sigma_i\}$ *be a set of representatives for* $H\backslash G$ *(with* $\sigma_1 = e$*) such that* $i \neq j$ *implies that* $\operatorname{Hom}_{\mathcal{O}_Z}(\sigma_i^*\mathcal{F}, \sigma_j^*\mathcal{F}) = 0$*, let* $I(\mathcal{F}) = \bigoplus \sigma_i^*\mathcal{F}$*, and suppose* $a(H,\mathcal{F}) \neq \emptyset$*. Then* I_H^G *induces a bijection* $a(H,\mathcal{F}) \xrightarrow{\simeq} a(G, I(\mathcal{F}))$*.*

(b) *If* $\mathcal{O}_Z$ *and the* H*-action on* Z *is* k*-linear for a field* k *and* $\operatorname{Aut}_{\mathcal{O}_Z}(\mathcal{F}) = k^\times$*, then* $a(H,\mathcal{F})$*, if non-empty, is a torsor for the character group* $\widehat{H} = \operatorname{Hom}(H, k^\times)$*.*

Proof. — For (a) let $\mathcal{F} \in \mathsf{mod}_{\mathcal{O}_Z}$ and suppose ϕ is a G-action on $I(\mathcal{F})$. By the assumptions in (a) $\phi_\tau : \tau^* I(\mathcal{F}) \to I(\mathcal{F})$ restricted to $\tau^*\mathcal{F}$ maps solely into the direct summand $\mathcal{F} = \sigma_1^*\mathcal{F}$ for all $\tau \in H$. Hence the G-action on $I(\mathcal{F})$ induces an H-action on $\mathcal{F}$, denoted $R_{\mathcal{F}}^H(I(\mathcal{F}),\phi)$. If $\mathcal{F} \in \mathsf{mod}_{\mathcal{O}_Z}^H$, clearly $R_{\mathcal{F}}^H(I_H^G(\mathcal{F})) = \mathcal{F}$.

Let $(I(\mathcal{F}),\phi^!) = I_H^G(R_{\mathcal{F}}^H(I(\mathcal{F}),\phi))$. By the assumptions in (a) $\phi_{\sigma_k} : \sigma_k^* I(\mathcal{F}) \to I(\mathcal{F})$ restricted to $\sigma_k^*\mathcal{F}$ maps solely into the direct summand $\sigma_k^*\mathcal{F}$. Let $\zeta_k \in \operatorname{Aut}_{\mathcal{O}_Z}(\sigma_k^*\mathcal{F})$ denote this restricted map. For $\phi^!$ the corresponding automorphism is the identity. Let $f \in \operatorname{Aut}_{\mathcal{O}_Z}(I(\mathcal{F}))$ be defined by $\xi_k = \zeta_k^{-1}$ on the summand $\sigma_k^*\mathcal{F}$. We claim that f induces a natural isomorphism $f : (I(\mathcal{F}),\phi) \xrightarrow{\simeq} (I(\mathcal{F}),\phi^!)$. We have to show that f commutes with ϕ and $\phi^!$, *i.e.*, $\xi\phi_\sigma = \phi^!_\sigma \sigma^*\xi$ for all $\sigma \in G$. Now $\sigma = \tau\sigma_j$ for unique j and $\tau \in H$. By associativity $\phi_\sigma = \phi_{\sigma_j} \circ \sigma_j^*\phi_\tau$, and since $\phi_\tau = \phi^!_\tau$, it is sufficient to show commutativity for $\sigma = \sigma_j$. Restricting ϕ_{σ_j} to $\sigma_j^*(\sigma_i^*\mathcal{F})$ gives $\phi_{\sigma_j} : \sigma_j^*(\sigma_i^*\mathcal{F}) \to \sigma_k^*\mathcal{F}$ where $\sigma_i\sigma_j = \tau\sigma_k$ for $\tau \in H$. Consider the following diagram of isomorphisms:

$$
\begin{array}{ccccccc}
\sigma_j^*(\sigma_i^*\mathcal{F}) & \xrightarrow{\sigma_j^*(\zeta_i)} & \sigma_j^*(\sigma_i^*\mathcal{F}) & \xrightarrow{\sigma_j^*(\xi_i)} & \sigma_j^*(\sigma_i^*\mathcal{F}) & \xleftarrow{=} & \sigma_k^*(\tau^*\mathcal{F}) \\
\downarrow{\scriptstyle \sigma_k^*\phi_\tau} & & \downarrow{\scriptstyle \phi_{\sigma_j}} & & \downarrow{\scriptstyle \phi^!_{\sigma_j}} & & \downarrow{\scriptstyle \sigma_k^*\phi_\tau} \\
\sigma_k^*\mathcal{F} & \xrightarrow[\phi_{\sigma_k}=\zeta_k]{} & \sigma_k^*\mathcal{F} & \xrightarrow[\xi_k]{} & \sigma_k^*\mathcal{F} & \xleftarrow[\phi^!_{\sigma_k}=\mathrm{id}]{} & \sigma_k^*\mathcal{F}
\end{array}
\tag{3.2}
$$

Since $\sigma_j^*(\sigma_i^*\mathcal{F}) = \sigma_k^*(\tau^*\mathcal{F})$ the left hand square commutes by the associativity condition. The right hand square commutes by definition of $\phi^!$. By inspection it follows that the central square commutes (identify the two outer vertical arrows).

For (b) let $\mathcal{F}$ be an $\mathcal{O}_Z$-module which admits an H-action. Then the character group $\widehat{H}$ acts on the set of isomorphism classes of H-actions on $\mathcal{F}$, *i.e.*, if $\phi_\tau : \tau^*\mathcal{F} \to \mathcal{F}$, $\tau \in H$, defines an action on $\mathcal{F}$, and $\chi \in \widehat{H}$, then $(\chi\phi)_\tau = \chi(\tau)\phi_\tau : \tau^*\mathcal{F} \to \mathcal{F}$ also defines an H-action $\mathcal{F}_\chi$ on $\mathcal{F}$. The action of $\widehat{H}$ is effective: An isomorphism $f : \mathcal{F} \cong \mathcal{F}_\chi$ satisfies $f\phi_\tau = \chi(\tau)\phi_\tau\tau^* f$, and since $f \in \operatorname{Aut}(\mathcal{F}) = k^\times$, $\chi(\tau) = 1$ for all $\tau \in H$. The action is also transitive:

Given two H-actions ϕ and ψ on $\mathcal{F}$, then $\chi(\tau) := \psi_\tau \phi_\tau^{-1} \in \operatorname{Aut}(\mathcal{F}) = k^\times$, and $\chi : H \to k^\times$ is a character by the associativity and identity conditions on ϕ and ψ. □

Example 3.2. — Let Z be a smooth projective (holomorphic) variety with G a subgroup of the automorphism group of Z and L a line bundle on Z. Let $H = G_{[L]}$, the isotropy group of $[L] \in \operatorname{Pic} Z$. Then $\mathcal{F} = L$ satisfies the conditions in Lemma 3.1 (a) and (b).

Remark 3.3. — In general it is not sufficient to have isomorphisms $\theta_\tau : \tau^*\mathcal{F} \cong \mathcal{F}$ for all $\tau \in H$ to define an H-action on $\mathcal{F}$ (and hence a G-action on $I(\mathcal{F})$). Suppose $\operatorname{Aut}_{\mathcal{O}_Z}(\mathcal{F}) = k^\times$. The commutativity defect in (3.1) gives a cocycle $c : H \times H \to k^\times$ with $c(\tau, \sigma) = \theta_{\tau\sigma}(\theta_\sigma \sigma^* \theta_\tau)^{-1}$, and hence a cohomology class $[\mathcal{F}] \in \mathrm{H}^2(H; k^\times)$. D. Ploog shows that $[\mathcal{F}] = 0$ if and only if $\mathcal{F}$ has an H-action. (Our Lemma 3.1 (b) is also contained in [23, Lemma 1].) In the case H is cyclic of order h, $\mathrm{H}^2(H; k^\times) = k^\times/(k^\times)^h$. If σ generates H, the only non-trivial relation is $\sigma^h = e$, so consider $(\sigma^h)^*\theta_\sigma \circ \cdots \circ \sigma^*\theta_\sigma = \zeta \in k^\times$. If there is a $\xi \in k^\times$ with $\xi^h = \zeta$ then $\phi_\sigma := \xi^{-1}\theta_\sigma$ and iterating defines an H-action on $\mathcal{F}$.

Atiyah [1] classified vector bundles on an elliptic curve E (depending on a choice of neutral element $P_0 \in E$ for the group law) by a two step procedure which gives a one-to-one correspondence between the set $\mathcal{B}_{\mathrm{ind}}(r, d)$ of isomorphism classes of indecomposable vector bundles of rank r and degree d and $\operatorname{Pic}^0 E$. For the first step, consider an indecomposable vector bundle F of rank r and degree d. If $0 < d < r$, then the natural map $\mu : \mathcal{O}_E \otimes \mathrm{H}^0(F) \to F$ is injective and the cokernel F' is an indecomposable vector bundle of rank $r - d$ and degree d. The map $F \mapsto F'$ sets up a bijection $\mathcal{B}_{\mathrm{ind}}(r, d) \cong \mathcal{B}_{\mathrm{ind}}(r - d, d)$. Twisting F by $\mathcal{O}_E(P_0)$ gives a bijection $\mathcal{B}_{\mathrm{ind}}(r, d) \cong \mathcal{B}_{\mathrm{ind}}(r, d + r)$. Combined, the bijections allow for an Euclidean algorithm which gives a bijection $\mathcal{B}_{\mathrm{ind}}(r, d) \cong \mathcal{B}_{\mathrm{ind}}(h, 0)$ where $h = (r, d)$. For the second step consider the indecomposable vector bundles $\mathcal{F}_r$ of rank r and degree 0 inductively defined by the extensions $\mathcal{O}_E \to \mathcal{F}_r \to \mathcal{F}_{r-1}$ with $\mathcal{F}_1 = \mathcal{O}_E$. Tensorisation with $\mathcal{F}_r$ gives a bijection $\operatorname{Pic}^0 E \cong \mathcal{B}_{\mathrm{ind}}(r, 0)$. The Atiyah map $\alpha_{r,d} : \operatorname{Pic}^0 E \to \mathcal{B}_{\mathrm{ind}}(r, d)$ is the induced map.

Let $G \subseteq \operatorname{Aut}(E)$ be a subgroup which we will assume fixes P_0. It is well-known that G is a finite cyclic group. We now classify all vector bundles F on E with a compatible G-action by extending Atiyah's classification. Suppose $F = \bigoplus_{i=1}^n F_i$, with F_i indecomposable as $\mathcal{O}_E$-module. If F is indecomposable as an $\mathcal{O}_E[G]$-module, we may assume that $\sigma^* F_i \cong F_{i+1}$ where σ is a generator for G. Hence the rank and degree of F_i is constant for all i. Let

$\mathcal{B}(r,d;n)$ (respectively $\mathcal{B}^G(r,d;n)$) denote the set of isomorphism classes of $\mathcal{O}_E$-modules (respectively $\mathcal{O}_E[G]$-modules) F such that $F \cong \bigoplus_{i=1}^n F_i$ as $\mathcal{O}_E$-modules, where F_i is an indecomposable vector bundle of rank r and degree d for all i. Clearly G acts on $\mathcal{B}(r,d;n)$ by pullback. Let $\mathcal{B}^G_{\mathrm{ind}}(r,d;n)$ denote the set of isomorphism classes of indecomposable $\mathcal{O}_E[G]$-modules in $\mathcal{B}^G(r,d;n)$. Forgetting the G-action gives a map $\mathcal{B}^G(r,d;n) \to \mathcal{B}(r,d;n)$. Since P_0 is fixed by G, the inclusion of the ideal sheaf $\mathcal{O}_E(-P_0) \subseteq \mathcal{O}_E$ is G-equivariant for any G-action on $\mathcal{O}_E$. The canonical action on $\mathcal{O}_E$ hence induces a canonical action on the dual $\mathcal{O}_E(P_0)$. Let C_n denote the cyclic group of order n.

Proposition 3.4. — *Suppose E is an elliptic curve with a faithful action of a group G with a fixed point P_0. Let $|G| = m$. For all $r, n > 0$ and any d the following holds:*

(i) *The Atiyah map induces a G-equivariant bijective map $\mathcal{B}(r,d;n) \to \mathcal{B}(1,0;n)$.*

(ii) *There is a canonical bijective map $\mathcal{B}^G_{\mathrm{ind}}(r,d;n) \to \mathcal{B}^G_{\mathrm{ind}}(1,0;n)$ commuting with the map in (i).*

(iii) *Each non-empty fiber of the forgetful map $\mathcal{B}^G_{\mathrm{ind}}(r,d;n) \to \mathcal{B}(r,d;n)^G \cong \mathcal{B}(1,0;n)^G$ is a torsor over the character group of the stabilizer subgroup $G_P \cong C_{m/n}$ for some $P \in E$. Moreover; if F is a vector bundle with G-action which is indecomposable in $\mathsf{mod}^G_{\mathcal{O}_E}$, then its isomorphism class is contained in $\mathcal{B}^G_{\mathrm{ind}}(r,d;n)$ for some $n|m$.*

Proof. — The map in (i) is given by applying the Atiyah map to the summands. Clearly $F \mapsto F \otimes \mathcal{O}_E(P_0) = F(P_0)$ commutes with pullback along $\sigma \in G$ since in general $\sigma^*\mathcal{O}_E(D) = \mathcal{O}_E(\sigma^{-1}(D))$ and P_0 is G-fixed, and hence $\mathcal{B}(r,d;n) \cong \mathcal{B}(r,d+r;n)$ G-equivariantly. With the canonical G-action on $\mathcal{O}_E(P_0)$, a G-action on F naturally induces a G-action on $F(P_0)$ and so twisting gives bijections $\mathcal{B}^G(r,d;n) \to \mathcal{B}^G(r,d+r;n)$. Likewise the map $F \mapsto F' = \operatorname{coker}\mu$ with $\mu : \mathcal{O}_E \otimes \mathrm{H}^0(F) \to F$ gives an equivariant map $\mathcal{B}(r,d;n) \to \mathcal{B}(r-d,d;n)$ in the case $0 < d < r$ and so $\mathcal{B}(r,d;n) \cong \mathcal{B}(h,0;n)$ G-equivariantly where $h = (r,d)$. Moreover, if F has a G-action, the induced action on the global sections $\mathrm{H}^0(F)$ tensorized with the natural G-action on $\mathcal{O}_E$ makes μ $\mathcal{O}_E[G]$-linear and so induces a G-action on F'. For the converse, given a G-action on a vector bundle F' of degree $d > 0$ and rank r'. Then the universal extension

$$0 \to \mathcal{O}_E \otimes \mathrm{Ext}^1_{\mathcal{O}_E}(F', \mathcal{O}_E)^\vee \to F \to F' \to 0 \tag{3.3}$$

is a G-sequence since it corresponds to the identity under the natural isomorphisms

$$\begin{aligned}\mathrm{id} \in \mathrm{End}_{\mathbb{C}}(\mathrm{H}^0(F'))^G &\cong \mathrm{End}_{\mathbb{C}}(\mathrm{H}^1(F'^{\vee})^{\vee})^G \\ &\cong \mathrm{H}^1(F'^{\vee} \otimes \mathrm{H}^1(F'^{\vee})^{\vee})^G \cong \mathrm{Ext}^1_{\mathcal{O}_E[G]}(F', \mathcal{O}_E \otimes \mathrm{Ext}^1_{\mathcal{O}_E}(F', \mathcal{O})^{\vee}).\end{aligned}$$

It follows from [1, Lemma 16] that this is the inverse operation to $F \mapsto \operatorname{coker} \mu$ and so $\mathcal{B}^G(r,d;n) \cong \mathcal{B}^G(h,0;n)$ for $h = (r,d)$.

The last argument also applies to the sheaves $\mathcal{F}_r$ of degree 0. Hence there is a standard G-action on the $\mathcal{F}_r$ induced by the standard action on $\mathcal{O}_E$ such that the defining sequence $\mathcal{O}_E \xrightarrow{\mu_r} \mathcal{F}_r \to \mathcal{F}_{r-1}$ is a short exact sequence of $\mathcal{O}_E[G]$-modules. In particular we have proved (i) since the isomorphism $\sigma^* \mathcal{F}_r \cong \mathcal{F}_r$ for $\sigma \in G$ implies that the map $\mathrm{Pic}^0 E \ni L \mapsto L \otimes \mathcal{F}_r$ commutes with pullback along σ.

We define maps $\mathcal{B}^G_{\mathrm{ind}}(r,0;n) \to \mathcal{B}^G_{\mathrm{ind}}(r-1,0;n)$: Let $F = \bigoplus F_i \in \mathcal{B}^G_{\mathrm{ind}}(r,0;n)$. We may assume $F_i = L_i \otimes \mathcal{F}_r$ with $\phi_\sigma : \sigma^* F_{i-1} \cong F_i$ which implies that $\sigma^* L_{i-1} \cong L_i$ (by [1, Lemma 21] $\mathcal{E}nd_{\mathcal{O}_E}(\mathcal{F}_r) \cong \bigoplus_{j \geqslant 1} \mathcal{F}_{2(r-j)+1}$ and $\mathrm{H}^0(L \otimes \mathcal{F}_r) = 0$ for $\mathrm{Pic}^0 E \ni L \ncong \mathcal{O}_E$). Since E is an elliptic curve, $L_0 \cong \mathcal{O}_E(P - P_0)$ for some $P \in E$, we may hence assume $L_i = \mathcal{O}_E(\sigma^{-i}(P) - P_0)$ for all i. The G-action, given by $\phi_\sigma : \sigma^* F \cong F$, induces a G-action on $\bigoplus L_i$ given by $\mathrm{id} \otimes \mathrm{H}^0(\phi_\sigma \otimes \mathrm{id}) : \sigma^* L_{i-1} \otimes \mathrm{H}^0(\sigma^* F_{i-1} \otimes L_i^{\vee}) \cong L_i \otimes \mathrm{H}^0(F_i \otimes L_i^{\vee})$. The natural map $\mu : \bigoplus L_i \otimes \mathrm{H}^0(F_i \otimes L_i^{\vee}) \to \bigoplus F_i$ is G-equivariant, and we get an induced G-action on the cokernel $F' = \operatorname{coker} \mu \cong \bigoplus L_i \otimes \mathcal{F}_{r-1}$. For the inverse operation, a G-action on $F' = \bigoplus L_i \otimes \mathcal{F}_{r-1}$ induces by the same argument a G-action on $\bigoplus L_i$. With these actions we find

$$\begin{aligned}&\mathrm{Ext}^1_{\mathcal{O}_E[G]}(\oplus L_i \otimes \mathcal{F}_{r-1}, \oplus L_j)^{\vee} \cong (\mathrm{H}^1(\oplus L_i^{\vee} \otimes L_j \otimes \mathcal{F}_{r-1}^{\vee})^{\vee})^G \\ &\cong (\oplus_{i=1}^n \mathrm{H}^0(\mathcal{F}_{r-1}))^G \cong (\mathbb{C}^n)^G \cong \mathbb{C},\end{aligned}$$

where the action on $\mathbb{C}^n$ is given by cyclic permutation. The corresponding extension gives the short exact G-sequence $\bigoplus L_i \to \bigoplus L_i \otimes \mathcal{F}_r \to \bigoplus L_i \otimes \mathcal{F}_{r-1}$. Hence $\mathcal{B}^G_{\mathrm{ind}}(r,0;n) \to \mathcal{B}^G_{\mathrm{ind}}(r-1,0;n)$ is bijective. The composition of these gives the inverse of $\mathcal{B}^G_{\mathrm{ind}}(1,0;n) \to \mathcal{B}^G_{\mathrm{ind}}(r,0;n)$ defined by tensorisation with $\mathcal{F}_r$ (with its standard action).

By (i) and (ii) we have reduced (iii) to the case $r = 1$ and $d = 0$ and Lemma 3.1 gives the rest of the statement. □

Remark 3.5. — Related results can be found in [17] and [24].

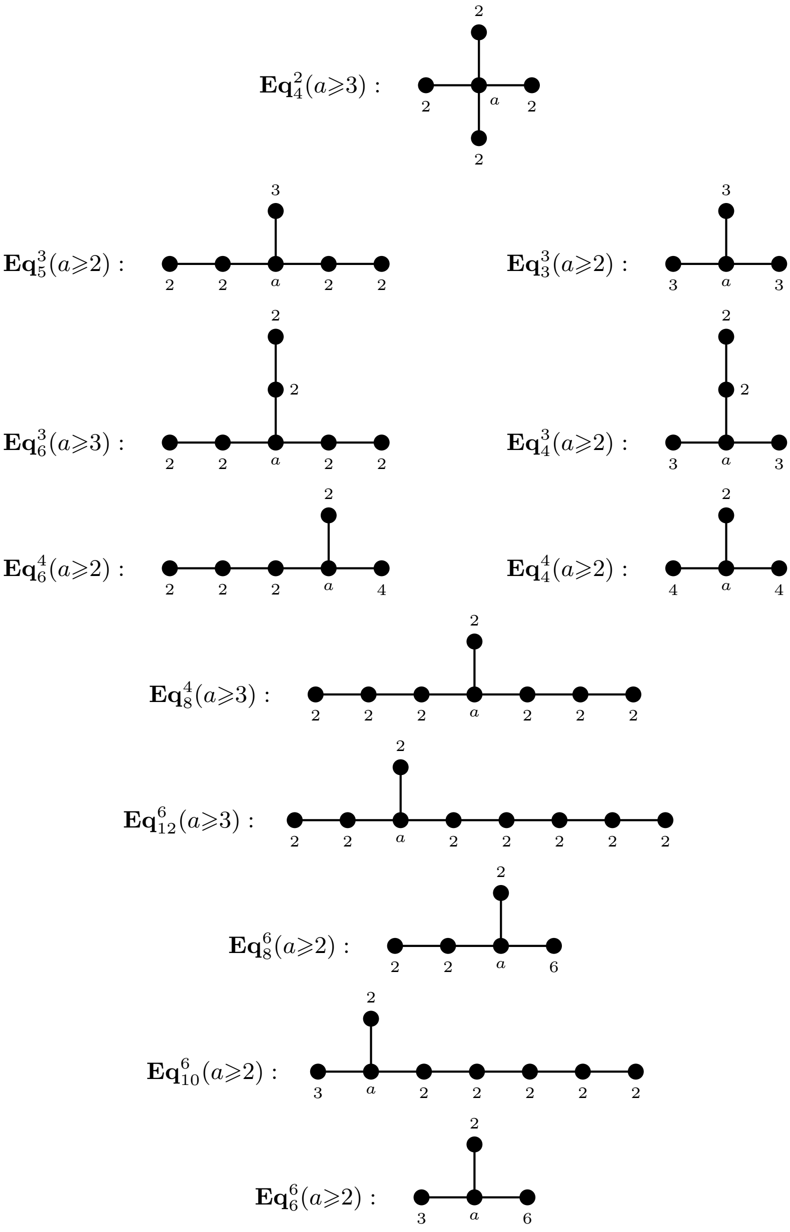


Figure 3.1. Classification of elliptic quotients.

4. Classification of reflexive modules on elliptic quotients

In this section we classify the reflexive modules on an elliptic quotient. The elliptic quotients constitute an large subclass of the log-canonical surface singularities.

Let $\omega = \omega_{X,x}$ be the canonical module on a log-canonical surface singularity (X,x). Then $\omega^{[m]} := (\omega^{\otimes m})^{\vee\vee} \cong \mathcal{O}_{X,x}$, for some m, and the *log-index* is the smallest positive integer m such that $\omega^{[m]} \cong \mathcal{O}_{X,x}$. The *canonical covering* Y of X is defined as $Y = \operatorname{Specan}(\oplus_{i=0}^{m-1}(\omega^{\otimes i})^\vee)$, see [28, 14], and also [20]. Here $\operatorname{Specan}(\oplus_{i=0}^{m-1}(\omega^{\otimes i})^\vee)$ denotes the analytic space associated to the sheaf $\oplus_{i=0}^{m-1}(\omega^{\otimes i})^\vee$ of $\mathcal{O}_X$ algebras, see II.§ 3 in [7]. We have that Y is a normal surface singularity.

We say that (X,x) is an *elliptic quotient* if the canonical covering is simple elliptic and if (X,x) is not a quotient singularity. In particular (X,x) is the quotient of the canonical covering (Y,y) under the covering group G. In fact, the action of the group is induced by an action on the associated elliptic curve E and on a line bundle $\mathcal{L}$. We will need this stronger characterization which follows from the next (essentially well known) theorem. See [22] and [14].

Theorem 4.1. — *Let (X,x) be an elliptic quotient with log-index m, canonical covering Y and let $G \cong C_m$ be the covering group of Y over X. Let $p : (\widetilde{Y}, E) \to (Y,y)$ be the minimal (good) resolution of Y, and let $\mathcal{L}$ be the conormal bundle of E in $\widetilde{Y}$, and let $b = -\deg_E \mathcal{L}$. Then the following holds:*

(1) *Y is simple elliptic of type $\mathrm{El}(b)$, the action of G lifts to $\widetilde{Y}$ and induces an action on the elliptic curve E and on the line bundle $\mathcal{L}$ on E.*
(2) *There is a G-equivariant isomorphism $(Y,y) \cong (Y',0)$, where*

$$Y' = \operatorname{Specan}(\oplus_{i\geqslant 0} \mathrm{H}^0(E, \mathcal{L}^i))$$

with the natural action induced from the action on $\mathcal{L}$ over E.
(3) *The log-index $m = 1,2,3,4$ or 6. The action of G on E is non-free at $3,4$ or 6 points $Q_i^{(j)}$ with orbits of length $\mathbf{d} = (d_1, \dots, d_n)$. The Seifert partial resolution $\overline{X} \cong \widetilde{Y}/G$ has n cyclic quotient singularities of type (d_i, e_i). Let $c = m\sum e_i/d_i$ and let a be determined by $ma = b + c$. The dual graph of (X,x) is $\mathbf{Eq}_c^m(a)$ in Figure 3.1. The corresponding values of m, n, $\mathbf{d}$ and $\mathbf{e}$ are given in the following table.*

m		*Graph*	*Exceptional orbits*	n	$\mathbf{d},\mathbf{e}$
1	*elliptic*	$\mathrm{El}(b)$	*None*	0	
2	*rational*	$\mathbf{Eq}_4^2(a)$	$\{Q_1\},\{Q_2\},\{Q_3\},\{Q_4\}$	4	$(2,2,2,2),(1,1,1,1)$
3	*rational*	$\mathbf{Eq}_c^3(a)$	$\{Q_1\},\{Q_2\},\{Q_3\}$	3	$(3,3,3),$ $(2,2,2)$ $(2,2,1)$ $(2,1,1)$ $(1,1,1)$
4	*rational*	$\mathbf{Eq}_c^4(a)$	$\{Q_1\},\{Q_2\},\{Q_3,Q_3'\}$	3	$(4,4,2),$ $(3,3,1)$ $(3,1,1)$ $(1,1,1)$
6	*rational*	$\mathbf{Eq}_c^6(a)$	$\{Q_1\},\{Q_2,Q_2'\},\{Q_3,Q_3',Q_3''\}$	3	$(6,3,2),$ $(5,2,1)$ $(5,1,1)$ $(1,2,1)$ $(1,1,1)$

Proof. — The theorem follows from [14, Th. 9.6.(3)] and [22]. To introduce notation necessary in the (classification) Theorem 4.7 and for completeness, we sketch the proof.

The action on (Y,y) by G, extends to an action on the minimal resolution $(\widetilde{Y},E)$, and hence on the conormal bundle $\mathcal{L}$ of E in $\widetilde{Y}$. The quotient of $\widetilde{Y}$ under this action, is the so-called Seifert partial resolution $\overline{X}$ of X, and restricting the map $\widetilde{\pi}:\widetilde{Y}\to\overline{X}$, we get a ramified covering $q:E\to\mathbb{P}^1$ with covering group G. There is a fixed point for the action of the group on E, see page 143 in [14]. It thus follows that G acts as stated. The Seifert partial resolution $\overline{X}$ has one cyclic quotient singularity P_i for each exceptional orbit $\{Q_i^{(j)}\}_j\subset E$ where the action of G is not free. The type of P_i is uniquely determined by the action of G on $\mathcal{L}$, since locally on $\widetilde{Y}$ the action of G may be linearized. Let d_i be the ramification index of $Q_i^{(j)}$. Then P_i is of type (d_i,e_i) for some e_i with $1\leqslant e_i<d_i$ and e_i and d_i relatively prime. By Hurwitz's $\sum_{i=1}^n\frac{d_i-1}{d_i}=2$. Using these restrictions and that the dual graphs of cyclic quotients are determined by developing $\frac{d_i}{e_i}$ as a continued fraction, one derive the graphs in Figure 3.1.

On the other hand, given an elliptic quotient (X,x) with a dual graph as in Figure 3.1, there is a central curve C in the exceptional set of the minimal resolution $\widetilde{X}\to X$. Let P_i be the intersection points with the other exceptional components. Define $\frac{d_i}{e_i}$ as the continued fraction $[b_{k_1},\dots,b_{k_{r_i}}]$ where b_{k_j} are the self intersection numbers of the exceptional components in the chain intersecting C in P_i. Then there is a unique ramified covering $q':E'\to C=\mathbb{P}^1$ such that the ramification index of any point $Q_i^{(j)}$ over P_i is d_i and such that the order of the covering group is m as in the theorem. Let D be a divisor on C of degree a, and let

$$D'=(q')^{-1}(D)-\sum_{i=1}^n\sum_j e_iQ_i^{(j)}.$$

Let $\mathcal{L}' = \mathcal{O}_{E'}(D')$. It follows from [22] that (X, x) is the quotient of

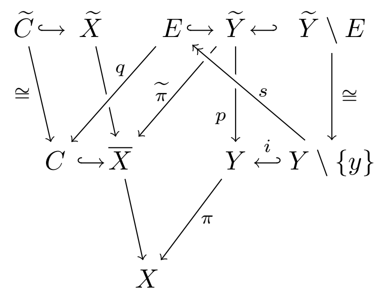
$$Y' = \operatorname{Specan}(\oplus_{i \geqslant 0} \mathrm{H}^0(E', (\mathcal{L}')^i))$$

by the natural action of G. Denote again by Y the canonical cover of X, and let E, $\mathcal{L}$ and π be as defined above. Since E' is uniquely determined by the P_i and $\mathbf{d}$, it follows that $E = E'$. The analytic type of a simple elliptic surface singularity depends only on the isomorphism class of the exceptional elliptic curve and the degree of the normal bundle since two line bundles of the same degree differs by a translation, *cf.* [25]. One checks that $\deg \mathcal{L} = \deg\, \mathcal{L}'$. It follows that $(Y, y) \cong (Y', 0)$, and from Proposition 2.5 that there is a G-equivariant isomorphism $(Y, y) \cong (Y', 0)$. □

Remark 4.2. — From Theorem 9.6 in [14] it follows that any normal surface singularity with a dual graph as in Figure 3.1 is an elliptic quotient.

Notation 4.3. — The notation in Theorem 4.1 and its proof is fixed and will be used freely for the rest of this article. In particular, we identify (Y, y) with $(Y', 0)$. The natural action of $\mathbb{C}^*$ on $\mathcal{L}$, makes $\widetilde{Y} \setminus E \to E$ into a principal $\mathbb{C}^*$-bundle, and we let s be the composition of the $\mathbb{C}^*$-equivariant isomorphism $\widetilde{Y} \setminus E \cong Y \setminus \{y\}$ and the bundle map. Let $\pi : Y \to X$ be the quotient map and choose a fixed point $P_0 \in E$ for the G-action. Put $S = \mathcal{O}_{Y,y}$ and $R = \mathcal{O}_{X,x}$.

The situation is summarized in the following diagram:



There is $\mathbb{C}^*$-action on X so that the quotient map π is $\mathbb{C}^*$-equivariant, and s^* induces an equivalence of categories between the category Vect_E of vector bundles on E and the category $\mathsf{Vect}^{\mathbb{C}^*}_{Y \setminus \{y\}}$ of vector bundles on $Y \setminus \{y\}$ with $\mathbb{C}^*$-action. If $i : Y \setminus \{y\} \to Y$ is the inclusion, i_* gives an equivalence of the category $\mathsf{Vect}^{\mathbb{C}^*}_{Y \setminus \{y\}}$ with the category $\mathsf{Ref}^{\mathbb{C}^*}_Y$ of reflexive sheaves on Y with $\mathbb{C}^*$-action. In fact, the $\mathbb{C}^*$-action implies that $\mathcal{N} = i_* \mathcal{E}$ is coherent for any $\mathcal{E}$ in $\mathsf{Vect}^{\mathbb{C}^*}_{Y \setminus \{y\}}$, and $\mathcal{N}^{\vee\vee} \cong \mathcal{H}om(\mathcal{H}om(i_* \mathcal{E}, i_* \mathcal{O}_{Y \setminus \{y\}}), i_* \mathcal{O}_{Y \setminus \{y\}}) \cong$

$i_*\mathcal{E} \cong \mathcal{N}$. Composing, we have an equivalence $H : \mathsf{Vect}_E \to \mathsf{Ref}_Y^{\mathbb{C}^*}$ given by $H(\mathcal{F}) = i_* s^* \mathcal{F}$. An inverse equivalence is given by $F(\mathcal{N}) = (s_* \mathcal{N}_{|Y\setminus\{y\}})^{\mathbb{C}^*}$.

Proposition 4.4. — *The functor F induces an equivalence*

$$F_y^G : \mathsf{Ref}_{Y,y}^{G\times\mathbb{C}^*} \to \mathsf{Vect}_E^G$$

between the category of $S[G\times\mathbb{C}^*]$-modules reflexive as S-modules and the category of vector bundles on E with G-action. An inverse equivalence H_y^G is given by $H_y^G(\mathcal{F}) = (i_* s^* \mathcal{F})_y$.

Proof. — If $N_1 \to N_2$ is a morphism in $\mathsf{Ref}_{Y,y}^{G\times\mathbb{C}^*}$ it is possible to choose representatives $\mathcal{N}_1$ and $\mathcal{N}_2$ so that the map $\mathcal{N}_1 \to \mathcal{N}_2$ is $G\times\mathbb{C}^*$-equivariant. We define $F_y^G(N_1 \to N_2) := F(\mathcal{N}_1 \to \mathcal{N}_2)$. Since s is $G\times\mathbb{C}^*$-equivariant, it follows that $F(\mathcal{N}_1 \to \mathcal{N}_2)$ is G-equivariant. Likewise, if $\mathcal{F}_1 \to \mathcal{F}_2$ is in Vect_E^G, it follows that $(i_* s^*(\mathcal{F}_1 \to \mathcal{F}_2))_y$ is $G\times\mathbb{C}^*$-equivariant. Since F and H are inverse equivalences, there are natural isomorphisms $F_y^G(H_y^G(\mathcal{F})) \cong \mathcal{F}$ of $\mathcal{O}_E$-modules and $H_y^G(F_y^G(N)) \cong N$ of $S[\mathbb{C}^*]$-modules. It follows by naturality that these isomorphisms respect the G-structures so that F_y^G and H_y^G are inverse equivalences. □

Theorem 4.5. — *A reflexive module on an elliptic quotient (X,x) is gradable. Equivalently; every $S[G]$-module N (reflexive as an S-module) is induced from an $S[\mathbb{C}^*\times G]$-module.*

Proof. — Let M be a reflexive R-module. We may assume that M is indecomposable, hence $N := (M\otimes_R S)^{\vee\vee}$ is an indecomposable $S[G]$-module, see Proposition 2.5. We first claim that N admits a $\mathbb{C}^*$-action and a (possibly different) G-action compatible with the $\mathbb{C}^*$-action, *i.e.*, that N admits an $S[\mathbb{C}^*\times G]$-structure.

Let $p : (\widetilde{Y}, E) \to (Y,y)$ be the minimal resolution. Then the action of G on Y extends to an action of G on $\widetilde{Y}$ such that p is G-equivariant. If $\mathcal{N}$ is a representative of N, we get that $\widetilde{\mathcal{N}} = (p^*\mathcal{N})^{\vee\vee}$ is an $\mathcal{O}_{\widetilde{Y}}[G]$-module which is locally free as $\mathcal{O}_{\widetilde{Y}}$-module. Since E is a G-subspace of $\widetilde{Y}$, we get that $\mathcal{O}_E \otimes_{\mathcal{O}_{\widetilde{Y}}} \widetilde{\mathcal{N}}$ is a vector bundle on E which is an $\mathcal{O}_E[G]$-module. Assume first that N is an indecomposable S-module. From Theorem 2.1 it follows that $\mathcal{O}_E \otimes_{\mathcal{O}_{\widetilde{Y}}} \widetilde{\mathcal{N}} = \alpha'_{r,d}(\lambda)$ for some $\lambda \in \mathrm{Pic}^0(E)$. Thus we have that $\alpha'_{r,d}(\lambda)$ admits an action of G. We claim that this implies that $\alpha_{r,d}(\lambda)$ also admits an action of G. In the case where $\alpha'_{r,d}(\lambda) = \alpha_{r,d}(\lambda)$ this is clear. In the other cases $\alpha'_{r,d}(\lambda) = \mathcal{O}_E^n \oplus \mathcal{G}$, with $\mathcal{G}$ indecomposable, then $\mathrm{H}^0(\mathcal{G}^\vee) = 0$, and this implies that the restriction of an isomorphism ϕ_σ :

$\sigma^*(\mathcal{O}_E \otimes_{\mathcal{O}_{\widetilde{Y}}} \widetilde{\mathcal{N}}) \cong \mathcal{O}_E \otimes_{\mathcal{O}_{\widetilde{Y}}} \widetilde{\mathcal{N}}$ to the non-free summands, maps into the non-free summands. Thus $\mathcal{G}$ is an $\mathcal{O}_E[G]$-submodule. From Proposition 3.4, we get that λ admits a G-action. From Proposition 3.4 again, $\alpha_{r,d}(\lambda)$ admits a G-action. With $\mathcal{F} = \alpha_{r,d}(\lambda)$ we get from Proposition 2.2 that $\mathcal{N} = i_* s^* \mathcal{F}$ where $s : Y \setminus \{y\} \to E$. From Proposition 4.4, it follows that N is an $S[\mathbb{C}^* \times G]$-module.

If $\mathcal{N} = \oplus \mathcal{N}_i$ (and $N = \oplus N_i$ correspondingly) as $\mathcal{O}_Y$-modules then $F(\mathcal{N}_i) = (s_* \mathcal{N}_{i|Y \setminus \{y\}})^{\mathbb{C}^*}$, all have the same rank and degree. From Theorem 2.1 it follows that $\mathcal{O}_E \otimes_{\mathcal{O}_{\widetilde{Y}}} \widetilde{\mathcal{N}} = \mathcal{O}_E^n \oplus \mathcal{G}$ with $\mathrm{H}^0(\mathcal{G}^\vee) = 0$, and this again implies that the restriction of an isomorphism $\phi_\sigma : \sigma^*(\mathcal{O}_E \otimes_{\mathcal{O}_{\widetilde{Y}}} \widetilde{\mathcal{N}}) \cong \mathcal{O}_E \otimes_{\mathcal{O}_{\widetilde{Y}}} \widetilde{\mathcal{N}}$ to the non-free summands, maps into the non-free summands. From Proposition 3.4, we get also in this case, that there is a vector bundle $\mathcal{F}$ on E that admits a G-action, such that $N = H^G_y(\mathcal{F}) = (i_* s^* \mathcal{F})_y$. From Proposition 4.4, it follows that N is an $S[\mathbb{C}^* \times G]$-module.

We have shown that N admits a $\mathbb{C}^*$-action and a possibly different G-action that is compatible with the $\mathbb{C}^*$-action. But Lemma 4.6 below implies that all compatible G-actions on N commute with the $\mathbb{C}^*$-action if one of them does. □

If N is an $S[G]$-module and χ is a character of G, then multiplication by χ defines a new compatible G-action N_χ on N (*i.e.*, $\sigma(n)_{\mathrm{new}} := \chi(\sigma)\sigma(n)_{\mathrm{old}}$, $\sigma \in G, n \in N$).

Lemma 4.6. — *Assume N is an indecomposable reflexive S-module admitting a compatible G-action, and assume G is abelian. Then the action of the character group $\widehat{G}$, on the set of isomorphism classes of compatible G-actions on N, is transitive.*

Proof. — Let N_i, $i = 1, 2$, be $S[G]$-module structures on N, and $M_i := N_i^G$. By Proposition 2.5 $N_i \cong (S \otimes_R M_i)^{\vee\vee}$ as $S[G]$-modules. As R-modules $S \cong \oplus_{\chi \in \widehat{G}} S^G_\chi$ by [2, Cor. 4.7], hence $N \cong \oplus_{\chi \in \widehat{G}} \left(S^G_\chi \otimes_R M_i\right)^{\vee\vee}$. It follows from Proposition 2.5 that all $\left(S^G_\chi \otimes_R M_i\right)^{\vee\vee}$ are indecomposable and by Krull-Schmidt $M_2 \cong \left(S^G_\chi \otimes_R M_1\right)^{\vee\vee}$ for some χ. Hence by Proposition 2.5,

$$N_2 \cong (S \otimes_R S^G_\chi \otimes_R M_1)^{\vee\vee} \cong (S_\chi \otimes_R M_1)^{\vee\vee} \cong (S_\chi \otimes_S N_1)^{\vee\vee} = (N_1)_\chi.$$

□

Recall Notation 4.3, and denote by $s_i = s_i(X, x)$ the number of orbits of length i on E under the action of G. We have:

Theorem 4.7. — *Assume that (X, x) is an elliptic quotient that is not simple elliptic. If $\cup C_i$ is the exceptional divisor of $\widetilde{X} \to X$, let $h =$*

$|\det(C_iC_j)|$ and let Ψ be the set of intersection points of the central curve $\widetilde{C} \cong C$ with the other C_i. Then the following holds:

(i) *Every indecomposable reflexive R-module is given as $M = N^G$ where N is an (any) indecomposable $S[G]$-module, reflexive as an S-module. Every such N is given as:*

$$N \cong I_H^G(N') = \bigoplus_{\overline{\sigma} \in H\backslash G} \sigma^* N'$$

where N' is the $S[H]$-module $N' := i_* s^*(\alpha_{r',d'}(\mathcal{O}_E(P-P_0)_\chi))_y$ for some r' and d', H is the stabilizer of some $P \in E$ and $\chi : H \to \mathbb{C}^*$ is a character. Moreover, $M = N^G \cong (N')^H$ as R-modules.

Let $r = \operatorname{rank} M$. There are two possibilities:

(ii) *Either M is one of $\operatorname{ind}^r(X,x)$ isomorphism classes of isolated indecomposable reflexive modules of rank r where $rbm = \frac{rh}{s_1}$ and*

$$\operatorname{ind}^r(X,x) = \frac{rh}{s_1} \sum_{n|(r,m)}^{n \neq m} \frac{s_n}{n^2}\,.$$

This is the only possibility if $m \nmid r$, and in this case H is non-trivial.

(iii) *Or H is trivial, and M sits in one of $rb/m = rhm^{-2}s_1^{-1}$ flat families of non-isomorphic indecomposable reflexive modules parametrized by $C\backslash\Psi$, i.e., there is a coherent sheaf $\mathcal{M}$ of $\mathcal{O}_{X\times(C\backslash\Psi)}$-modules such that $(pr_2)_*\mathcal{M}$ is $\mathcal{O}_{C\backslash\Psi}$-flat, all fibers $\mathcal{M} \otimes k(t)$ are non-isomorphic and $\mathcal{M} \otimes k(t_0) \cong M$ for some $t_0 \in C \setminus \Psi$.*

Proof. — By Proposition 2.5 every indecomposable reflexive R-module is given as $M = N^G$. By Theorem 4.5 and Proposition 4.4 $N = (i_* s^* \mathcal{F})_y$ for some $\mathcal{F}$ with a G-action and by Proposition 3.4 and Lemma 3.3 it follows that $\mathcal{F} = \oplus_{\overline{\sigma}\in H\backslash G} \sigma^* \alpha_{r',d'}(\mathcal{O}_E(P-P_0)_\chi)$ for some r' and d'. Thus $N^G = (\oplus_{\overline{\sigma}\in H\backslash G}\sigma^*(N'))^G$. By definition, the $\sigma^* N'$ are all isomorphic as R-modules. Thus N^G is a summand of N'. Furthermore H acts on N', so N^G is a summand of $(N')^H$. Since $(N')^H$ and N^G have equal ranks as R-modules, they are isomorphic. This proves (i).

By Proposition 2.5 there is a one-to-one-correspondence between the set of isomorphism classes of indecomposable R-modules and the set of indecomposable $S[G]$-modules. We have that $\mathcal{B}(1,0;n) \cong \operatorname{Hilb}_E^n$. Let $U_n \subset \operatorname{Hilb}_E^n$ be the image of $\mathcal{B}_{\text{ind}}^G(r,d;n)$ by the G-Atiyah-map, see Proposition 3.4. Let $\widehat{H}$ be the character group of the subgroup $H = \langle\sigma^n\rangle$ of $G = \langle\sigma\rangle$ and let $\operatorname{Ind}^{n,r,G}(Y,y)$ be the set of indecomposable $S[G]$-modules that have rank r and n indecomposable S-summands. By Proposition 2.2, Theorem 4.5 and Proposition 4.4 there is an one-to-one-correspondence

$$\mathrm{Ind}^{n,r,G}(Y,y) \longleftrightarrow \{(d,\lambda,\chi) \in \mathbb{N} \times U_n \times \widehat{H} \mid r/n \leqslant d < (b+1)r/n\}$$

if $n \leqslant m$. When $n < m$, we have that the set U_n of n-orbits in E is finite, and one finds that $\mathrm{Ind}^{n,r,G}(Y,y)$ has $(r/n)b(m/n)s_n$ elements. In particular bms_1 gives the number of rank one reflexive modules on the rational singularity (X,x) which is also given as h, see [21] or [19].

Let $U = E$ minus the points with non-trivial stabilizer. In the case $n = m$ there is a map $U \to U_n \cong C \setminus \Psi$, $u \mapsto \{\sigma u\}_{\sigma\in G}$, which is unramified and m-to-one. There is a sheaf F on $E \times U$, flat over U, such that $F \otimes k(P) \cong \alpha_{r',d'}(\mathcal{O}_E(P - P_0))$. We define $\mathcal{F} = \oplus_{\sigma\in G}\sigma^*F$. The family is then given as $(\pi_*(i_*s^*\mathcal{F}))^G$ on $X \times U_n$. There are $rb/m = rhm^{-2}s_1^{-1}$ such families. For flatness, see comments on page 153 in [13]. □

5. Connections on reflexive modules over elliptic quotients

In this section we prove the following theorem.

Theorem 5.1. — *Let (X,x) be an elliptic quotient surface singularity, and let M be a reflexive $R = \mathcal{O}_{X,x}$-module. Then there exists an integrable connection on M;*

$$\nabla : \mathrm{Der}_{\mathbb{C}}(R) \to \mathrm{End}_{\mathbb{C}}(M).$$

Keep the notation introduced in the previous section. Let N be an $S[G]$-module, reflexive as an S-module, and recall that when N is indecomposable as S-module, it is given as $(i_*s^*\alpha_{r,d}(\lambda))_y$, see Proposition 2.2.

The proof is organized as follows: We first note that we may reduce to the two cases $d = 0$ and $(r,d) = 1$. When $d = 0$, $\alpha_{r,d}(\lambda)$ admits an integrable connection, we work on the elliptic curve and show that an action of G descends to the local system. When $(r,d) = 1$ we show that $N = (i_*s^*\alpha_{r,d}(\lambda))_y$ is an *invariant module*, see [10, Def. 5.6], *i.e.*, there exists a Galois covering $\pi' : (Y',y') \to (Y,y)$ such that $(\pi^*N)^{\vee\vee}$ is free.

Recall from Section 3, the rank r and degree 0 vector bundle $\mathcal{F}_r$ with global sections on an elliptic curve.

Lemma 5.2. — *Let $G \subset \mathrm{Aut}(E)$ act on E with at least one fixed point. Then there is a sequence*

$$0 \to \mathbb{C}_E \to \mathbb{V}_r \to \mathbb{V}_{r-1} \to 0 \tag{5.1}$$

of local systems on E with G-actions inducing the defining extension

$$0 \to \mathcal{O}_E \to \mathcal{F}_r \to \mathcal{F}_{r-1} \to 0.$$

Proof. — From Weil's theorem we know that degree zero indecomposable vector bundles admit integrable connections. We prove by induction that there is a local system $\mathbb{V}_r$ on E with G-action such that $\mathbb{V}_r \otimes_{\mathbb{C}} \mathcal{O}_E \cong \mathcal{F}_r$ and such that there are G-equivariant exact sequences (5.1). Define $\mathbb{V}_1 = \mathbb{C}_E$ and $\mathbb{V}_0 = 0$. The induction hypothesis is satisfied for $r = 1$. Assume that it holds up to r. There is an exact sequence

$$0 \to \mathbb{V}_r^\vee \to \mathcal{F}_r^\vee \xrightarrow{d \otimes \mathrm{id}} \Omega_E^1 \otimes \mathcal{F}_r^\vee \to 0 \tag{5.2}$$

where $\mathcal{O}_E \otimes_{\mathbb{C}} \mathbb{V}_r^\vee \cong \mathcal{F}_r^\vee$. We have $\Omega_E^1 \cong \omega \cong \mathcal{O}_E$. From the long exact sequence in cohomology we get a surjection $\mathrm{H}^1(\mathcal{F}_r^\vee) \twoheadrightarrow \mathrm{H}^2(\mathbb{V}_r^\vee)$. Dualizing (5.1) and taking cohomology we get

$$\cdots \to \mathrm{H}^2(\mathbb{V}_{r-1}^\vee) \to \mathrm{H}^2(\mathbb{V}_r^\vee) \to \mathrm{H}^2(\mathbb{C}_E) \to \mathrm{H}^3(\mathbb{V}_{r-1}^\vee) \to \cdots$$

Since $\mathrm{H}^3(\mathbb{V}_{r-1}^\vee) = 0$ and $\mathrm{H}^2(\mathbb{C}_E) \neq 0$ it follows that $\mathrm{H}^2(\mathbb{V}_r^\vee) \neq 0$. Since $\mathrm{H}^1(\mathcal{F}_r^\vee) \cong \mathbb{C}$, the connecting map $\mathrm{H}^1(\mathcal{F}_r^\vee) \to \mathrm{H}^2(\mathbb{V}_r^\vee)$ from (5.2) is an isomorphism, hence $\mathrm{H}^1(\mathbb{V}_r^\vee) \to \mathrm{H}^1(\mathcal{F}_r^\vee)$ is surjective. By Proposition 3.4, $\mathcal{F}_r$ has a canonical G-action corresponding to the canonical G-action on $\mathcal{O}_E$. It follows that (5.2) and its long exact sequence are G-sequences, and because G is finite, $\mathrm{H}^1(\mathbb{V}_r^\vee)^G \to \mathrm{H}^1(\mathcal{F}_r^\vee)^G$ is surjective. Since $\mathrm{Ext}^1_{\mathbb{C}_E[G]}(\mathbb{V}_r, \mathbb{C}_E) \cong \mathrm{H}^1(\mathbb{V}_r^\vee)^G$ and $\mathrm{Ext}^1_{\mathcal{O}_E[G]}(\mathcal{F}_r, \mathcal{O}_E) \cong \mathrm{H}^1(\mathcal{F}_r^\vee)^G$, the result follows since we have that $\mathrm{H}^1(\mathcal{F}_r^\vee)^G \neq 0$ by the proof of Proposition 3.4. □

PROPOSITION 5.3. — *Assume that r and d are such that $(r, d) = 1$ and that G is non-trivial. If N is an $S[G]$-module that is reflexive as S-module and $N \cong (i_* s^* \alpha_{r,d}(\lambda))_y$ as in Proposition 2.2, then N and $M = N^G$ are invariant modules and M admits an integrable connection.*

Proof. — Since $(r, d) = 1$, we may consider the representation $\tau = \tau_{r,d} : H_b = \pi_1^{\mathrm{loc}}(Y, y) \to \mathrm{Gl}(r, \mathbb{C})$, see Theorem 2.3. We claim that this representation is profinite, *i.e.*, that it factors through a finite quotient. Because of the group structure of H_b, it is enough to show that $\tau_{r,d}(\alpha)$ and $\tau_{r,d}(\beta)$ have finite orders, and this is easily checked. There is a corresponding vector bundle F_τ on E, see Proposition 2.2, with rank r and degree d such that $s^* F_\tau = \mathbb{V} \otimes \mathcal{O}_{Y \setminus \{y\}}$ where $\mathbb{V}$ is the local system corresponding to $\tau_{r,d}$.

We have that $\alpha_{r,d}(\lambda) \cong F_\tau \otimes \mathcal{L}$, ([1, Cor. 7.ii)]), where $\mathcal{L}$ is a line bundle with degree 0. We prove that $\mathcal{L}$ is induced from a profinite character $\chi : \pi_1(E) \to \mathbb{C}^*$. From Theorem 4.5 and Proposition 4.4, $\alpha_{r,d}(\lambda)$ and hence $\det \alpha_{r,d}(\lambda) \cong \det F_\tau \otimes \mathcal{L}^r$ admits a G-action. From Proposition 3.4, $\det \alpha_{r,d}(\lambda)$ corresponds to a fixed point in $\mathrm{Pic}^0(E) \cong E$. Since every $\sigma \in G$ is an isogeny, the fixed points are closed under the group law and thus form a finite group, and hence $\det \alpha_{r,d}(\lambda)$ has finite order. Since

τ is profinite, $\det\tau$ is profinite, and hence has finite order. It follows that $\det(s^*F_\tau) \cong s^*\det(F_\tau)$ has finite order. By Theorem 2.1, it follows that $\det(F_\tau)$, and hence $\mathcal{L}$, has finite order. If $\mathcal{L}^n = \mathcal{O}_E$, it follows that $\mathcal{L}$ maps to the the identity in $\mathrm{H}^1(\mathcal{O}_E^*) \xrightarrow{n} \mathrm{H}^0(\mathcal{O}_E^*)$. The kernel is $\mathrm{H}^1(\mu_n)$ where μ_n are the nth roots of 1. Thus there is a local system $\mathbb{L}$ on E such that $\mathcal{L} \cong \mathcal{O}_E \otimes \mathbb{L}$ where $\mathbb{L}$ corresponds to a profinite character χ. We can extend χ to a character of H_b which corresponds to the local system $s^*(\mathbb{L})$. Thus $\tau_{r,d} \otimes_{\mathbb{C}} \chi$ is a profinite representation and $N = (i_*(\mathbb{V} \otimes s^*(\mathbb{L}) \otimes \mathcal{O}_{Y\setminus\{y\}}))_y$. By Theorem 5.5 in [10], we conclude that N is an invariant module on Y with G-action. Since the composition of two Galois coverings is Galois, it follows that $M = N^G$ is an invariant $R = S^G$-module. Hence by Theorem 5.5 in [10], M admits an integrable connection. □

Proof of Theorem 5.1. — We may assume that $M = N^G$ is indecomposable. Hence N is indecomposable as $S[G]$-module, see Proposition 2.5.

First we assume that N is indecomposable as an S-module: From Proposition 2.2, we know that $N \cong i_*(s^*\alpha_{r,d}(\lambda))_y$, for some $\lambda \in \mathrm{Pic}^0(E)$ and some r and $d \geqslant r \geqslant 1$. From [1, Lemma 24], we have $\alpha_{r,d}(\lambda) \cong \alpha_{r',d'}(\lambda) \otimes \mathcal{F}_h$ where $h = (r,d)$ and $(r',d') = 1$, hence $N \cong i_*(s^*\alpha_{r',d'}(\lambda)) \otimes s^*\mathcal{F}_h)_y \cong (i_*(s^*\alpha_{r',d'}(\lambda))_y \otimes i_*(s^*\mathcal{F}_h)_y)^{\vee\vee}$. From Theorem 4.5 and Proposition 4.4, we know that $\alpha_{r,d}(\lambda)$ admits an action of G, and from Proposition 3.4 it follows that $\alpha_{r',d'}(\lambda)$ has a unique G-action such that $\alpha_{r,d}(\lambda) \cong \alpha_{r',d'}(\lambda) \otimes \mathcal{F}_h$ in $\mathcal{B}^G(r,d;1)$ where $\mathcal{F}_h$ is assumed to have the canonical G-action in Lemma 5.2. By Proposition 5.3, $i_*(s^*\alpha_{r',d'}(\lambda))_y$ admits a G-equivariant integrable connection, and by Lemma 5.2, $(i_*(s^*\mathcal{F}_h))_y$ admits a G-equivariant integrable connection. From [10, Sec. 3.3] it follows that N admits a G-equivariant integrable connection, and hence $M = N^G$ admits an integrable connection.

Assume N is decomposable. From Theorem 4.7, $M = (N')^H$ for a subgroup H of G and an indecomposable S-module N' that admits H-action. By the first part of the proof we have that M admits an integrable connection $\nabla_H : \mathrm{Der}_{\mathbb{C}}(S^H) \to \mathrm{End}_{\mathbb{C}}(M)$. By [26], $\mathrm{Der}_{\mathbb{C}}(S^H)^G = \mathrm{Der}_{\mathbb{C}}((S^H)^G) = \mathrm{Der}_{\mathbb{C}}(R)$, so that $\mathrm{Der}_{\mathbb{C}}(R)$ is an R-summand of $\mathrm{Der}_{\mathbb{C}}(S^H)$. An integrable connection on M is obtained by restricting ∇_H to $\mathrm{Der}_{\mathbb{C}}(R)$. □

Corollary 5.4. — *Keep the notation of Theorem 4.7. If $m|r$, the dense image under $q : E \to C$ of the torsion points, parametrizes the invariant modules in $\phi(\frac{r}{m})hm^{-1}s_1^{-1}$ of the $rhm^{-2}s_1^{-1}$ flat families, where ϕ is Euler's phi function. The other families do not contain invariant modules.*

The number of isolated invariant modules of rank r is

$$\frac{h}{s_1}\sum_{n|(r,m)}^{n\neq m}\phi(\frac{r}{n})\frac{s_n}{n}.$$

Proof. — By Theorem 4.7 any reflexive R-module M is given as $M = (N')^H$ where H is a subgroup of G and N' is the $S[H]$-module $N' = i_*s^*(\alpha_{r',d'}(\mathcal{O}_E(P-P_0)_\chi))_y$. Assume first that H is non-trivial. Then we claim that N' is an invariant module if and only if $(r',d')=1$. One direction follows from Proposition 5.3. Assume on the other hand that N' is an invariant module. As in the proof of Theorem 5.1, $N' \cong (i_*(s^*\alpha_{r'',d''}(\lambda))_y \otimes i_*(s^*\mathcal{F}_h)_y)^{\vee\vee}$ where $(r'',d'')=1$. From the proof of Theorem 5.1 and from Proposition 5.3, $(i_*(s^*\alpha_{r'',d''}(\lambda))_y$ is an invariant module. It follows from Lemma 5.2 and Theorem 5.9 of [10] that $i_*(s^*\mathcal{F}_h)_y$ is an invariant module if and only if $h=(r',d')=1$. We conclude that $(r',d')=1$. This establishes the claim, and from this it follows that among the isolated reflexive modules, it is precisely the modules corresponding to $(r',d')=1$ that are invariant modules, and we thus arrive at the number $\frac{h}{s_1}\sum_{n|(r,m)}^{n\neq m}\phi(\frac{r}{n})\frac{s_n}{n}$.

If H is trivial, $M=N'$. We claim that $N'=i_*s^*(\alpha_{r',d'}(\mathcal{O}_E(P-P_0)_\chi))_y$ is an invariant module if and only if $(r',d')=1$ and P is a torsion point: If $(r',d')\neq 1$, then $N' \cong (i_*(s^*\alpha_{r'',d''}(\lambda))_y \otimes i_*(s^*\mathcal{F}_h)_y)^{\vee\vee}$ with $h\neq 1$. In this case $i_*(s^*\mathcal{F}_h)_y$ is not an invariant module, so N' cannot be an invariant module. Hence; if N' is an invariant module, $N' \cong i_*(s^*\alpha_{r'',d''}(\lambda))_y{}^{\vee\vee}$ with $(r'',d'')=1$. By Theorem 6 in [1], $\det s^*\alpha_{r'',d''}(\lambda) \cong s^*(\mathcal{O}_E(P-P_0)_\chi \otimes \mathcal{O}_E(d''P_0))$. Since a rank one reflexive module L is an invariant module if and only if $(L^{\otimes n})^{\vee\vee}$ is trivial for some n, $\det(s^*\alpha_{r'',d''}(\lambda))$ must have finite order. By Theorem 2.1, $s^*\mathcal{O}_E(d''P_0)$ has finite order, and this implies that $s^*\mathcal{O}_E(P-P_0)_\chi$ has finite order. By Theorem 2.1, P is a torsion point.

So, assume that $(r',d')=1$ and that P is a torsion point. Then as in the proof of Proposition 5.3, $N' = (i_*s^*\alpha_{r',d'}(\mathcal{O}_E(P-P_0)))_y^{\vee\vee} \cong (i_*s^*(F_\tau \otimes \mathcal{L}))_y^{\vee\vee}$ for some line bundle $\mathcal{L}$ on E . We have that $\det s^*\alpha_{r',d'}(\mathcal{O}_E(P-P_0))$ has finite order, so $\det(s^*(F_\tau\otimes\mathcal{L}))$ has finite order. Since $\det(s^*F_\tau)$ has finite order, it follows that $\mathcal{L}$ has finite order, and from this we conclude that $(i_*s^*\mathcal{L})_y^{\vee\vee}$ is an invariant module. Since $(i_*s^*F_\tau)_y^{\vee\vee}$ is an invariant module, it follows that $N' \cong ((i_*s^*(F_\tau)_y \otimes (i_*s^*\mathcal{L})_y)^{\vee\vee}$ is an invariant module. This establishes the claim, and from this we arrive at the stated number of families satisfying $(r',d')=1$. (Note $r'=\frac{r}{m}$ and $d=d'$). □

6. An example

Example 6.1. — Let Y be the cone in $\mathbb{C}^3$ given by $x^3 + y^3 + z^3 = 0$, and let E be the elliptic curve in $\mathbb{P}^2$ given by the same equation. Let $G = \mathbb{Z}/(3)$, and let $\omega \neq 1$ be a third root of unity. Let G act on Y by $(x, y, z) \mapsto (\omega x, \omega^2 y, \omega z)$ and let X be the quotient of Y under this action. The action on Y is compatible with the grading and so gives an action on E.

The quotient X is an elliptic quotient and Y is the canonical covering. The graph of X is $\mathbf{Eq}_3^3(2)$, see Figure 3.1. On the quotient (X, x), the isolated indecomposable reflexive modules of rank r are given as N^G where N is a reflexive module on (Y, y) given as $N = i_*(s^*\alpha_{r,d}(\lambda)))_y$ with $\lambda \in E^G$, see Theorem 4.7. There are three fixed points and three characters of G. There are $3r$ possibilities for d, since $r \leqslant d < (b+1)r = 4r$. Thus we arrive at $27r$ isomorphism classes of indecomposable reflexive modules of rank r on (X, x), and we have that N, and hence N^G, is an invariant module if and only if $(r, d) = 1$. Among the $27r$ modules, the number of isolated invariant modules is given as $27\phi(r)$, where ϕ is Euler's ϕ-function.

(1) If $r \not\equiv 0 \bmod 3$ these are the only indecomposable reflexive modules on X of rank r.

(2) If $r \equiv 0 \bmod 3$ there are in addition r one-parameter families (one for each d satisfying $r/3 \leqslant d < 4r/3$) of non-isomorphic indecomposable reflexive modules on X of rank r. The parameter space for these one-parameter families is $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Such a family contains invariant modules if and only if $(r/3, d) = 1$, and in this case the invariant modules are parameterized by the countable, dense image of the torsion points $U \cap E_{\mathrm{tors}}$ under the quotient map $U \to U/G \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$ where $U = \{x \in E | G_x = (0)\}$.

From [21], $\pi_1^{\mathrm{loc}}(X, x)$ is generated by four elements $a_0, \ldots, a_3$ subject to the relations $a_0 a_j = a_j a_0$, $a_0 = a_j^3$, $j = 1, 2, 3$ and $a_2 a_1 a_3 = a_0^2$. Setting $a_0 = a_2^3$ and $a_3 = a_1^2 a_2^2$, one finds that $\pi_1^{\mathrm{loc}}(X, x)$ is the group given by two generators a_1 and a_2 and the relations $a_1^3 = a_2^3 = (a_1^2 a_2^2)^3$. In Appendix A, we find the representations of rank one and two. There are 27 characters and 54 rank two indecomposable representations. Thus in rank one and two, there is a one-to-one correspondence between indecomposable representations of the local fundamental group and indecomposable reflexive modules. In particular; rank one and rank two reflexive modules admit *unique integrable connections.* On (Y, y), in contrast, there is a positive dimensional family of connections on each indecomposable reflexive module, see [12, Th. 6.30].

Appendix A. Representations of a non-finite group

PROPOSITION A.1. — *Let G be the group given by two generators a_1 and a_2 and the relations $a_1^3 = a_2^3 = (a_1^2 a_2^2)^3$. Then we have:*

1. *The group of characters is $\mathbb{Z}/(3) \oplus \mathbb{Z}/(9)$.*
2. *There is exactly one simple two dimensional representation with trivial determinant. This representation is profinite.*
3. *There is exactly one non-simple, indecomposable two dimensional representation with trivial determinant. This representation is not profinite.*
4. *There are exactly 27 simple two dimensional representations. These are profinite.*
5. *There are exactly 27 non-simple, indecomposable two dimensional representations. These are not profinite.*

Proof. — We leave the proof of (1) to the reader, and we first show that (4) and (5) may be reduced to (2) and (3). Note that for $y \in \mathbb{Z}/(3)\times\mathbb{Z}/(9)$ there exists $x \in \mathbb{Z}/(3)\times\mathbb{Z}/(9)$ such that $2x = y$. Hence for any $\rho : G \to \mathrm{Gl}(2,\mathbb{C})$ there is a character ξ such that $\det(\rho\otimes\xi) = \mathrm{id}$. Since also $\rho\otimes\xi = \rho$ gives that ξ is trivial, we see that it is enough to prove (2) and (3).

If $\rho : G \to \mathrm{Gl}(2,\mathbb{C})$ is a representation, let $A_i := \rho(a_i) \in \mathrm{Gl}(2,\mathbb{C})$. For ρ to be indecomposable, A_1 and A_2 must have one-dimensional eigenspaces where maximally one of these are common for A_1 and A_2. In particular each A_i must have two distinct eigenvalues.

We prove that if A_1 and A_2 have one common eigenvector, we get (3), and if not, we get (2).

Assume that A_1 and A_2 have one common eigenvector. Let v_1 and v_2 be the eigenvectors of A_1, and let v_1 and v_3 be the eigenvectors of A_2. We may assume that $v_3 = v_1 + v_2$, and assuming $\det(A_i) = 1$, we have in the basis $\{v_1, v_2\}$,

$$A_1 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & \lambda^{-1}-\lambda \\ 0 & \lambda^{-1} \end{pmatrix}$$

for $\lambda, \omega \in \mathbb{C}$. From $A_1^3 = A_2^3$ we get $\omega^3 = \lambda^3$ and $\lambda^6 = \omega^6 = 1$. From $(A_1^2 A_2^2)^3 = A_1^3$, we then get that $\lambda^3 = \omega^3 = 1$ and from the $(1,2)$-entry

$$\omega^{-2}\lambda^{-2}\left(\omega^2 + \lambda^2\omega^2 + 1\right)(\omega+1)(\omega-1)(\lambda+1)(\lambda-1) = 0.$$

We cannot have ω (resp. λ) equal to -1 or 1 since this would give $\omega = \omega^{-1}$ (resp. $\lambda = \lambda^{-1}$). Thus $\omega^2 + \lambda^2\omega^2 + 1 = 0$. Since $\omega^3 = 1$, we get $\omega = \lambda = -\frac{1}{2}\pm\frac{1}{2}i\sqrt{3}$. These two possibilities give however isomorphic representations.

Moreover;

$$A_1A_2 = \begin{pmatrix} 1 & \frac{1}{2}i\sqrt{3}+\frac{3}{2} \\ 0 & 1 \end{pmatrix}$$

has infinite order. Because of the common eigenvector it is not simple, but it is indecomposable.

We now assume that the set of eigenvectors for A_1 and A_2 consists of four different vectors. Choosing basis we may assume that e_1 and e_2 are eigenvectors for A_1 and that $e_1 + e_2$ and $ae_1 + e_2$ are the eigenvectors for A_2, $a \neq 0, 1$. Assuming $\det(A_i) = 1$, we have

$$A_1 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \; A_2 = \begin{pmatrix} 1 & a \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} 1 & a \\ 1 & 1 \end{pmatrix}^{-1}.$$

Since the off-diagonal entries of A_2^3 must be zero, it follows that $\lambda^6 = 1$. Substituting this in $A_2^3 = A_1^3$, we get $\left(\frac{\omega}{\lambda}\right)^3 = 1$, *i.e.*, $\lambda = \omega$ or $\lambda = \overline{\omega}$. Since we cannot have $\omega = \pm 1$, it follows that $\omega^4 + \omega^2 + 1 = 0$. Zero on the off-diagonal entries of $(A_1^2 A_2^2)^3$, implies $f(\lambda, \omega) = 0$ where

$$\begin{aligned} f(\lambda,\omega) := &\left(a\lambda^4 + a\omega^4 + \lambda^2\omega^2 - \lambda^4\omega^4 - a\lambda^2\omega^2 - 1\right) \\ &\left(a\lambda^4 + a\omega^4 - \lambda^2\omega^2 - \lambda^4\omega^4 + a\lambda^2\omega^2 - 1\right). \end{aligned}$$

Using $\omega^4 + \omega^2 + 1 = 0$ we find that $f(\omega, \omega) = 3a\omega^2\,(a+2)$ and $f(\overline{\omega}, \omega) = (-2a-1)(-3)$. We thus get $a = -2$ if $\lambda = \omega$ and $a = -\frac{1}{2}$ if $\lambda = \overline{\omega}$. Again checking the relations, we get that $\omega^3 = -1$ (and thus $\omega = \frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$) in both cases. Straight forward simultaneous conjugations show however that the four representations are isomorphic. Finally one checks that the group generated by A_1 and A_2 is isomorphic to the group $\mathrm{SL}(2, \mathbb{F}_3)$ and hence the representation is profinite and simple. An isomorphism from $\mathrm{SL}(2, \mathbb{F}_3) = \langle k, r \mid k^6, r^4, (kr)^3 \rangle$ can for instance be given by

$$k = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \mapsto A_1$$

$$r = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \mapsto A_1^2 A_2.$$

□

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Trond Stølen GUSTAVSEN
Buskerud University College
Department of Teacher Education
Pb. 7053
3007 Drammen (Norvège)
trond.gustavsen@hibu.no

Runar ILE
University of Bergen
Department of Mathematics
Johs. Brunsgt. 12
5008 Bergen (Norvège)
runar.ile@math.uib.no