



# ANNALES

DE

# L'INSTITUT FOURIER

Katsutoshi YAMANOI

**On fundamental groups of algebraic varieties and value distribution theory**

Tome 60, n° 2 (2010), p. 551-563.

[http://aif.cedram.org/item?id=AIF\\_2010\\_\\_60\\_2\\_551\\_0](http://aif.cedram.org/item?id=AIF_2010__60_2_551_0)

© Association des Annales de l'institut Fourier, 2010, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (<http://aif.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://aif.cedram.org/legal/>). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

*Article mis en ligne dans le cadre du*  
*Centre de diffusion des revues académiques de mathématiques*  
<http://www.cedram.org/>

# ON FUNDAMENTAL GROUPS OF ALGEBRAIC VARIETIES AND VALUE DISTRIBUTION THEORY

by Katsutoshi YAMANOI

*Dedicated to Professor Junjiro Noguchi on his 60th birthday*

---

ABSTRACT. — If a smooth projective variety  $X$  admits a non-degenerate holomorphic map  $\mathbb{C} \rightarrow X$  from the complex plane  $\mathbb{C}$ , then for any finite dimensional linear representation of the fundamental group of  $X$  the image of this representation is almost abelian. This supports a conjecture proposed by F. Campana, published in this journal in 2004.

RÉSUMÉ. — Si une variété  $X$  projective lisse admet une application holomorphe non-dégénérée  $\mathbb{C} \rightarrow X$  du plan complexe  $\mathbb{C}$ , alors pour chaque représentation linéaire de dimension finie du groupe fondamental de  $X$  l'image de cette représentation est presque abélienne. Cela soutient une conjecture proposée par F. Campana, parue dans ce même journal en 2004.

## 1. Main results

Let  $X$  be a smooth projective variety. We say that a holomorphic map  $f: \mathbb{C} \rightarrow X$  is non-degenerate if the image  $f(\mathbb{C})$  is Zariski dense in  $X$ . A group  $G$  is called almost abelian if  $G$  has a finite index subgroup which is abelian. In this paper, we prove the following theorem.

**THEOREM 1.1.** — *Let  $X$  be a smooth projective variety which admits a non-degenerate holomorphic map  $f: \mathbb{C} \rightarrow X$ . Then for any representation  $\varrho: \pi_1(X) \rightarrow \mathrm{GL}_n(\mathbb{C})$ , the image  $\varrho(\pi_1(X))$  is almost abelian.*

This theorem shows that the following conjecture proposed by F. Campana [4, Conjecture 9.8] is true in the special case that  $\pi_1(X)$  is linear.

---

*Keywords:* Value distribution theory, holomorphic map, fundamental group, algebraic variety.

*Math. classification:* 32H30, 14F35.

CONJECTURE 1.2. — *Let  $X$  be a smooth projective variety which admits a non-degenerate holomorphic map  $f: \mathbb{C} \rightarrow X$ . Then the fundamental group  $\pi_1(X)$  is almost abelian.*

This conjecture comes from Campana’s theory of “special” variety (cf. [4]). A complex manifold  $X$  which admits a holomorphic map  $f: \mathbb{C} \rightarrow X$  with metrically dense image has vanishing Kobayashi pseudo-metric. It is Campana’s view that a smooth projective variety  $X$  would have vanishing Kobayashi pseudo-metric if and only if  $X$  is “special” (cf. [4, Conjecture 9.2]), and that the fundamental group of a “special” variety would be almost abelian (cf. [4, Conjecture 7.1]). For more discussion about Conjecture 1.2, we refer the reader to [4].

A representation  $\varrho: \pi_1(X) \rightarrow \mathrm{GL}_n(\mathbb{C})$  is called big if the following condition is satisfied (cf. [11]):

If  $Z \subset X$  is a positive dimensional subvariety containing a very general point of  $X$ , then the image  $\varrho(\mathrm{Im}(\pi_1(\tilde{Z}) \rightarrow \pi_1(X)))$  is infinite. Here  $\tilde{Z}$  is a desingularization of  $Z$ .

For example, if there exists an unramified Galois covering  $\tilde{X} \rightarrow X$  such that  $\tilde{X}$  is a Stein space and its Galois transformation group  $\Gamma$  is a linear group, then the corresponding surjection  $\pi_1(X) \rightarrow \Gamma$  is a big representation.

COROLLARY 1.3. — *Let  $X$  be a smooth projective variety with a big representation  $\varrho: \pi_1(X) \rightarrow \mathrm{GL}_n(\mathbb{C})$ . If  $X$  admits a non-degenerate holomorphic map  $f: \mathbb{C} \rightarrow X$ , then there exists a finite unramified covering  $X'$  of  $X$  which is birationally equivalent to an Abelian variety.*

The strategy of the proof of Theorem 1.1 is roughly as follows. Based on results of [4] and [20], Campana proved the following ([4]): If there exists a representation  $\varrho: \pi_1(X) \rightarrow \mathrm{GL}_n(\mathbb{C})$  such that the image  $\varrho(\pi_1(X))$  is not almost abelian, then there exist a finite unramified covering  $X' \rightarrow X$  and a dominant rational map  $X' \dashrightarrow Z$  with  $Z$  of general type and positive dimensional. The proof of this result shows that  $Z$  is not only of general type, but has more precise structure. Thanks to this precise structure, we can show that every holomorphic map  $g: \mathbb{C} \rightarrow Z$  is degenerate, i.e. the image  $g(\mathbb{C})$  is not Zariski dense in  $Z$ . This implies our theorem.

## 2. A reduction of the proof of Theorem 1.1

The proof of Theorem 1.1 is based on the following proposition, which is a special case of the theorem.

PROPOSITION 2.1. — *Let  $X$  be a smooth projective variety, and let  $G$  be an almost simple algebraic group defined over the complex number field. Assume that there exists a representation  $\varrho: \pi_1(X) \rightarrow G(\mathbb{C})$  whose image  $\varrho(\pi_1(X))$  is Zariski dense in  $G$ . Then every holomorphic map  $f: \mathbb{C} \rightarrow X$  is degenerate.*

*Proposition 2.1 implies Theorem 1.1.* — Let  $X$  be a smooth projective variety which admits a non-degenerate holomorphic map  $f: \mathbb{C} \rightarrow X$ . Let  $\varrho: \pi_1(X) \rightarrow \mathrm{GL}_n(\mathbb{C})$  be a representation. We shall prove that  $\varrho(\pi_1(X))$  is almost abelian.

Let  $H \subset \mathrm{GL}_n(\mathbb{C})$  be the Zariski closure of the image  $\varrho(\pi_1(X))$ . Let  $H_0 \subset H$  be the connected component of  $H$  containing the identity element of  $H$ . Then  $\Gamma = \varrho^{-1}(\varrho(\pi_1(X)) \cap H_0)$  is a finite index subgroup of  $\pi_1(X)$ . Replacing  $X$  by the finite unramified covering  $X' \rightarrow X$  which corresponds to  $\Gamma$ , we may assume that  $H$  is connected.

Let  $R(H) \subset H$  be the radical of  $H$ , i.e.  $R(H)$  is the maximal connected solvable closed normal subgroup of  $H$ . Put  $H_{s.s.} = H/R(H)$ . We first prove that  $H_{s.s.}$  is trivial.

Assume, for the sake of contradiction, that  $H_{s.s.}$  is not trivial. Then  $H_{s.s.}$  is a semi-simple algebraic group. Hence  $H_{s.s.}$  is an almost direct product of almost simple algebraic groups  $G_1, \dots, G_l$ . Let  $H_{s.s.} \rightarrow G_1$  be a projection, and let  $\varrho': \pi_1(X) \rightarrow G_1$  be the composition of  $\varrho$  and the two projections  $H \rightarrow H_{s.s.} \rightarrow G_1$ . Since the image  $\varrho'(\pi_1(X))$  is Zariski dense in  $G_1$ , we may apply Proposition 2.1 to conclude that every holomorphic map  $\mathbb{C} \rightarrow X$  is degenerate. This contradicts to our assumption that  $X$  admits a non-degenerate holomorphic map  $f: \mathbb{C} \rightarrow X$ . Hence we have proved that  $H_{s.s.}$  is trivial, i.e.  $H = R(H)$ .

Now the image  $\varrho(\pi_1(X))$  is a solvable group. We note that every finite unramified covering  $X'$  of  $X$  admits a non-degenerate holomorphic map  $f': \mathbb{C} \rightarrow X'$  coming from a lifting of  $f: \mathbb{C} \rightarrow X$ . Hence by [14, Theorem 6.4.1], the Albanese map of  $X'$  is surjective for every finite unramified covering  $X'$  of  $X$ . Hence by [3, Théorème 2.9], there exists a finite unramified covering  $X'$  of  $X$  such that  $\varrho$  factors the induced group homomorphism  $\pi_1(X') \rightarrow \pi_1(\mathrm{Alb}(X'))$ . From this, we conclude that  $\varrho(\pi_1(X'))$  is abelian. Hence  $\varrho(\pi_1(X))$  is almost abelian. □

### 3. Representations over non-archimedean local fields

Let  $K$  be a number field, and let  $\mathcal{O}_K$  be the ring of integers in  $K$ . Given a prime ideal  $p$  from  $\mathcal{O}_K$ , we denote by  $K_p$  the completion of  $K$  with

respect to the natural discrete valuation defined by  $p$ . Let  $G$  be an almost simple algebraic group defined over  $K_p$ , and let  $\varrho: \pi_1(X) \rightarrow G(K_p)$  be a  $p$ -adic representation. We say that  $\varrho$  is  $p$ -bounded if the image  $\varrho(\pi_1(X))$  is contained in a maximal compact subgroup of  $G(K_p)$ . If  $\varrho$  is not  $p$ -bounded, then we say that  $\varrho$  is  $p$ -unbounded.

In this section, we prove the following:

**PROPOSITION 3.1.** — *Let  $X$  be a smooth projective variety. Let  $G$  be an almost simple algebraic group defined over the  $p$ -adic field  $K_p$ . Assume that there exists a  $p$ -unbounded representation  $\varrho: \pi_1(X) \rightarrow G(K_p)$  whose image is Zariski dense in  $G$ . Then every holomorphic map  $f: \mathbb{C} \rightarrow X$  is degenerate.*

The proof of this proposition is based on the consideration of the spectral covering  $\pi: X^s \rightarrow X$ . We follow the exposition of [20, Section 1]. The construction of  $X^s$  is based on the theory of equivariant harmonic maps to buildings due to Gromov and Schoen [8]; Since  $\varrho$  is reductive, there exists a non-constant  $\varrho$ -equivariant pluriharmonic map  $u: \tilde{X} \rightarrow \Delta(G)$  from the universal covering of  $X$  to the Bruhat-Tits building of  $G$ . Considering the complexified differential of  $u$ , we get a multi-valued holomorphic one form  $\omega$  on  $X$ . We consider a finite ramified Galois covering  $\pi: X^s \rightarrow X$  such that  $\pi^*\omega$  splits into single-valued holomorphic one forms  $\omega_1, \dots, \omega_l$ . All the forms  $\omega_1, \dots, \omega_l$  are contained in the space  $H^0(X^s, \pi^*\Omega_X^1)$ . The covering  $\pi: X^s \rightarrow X$  is unramified outside  $\cup_{\omega_i \neq \omega_j} (\omega_i - \omega_j)_0$  where  $\omega_i - \omega_j$  are considered as forms from  $H^0(X^s, \pi^*\Omega_X^1)$  (cf. [9, Lemma 2.1]). For more detail about the construction of the spectral covering, we refer the reader to [20], [21], [5] and [10].

We construct the Albanese map  $\Phi: X^s \rightarrow A$  with respect to  $\omega_1, \dots, \omega_l$  as follows (cf. [21, p. 64]): Let  $a: \widehat{X^s} \rightarrow A(\widehat{X^s})$  be the Albanese map, where  $\psi: \widehat{X^s} \rightarrow X^s$  is a desingularization of  $X^s$ . For  $i = 1, \dots, l$ , let  $\tilde{\omega}_i$  be the holomorphic one form on  $A(\widehat{X^s})$  such that  $\psi^*(\omega_i) = a^*\tilde{\omega}_i$ . Let  $B \subset A(\widehat{X^s})$  be the maximal Abelian subvariety such that all  $\tilde{\omega}_i$  vanish on  $B$ . We set  $A = A(\widehat{X^s})/B$ . Then since  $X^s$  is normal, the composition of  $\widehat{X^s} \rightarrow A(\widehat{X^s}) \rightarrow A$  factors through  $\psi: \widehat{X^s} \rightarrow X^s$ . This induces the desired map  $\Phi: X^s \rightarrow A$ .

We summarize the needed properties of the spectral covering from [20, Section 1].

**PROPOSITION 3.2.** — *Assume furthermore that  $\varrho$  is big. Then:*

- (1)  $\Phi: X^s \rightarrow A$  is generically finite.
- (2)  $X^s$  is of general type.

The proof of (1) can be found in [20, p. 148]. Indeed the following stronger result is proved in [20, p. 148]: The Stein factorization of  $\Phi: X^s \rightarrow A$  is a Shafarevich map for the pull-back representation  $\pi^*\varrho: \pi_1(X^s) \rightarrow G(K_p)$ . The implication of (1) is immediate; Since  $\varrho$  is big,  $\pi^*\varrho$  is also big. Hence the Shafarevich map for the representation  $\pi^*\varrho$  is birational, which implies (1). The proof of (2) can be found in [20, p. 151].

*Notation.* — Before going to prove Proposition 3.1, we introduce the notations of Nevanlinna theory (cf. [14], [13]). Let  $Y$  be a Riemann surface with a proper surjective holomorphic map  $p_Y: Y \rightarrow \mathbb{C}$ . For  $r > 0$ , we set  $Y(r) = p_Y^{-1}(\{z; |z| < r\})$ . We put

$$N_{\text{ram } p_Y}(r) = \frac{1}{\deg p_Y} \int_1^r \left[ \sum_{y \in Y(t)} \text{ord}_y \text{ram } p_Y \right] \frac{dt}{t},$$

where  $\text{ram } p_Y$  is the ramification divisor of  $p_Y$ .

Let  $X$  be a projective variety and let  $Z$  be a closed subscheme of  $X$ . Let  $g: Y \rightarrow X$  be a holomorphic map with Zariski dense image. Since  $Y$  is one dimensional, the pull-back  $g^*Z$  is a divisor on  $Y$ . We set

$$N(r, g, Z) = \frac{1}{\deg p_Y} \int_1^r \left[ \sum_{y \in Y(t)} \text{ord}_y g^*Z \right] \frac{dt}{t},$$

$$\bar{N}(r, g, Z) = \frac{1}{\deg p_Y} \int_1^r \left[ \sum_{y \in Y(t)} \min\{1, \text{ord}_y g^*Z\} \right] \frac{dt}{t}.$$

Let  $\psi: \hat{X} \rightarrow X$  be a desingularization, let  $\hat{g}: Y \rightarrow \hat{X}$  be the lifting of  $g$ . Let  $M$  be a line bundle on  $X$ . Let  $\|\cdot\|$  be a smooth Hermitian metric on  $\psi^*M$ , let  $\Omega$  be the curvature form of  $(M, \|\cdot\|)$ . We define

$$T(r, g, M) = \frac{1}{\deg p_Y} \int_1^r \left[ \int_{Y(t)} \hat{g}^*\Omega \right] \frac{dt}{t} + O(1).$$

This definition is independent of the choice of desingularization and Hermitian metric up to bounded function  $O(1)$ . Given a divisor  $D \in H^0(X, M)$ , we have the following Nevanlinna inequality (cf. [14, p. 180], [12, p. 269]):

$$(3.1) \quad N(r, g, D) \leq T(r, g, M) + O(1).$$

Let  $M$  be an ample line bundle on  $X$ . Let  $\omega \in H^0(X, \Omega_X^1)$  be a holomorphic one form. Set  $\eta = g^*\omega/p_Y^*(dz)$ . Then  $\eta$  is a meromorphic function on  $Y$ . We set

$$m(r, \eta) = \frac{1}{\deg p_Y} \int_{\partial Y(r)} \max\{\log |\eta(y)|, 0\} \frac{d \arg p_Y(y)}{2\pi}.$$

Then by the lemma on logarithmic derivative ([13, Lemma 1.6]), we have

$$m(r, \eta) = o(T(r, g, M)) \parallel.$$

Here the symbol  $\parallel$  means that the stated estimate holds for  $r > 0$  outside some exceptional interval with finite Lebesgue measure. By the first main theorem (cf. [12, p. 269]), we have

$$T(r, \eta, \mathcal{O}_{\mathbb{P}^1}(1)) = N(r, \eta, \infty) + m(r, \eta) + O(1),$$

where we consider  $\eta$  as a holomorphic map from  $Y$  into  $\mathbb{P}^1$ . Thus we have

$$(3.2) \quad T(r, \eta, \mathcal{O}_{\mathbb{P}^1}(1)) \leq N(r, \eta, \infty) + o(T(r, g, M)) \parallel.$$

*Proof of Proposition 3.1.* — First we shall reduce to the case that  $\varrho$  is big. Put  $H = \ker \rho$  and consider the  $H$ -Shafarevich map  $\text{sh}_X^H: X \dashrightarrow \text{Sh}^H(X)$  ([11, p. 185]). We remark that  $\text{Sh}^H(X)$  is only defined up to birationally equivalent class. Replacing  $X$  and  $\text{Sh}^H(X)$  by suitable models, we may assume that  $\text{sh}_X^H: X \rightarrow \text{Sh}^H(X)$  is a morphism. Let  $F$  be a general fiber of  $\text{sh}_X^H$  and let  $\pi_1(F)_X$  be the image of the natural map  $\pi_1(F) \rightarrow \pi_1(X)$ . Then by the definition of the  $H$ -Shafarevich map, the image  $\varrho(\pi_1(F)_X) \subset G(K_p)$  is finite. We apply [21, Lemma 2.2.3]. The conclusion is as follows: After passing to a blowing-up and a finite unramified covering  $e: X' \rightarrow X$ , and denoting  $s: X' \rightarrow \Sigma$  the Stein factorization of  $\text{sh}_X^H \circ e$ , there exists a representation  $\varrho_\Sigma: \pi_1(\Sigma) \rightarrow G(K_p)$  such that the pullback representation  $e^* \varrho: \pi_1(X') \rightarrow G(K_p)$  factors through  $\varrho_\Sigma$ . Replacing  $X'$  and  $\Sigma$  by suitable models, we may assume that  $\Sigma$  is smooth. By the construction of  $\Sigma$ , we remark that the representation  $\varrho_\Sigma$  is big and Zariski dense (cf. [21, Proposition 2.2.2]). Given a holomorphic map  $f: \mathbb{C} \rightarrow X$ , we may take a lifting  $f': \mathbb{C} \rightarrow X'$  of  $f$ . If the composite holomorphic map  $s \circ f': \mathbb{C} \rightarrow \Sigma$  is degenerate, then  $f$  is also degenerate. Thus replacing  $X$  by  $\Sigma$ ,  $\varrho$  by  $\varrho_\Sigma$  and  $f$  by  $s \circ f'$ , we have reduced to the case when  $\varrho$  is big.

Now assume, for the sake of contradiction, that there exists non-degenerate holomorphic map  $f: \mathbb{C} \rightarrow X$ . Then we may construct a Riemann surface  $Y$  with proper, surjective holomorphic map  $p_Y: Y \rightarrow \mathbb{C}$  such that:

- the lifting  $g: Y \rightarrow X^s$  of  $f$  exists, and
- $p_Y$  is unramified outside the discrete set  $g^{-1}(R) \subset Y$ , where  $R$  is the ramification divisor of  $\pi: X^s \rightarrow X$ . Hence we have

$$(3.3) \quad N_{\text{ram } p_Y}(r) \leq (\deg p_Y) \bar{N}(r, g, R).$$

Since we are assuming that  $f$  is non-degenerate, we remark that

$$(3.4) \quad \text{the image } g(Y) \text{ is Zariski dense in } X^s.$$

For  $\omega_i \neq \omega_j$ , we set  $\Xi_{ij} = (\omega_i - \omega_j)_0$ , where  $\omega_i - \omega_j$  is considered as a form from  $H^0(X^s, \pi^* \Omega_X^1)$ . We have

$$(3.5) \quad R \subset \cup_{i,j} \Xi_{ij}.$$

□

Let  $M$  be an ample line bundle on  $X^s$ .

CLAIM. —  $\bar{N}(r, g, \Xi_{ij}) \leq \varepsilon T(r, g, M) \parallel$  for all  $\varepsilon > 0$ .

*Proof of Claim.* — We prove the claim in the two possible cases:

Case 1. —  $g^* \omega_i \neq g^* \omega_j$ . Since  $\omega_i \in H^0(X^s, \pi^* \Omega_X^1)$ , we may consider  $g^* \omega_i$  as a holomorphic section of  $p_Y^* \Omega_C^1$ . Thus  $\eta_i = g^* \omega_i / p_Y^*(dz)$  is a holomorphic function on  $Y$ . Since  $g^* \omega_i \neq g^* \omega_j$ , we have  $\eta_i \neq \eta_j$ . Note that if  $g(y) \in \Xi_{ij}$ , we have  $\eta_i(y) = \eta_j(y)$ . Hence using the Nevanlinna inequality (3.1), we have

$$\begin{aligned} \bar{N}(r, g, \Xi_{ij}) &\leq N(r, \eta_i - \eta_j, 0) \\ &\leq T(r, \eta_i - \eta_j, \mathcal{O}_{\mathbb{P}^1}(1)) + O(1). \end{aligned}$$

Since  $\eta_i - \eta_j$  has no poles, we have  $N(r, \eta_i - \eta_j, \infty) = 0$ . Thus, applying (3.2) to  $\eta_i - \eta_j = g^*(\omega_i - \omega_j) / p_Y^*(dz)$ , we have

$$T(r, \eta_i - \eta_j, \mathcal{O}_{\mathbb{P}^1}(1)) = o(T(r, g, M)) \parallel.$$

We conclude that

$$\bar{N}(r, g, \Xi_{ij}) = o(T(r, g, M)) \parallel.$$

Case 2. —  $g^* \omega_i = g^* \omega_j$ . Let  $b: X^s \rightarrow B$  be the Albanese map with respect to  $\omega_i - \omega_j$ , which is constructed as follows: Let  $\Phi: X^s \rightarrow A$  be the Albanese map with respect to  $\omega_1, \dots, \omega_l$ . For  $k = 1, \dots, l$ , let  $\tilde{\omega}_k$  be the holomorphic one form on  $A$  such that  $\Phi^* \tilde{\omega}_k = \omega_k$ . Let  $C \subset A$  be the maximal Abelian subvariety such that  $\tilde{\omega}_i - \tilde{\omega}_j$  vanishes on  $C$ . Put  $B = A/C$ . We define the map  $b: X^s \rightarrow B$  by the composition of  $\Phi: X^s \rightarrow A$  and the quotient  $A \rightarrow B$ .

Let  $\Xi'_{ij}$  be an irreducible component of  $\Xi_{ij}$ . Since there are only finitely many irreducible components of  $\Xi_{ij}$ , it is enough to prove

$$\bar{N}(r, g, \Xi'_{ij}) \leq \varepsilon T(r, g, M) \parallel \quad \text{for all } \varepsilon > 0.$$

Since  $\omega_i - \omega_j$  vanishes on  $\Xi'_{ij}$ , we see that  $b(\Xi'_{ij})$  is a point on  $B$ . We take an open subset  $U \subset B$  and a holomorphic function  $\varphi$  on  $U$  such that  $\varphi(b(\Xi'_{ij})) = 0$  and  $\omega_i - \omega_j = b^*(d\varphi)$  on  $b^{-1}(U)$ .

Let  $S \rightarrow b(X^s)$  be the normalization. Since  $X^s$  is normal,  $b$  factors as

$$X^s \xrightarrow{c} S \xrightarrow{\psi} B.$$



Since  $\psi$  is finite,  $c(\Xi'_{ij})$  is a point on  $S$ ; We denote this point by  $P$ . Let  $\mathcal{O}_{S,P}^{\text{an}}$  be the stalk at  $P$  in the sense of analytic space, and let  $\mathfrak{m} \subset \mathcal{O}_{S,P}^{\text{an}}$  be the maximal ideal. Since  $S$  is normal,  $\mathcal{O}_{S,P}^{\text{an}}$  is integral. We remark that  $\varphi \circ \psi \in \mathcal{O}_{S,P}^{\text{an}}$  is neither zero nor a unit, which follows from  $\omega_i - \omega_j = b^*(d\varphi)$  and  $\varphi(b(\Xi'_{ij})) = 0$ . Hence we have

$$(3.6) \quad \dim \mathcal{O}_{S,P}^{\text{an}}/(\varphi \circ \psi) = \dim S - 1.$$

Set

$$V_n = \text{Spec } \mathcal{O}_{S,P}^{\text{an}}/((\varphi \circ \psi) + \mathfrak{m}^n).$$

Then  $V_n$  is a closed subscheme of  $S$  with  $\text{supp } V_n = P$ .

Let  $L$  be an ample line bundle on  $S$ . Using (3.6), we have

$$h^0(V_n, \mathcal{O}_{V_n} \otimes L^{\otimes \ell}) = h^0(V_n, \mathcal{O}_{V_n}) = O(n^{\dim S - 1}).$$

On the other hand, there are positive constants  $c > 0$  and  $\ell_0 > 0$  such that

$$h^0(S, L^{\otimes \ell}) > c\ell^{\dim S}$$

for  $\ell > \ell_0$ . Thus we may take a positive integer  $\ell(n)$  such that  $\ell(n) = o(n)$  as  $n \rightarrow \infty$ , and that  $h^0(V_n, \mathcal{O}_{V_n} \otimes L^{\otimes \ell(n)}) < h^0(S, L^{\otimes \ell(n)})$ . For example,  $\ell(n) \sim n^{1 - \frac{1}{2 \dim S}}$ . Thus we may take a divisor  $D_n$  from  $H^0(S, L^{\otimes \ell(n)})$  such that  $V_n \subset D_n$ .

Now we claim that if  $c \circ g(y) = P$  for  $y \in Y$ , then  $\text{ord}_y(c \circ g)^* D_n \geq n$ . To see this, we take  $y \in Y$  such that  $c \circ g(y) = P$ . Let  $O \subset Y$  be the connected component of  $(b \circ g)^{-1}(U)$  containing  $y$ . By the assumption  $g^*(\omega_i - \omega_j) = 0$ , we have  $\varphi \circ b \circ g = 0$  on  $O$ . Hence  $(\varphi \circ \psi) \circ (c \circ g) = 0$  on  $O$ . Thus by the construction of  $V_n$ , we have  $\text{ord}_y(c \circ g)^* V_n \geq n$ . Hence by  $V_n \subset D_n$ , we have  $\text{ord}_y(c \circ g)^* D_n \geq n$ .

By (3.4),  $c \circ g(Y)$  is Zariski dense in  $S$ . Hence we have

$$\begin{aligned} n\bar{N}(r, g, \Xi'_{ij}) &\leq n\bar{N}(r, c \circ g, P) \\ &\leq N(r, c \circ g, D_n) \\ &\leq l(n)T(r, c \circ g, L) + O(1), \end{aligned}$$

where the last estimate follows from the Nevanlinna inequality (3.1). Thus, by  $l(n) = o(n)$  and  $T(r, c \circ g, L) = O(T(r, g, M))$ , we have

$$\bar{N}(r, g, \Xi'_{ij}) \leq \varepsilon T(r, g, M) \parallel$$

for all  $\varepsilon > 0$ . We have proved our claim.

Now we go back to the proof of Proposition 3.1. By (3.3) and (3.5), we have

$$N_{\text{ram } p_Y}(r) \leq \varepsilon T(r, g, M) \parallel$$

for all  $\varepsilon > 0$ . Hence by Proposition 3.2 and Proposition 3.3 below, we conclude that the image  $g(Y)$  is not Zariski dense in  $X^s$ , which contradicts to (3.4). This concludes the proof of Proposition 3.1.  $\square$

**PROPOSITION 3.3.** — *Let  $X$  be a smooth projective variety such that (1) the Albanese map is generically finite, and (2)  $X$  is of general type. Let  $M$  be an ample line bundle on  $X$ . Let  $g: Y \rightarrow X$  be a holomorphic map from a Riemann surface  $Y$  with a proper surjective holomorphic map  $p_Y: Y \rightarrow \mathbb{C}$ . Assume that*

$$N_{\text{ram } p_Y}(r) \leq \varepsilon T(r, g, M) \parallel$$

for all  $\varepsilon > 0$ . Then the image of  $g$  is not Zariski dense in  $X$ .

This is a generalization of [18, Corollary 3.1.14]. The proof is parallel to that of [18, Corollary 3.1.14]. See also [17] for a generalization of Proposition 3.3.

### 4. Proof of Proposition 2.1

In this section, we prove Proposition 2.1. A representation of the fundamental group into an algebraic group  $G$  is called rigid if every nearby representation is conjugate to it. A representation which is not rigid is called non-rigid. The proof of Proposition 2.1 divides into two cases according to whether the representation  $\varrho: \pi_1(X) \rightarrow G$  is rigid or non-rigid.

#### 4.1. Case 1: $\varrho$ is rigid

In this case  $\varrho$  is defined over some number field  $K$ . Given a prime ideal  $p$  from  $\mathcal{O}_p$ , we denote by  $\varrho_p: \pi_1(X) \rightarrow G(K_p)$  the composition of  $\varrho: \pi_1(X) \rightarrow G(K)$  and the inclusion  $G(K) \subset G(K_p)$ . If there exists a prime ideal  $p$  such that  $\varrho_p$  is  $p$ -unbounded, then Proposition 2.1 is a direct consequence of Proposition 3.1. Hence in the following, we consider the case that  $\varrho_p$  is  $p$ -bounded for every prime ideal  $p$ .

In this case, we remark that  $\varrho^{-1}(G(\mathcal{O}_K))$  is of finite index in  $\pi_1(X)$  (cf. [19, p. 120]). This can be proved as follows: Since  $\pi_1(X)$  is finitely generated, there are only finite prime ideals  $p_1, \dots, p_k$  such that  $\varrho(\pi_1(X))$  is not contained in  $G(\mathcal{O}_{K_{p_i}})$ . Since  $\varrho(\pi_1(X))$  is  $p_i$ -bounded for all  $p_i$ , the image of  $\varrho(\pi_1(X))$  in  $G(K_p)/G(\mathcal{O}_{K_p})$  is finite for all  $p_i$ . This shows our assertion.

Thus, after passing to a finite unramified covering, we may assume that  $\varrho(\pi_1(X)) \subset G(\mathcal{O}_K)$ .

Now by a result of Simpson (cf. [16, p. 58]),  $\varrho$  is a complex direct factor of a  $\mathbb{Z}$ -variation of Hodge structure. In particular, there is the period mapping  $c: X \rightarrow \Gamma \backslash \mathcal{D}$  of this variation of Hodge structure (cf. [7, p. 57]). Here  $\mathcal{D}$  is the classifying space and  $\Gamma$  is the arithmetic group which preserves the polarization and the lattice of the variation of Hodge structure. Then  $c$  is a horizontal locally liftable holomorphic map (cf. [7, 3.13]). Hence, for a holomorphic map  $f: \mathbb{C} \rightarrow X$ ,  $c \circ f: \mathbb{C} \rightarrow \Gamma \backslash \mathcal{D}$  is also a horizontal locally liftable holomorphic map. Since  $\mathcal{D}$  has negative curvature in the horizontal direction,  $c \circ f$  is constant (cf. [6, Corollary 9.7]). Since  $\Gamma \backslash \mathcal{D}$  has the structure of a normal analytic space (cf. [7, p. 56]), the fibers of  $c$  are Zariski closed subsets on  $X$ . This shows that  $f$  is degenerate. Hence we have proved Proposition 2.1 when  $\varrho$  is rigid.

#### 4.2. Case 2: $\varrho$ is non-rigid

It suffices to prove the following:

LEMMA 4.1. — *Let  $G$  be an almost simple algebraic group defined over the complex number field. Assume that there exists a Zariski dense, non-rigid representation  $\varrho: \pi_1(X) \rightarrow G(\mathbb{C})$ . Then every holomorphic map  $f: \mathbb{C} \rightarrow X$  is degenerate.*

*Proof.* — We remark that  $G$  is defined over some number field  $K$  after some conjugations. Since  $\pi_1(X)$  is finitely presented, there exists an affine scheme  $R$  over  $K$  such that

$$R(L) = \text{Hom}(\pi_1(X), G(L))$$

for every field extension  $L/K$ . This space is defined as follows: We choose generators  $\gamma_1, \dots, \gamma_k$  for  $\pi_1(X)$ . Let  $\mathcal{R}$  be the set of relations among the generators  $\gamma_i$ . Then

$$R \subset \underbrace{G \times \cdots \times G}_{k \text{ times}}$$

is the closed subscheme defined by the equations  $r(m_1, \dots, m_k) = 1$  for  $r \in \mathcal{R}$ . A representation  $\tau: \pi_1(X) \rightarrow G(L)$  corresponds to the point  $(m_1, \dots, m_k) \in R(L)$  with  $m_i = \tau(\gamma_i)$ . Note that  $R$  is an affine scheme, since it is a closed subscheme of an affine variety. Let  $R_{Z.D.} \subset R$  be the space of Zariski dense representations. Then by [1, Proposition 8.2],  $R_{Z.D.}$

is a Zariski open subset of  $R$ . The group  $G$  acts on  $R$  by simultaneous conjugation. Put  $M = R//G$ , and let  $p: R \rightarrow M$  be the quotient map. Then  $M$  is an affine scheme defined over  $K$ . Let  $[\varrho] \in R_{Z.D.}(\mathbb{C})$  be the point which correspond to the Zariski dense representation  $\varrho: \pi_1(X) \rightarrow G(\mathbb{C})$ .

Since  $R_{Z.D.}(\bar{\mathbb{Q}})$  is dense in  $R_{Z.D.}(\mathbb{C})$ , by deforming  $\varrho$  slightly and replacing  $K$  by its finite extension, we may assume that  $\varrho$  is defined over  $K$ . Let  $\mathfrak{p}$  be a prime ideal from  $\mathcal{O}_K$  and let  $K_{\mathfrak{p}}$  be the completion. In the following, we shall work over this  $K_{\mathfrak{p}}$ .

Since  $\varrho$  is non-rigid, we have  $\dim M > 0$ . Hence there exists a morphism  $\psi: M \rightarrow \mathbb{A}^1$  such that the image  $\psi(M)$  is Zariski dense in  $\mathbb{A}^1$ . Since the image  $\psi \circ p(R_{Z.D.})$  is also Zariski dense in  $\mathbb{A}^1$ , there exists an affine curve  $C \subset R_{Z.D.}$  such that the restriction  $\psi \circ p|_C: C \rightarrow \mathbb{A}^1$  is generically finite. We may take a Zariski open subset  $U \subset \mathbb{A}^1$  such that  $\psi \circ p|_C$  is finite over  $U$ . Let  $x \in U(K_{\mathfrak{p}})$  be a point, and let  $y \in C(\bar{K}_{\mathfrak{p}})$  be a point over  $x$ . Then  $y$  is defined over some extension of  $K_{\mathfrak{p}}$  whose extension degree is bounded by the degree of  $\psi \circ p|_C: C \rightarrow \mathbb{A}^1$ . Note that there are only finitely many such field extensions. Hence there exists a finite extension  $L/K_{\mathfrak{p}}$  such that the points over  $U(K_{\mathfrak{p}})$  are all contained in  $C(L)$ . Since  $U(K_{\mathfrak{p}}) \subset \mathbb{A}^1(L)$  is unbounded, the image  $\psi \circ p(R_{Z.D.}(L)) \subset \mathbb{A}^1(L)$  is unbounded.

Let  $R_0 \subset R(L)$  be the subset whose points correspond to  $p$ -bounded representations. Let  $M_0 \subset M(L)$  be the image of  $R_0$  under the quotient  $p: R \rightarrow M$ . Then by Lemma 4.2 below,  $M_0$  is compact. Hence  $\psi(M_0)$  is compact. In particular it is bounded. On the other hand,  $\psi \circ p(R_{Z.D.}(L)) \subset \mathbb{A}^1(L)$  is unbounded. Hence we have  $R_{Z.D.}(L) \not\subset R_0$ . Thus we may take a Zariski dense,  $p$ -unbounded representation  $\bar{\varrho}: \pi_1(X) \rightarrow G(L)$ . By Proposition 3.1, every holomorphic map  $f: \mathbb{C} \rightarrow X$  is degenerate. □

LEMMA 4.2. —  $M_0$  is compact.

*Proof.* — Note that there are only finitely many conjugacy classes of maximal compact subgroups in  $G(L)$ . Hence all maximal compact subgroups are conjugate to one of maximal compact subgroups  $H_1, \dots, H_k \subset G(L)$ . Hence given a  $p$ -bounded representation  $\tau: \pi_1(X) \rightarrow G(L)$ , there is a  $G(L)$ -conjugation  $\tilde{\tau}: \pi_1(X) \rightarrow G(L)$  of  $\tau$  such that the image  $\tilde{\tau}(\pi_1(X))$  is contained in one of  $H_1, \dots, H_k$ .

Now take a sequence  $[\tau_1], [\tau_2], \dots \in M_0$ . Then we may take representations  $\tau_1, \tau_2, \dots$  from  $R_0$  such that  $\tau_j(\pi_1(X))$  is contained in one of  $H_1, \dots, H_k$ . By taking subsequence, we may assume that  $\tau_j(\pi_1(X)) \subset H_i$  for all  $j$ . Now since  $H_i$  is compact, some subsequence  $\tau_j$  should converge to  $\tau_{\infty}: \pi_1(X) \rightarrow H_i$ . Then the sequence  $[\tau_j]$  converges to  $[\tau_{\infty}] \in M_0$ . This shows that  $M_0$  is compact. □

## 5. Proof of Corollary 1.3

Let  $X$  be a smooth projective variety with a big representation  $\varrho: \pi_1(X) \rightarrow \mathrm{GL}_n(\mathbb{C})$ . Assume that  $X$  admits a non-degenerate holomorphic map  $f: \mathbb{C} \rightarrow X$ . Then by Theorem 1.1, the image  $\varrho(\pi_1(X))$  is almost abelian. Hence after passing to a finite unramified covering  $X'$  of  $X$  we may assume that  $\varrho(\pi_1(X))$  is a free abelian group, i.e.  $\varrho$  factors the Albanese map  $a_X: X \rightarrow \mathrm{Alb}(X)$ . We shall prove that the Albanese map  $a_X$  is birational.

Since  $X$  admits a non-degenerate holomorphic map, the Albanese map  $a_X$  is surjective ([14, Theorem 6.4.1]) and has connected fibers ([15]). Let  $F$  be a general fiber of  $a_X$ . Then  $\varrho(\mathrm{Im}(\pi_1(F) \rightarrow \pi_1(X)))$  is trivial. Since  $\varrho$  is big,  $F$  should be a point. Hence the Albanese map  $a_X$  is birational. This concludes the proof of the corollary.

## BIBLIOGRAPHY

- [1] N. A'CAMPO & M. BURGER, "Réseaux arithmétiques et commensurateur d'après G. A. Margulis", *Invent. Math.* **116** (1994), no. 1-3, p. 1-25.
- [2] G. T. BUZZARD & S. S. Y. LU, "Algebraic surfaces holomorphically dominable by  $\mathbb{C}^2$ ", *Invent. Math.* **139** (2000), no. 3, p. 617-659.
- [3] F. CAMPANA, "Ensembles de Green-Lazarsfeld et quotients résolubles des groupes de Kähler", *J. Algebraic Geom.* **10** (2001), no. 4, p. 599-622.
- [4] F. CAMPANA, "Orbifolds, special varieties and classification theory", *Ann. Inst. Fourier (Grenoble)* **54** (2004), no. 3, p. 499-630.
- [5] P. EYSSIDIEUX, "Sur la convexité holomorphe des revêtements linéaires réductifs d'une variété projective algébrique complexe", *Invent. Math.* **156** (2004), no. 3, p. 503-564.
- [6] P. GRIFFITHS & W. SCHMID, "Locally homogeneous complex manifolds", *Acta Math.* **123** (1969), p. 253-302.
- [7] ———, "Recent developments in Hodge theory: a discussion of techniques and results", in *Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973)*, Oxford Univ. Press, Bombay, 1975, p. 31-127.
- [8] M. GROMOV & R. SCHOEN, "Harmonic maps into singular spaces and  $p$ -adic super-rigidity for lattices in groups of rank one", *Inst. Hautes Études Sci. Publ. Math.* (1992), no. 76, p. 165-246.
- [9] J. JOST & K. ZUO, "Harmonic maps into Bruhat-Tits buildings and factorizations of  $p$ -adically unbounded representations of  $\pi_1$  of algebraic varieties. I", *J. Algebraic Geom.* **9** (2000), no. 1, p. 1-42.
- [10] L. KATZARKOV, "On the Shafarevich maps", in *Algebraic geometry—Santa Cruz 1995*, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, p. 173-216.
- [11] J. KOLLÁR, "Shafarevich maps and plurigenera of algebraic varieties", *Invent. Math.* **113** (1993), no. 1, p. 177-215.
- [12] J. NOGUCHI, "Meromorphic mappings of a covering space over  $\mathbb{C}^m$  into a projective variety and defect relations", *Hiroshima Math. J.* **6** (1976), no. 2, p. 265-280.

- [13] ———, “On the value distribution of meromorphic mappings of covering spaces over  $\mathbf{C}^m$  into algebraic varieties”, *J. Math. Soc. Japan* **37** (1985), no. 2, p. 295-313.
- [14] J. NOGUCHI & T. OCHIAI, *Geometric function theory in several complex variables*, Translations of Mathematical Monographs, vol. 80, American Mathematical Society, Providence, RI, 1990, Translated from the Japanese by Noguchi, xii+283 pages.
- [15] J. NOGUCHI, J. WINKELMANN & K. YAMANOI, “Degeneracy of holomorphic curves into algebraic varieties”, *J. Math. Pures Appl. (9)* **88** (2007), no. 3, p. 293-306.
- [16] C. T. SIMPSON, “Higgs bundles and local systems”, *Inst. Hautes Études Sci. Publ. Math.* (1992), no. 75, p. 5-95.
- [17] K. YAMANOI, “Holomorphic curves in abelian varieties and intersections with higher codimensional subvarieties II”, preprint.
- [18] ———, “Holomorphic curves in abelian varieties and intersections with higher codimensional subvarieties”, *Forum Math.* **16** (2004), no. 5, p. 749-788.
- [19] R. J. ZIMMER, *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81, Birkhäuser Verlag, Basel, 1984, x+209 pages.
- [20] K. ZUO, “Kodaira dimension and Chern hyperbolicity of the Shafarevich maps for representations of  $\pi_1$  of compact Kähler manifolds”, *J. Reine Angew. Math.* **472** (1996), p. 139-156.
- [21] ———, *Representations of fundamental groups of algebraic varieties*, Lecture Notes in Mathematics, vol. 1708, Springer-Verlag, Berlin, 1999, viii+135 pages.

Manuscrit reçu le 4 décembre 2008,  
 accepté le 27 mars 2009.

Katsutoshi YAMANOI  
 Kumamoto University  
 Graduate School of Science and Technology  
 Kurokami, Kumamoto 860-8555 (Japan)  
 yamanoi@kumamoto-u.ac.jp