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GEOMETRIC INVARIANT THEORY AND GENERALIZED EIGENVALUE PROBLEM II

by Nicolas RESSAYRE (*)

Abstract. — Let G be a connected reductive subgroup of a complex connected reductive group $\hat{G}$. Fix maximal tori and Borel subgroups of G and $\hat{G}$. Consider the cone $\mathcal{LR}^\circ(G,\hat{G})$ generated by the pairs $(\nu,\hat{\nu})$ of strictly dominant characters such that V_ν^* is a submodule of $V_{\hat{\nu}}$. We obtain a bijective parametrization of the faces of $\mathcal{LR}^\circ(G,\hat{G})$ as a consequence of general results on GIT-cones. We show how to read the inclusion of faces off this parametrization.

Résumé. — Soit G un sous-groupe fermé réductif et connexe d'un groupe réductif complexe et connexe $\hat{G}$. On fixe des tores maximaux et des sous-groupes de Borel de G et $\hat{G}$. De cette manière les représentations irréductibles de G et $\hat{G}$ sont paramétrées par des poids dominants. On s'intéresse au cône $\mathcal{L}R^\circ(G,\hat{G})$ engendré par les paires $(\nu,\hat{\nu})$ de poids dominants réguliers tels que V_ν^* est un sous-G-module de $V_{\hat{\nu}}$. Nous obtenons ici une paramétrisation bijective des faces de $\mathcal{L}R^\circ(G,\hat{G})$, en étudiant plus généralement les GIT-cônes des G-variétés projectives. Nous montrons aussi comment les relations d'inclusions entre les faces de $\mathcal{L}R^\circ(G,\hat{G})$ se lisent sur notre paramétrisation.

1. Introduction

The ground field $\mathbb{K}$ is assumed to be algebraically closed of characteristic zero. Consider a connected reductive group G acting on a projective variety X. To any ample G-linearized line bundle $\mathcal{L}$ on X we associate the following open subset $X^{\rm ss}(\mathcal{L})$ of X:

$$X^{\rm ss}(\mathcal{L}) = \left\{ x \in X \,:\, \exists n > 0 \text{ and } \sigma \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \sigma(x) \neq 0 \right\}.$$

The points of $X^{\rm ss}(\mathcal{L})$ are said to be *semistable* with respect to $\mathcal{L}$. There exists a good quotient $\pi : X^{\rm ss}(\mathcal{L}) \longrightarrow X^{\rm ss}(\mathcal{L})/\!/G$. A natural question is:

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What are the $\mathcal{L}$'s with nonempty set $X^{\rm ss}(\mathcal{L})$?

Let us fix a free finitely generated subgroup Λ of the group $\mathrm{Pic}^G(X)$ of G-linearized line bundles on X. Set $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that the classes of ample G-linearized line bundles in Λ generate an open convex cone $\Lambda_{\mathbb{Q}}^{++}$ in $\Lambda_{\mathbb{Q}}$. Define the G-ample cone $\mathcal{AC}_\Lambda^G(X)$ as the locally closed subcone generated by the ample $\mathcal{L} \in \Lambda$ such that $X^{\rm ss}(\mathcal{L})$ is not empty. By [5] (see also [14]), $\mathcal{AC}_\Lambda^G(X)$ is characterized as a subset of $\Lambda_{\mathbb{Q}}^{++}$ by finitely many rational linear inequalities: we will say that $\mathcal{AC}_\Lambda^G(X)$ is a closed convex rational polyhedral cone in $\Lambda_{\mathbb{Q}}^{++}$. We are interested in the faces of $\mathcal{AC}_\Lambda^G(X)$; that is, the subcones obtained by intersecting $\mathcal{AC}_\Lambda^G(X)$ with a supporting hyperplane.

The following statement associates a G-variety to any nonempty face of $\mathcal{AC}_\Lambda^G(X)$. It will be useful to prove the injectivity of our parametrization of the faces of $\mathcal{LR}^\circ(G,\hat{G})$.

Proposition 1.1. — *Let $\mathcal{F}$ be a face of $\mathcal{AC}_\Lambda^G(X)$. Let $\mathcal{L}$ and $\mathcal{L}'$ be two points in the relative interior of $\mathcal{F}$. Let π and π' denote the corresponding quotient maps.*

Then, for general $\xi \in X^{\rm ss}(\mathcal{L})/\!/G$ and general $\xi' \in X^{\rm ss}(\mathcal{L}')/\!/G$), the G-varieties $\pi^{-1}(\xi')$ and $\pi^{-1}(\xi)$ are isomorphic.

Let $\mathcal{L}_0$ be any point in the relative interior of a face $\mathcal{F}$ of $\mathcal{AC}^G(X)$. The local geometry of $\mathcal{AC}_\Lambda^G(X)$ along $\mathcal{F}$ is described by the convex cone $\mathcal{C}_\mathcal{F}$ generated by the vectors $p - \mathcal{L}_0$ for $p \in \mathcal{AC}_\Lambda^G(X)$. This cone only depends on $\mathcal{F}$ and not on $\mathcal{L}_0$.

To describe $\mathcal{C}_\mathcal{F}$, we introduce some notation useful to describe. Consider the quotient $\pi\,:\,X^{\rm ss}(\mathcal{L}_0) \longrightarrow X^{\rm ss}(\mathcal{L}_0)/\!/G$. Let x be any point in $X^{\rm ss}(\mathcal{L}_0)$ with closed orbit in $X^{\rm ss}(\mathcal{L}_0)$; then the isotropy subgroup G_x is reductive. Consider the set V of points $y \in X$ such that the closure of the orbit $G_x.y$ contains x. Then V is an affine closed G_x-stable subvariety of $X^{\rm ss}(\mathcal{L}_0)$ containing $\{x\}$ as its unique closed G_x-orbit. Moreover, the fiber $\pi^{-1}(\pi(x))$ is isomorphic to the fiber product $G \times_{G_x} V$. Let $X(G_x)$ denote the group of characters of G_x. Consider the convex cone $\mathcal{C}_x$ in $X(G_x) \otimes \mathbb{Q}$ generated by the weights of G_x acting on the algebra of regular functions on L. Finally, we consider the linear map $\mu\,:\,\Lambda_{\mathbb{Q}} \longrightarrow X(G_x) \otimes \mathbb{Q}$, that associates with any $\mathcal{L}$ the weight of the action of G_x on $\mathcal{L}_x$.

Proposition 1.2. — *With above notation, we have:*

$$\mathcal{C}_\mathcal{F} = \mu^{-1}(\mathcal{C}_x).$$

Proposition 1.2 is closely related to R. Sjamaar's description of the local structure of the moment polytope in the symplectic setting (see [16]). In Sjamaar's situation, G_x° is a torus which simplifies the description of $\mathcal{C}_x$.

Now, assume that G is embedded in another connected reductive group $\hat{G}$. We fix maximal tori $T \subset \hat{T}$ and Borel subgroups $B \supset T$ and $\hat{B} \supset \hat{T}$ of G and $\hat{G}$. Consider the complete flag variety $\mathcal{B} = G/B \times \hat{G}/\hat{B}$ endowed with the diagonal G-action. We set $\Lambda = \mathrm{Pic}^G(\mathcal{B})$ and denote $\mathcal{AC}^G_\Lambda(\mathcal{B})$ by $\mathcal{AC}^G(\mathcal{B})$. To any $(\nu, \hat{\nu}) \in X(T) \times X(\hat{T})$ we associate the usual $G \times \hat{G}$-linearized line bundle $\mathcal{L}_{(\nu,\hat{\nu})}$ over $\mathcal{B}$: this defines an isomorphism $X(T) \times X(\hat{T}) \xrightarrow{\sim} \mathrm{Pic}^{G\times\hat{G}}(\mathcal{B})$. We also consider the map $r_G : \mathrm{Pic}^{G\times\hat{G}}(\mathcal{B}) \longrightarrow \mathrm{Pic}^G(\mathcal{B})$ obtained by restricting the action to $G \times \hat{G}$ to G. Note that $\mathcal{L}_{(\nu,\hat{\nu})}$ is ample if and only if ν and $\hat{\nu}$ are strictly dominant. Let us denote by $\mathcal{LR}^\circ(G, \hat{G})$ the set of pairs $(\nu, \hat{\nu})$ of rational strictly dominant weights such that $V^*_{n\nu}$ is a G-submodule of $V_{n\hat{\nu}}$ for some positive n. A direct application of the Borel-Weil theorem shows that $(\nu, \hat{\nu}) \in \mathcal{LR}^\circ(G, \hat{G})$ if and only if $r_G(\mathcal{L}_{(\nu,\hat{\nu})}) \in \mathcal{AC}^G(\mathcal{B})$.

The interior of $\mathcal{AC}^G(\mathcal{B})$ is nonempty if and only if no nontrivial connected normal subgroup of G is normal in $\hat{G}$: we assume, from now on that $\mathcal{AC}^G(\mathcal{B})$ has nonempty interior. In Theorem 7.2 below, we obtain a bijective parametrization of the faces of $\mathcal{AC}^G(\mathcal{B})$ and a description of their inclusion relations. To avoid too many notation, in this introduction, we will only state our results in the case when G is semisimple and diagonally embedded in $\hat{G} = G \times G$.

For any standard parabolic subgroup P of G, we consider the cohomology group $\mathrm{H}^*(G/P, \mathbb{Z})$ and its basis consisting of classes of Schubert varieties. We consider on this group the Belkale-Kumar product $\odot_0$ defined in [2]. The structure-coefficient of this product in this basis are either 0 or the structure-coefficient of the usual cup product. These coefficients are parametrized by the triples of Schubert classes.

Theorem 1.3. — *The faces of $\mathcal{AC}^G((G/B)^3)$ correspond bijectively to the triples of Schubert classes in G/P with structure-coefficients of the ring $(\mathrm{H}^*(G/P, \mathbb{Z}), \odot_0)$ equal to one, for the various standard parabolic subgroups P of G.*

We will now explain how to read the inclusions of faces off this parametrization. Let P and P' be two standard parabolic subgroups. Let X_1, X_2 and X_3 (resp. X'_1, X'_2 and X'_3) be three Schubert varieties in G/P (resp. G/P') such that the corresponding structure-coefficients for $\odot_0$ are equal to one. Let $\mathcal{F}$ and $\mathcal{F}'$ denote the corresponding faces of $\mathcal{AC}^G_\Lambda((G/B)^3)$.

THEOREM 1.4. — *The following are equivalent:*

(1) $\mathcal{F} \subset \mathcal{F}'$;
(2) $P \subset P'$ *and* $p(X_i) = X_i'$ *for* $i = 1, 2$ *and* 3, *where* $p : G/P \longrightarrow G/P'$ *is the projection.*

In [15], we already obtained a partial description of the faces of $\mathcal{LR}^\circ(G,\hat{G})$. Here we make three improvements. First, we prove that each face of the closure of $\mathcal{LR}^\circ(G,\hat{G})$ coming from a well-covering pair (see Section 5.2) is a face of $\mathcal{LR}^\circ(G,\hat{G})$; that is, contains strictly dominant weights. Moreover, defining the notion of an admissible pair $(S,\hat{w})$ (see Section 7), we obtain an injective parametrization of the faces. The proof of this injectivity uses Propositions 1.1 and 1.2. Finally, we describe the inclusion relations between the faces.

Let us explain some of the motivations to understand these faces. The first motivation comes a posteriori. Theorem 1.3 and 1.4 show that this set has a rich structure. A strategy to understand examples is to study the smallest faces (whose codimension is the rank of G) and to understand the local geometry around theses faces: this should be interesting since the Belkale-Kumar product on G/B is particularly simple. The results obtained here would allow to apply such a strategy.

Let V be a complex finite-dimensional vector space. The closure of the cone $\mathcal{AC}^{\mathrm{GL}(V)}(\mathcal{F}\updownarrow(V)^3)$ is the object of the famous Horn conjecture [6]. We will call $\mathcal{AC}^{\mathrm{GL}(V)}(\mathcal{F}\updownarrow(V)^3)$ the Horn cone. In [7], Knutson-Tao-Woodward proved that the codimension one faces of the Horn cone correspond bijectively to the Littlewood-Richardson coefficients (LR-coefficients for short) $c_{\lambda\mu}^\nu$ equal to one, for partitions λ, μ and ν of a given size (Theorem 1.3 generalizes this result). The Fulton conjecture (see [7, 1, 13]) asserts that if $c_{\lambda\mu}^\nu = 1$ then for any positive integer k, $c_{k\lambda\,k\mu}^{k\nu} = 1$. This implies easily that the set of LR-coefficients equal to one is parametrized by the integral points on a union of faces of the Horn cone.

The relations between the geometry of $\mathcal{LR}^\circ(G,G^2)$ and the Belkale-Kumar cohomology rings will be also applied elsewhere to obtain results about this product.

Let now Y be a normal projective G-variety endowed with an ample G-linearized line bundle $\mathcal{L}$. We now want to explain how to recover the moment polytope of Y from some G-ample cone. Consider the dominant chamber $X(T)^+_{\mathbb{Q}}$ in $X(T)_{\mathbb{Q}} = X(T)\otimes\mathbb{Q}$ corresponding to B. The interior of $X(T)^+_{\mathbb{Q}}$ is denoted by $X(T)^{++}_{\mathbb{Q}}$. We are interested in the moment polytope $P(Y,\mathcal{L})$ as defined in [4]: it is a convex polytope contained in $X(T)^+_{\mathbb{Q}}$. Set $P^\circ(Y,\mathcal{L}) := P(Y,\mathcal{L}) \cap X(T)^{++}_{\mathbb{Q}}$.

Set $X = G/B \times Y$. Let Λ be the subgroup of $\mathrm{Pic}^G(X)$ generated by the pullback of $\mathcal{L}$ and the pullbacks of G-linearized line bundles on G/B. Now, we have:

$$P^\circ(Y,\mathcal{L}) = \mathcal{AC}^G_\Lambda(X) \cap (\mathcal{L} + X(T)_{\mathbb{Q}}).$$

In particular, the faces of $\mathcal{AC}^G_\Lambda(X)$ correspond bijectively to the faces of $P^\circ(Y,\mathcal{L})$.

We assume that $P^\circ(Y,\mathcal{L})$ is nonempty. Actually, in [4] it is proved that any moment polytope $P(Y,\mathcal{L})$ can be described in terms of one which intersects the interior of the dominant chamber. M. Brion associated in [4, Theorem 1 and 2] a subtorus S of T and an irreducible component C of the T-fixed points to any face of $P^\circ(Y,\mathcal{L})$. He proved that the face can be recovered from the pair (C,S) and obtained some geometrical properties of (C,S). In Proposition 8.4 below, we improve these results by showing that (C,S) is well-B-covering (see Definition 8.2).

CONVENTION. — *The notation introduced in the environments* "NOTATION" *are fixed for all the sequence of the paper.*

2. An example of GIT-cone

Let us fix a connected reductive group G acting on an irreducible projective algebraic variety X.

2.1. The G-ample cone

As in the introduction, for an ample G-linearized line bundle $\mathcal{L}$ on X, we consider the set of *semistable points*:

$$X^{\mathrm{ss}}(\mathcal{L}) = \left\{x \in X \,:\, \exists n > 0 \text{ and } \sigma \in \mathrm{H}^0(X,\mathcal{L}^{\otimes n})^G \text{ such that } \sigma(x) \neq 0\right\}.$$

To specify the acting group, we sometimes denote $X^{\mathrm{ss}}(\mathcal{L})$ by $X^{\mathrm{ss}}(\mathcal{L},G)$. There exists a good quotient:

$$\pi \,:\, X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})/\!/G,$$

such that $X^{\mathrm{ss}}(\mathcal{L})/\!/G$ is a projective variety. A point $x \in X^{\mathrm{ss}}(\mathcal{L})$ is said to be *stable* if G_x is finite and $G.x$ is closed in $X^{\mathrm{ss}}(\mathcal{L})$. Then, for any stable point x we have $\pi^{-1}(\pi(x)) = G.x$ and the set $X^{\mathrm{s}}(\mathcal{L})$ of stable points is open in X.

Let Λ be a free finitely generated subgroup of $\mathrm{Pic}^G(X)$ and set $\Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q}$. Let $\Lambda^{++}_{\mathbb{Q}}$ denote the convex cone generated by the ample elements of Λ. Since $X^{\rm ss}(\mathcal{L})=X^{\rm ss}(\mathcal{L}^{\otimes n})$, for any ample $\mathcal{L}$ and any positive integer n, we can define $X^{\rm ss}(\mathcal{L})$ for any $\mathcal{L}\in\Lambda_{\mathbb{Q}}^{++}$. We consider the *G-ample cone*:

$$\mathcal{AC}^G_\Lambda(X)=\{\mathcal{L}\in\Lambda_{\mathbb{Q}}^{++}\ :\ X^{\rm ss}(\mathcal{L})\text{ is not empty}\}.$$

Since the tensor product of two nonzero G-invariant sections is a nonzero G-invariant section, $\mathcal{AC}^G_\Lambda(X)$ is a convex cone. By [5] (see also [14]), $\mathcal{AC}^G_\Lambda(X)$ is a closed convex rational polyhedral cone when viewed as a subcone of $\Lambda_{\mathbb{Q}}^{++}$. This cone is the central object of this paper.

Two points $\mathcal{L}$ and $\mathcal{L}'$ in $\mathcal{AC}^G_\Lambda(X)$ are said to be *GIT-equivalent* if $X^{\rm ss}(\mathcal{L})=X^{\rm ss}(\mathcal{L}')$. An equivalence class is simply called a *GIT-class*.

For $x\in X$, the *stability set of x* is the set of $\mathcal{L}\in\Lambda_{\mathbb{Q}}^{++}$ such that $X^{\rm ss}(\mathcal{L})$ contains x; it is denoted by $\mathcal{AC}^G_\Lambda(x)$. In [14] following [5], we have studied the geometry of the GIT-classes and the stability sets. Note that [14] only considers the case where $\Lambda=\mathrm{Pic}^G(X)$; but all the results and proofs of [14] remain valid here. In particular, there are only finitely many GIT-classes and each GIT-class is the relative interior of a closed convex polyhedral cone of $\Lambda_{\mathbb{Q}}^{++}$. Finally, the closures of GIT-classes form a fan in $\Lambda_{\mathbb{Q}}^{++}$.

2.2. The G-ample cone of an affine variety

Notation 2.1. — If Γ is an affine algebraic group, $[\Gamma,\Gamma]$ will denote its derived subgroup and $X(\Gamma)$ will denote its character group.

For later use, we consider here a G-ample cone for the action of G on an affine variety. More precisely, let V be an affine G-variety containing a fixed point O as its unique closed orbit. The action of G over the fiber at O gives a morphism $\mu^\bullet(O,G)\ :\ \mathrm{Pic}^G(V)\longrightarrow X(G)$ satisfying:

$$\forall\mathcal{L}\in\mathrm{Pic}^G(V)\ \forall g\in G,\, x\in\mathcal{L}_O\ \ g.x=\mu^{\mathcal{L}}(O,G)(g^{-1})x,$$

where $\mathcal{L}_O$ is the fiber over O in $\mathcal{L}$. By [14, Lemma 7], $\mu^\bullet(O,G)$ is an isomorphism. The pullback $\mathcal{L}_\chi$ of a character χ is the trivial bundle endowed with the following action:

$$\forall g\in G\ \forall(v,t)\in V\times\mathbb{K}\ \ g.(v,t)=(v,\chi(g^{-1})t).$$

For any $\chi\in X(G)$, we have:

$$\mathrm{H}^0(V,\mathcal{L}_\chi)^G=\{f\in\mathbb{K}[V]\ :\ \forall x\in V\ \ (g.f)(x)=\chi(g)f(x)\}=\mathbb{K}[V]_\chi.$$

Note that $\mathrm{H}^0(V,\mathcal{L}_\chi)^G$ is contained in the algebra $\mathbb{K}[V]^{[G,G]}$ of the regular $[G,G]$-invariant functions on V. Set

$$S=\{\chi\in X(G)\ :\ \mathrm{H}^0(V,\mathcal{L}_\chi)^G\ \text{ is non trivial}\}.$$

This is the set of weights of the torus $G/[G,G]$ in $\mathbb{K}[V]^{[G,G]}$.

As in the projective case, we set

$$V^{\mathrm{ss}}(\chi)=\{v\in V\ :\ \exists n>0 \text{ and } f\in\mathbb{K}[V]_{n\chi}\ \text{ s.t. } f(v)\neq 0\}.$$

We consider the G-cone $\mathcal{AC}^G(V)$ in $X(G)\otimes\mathbb{Q}$ generated by the characters χ such that $V^{\mathrm{ss}}(\chi)\neq\emptyset$. Note that any $\mathcal{L}_\chi$ is ample.

Lemma 2.2. — *We assume that V is irreducible. Then the set S is a finitely generated semigroup in $X(G)$. Moreover, $\mathcal{AC}^G(V)$ is the convex cone generated by S; it is strictly convex.*

Proof. — Since $\mathbb{K}[V]^{[G,G]}$ is a finitely generated algebra without zero divisors, S is a finitely generated semigroup. The fact that $\mathcal{AC}^G(V)$ is generated by S is obvious. Finally, $\mathcal{AC}^G(V)$ is strictly convex since $\mathrm{H}^0(V,\mathcal{L}_0)^G=\mathbb{K}$, where $\mathcal{L}_0$ denotes the trivial line bundle linearized with the trivial action. □

3. Etale Slice Theorem

In this section, we recall some very useful results of D. Luna. We fix an ample G-linearized line bundle $\mathcal{L}$ on the irreducible projective G-variety X.

3.1. Closed orbits in general position

Notation 3.1. — If H is a subgroup of G then $N_G(H)$ denotes the normalizer of H in G. Consider the quotient $\pi: X^{\mathrm{ss}}(\mathcal{L})\longrightarrow X^{\mathrm{ss}}(\mathcal{L})/\!/G$. For all $\xi\in X^{\mathrm{ss}}(\mathcal{L})/\!/G$, we denote by $T(\xi)$ the unique closed G-orbit in $\pi^{-1}(\xi)$. We denote by $(X^{\mathrm{ss}}(\mathcal{L})/\!/G)_{\mathrm{pr}}$ the set of those ξ such that there exists an open neighborhood Ω of ξ in $X^{\mathrm{ss}}(\mathcal{L})/\!/G$ such that the orbit $T(\xi')$ is isomorphic to $T(\xi)$, for all $\xi'\in\Omega$. We will denote by X^H the H-fixed point set. If Y is a locally closed subvariety of X, $\mathcal{L}_{|Y}$ denotes the restriction of $\mathcal{L}$ to Y.

Since π is a gluing of affine quotients, many results on the actions of G on affine variety remain true for $X^{\mathrm{ss}}(\mathcal{L})$. For example, the following theorem is a result of Luna and Richardson (see [10, Section 3] and [9, Corollary 4] or [12, Section 7]):

Theorem 3.2. — *Keep the above notation and assume that X is normal. Then, the set $(X^{\mathrm{ss}}(\mathcal{L})/\!/G)_{\mathrm{pr}}$ is nonempty and open in $X^{\mathrm{ss}}(\mathcal{L})/\!/G$. Let H be the isotropy group of a point in $T(\xi_0)$ with $\xi_0\in(X^{\mathrm{ss}}(\mathcal{L})/\!/G)_{\mathrm{pr}}$. We have:*

(1) *The group H has fixed points in $T(\xi)$ for any $\xi \in X^{\rm ss}(\mathcal{L})/\!/G$.*
(2) *Let Y be the closure of $\pi^{-1}\left((X^{\rm ss}(\mathcal{L})/\!/G)_{\rm pr}\right)^H$ in X. Then Y is the union of some components of X^H. Moreover, H acts trivially on some positive power $\mathcal{L}_{|Y}^{\otimes n}$ of $\mathcal{L}_{|Y}$ and the natural map*

$$Y^{\rm ss}(\mathcal{L}_{|Y}^{\otimes n})/\!/(N_G(H)/H) \longrightarrow X^{\rm ss}(\mathcal{L})/\!/G$$

is an isomorphism. Finally, Y contains stable points for the action of $N_G(H)/H$ and for the line bundle $\mathcal{L}_{|Y}^{\otimes n}$.

A subgroup H as in Theorem 3.2 will be called *a principal isotropy group of $X^{\rm ss}(\mathcal{L})$*. The conjugacy class of H is called *the principal isotropy group of $X^{\rm ss}(\mathcal{L})$*.

3.2. The principal Luna stratum

Let H be a reductive subgroup of G and let Y be an affine H-variety. We endow $G \times Y$ with the $G \times H$-action given by the formula (with obvious notation):

$$(g,h).(g_1,y) = (gg_1h^{-1}, hy).$$

Then the GIT-quotient of $G \times Y$ by $\{e\} \times H$ is denoted by $G \times_H Y$. Since this action is free, $G \times_H Y$ is an affine variety (smooth if Y is) whose closed points parametrize the H-orbits in $G \times Y$. The class of $(g,y) \in G \times Y$ will be denoted by $[g : y]$. The action of $G \times \{e\}$ on $G \times Y$ induces a G-action on $G \times_H Y$ and the first projection, a G-equivariant morphism $G \times_H Y \longrightarrow G/H$.

When X is smooth, the open subset $(X^{\rm ss}(\mathcal{L})/\!/G)_{\rm pr}$ is called the *principal Luna stratum* and has very useful properties (see [8] or [12]):

Theorem 3.3 (Luna). — *We assume that X is smooth. Let H be a principal isotropy group of $X^{\rm ss}(\mathcal{L})$.*

Then, there exists a H-module L such that for any $\xi \in (X^{\rm ss}(\mathcal{L})/\!/G)_{\rm pr}$ and points x in $T(\xi)$ satisfying:

(1) *$G_x = H$;*
(2) *the H-module $T_xX/T_x(G.x)$ is isomorphic to the direct sum of its fixed points and of L; in particular, its isomorphism class is independent of ξ and x;*
(3) *for any $v \in L$, 0 belongs to the closure of $H.v$;*
(4) *the fiber $\pi^{-1}(\xi)$ is isomorphic to $G \times_H L$.*

3.3. The fibers of quotient morphisms

Theorem 3.3 describes the general fiber of a GIT-quotient of a smooth variety. We also have the following general (but less precise) description of any fiber of a GIT-quotient (see [8] or [12]):

Proposition 3.4. — *Let x be a semistable point with respect to $\mathcal{L}$ whose orbit is closed in $X^{\rm ss}(\mathcal{L})$. Set $V = \{y \in X : x \in \overline{G_x.y}\}$. Then, V is an affine G_x-variety, containing x as the unique closed G_x-orbit. Moreover, $\pi^{-1}(\pi(x))$ is isomorphic to $G \times_{G_x} V$.*

4. On the faces of the G-ample cone

4.1. Isotropy subgroups associated to faces of $\mathcal{AC}^G_\Lambda(X)$

Let φ be a linear form on $\Lambda_\mathbb{Q}$ which is nonnegative on $\mathcal{AC}^G_\Lambda(X)$. If the set $\mathcal{F}$ of $\mathcal{L} \in \mathcal{AC}^G_\Lambda(X)$ such that $\varphi(\mathcal{L}) = 0$ is nonempty then $\mathcal{F}$ is called *a face of* $\mathcal{AC}^G_\Lambda(X)$. Now, we associate two invariants to a face of $\mathcal{AC}^G_\Lambda(X)$.

Theorem 4.1. — *Let $\mathcal{F}$ be a face of $\mathcal{AC}^G_\Lambda(X)$. Then, we have:*

1. *The principal isotropy group of $X^{\rm ss}(\mathcal{L})$ does not depend on the point $\mathcal{L}$ in the relative interior of $\mathcal{F}$, but only on $\mathcal{F}$. We call this isotropy group the principal isotropy group of $\mathcal{F}$.*

   Let us fix a principal isotropy group H of $\mathcal{F}$.
2. *For any $\mathcal{M} \in \mathcal{F}$, H fixes points in any closed orbit of G in $X^{\rm ss}(\mathcal{M})$.*
3. *The closure Y of $\left(\pi_\mathcal{L}^{-1}\left((X^{\rm ss}(\mathcal{L})/\!/G)_{\rm pr}\right)\right)^H$ in X does not depend on a choice of $\mathcal{L}$ in the relative interior of $\mathcal{F}$. Let $Y_\mathcal{F}$ denote this subvariety of X^H; then $Y_\mathcal{F}$ is the union of some components of X^H.*
4. *The group H acts trivially on some positive power $\mathcal{L}^{\otimes n}_{|Y_\mathcal{F}}$ of $\mathcal{L}_{|Y_\mathcal{F}}$. Moreover, the natural map*
$$Y^{\rm ss}_\mathcal{F}(\mathcal{L}^{\otimes n}_{|Y_\mathcal{F}})/\!/(N_G(H)/H) \longrightarrow X^{\rm ss}(\mathcal{L})/\!/G$$
   is an isomorphism and $Y_\mathcal{F}$ contains stable points for the action of $N_G(H)/H$ and the line bundle $\mathcal{L}^{\otimes n}_{|Y_\mathcal{F}}$.

Proof. — Let $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathcal{AC}^G_\Lambda(X)$. By an easy argument of convexity, to prove Assertion 1 it is sufficient to prove that the principal isotropy group of $X^{\rm ss}(\mathcal{L})$ does not depend on $\mathcal{L}$ in the open segment $]\mathcal{L}_1, \mathcal{L}_2[$. Let us fix $\mathcal{L}$, $\mathcal{L}' \in]\mathcal{L}_1, \mathcal{L}_2[$. Let $x \in X$ lie in the preimage of $(X^{\rm ss}(\mathcal{M})/\!/G)_{\rm pr}$, for $\mathcal{M} = \mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}$ and $\mathcal{L}'$ under the quotient maps.

Recall that $\Omega_\Lambda(x)$ is a closed polyhedral convex cone in $\Lambda_{\mathbb{Q}}^{++}$. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ belong to $\Omega_\Lambda(x)$, $\mathcal{L}$ and $\mathcal{L}'$ belong to the relative interior of the same face of $\Omega_\Lambda(x)$. By [14, Proposition 6, Assertion (iii)], there exists $x' \in \overline{G.x}$ such that this face is $\Omega_\Lambda(x')$. But, [14, Proposition 6, Assertion (i)] shows that the closed orbits in $X^{\rm ss}(\mathcal{L}) \cap \overline{G.x'}$ and $X^{\rm ss}(\mathcal{L}') \cap \overline{G.x'}$ are equal. Now, our choice of the point x implies that the principal isotropy groups of $X^{\rm ss}(\mathcal{L})$ and $X^{\rm ss}(\mathcal{L}')$ are equal.

Let H be a principal isotropy group of $X^{\rm ss}(\mathcal{L})$. Let Y be the closure of $X^H \cap \pi_{\mathcal{L}}^{-1}((X^{\rm ss}(\mathcal{L})/\!/G)_{\rm pr})$. By Theorem 3.2, $N_G(H)$ acts transitively on the set of irreducible components of Y. Let Y_1 be such a component of X^H. Again by Theorem 3.2, $\pi_{\mathcal{L}}(Y_1 \cap X^{\rm ss}(\mathcal{L})) = X^{\rm ss}(\mathcal{L})/\!/G$; that is, any closed G-orbit in $X^{\rm ss}(\mathcal{L})$ intersects Y_1. Finally, Y is the union of irreducible components of X^H which intersect a general closed G-orbit in $X^{\rm ss}(\mathcal{L})$. But, the above proof of Assertion 1 shows that a general closed orbit in $X^{\rm ss}(\mathcal{L})$ is also a closed orbit in $X^{\rm ss}(\mathcal{L}')$. In particular, Y is the closure of $X^H \cap \pi_{\mathcal{L}'}^{-1}((X^{\rm ss}(\mathcal{L}')/\!/G)_{\rm pr})$. Assertion 3 follows.

Let us now fix a principal isotropy group H of $\mathcal{F}$. Let $\mathcal{M}_1 \in \mathcal{F}$. By assertion 1 of Theorem 3.2, to prove the second assertion, it is sufficient to prove that the principal isotropy group of $X^{\rm ss}(\mathcal{M}_1)$ contains H. By [14, Theorem 4], there exists a point $\mathcal{M}_2$ in the relative interior of $\mathcal{F}$ such that $X^{\rm ss}(\mathcal{M}_1)$ contains $X^{\rm ss}(\mathcal{M}_2)$. The inclusion $X^{\rm ss}(\mathcal{M}_2) \subset X^{\rm ss}(\mathcal{M}_1)$ induces a surjective morphism $\eta : X^{\rm ss}(\mathcal{M}_2)/\!/G \longrightarrow X^{\rm ss}(\mathcal{M}_1)/\!/G$. Let $\xi' \in (X^{\rm ss}(\mathcal{M}_2)/\!/G)_{\rm pr}$ such that $\xi = \eta(\xi') \in (X^{\rm ss}(\mathcal{M}_1)/\!/G)_{\rm pr}$. Let x be a point in the closed G-orbit in $X^{\rm ss}(\mathcal{M}_1)$ over ξ. The fiber in $X^{\rm ss}(\mathcal{M}_1)$ over ξ is fibered over $G.x$; hence, for any y in this fiber, G_y is conjugated to a subgroup of G_x. Since this fiber contains the fiber in $X^{\rm ss}(\mathcal{M}_2)$ over ξ', then H is conjugated to a subgroup of G_x. The second assertion is proved. □

4.2. Local structure of the G-ample cone around a face

Notation 4.2. — Let E be a prime Cartier divisor on a variety X endowed with a line bundle $\mathcal{L}$ and let σ be a rational section of $\mathcal{L}$. We will denote by $\nu_E(\sigma) \in \mathbb{Z}$ the order of vanishing of σ along E.

Let $\mathcal{P}$ be a polyhedron in a rational vector space V and $\mathcal{F}$ be a face of $\mathcal{P}$. The cone of V generated by the vectors $y - x$ for $y \in \mathcal{P}$ does not depend on the choice of x in the relative interior of $\mathcal{F}$. This cone will be called *the cone of $\mathcal{P}$ viewed from $\mathcal{F}$*. It encodes the geometry of $\mathcal{P}$ in a neighborhood of x.

Let $\mathcal{F}$ be a face of $\mathcal{AC}^G_\Lambda(X)$. Let $\mathcal{L}$ belong to the relative interior of $\mathcal{F}$. Let x be a semistable point whose G-orbit is closed in $X^{\rm ss}(\mathcal{L})$. Let V be the affine G_x-variety defined in Proposition 3.4. Consider the cone $\mathcal{AC}^{G_x}(V)$ as in Section 2.2. Notice that V is not necessarily irreducible, and so $\mathcal{AC}^{G_x}(V)$ is not necessarily convex.

As in Section 2.2, the G_x-action on the fiber over x defines a morphism $\mu^\bullet(x, G_x) : \Lambda \longrightarrow X(G_x)$ and a linear map from $\Lambda_\mathbb{Q}$ on $X(G_x)_\mathbb{Q}$ also denoted by $\mu^\bullet(x, G_x)$.

Theorem 4.3. — *The cone of $\mathcal{AC}^G_\Lambda(X)$ viewed from $\mathcal{F}$ is the preimage by $\mu^\bullet(x, G_x)$ of $\mathcal{AC}^{G_x}(V)$. In particular, if $\mu^\bullet(x, G_x)$ is surjective then $\mathcal{AC}^{G_x}(V)$ is convex.*

Proof. — Let $\mathcal{L}_0$ and $\mathcal{L}$ be two ample G-linearized line bundles in Λ. We assume that $\mathcal{L}_0$ is the only point in the segment $[\mathcal{L}; \mathcal{L}_0]$ which belongs to $\mathcal{AC}^G_\Lambda(X)$. For convenience, we set $U = X^{\rm ss}(\mathcal{L}_0)$. By assumption, there is no G-invariant rational section of $\mathcal{L}$ which is regular on X; we claim that there is no such section which is regular on U.

Let us prove the claim. Fix a nonzero regular G-invariant section σ_0 of $\mathcal{L}_0^{\otimes m}$ for some positive integer m. Let σ be a G-invariant rational section of $\mathcal{L}$ which is regular on U. For any positive integer k, $\sigma\otimes\sigma_0^{\otimes k}$ is a rational G-invariant section of $\mathcal{L}\otimes\mathcal{L}_0^{\otimes mk}$ which is regular on U. Let E be an irreducible component of codimension one of $X-U$. By definition of U, σ_0 is zero along E and $\nu_E(\sigma_0) > 0$. Then, $\nu_E(\sigma \otimes \sigma_0^{\otimes k}) = \nu_E(\sigma) + k.\nu_E(\sigma_0)$ is positive for k big enough. We deduce that $\sigma \otimes \sigma_0^{\otimes k}$ is regular on X for k big enough. Since by assumption $\mathcal{L} \otimes \mathcal{L}_0^{\otimes mk}$ does not belong to $\mathcal{AC}^G_\Lambda(X)$, this implies that $\sigma \otimes \sigma_0^{\otimes k}$ and finally σ are zero. The claim is proved.

We now fix a point $\mathcal{L}_0$ in the relative interior of $\mathcal{F}$. By an elementary argument of convexity, there exists an open neighborhood Ω of $\mathcal{L}_0$ in $\Lambda^{++}_\mathbb{Q}$ such that for any $\mathcal{L} \in \Omega$, if $\mathcal{L}$ does not belong to $\mathcal{AC}^G_\Lambda(X)$ then $\mathcal{L}_0$ is the only point in $[\mathcal{L}, \mathcal{L}_0] \cap \mathcal{AC}^G_\Lambda(X)$.

By [14, Proposition 2.3], we may also assume that for all $\mathcal{L} \in \Omega$, $X^{\rm ss}(\mathcal{L})$ is contained in $X^{\rm ss}(\mathcal{L}_0)$. It remains to prove that for any $\mathcal{L} \in \Omega$, $\mathcal{L} \in \mathcal{AC}^G_\Lambda(X)$ if and only if $\mu^\mathcal{L}(x, G_x) \in \mathcal{AC}^{G_x}(V)$.

Let $\mathcal{L} \in \Omega$ which does not belong to $\mathcal{AC}^G_\Lambda(X)$. Set $\xi = \pi_{\mathcal{L}_0}(x)$. By the beginning of the proof, for any positive n, $\mathrm{H}^0(U, \mathcal{L}^{\otimes n})^G = \{0\}$. Since $\pi_{\mathcal{L}_0}^{-1}(\xi)$ is closed in U, this implies that $\mathrm{H}^0(\pi_{\mathcal{L}_0}^{-1}(\xi), \mathcal{L}^{\otimes n})^G = \{0\}$ for all positive n. So, for all positive n, $\mathrm{H}^0(V, \mathcal{L}_{|V}^{\otimes n})^{G_x} = \{0\}$. Then $\mu^\mathcal{L}(x, G_x)$ does not belong to $\mathcal{AC}^{G_x}(V)$.

Let now $\mathcal{L} \in \Omega \cap \mathcal{AC}_\Lambda^G(X)$. Since the map $\phi : X^{\rm ss}(\mathcal{L})/\!/G \longrightarrow X^{\rm ss}(\mathcal{L}_0)/\!/G$ induced by the inclusion $X^{\rm ss}(\mathcal{L}) \subset X^{\rm ss}(\mathcal{L}_0)$ is surjective, there exists $y \in X^{\rm ss}(\mathcal{L})$ such that $\phi \circ \pi_{\mathcal{L}}(y) = \xi$. Up to changing y by $g.y$ for some $g \in G$, one may assume that $y \in V$. Let σ be a G-invariant section of $\mathcal{L}$ which is nonzero at y. Obviously, the restriction of σ is a G_x-invariant section of $\mathcal{L}_{|V}$ which is nonzero. It follows that $\mu^{\mathcal{L}}(x, G_x)$ belongs to $\mathcal{AC}^{G_x}(V)$.

The last assertion follows from an obvious argument of convexity. □

5. Well-covering pairs

5.1. The functions $\mu^\bullet(x,\lambda)$

Let $\mathcal{L} \in \mathrm{Pic}^G(X)$. Let x be a point in X and λ be a one-parameter subgroup of G. Since X is complete, $\lim_{t\to 0}\lambda(t)x$ exists; let z denote this limit. The image of λ fixes z and so acts on the fiber $\mathcal{L}_z$: there exists $\mu^{\mathcal{L}}(x,\lambda) \in \mathbb{Z}$ such that:

$$\forall \tilde{z} \in \mathcal{L}_z \ \forall t \in \mathbb{K}^* \quad \lambda(t)\tilde{z} = t^{-\mu^{\mathcal{L}}(x,\lambda)}\tilde{z}.$$

The numbers $\mu^{\mathcal{L}}(x,\lambda)$ are used in [11] to give a numerical criterion for stability with respect to an ample $G-$linearized line bundle $\mathcal{L}$:

$$x \in X^{\rm ss}(\mathcal{L}) \iff \mu^{\mathcal{L}}(x,\lambda) \leqslant 0 \text{ for all one-parameter subgroup } \lambda,$$
$$x \in X^{\rm s}(\mathcal{L}) \iff \mu^{\mathcal{L}}(x,\lambda) < 0 \text{ for all nontrivial } \lambda.$$

5.2. Definition

Notation 5.1. — The set of fixed points of the image of λ will be denoted by X^λ; the centralizer of this image will be denoted by G^λ. We consider the parabolic subgroup of G:

$$P(\lambda) = \left\{ g \in G \ : \ \lim_{t\to 0} \lambda(t).g.\lambda(t)^{-1} \text{ exists in } G \right\}.$$

Let C be an irreducible component of X^λ. Since G^λ is connected, C is a G^λ-stable closed subvariety of X. We set:

$$C^+ := \{x \in X \ : \ \lim_{t\to 0} \lambda(t)x \in C\}.$$

Then, C^+ is a locally closed subvariety of X stable by $P(\lambda)$. Consider over $G\times C^+$ the action of $G\times P(\lambda)$ given by the formula (with obvious notation):

$$(g,p).(g',y) = (gg'p^{-1}, py).$$

Since the quotient map $G \longrightarrow G/P(\lambda)$ is a Zariski-locally trivial principal $P(\lambda)$-bundle, one can easily construct a quotient $G \times_{P(\lambda)} C^+$ of $G \times C^+$ by the action of $\{e\} \times P(\lambda)$. The action of $G \times \{e\}$ induces an action of G on $G \times_{P(\lambda)} C^+$. We recall the following definition from [15]:

Definition 5.2. — *Consider the G-equivariant map*

$$\begin{array}{cccc} \eta : & G \times_{P(\lambda)} C^+ & \longrightarrow & X \\ & [g:x] & \longmapsto & g.x. \end{array}$$

The pair (C, λ) is said to be covering (resp. dominant) if η is birational (resp. dominant). It is said to be well-covering if η induces an isomorphism from $G \times_{P(\lambda)} \Omega$ onto an open subset of X, where Ω is a $P(\lambda)$-stable open subset of C^+ intersecting C.

Let us denote by $\mu^\bullet(C, \lambda)$, the common value of the $\mu^\bullet(x, \lambda)$, for $x \in C^+$. We assume that (C, λ) is a dominant pair. By [15, Lemma 3], the set of $\mathcal{L} \in \mathcal{AC}^G_\Lambda(X)$ such that $\mu^{\mathcal{L}}(C, \lambda) = 0$ is either empty or a face $\mathcal{F}$ of $\mathcal{AC}^G_\Lambda(X)$. Moreover, $\mathcal{F}$ is the set of $\mathcal{L} \in \mathcal{AC}^G_\Lambda(X)$ such that $X^{\rm ss}(\mathcal{L})$ intersects C. >From now on, $\mathcal{F}$ which only depends on C is denoted by $\mathcal{F}(C)$.

6. The case where $X = Y \times G/B$

Assumption. — *We assume in this section that X has diagonalizable reductive isotropy groups and that Λ is abundant in the sense of the definitions of* [15, Section 3.3] *(this means, that for any $x \in X$ with a reductive isotropy group G_x, the morphism $\Lambda \longrightarrow X(G_x)$ given by the action of G_x on fibers over x has image of finite index).*

The main example satisfying these assumptions is $X = G/B \times Y$, where Y is a projective G-variety Y, and Λ contains the pullback of $\mathrm{Pic}^G(G/B)$.

6.1. Principal isotropy group and well-covering pairs

Let S be a subtorus of G. Let C be an irreducible component of X^S.

Definition 6.1. — *The pair (C, S) is said to be admissible if there exists $x \in C$ such that $G_x^\circ = S$. The pair is said to be well-covering if there exists a one-parameter subgroup λ of S, such that C is an irreducible component of X^λ and (C, λ) is well-covering.*

A rephrasing of [15, Corollary 3] is

Proposition 6.2. — *Let $\mathcal{F}$ be a face of $\mathcal{AC}_\Lambda^G(X)$ of codimension r. Then, there exists an admissible well-covering pair (C, S) with S of dimension r such that $\mathcal{F} = \mathcal{F}(C)$.*

We are now interested in the principal isotropy group of the faces of $\mathcal{AC}_\Lambda^G(X)$:

Proposition 6.3. — *Let $\mathcal{F}$ be a face of codimension r. Let (C, S) be a well-covering pair with a r-dimensional torus S such that $\mathcal{F}(C) = \mathcal{F}$.*

Then, (C, S) is admissible and there exists a principal isotropy group H of $\mathcal{F}$ such that $H^\circ = S$.

Proof. — By [15, Lemma 3], $\mathcal{F}$ is an union of GIT-classes. By [14], there are only finitely many such classes and they are convex; so, there exists a GIT-class F which has nonempty interior in $\mathcal{F}$. Let $\mathcal{L} \in F$. Let $\xi \in (X^{\rm ss}(\mathcal{L})/\!/G)_{\rm pr}$ and $T(\xi)$ be the corresponding closed G-orbit in $X^{\rm ss}(\mathcal{L})$.

By [15, Proposition 9], $T(\xi)$ intersects C. Let us fix $x \in T(\xi) \cap C$. Now, consider the morphism $\mu^\bullet(x, G_x) : \Lambda_\mathbb{Q} \longrightarrow X(G_x) \otimes \mathbb{Q}$ induced by restriction and the isomorphism $X(G_x) \simeq \mathrm{Pic}^G(T(\xi))$. By Theorem 4.3, $\mathrm{Ker}\,\mu^\bullet(x, G_x)$ is contained in $\mathrm{Span}(\mathcal{F})$. On the other hand, the GIT-class of $\mathcal{L}$ is contained in $\mathrm{Ker}\,\mu^\bullet(x, G_x)$. Finally, $\mathrm{Ker}\,\mu^\bullet(x, G_x) = \mathrm{Span}(\mathcal{F})$. Since Λ is abundant, this implies that the rank of $X(G_x)$ equals r.

Since $G.x$ is affine, G_x is reductive. Since X has diagonalizable reductive isotropy groups, G_x is diagonalizable. But the rank of $X(G_x)$ equals r and G_x° is a r-dimensional torus. Since $x \in C$, S is contained in G_x; it follows that $G_x^\circ = S$. In particular, (C, S) is admissible. □

6.2. Unicity

Notation 6.4. — Let S be a torus. We will denote by $Y(S)$ the group of one-parameter subgroups of S. There is a natural perfect paring $Y(S) \times X(S) \longrightarrow \mathbb{Z}$ denoted by $\langle \cdot, \cdot \rangle$.

The following lemma is a first statement of unicity.

Lemma 6.5. — *We assume in addition that X is smooth. Let $\mathcal{F}$ be a face of codimension r. Let (C_1, S_1) and (C_2, S_2) be two well-covering pairs, where S_1 and S_2 are two r-dimensional tori such that $\mathcal{F}(C_1) = \mathcal{F}(C_2) = \mathcal{F}$.*

Then, there exists $g \in G$ such that $g.C_2 = C_1$ and $g.S_2.g^{-1} = S_1$.

Proof. — Arguing as in the proof of Proposition 6.3, we obtain that $T(\xi)$ intersects C_1 and C_2. Up to conjugacy, we may assume that x belongs to

$T(\xi)\cap C_1\cap C_2$. So, we obtain that $G_x^\circ = S_1 = S_2$. Then, C_1 equals C_2 since they are the irreducible component of $X^{S_1} = X^{S_2}$ containing x. □

In contrast with the unicity of Lemma 6.5, for a given well-covering pair (C, S), the set of pairs (C^+, P) such that $G\times_P C^+ \longrightarrow X$ is well-covering is not unique. We are now interested in the set of $\lambda\in Y(S)$ such that the associated morphism η is dominant.

Let us fix a face $\mathcal{F}$ of codimension r. The set of linear forms $\varphi\in \mathrm{Hom}(\Lambda_{\mathbb{Q}},\mathbb{Q})$ such that φ is nonnegative on $\mathcal{AC}^G_\Lambda(X)$ and zero on $\mathcal{F}$ is denoted by $\mathcal{F}^\vee$.

Let us fix a well-covering pair (C, S) where S has dimension r and such that $\mathcal{F} = \mathcal{F}(C)$. Let $\mathcal{C}$ denote the set of $\lambda\in Y(S)\otimes\mathbb{Q}$ such that for some positive n, the pair $(C_{n\lambda}, n\lambda)$ is dominant, where $C_{n\lambda}$ denotes the irreducible component of $X^{n\lambda}$ containing C. Note that, since S acts trivially on C, the map $Y(S)\otimes\mathbb{Q}\longrightarrow \mathrm{Hom}(\Lambda_{\mathbb{Q}},\mathbb{Q})$, $\lambda\longmapsto \mu^\bullet(C,\lambda)$ is linear.

Lemma 6.6. — *We assume that X is smooth.*
Then, $\lambda\in\mathcal{C}$ if and only if $\mu^\bullet(C,\lambda)\in\mathcal{F}^\vee$.

Proof. — Let λ be a rational one-parameter subgroup of S, and let n be a positive integer such that $n\lambda\in Y(S)$. First assume that $\lambda\in\mathcal{C}$. Since $(C_{n\lambda}, n\lambda)$ is dominant, [15, Lemma 3] implies that $\mu^\bullet(C,\lambda)$ is nonnegative on $\mathcal{AC}^G_\Lambda(X)$. Moreover, for any $\mathcal{L}\in\mathcal{F}$, $X^{\mathrm{ss}}(\mathcal{L})$ intersects C. This implies that $\mu^{\mathcal{L}}(C,\lambda)=0$. Finally, $\mu^\bullet(C,\lambda)\in\mathcal{F}^\vee$.

Conversely, assume that $\mu^\bullet(C,\lambda)\in\mathcal{F}^\vee$. Set

$$C^+ = \{x\in X\ :\ \lim_{t\to 0} n\lambda(t)x\in C_{n\lambda}\}$$

and $\eta : G\times_{P(n\lambda)} C^+\longrightarrow X$. Let us fix a general point $x\in C$. Then, G_x is the principal isotropy group of $\mathcal{F}$, its neutral component is S and the G_x-module T_xX/T_xC is the type of $\mathcal{F}$. Theorem 4.3 implies that $\mu^\bullet(C,\lambda)\in\mathcal{F}^\vee$ if and only if $\langle n\lambda,\cdot\rangle$ is nonnegative on all weights of S in $T_xX/T_xG.x$. We deduce that the differential $T\eta_x$ is surjective. Since X is smooth, this implies that η is dominant. □

6.3. Inclusion of faces

We now assume that $X = G/B\times Y$ for a smooth projective G-variety Y and that Λ is abundant.

Proposition 6.7. — *Let (C_1, S_1) and (C_2, S_2) be two admissible well-covering pairs with two subtori S_1 and S_2 of T of dimensions r_1 and r_2*

such that $B/B \in p_{G/B}(C_i)$ for $i = 1, 2$. We assume that $\mathcal{F}(C_1)$ and $\mathcal{F}(C_2)$ have respectively codimension r_1 and r_2.

Then, the following are equivalent:

1. *$\mathcal{F}(C_1) \subset \mathcal{F}(C_2)$;*
2. *$C_1 \subset C_2$ and $S_2 \subset S_1$.*

Proof. — The second assertion implies the first one by [15, Lemma 3]. Conversely, we assume that $\mathcal{F}(C_1) \subset \mathcal{F}(C_2)$.

By Proposition 6.3, there exists $\mathcal{L} \in \mathcal{F}(C_1)$ and $x \in C_1$ such that $G_x^\circ = S_1$ and $G.x$ is closed in $X^{\rm ss}(\mathcal{L})$. Since C_2 intersects $G.x$, there exists $g \in G$ such that $g.x \in C_2$. Since S_2 fixes $g.x$, $S_2 \subset gS_1g^{-1}$. In particular, S_2 is contained in T and gTg^{-1}; so, T and gTg^{-1} are maximal tori in G^{S_2}. There exists $l \in G^{S_2}$ such that $lTl^{-1} = gTg^{-1}$. The element $n = l^{-1}g$ normalizes T. Since C_2 is stable by G^{S_2} (which is connected), x belongs to $n^{-1}C_2$. Applying $p_{G/B}$ we obtain that $p_{G/B}(x)$ belongs to $n^{-1}G^{S_2}B/B \cap G^{S_1}B/B$. In particular, $G^{S_2}B/B \cap nG^{S_1}B/B \neq \emptyset$.

We claim that $nB/B \in G^{S_2}B/B$. Since $n^{-1}S_2n = g^{-1}S_2g \subset S_1$, we have $G^{n^{-1}S_2n} \supset G^{S_1}$. In particular, $G^{S_2}n \supset nG^{S_1}$ and $G^{S_2}nB/B \supset nG^{S_1}B/B$. It follows that $G^{S_2}B/B$ intersects $G^{S_2}nB/B$; since they are G^{S_2}-orbits $G^{S_2}B/B = G^{S_2}nB/B$.

Since n normalizes T, this implies that $n \in G^{S_2}$. So, $n^{-1}S_2n = S_2 \subset S_1$.

Since $n \in G^{S_2}$ and $nx \in C_2$, $x \in C_2$. But $x \in C_1$. It follows that C_1 (resp. C_2) is the irreducible component of X^{S_1} (resp. X^{S_2}) containing x. Now, $S_2 \subset S_1$ implies $C_1 \subset C_2$. □

Remark 6.8. — Proposition 6.7 shows that $\mathcal{F}(C_1) = \mathcal{F}(C_2)$ if and only if $C_1 = C_2$ and $S_2 = S_1$. With the assumption of the proposition this is an improvement of Lemma 6.5.

7. Application to the branching rule cones

7.1. The branching rule cone in terms of GIT-cone

From now on, we assume that G is a connected reductive subgroup of a connected reductive group $\hat{G}$. Let us fix maximal tori T (resp. $\hat{T}$) and Borel subgroups B (resp. $\hat{B}$) of G (resp. $\hat{G}$) such that $T \subset B \subset \hat{B} \supset \hat{T} \supset T$.

Let $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ denote the Lie algebras of G and $\hat{G}$ respectively.

We denote by $\mathcal{LR}(G, \hat{G})$ (resp. $\mathcal{LR}^\circ(G, \hat{G})$) the cone of pairs $(\nu, \hat{\nu}) \in X(T)_{\mathbb{Q}} \times X(\hat{T})_{\mathbb{Q}}$ such that for some positive integer n, $n\hat{\nu}$ and $n\nu$ are dominant (resp. strictly dominant) weights such that $V_{n\nu} \otimes V_{n\hat{\nu}}$ contains nonzero G-invariant vectors.

In this section, X denotes the variety $G/B \times \hat{G}/\hat{B}$ endowed with the diagonal action of G. We will apply the results of Section 4 to X with $\Lambda = \mathrm{Pic}^G(X)$. The cone $\mathcal{AC}^G_\Lambda(X)$ will be denoted by $\mathcal{AC}^G(X)$. It is well known (see [15, Proposition 10] for a proof) that $\mathcal{LR}^\circ(G, \hat{G})$ is the set of pairs $(\nu, \hat{\nu})$ of rational weights such that $\mathcal{L}_{(\nu,\hat{\nu})} \in \mathcal{AC}^G(X)$. Moreover, if no ideal of $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$, by [15, Proposition 12] $\mathcal{LR}^\circ(G, \hat{G})$ has a nonempty interior.

7.2. A first parametrization of the faces

7.2.1. Admissible subtori

Consider the G-module $\hat{\mathfrak{g}}/\mathfrak{g}$. Let $Wt_T(\hat{\mathfrak{g}}/\mathfrak{g})$ be the set of nontrivial weights for the T-action on $\hat{\mathfrak{g}}/\mathfrak{g}$. For $I \subset Wt_T(\hat{\mathfrak{g}}/\mathfrak{g})$, we will denote by T_I the neutral component of the intersection of kernels of characters in I. A subtorus of the form T_I is said to be *admissible*.

Let $\lambda \in Y(T)$ or $Y(T)_\mathbb{Q}$. We denote by $I(\lambda)$ the set of characters $\chi \in Wt_T(\hat{\mathfrak{g}}/\mathfrak{g})$ such that $\langle \lambda, \chi \rangle = 0$. Let S be an admissible torus. We consider the set of $\lambda \in Y(S)_\mathbb{Q}$ such that

$$\forall \chi \in Wt_T(\hat{\mathfrak{g}}/\mathfrak{g}) \;\; \langle \lambda, \chi \rangle = 0 \;\Rightarrow\; \chi_{|S} \text{ is trivial;}$$

or equivalently, $T_{I(\lambda)} = S$. The set of such rational one-parameter subgroups is the complementary of the union of hyperplanes $\langle \cdot, \chi \rangle = 0$ for $\chi \in Wt_T(\hat{\mathfrak{g}}/\mathfrak{g})$ nontrivial on S. A connected component of this set will be called a *chamber of* $Y(S)_\mathbb{Q}$.

A one-parameter subgroup $\lambda \in Y(S)_\mathbb{Q}$ is said to be *S-regular* if

$$\forall \chi \in Wt_T(\hat{\mathfrak{g}}) \;\; \langle \lambda, \chi \rangle = 0 \;\Rightarrow\; \chi_{|S} \text{ is trivial.}$$

Note that in the definition of S-regularity we consider $Wt_T(\hat{\mathfrak{g}})$ and not $Wt_T(\hat{\mathfrak{g}}/\mathfrak{g})$. In fact, λ is S-regular if and only if $\hat{G}^\lambda = \hat{G}^S$ (and so $G^\lambda = G^S$).

Let $Y(T)^+$ denote the set of dominant one-parameter subgroups of T. A subtorus S of T is said to be *dominant* if $Y(S)_\mathbb{Q}$ is spanned by its intersection with $Y(T)^+$. A chamber $Y(S)_\mathbb{Q}$ is said to be *dominant* if it spans the same subspace than its its intersection with $Y(T)^+$.

7.2.2. Admissible pairs

Consider the parabolic subgroups P and $\hat{P}$ of G and $\hat{G}$ associated to λ. Let W_P be the Weyl group of P. The cohomology group $\mathrm{H}^*(G/P,\mathbb{Z})$ is freely generated by the Schubert classes $[\overline{BwP/P}]$ parametrized by the elements $w\in W/W_P$. Since $\hat{P}\cap G=P$, we have a canonical G-equivariant immersion $\iota\,:\,G/P(\lambda)\longrightarrow \hat{G}/\hat{P}(\lambda)$; and the corresponding morphism ι^* in cohomology.

Let ρ (resp. ρ^S) denote the half-sum of positive roots of G (resp. G^S). Let Φ^+ and $\Phi(P^u)$ denote the set of roots of the groups B and P^u for the torus T. In the same way, we define $\hat{\Phi}^+$ and $\Phi(\hat{P}^u)$. For $w\in W$ and $\hat{w}\in\hat{W}$, we consider the following characters of T and $\hat{T}$:

$$\theta^P_w:=\sum_{\alpha\in w\Phi^+\cap\Phi(P^u)}\alpha\quad\text{and}\quad\theta^{\hat{P}}_{\hat{w}}:=\sum_{\alpha\in\hat{w}\hat{\Phi}^+\cap\Phi(\hat{P}^u)}\alpha.$$

Let S be an admissible subtorus of T and $\hat{w}\in\hat{W}/\hat{W}_{\hat{G}^S}$. The pair $(S,\hat{w})$ is said to be *admissible* if there exists a S-regular one-parameter subgroup λ of S such that:

(1) $\iota^*([\overline{\hat{B}\hat{w}\hat{P}(\lambda)/\hat{P}(\lambda)}]).[\overline{BP(\lambda)/P(\lambda)}]=[\mathrm{pt}]\in\mathrm{H}^*(G/P(\lambda),\mathbb{Z})$;
(2) $(\theta^{\hat{P}(\lambda)}_{\hat{w}})_{|S}=(\theta^{P(\lambda)}-2(\rho-\rho^S))_{|S}$.

The following lemma explains these two conditions geometrically:

Lemma 7.1. — *Let S be an admissible subtorus and $\hat{w}\in\hat{W}/\hat{W}_{\hat{G}^S}$. Let λ be a S–regular one-parameter subgroup. Set $C(\hat{w})=G^SB/B\times\hat{G}^S\hat{w}^{-1}B/B$. Then the two above conditions are fulfilled if and only if $(C(\hat{w}),\lambda)$ is a well-covering pair.*

Proof. — The proof is very similar to [15, Proposition 11]: we leave the details to the reader. One can prove (using mainly Kleiman's transversality Theorem) that $\iota^*([\overline{\hat{B}\hat{w}\hat{P}/\hat{P}}]).[\overline{BP/P}]=[\mathrm{pt}]\in\mathrm{H}^*(G/P,\mathbb{Z})$ if and only if the morphism η as in Definition 5.2 is birational. Now, the condition $(\theta^{\hat{P}}_{\hat{w}})_{|S}=(\theta^P-2(\rho-\rho^S))_{|S}$ means that S acts trivially on the restriction over C of the determinant bundle of η. □

We may now give a first parametrization of the faces of $\mathcal{LR}^\circ(G,\hat{G})$:

Theorem 7.2. — *We assume that no ideal of $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$.*
The map which associates to $(S,\hat{w})$ the set

$$\mathcal{F}(S,\hat{w})=\{(\nu,\hat{\nu})\in\mathcal{LR}^\circ(G,\hat{G})\,:\,\hat{w}\hat{\nu}_{|S}+\nu_{|S}\text{ is trivial}\}$$

is a bijection from the set of admissible pairs onto the set of faces of $\mathcal{LR}^\circ(G,\hat{G})$. Moreover, the codimension of $\mathcal{F}(S,\hat{w})$ equals the dimension of S; and the following are equivalent:

(1) $\mathcal{F}(S,\hat{w}) \subset \mathcal{F}(S',\hat{w}')$;
(2) $S' \subset S$ and $\hat{w}W_{G^{S'}} = \hat{w}'W_{G^{S'}}$.

Proof. — Let $(S,\hat{w})$ be an admissible pair. Set $\overline{\mathcal{F}}(S,\hat{w}) = \{(\nu,\hat{\nu}) \in \mathcal{LR}(G,\hat{G}) : \hat{w}\hat{\nu}_{|S} = -\nu_{|S}\}$. By [15, Theorem 11], $\overline{\mathcal{F}}(S,\hat{w})$ is a face of $\mathcal{LR}(G,\hat{G})$ of codimension $\dim(S)$. In particular, $\widetilde{\mathcal{F}}(S,\hat{w})$ spans the subspace of the $(\nu,\hat{\nu}) \in X(T)\times X(\hat{T})$ such that $\hat{w}\hat{\nu}_{|S} = -\nu_{|S}$. To prove that the map in the theorem is well defined, it is enough to prove that $\overline{\mathcal{F}}(S,\hat{w})$ intersects $\mathcal{LR}^\circ(G,\hat{G})$. Assume that $\overline{\mathcal{F}}(S,\hat{w})$ is contained in the boundary of the dominant chamber. Then, its projection on $X(\hat{T})_{\mathbb{Q}}$ or $X(T)_{\mathbb{Q}}$ is contained in an hyperplane; this is a contradiction.

Let us prove the surjectivity. Let $\mathcal{F}$ be a face of $\mathcal{LR}^\circ(G,\hat{G})$. Let (C,λ) be a pair satisfying Proposition 6.2 for $\mathcal{F}$. Up to translate C and conjugacy S accordingly, we may assume that $S\subset T$ and C intersects $B/B\times\hat{G}/\hat{B}$. Then $C = C(\hat{w})$ for some $\hat{w}\in\hat{W}$. We know that there exists a one-parameter subgroup λ of S such that (C,λ) is well covering and we need to find such a λ which is S-regular. But Lemma 6.6 shows that there exists a S-regular λ such that (C,λ) is dominant. By [15, Theorem 12] (C,λ) is well-covering.

The injectivity and the assertion about inclusion of the faces are direct applications of Proposition 6.7. □

7.3. A second parametrization

From now on we assume that $Wt_T(\hat{\mathfrak{g}}/\mathfrak{g}) = Wt_T(\hat{\mathfrak{g}})$. The following observation explains the role of this assumption:

LEMMA 7.3. — *Let S be an admissible subtorus of T and $\mathcal{C}$ be a chamber of $Y(S)_{\mathbb{Q}}$. If $Wt_T(\hat{\mathfrak{g}}/\mathfrak{g}) = Wt_T(\hat{\mathfrak{g}})$ then the parabolic subgroups $\hat{P}(\lambda)$ and so $P(\lambda)$ do not depend on $\lambda\in\mathcal{C}$.*

Proof. — The parabolic subgroup $\hat{P}(\lambda)$ only depends on the signs of the $\langle\lambda,\hat{\alpha}\rangle$'s for the roots $\hat{\alpha}$ of $\hat{G}$. The lemma follows. □

The parabolic subgroups of Lemma 7.3 will be denoted by $\hat{P}(\mathcal{C})$ and $P(\mathcal{C})$.

Remark 7.4. — In [3], Berenstein-Sjamaar also consider a chamber decomposition of $Y(T)_{\mathbb{Q}}$. Our decomposition is the same if $Wt_T(\hat{\mathfrak{g}}/\mathfrak{g}) = Wt_T(\hat{\mathfrak{g}})$ but not in general. An easy example to see a difference is $G = \mathrm{GL}_2 \subset \mathrm{SL}_3 = \hat{G}$.

We consider the set Θ of quadruples $(S,\mathcal{C},w,\hat{w})$ such that

(1) S is a dominant admissible subtorus of T,
(2) $\mathcal{C}$ is a dominant chamber of $Y(S)_{\mathbb{Q}}$,
(3) $w \in W/W_{G^S}$ and $\hat{w} \in \hat{W}/\hat{W}_{\hat{G}^S}$,
(4) $\iota^*([\overline{\hat{B}\hat{w}\hat{P}(\mathcal{C})/\hat{P}(\mathcal{C})}]).[\overline{BwP(\mathcal{C})/P(\mathcal{C})}] = [\mathrm{pt}] \in \mathrm{H}^*(G/P(\mathcal{C}), \mathbb{Z})$;
(5) $(\theta_{\hat{w}}^{\hat{P}(\mathcal{C})})_{|S} = (\theta_w^{P(\mathcal{C})} - 2(\rho - \rho^S))_{|S}$.

We may now give a second parametrization of the faces of $\mathcal{LR}^\circ(G, \hat{G})$.

Theorem 7.5. — *We assume that no ideal of $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$ and that $Wt_T(\hat{\mathfrak{g}}/\mathfrak{g}) = Wt_T(\hat{\mathfrak{g}})$.*

The map which associates to $(S, \mathcal{C}, w, \hat{w}) \in \Theta$ the set

$$\mathcal{F}(S, \mathcal{C}, w, \hat{w}) = \{(\nu, \hat{\nu}) \in \mathcal{LR}^\circ(G, \hat{G}) \,:\, \hat{w}\hat{\nu}_{|S} + w\nu_{|S}\}$$

is a bijection from Θ onto the set of faces of $\mathcal{LR}^\circ(G, \hat{G})$. Moreover, the codimension of $\mathcal{F}(S, \mathcal{C}, w, \hat{w})$ equals the dimension of S and the following are equivalent:

(1) $\mathcal{F}(S, \mathcal{C}, w, \hat{w}) \subset \mathcal{F}(S', \mathcal{C}', w', \hat{w}')$;
(2) $w'S'w'^{-1} \subset wSw^{-1}$ *and* $\hat{w}w^{-1}w'W_{G^{S'}} = \hat{w}'W_{G^{S'}}$.

We now have to understand better the admissible pairs $(S, \hat{w})$.

Lemma 7.6. — *Let $(S, \hat{w})$ be an admissible pair. Then, the set of rational one-parameter subgroups λ of S such that $(C(\hat{w}), \lambda)$ is well-covering, is a chamber $\mathcal{C}(S, \hat{w})$ of $Y(S)$.*

Proof. — Let $\mathcal{F}$ denote the face of $\mathcal{LR}^\circ(G, \hat{G})$ associated to $(S, \hat{w})$ in Theorem 7.2. Then, S is the neutral component of a principal isotropy group of $\mathcal{F}$. Let L be the S-module such that (S, L) is the type of $\mathcal{F}$. By Theorem 4.3 and Lemma 2.2, the set of S-regular one-parameter subgroups λ such that $(C(\hat{w}), \lambda)$ is well-covering is contained in the interior $\mathcal{C}'$ of the dual of the cone generated by the weights $Wt_S(L)$ of S in L. But, there exists $x \in C$ such that $Wt_S(L) = Wt_S(T_xX/T_xG.x)$ and $G_x^\circ = S$.

Let U and $\hat{U}$ denote the unipotent radical of B and $\hat{B}$. Let $\mathfrak{b}$, $\hat{\mathfrak{b}}$, $\mathfrak{u}$, $\hat{\mathfrak{u}}$, $\mathfrak{s}$ denote the Lie algebras of B, $\hat{B}$, U, $\hat{U}$, S. We have:

$$\begin{aligned} Wt_S(L) &= Wt_S(T_xX/T_xG.x) \\ &= Wt_S((\mathfrak{g}/\mathfrak{b} \oplus \hat{\mathfrak{g}}/\hat{w}\hat{\mathfrak{b}})/(\mathfrak{g}/\mathfrak{s})) \\ &= Wt_S((\hat{\mathfrak{g}}/\hat{w}\hat{\mathfrak{b}})/\mathfrak{u}) \\ &= Wt_S(\hat{w}\hat{\mathfrak{u}}/\mathfrak{u}). \end{aligned}$$

In particular, for any $\chi \in Wt_S(\hat{\mathfrak{g}}/\mathfrak{g})$, $\pm\chi_{|S}$ belongs to $Wt_S(L)$. This implies that for any $\chi \in Wt_S(\hat{\mathfrak{g}}/\mathfrak{g})$, the sign of $\langle \lambda, \chi \rangle$ does not depend on $\lambda \in \mathcal{C}'$. It follows that $\mathcal{C}'$ is a chamber of $Y(S)_{\mathbb{Q}}$. □

We can now prove Theorem 7.5.

Proof. — We are going to construct a bijection φ between the admissible pairs $(S,\hat{w})$ and the elements $(S',\mathcal{C}',w',\hat{w}')\in\Theta$. This bijection will satisfy $C(\hat{w})=w'C(w',\hat{w}')$ and so $\mathcal{F}(S,\hat{w})=\mathcal{F}(S',\mathcal{C}',w',\hat{w}')$.

Let $(S,\hat{w})$ be an admissible pair. Consider the chamber $\mathcal{C}(\hat{w},S)$ defined by Lemma 7.6. Let $\lambda\in\mathcal{C}$. Then there exists a unique $w\in W/W_{G^S}$ such that $w\lambda$ is dominant. Since $Wt_T(\hat{\mathfrak{g}}/\mathfrak{g})=Wt_T(\hat{\mathfrak{g}})$, then λ is S-regular, $W_\lambda=W_{G^S}$ and wSw^{-1} is dominant. Moreover, $\mathcal{C}$ is contained in a Weyl chamber and so, $w\in W/W_{G^S}$ does not depend on the choice of λ. We set

$$\varphi(S,\hat{w})=(wSw^{-1},\,w\mathcal{C},\,w^{-1},\,\hat{w}w^{-1}).$$

One easily checks that φ is bijective and that its inverse is $(S,\mathcal{C},w,\hat{w})\mapsto(wSw^{-1},\hat{w}w^{-1})$. □

Remark 7.7. — If $Wt_T(\hat{\mathfrak{g}}/\mathfrak{g})=Wt_T(\hat{\mathfrak{g}})$, the set of pairs $(S,\mathcal{C})$ of dominant admissible tori and dominant chamber corresponds bijectively with the set of conjugacy classes of inclusions of the form $G/P(\lambda)\subset\hat{G}/\hat{P}(\lambda)$.

7.4. Application to the tensor product cone

In this section, G is assumed to be semisimple. As above, $T\subset B$ are fixed maximal torus and Borel subgroup of G. We also fix an integer $s\geqslant 2$ and set $\hat{G}=G^s$, $\hat{T}=T^s$ and $\hat{B}=B^s$. We embed G diagonally in $\hat{G}$. Then $\mathcal{LR}(G,\hat{G})$ is the set of $(s+1)$-uples $(\nu_0,\cdots,\nu_s)\in X(T)_{\mathbb{Q}}^{s+1}$ such that for some positive n the $n\nu_i$'s are strictly dominant weights and $V_{n\nu_0}\otimes\cdots\otimes V_{n\nu_s}$ contains nonzero G-invariant vectors.

Theorem 7.5 can be simplified in this case for at least two reasons. First, $Wt_T(\hat{\mathfrak{g}}/\mathfrak{g})$ is just the root system of G. Moreover, the Belkale-Kumar product allows to express the two conditions of the definition of admissible pairs (see Section 7.2.2) in an unified and beautiful way.

We will denote by S_P the neutral component of the center of the Levi subgroup of P containing T.

In [2], Belkale-Kumar defined a new product $\odot_0$ on the cohomology groups $H^*(G/P,\mathbb{Z})$ for any parabolic subgroup P of G. We consider the set Θ of $(P,X_{w_0},\cdots,X_{w_s})$ where P is a standard parabolic subgroup of G and the X_{w_i}'s are $s+1$ Schubert varieties of G/P such that

$$[X_{w_0}]\odot_0\cdots\odot_0[X_{w_s}]=[\mathrm{pt}].$$

By applying Theorem 7.5 to $\hat{G}=G^s$ as in [15], we obtain the following

Theorem 7.8. — *The map which associates to* $(P, X_{w_0}, \cdots, X_{w_s}) \in \Theta$ *the set* $\mathcal{F}(P, X_{w_0}, \cdots, X_{w_s})$ *of* $(\nu_0, \cdots, \nu_s) \in \mathcal{AC}^G(X)$ *such that the restriction of* $\sum_i w_i^{-1}\nu_i$ *to* S_P *is trivial is a bijection from* Θ *onto the set of faces of* $\mathcal{LR}(G, G^s)^\circ$. *Moreover, the codimension of* $\mathcal{F}(P, X_{w_0}, \cdots, X_{w_s})$ *equals the dimension of* S_P *and the following are equivalent:*

(1) $\mathcal{F}(P, X_{w_0}, \cdots, X_{w_s}) \subset \mathcal{F}(P', X_{w'_0}, \cdots, X_{w'_s})$*;*

(2) $P \subset P'$ *and* $\pi(X_{w_i}) = X_{w'_i}$ *for all* $i = 0, \cdots, s$ *(here,* $\pi : G/P \longrightarrow G/P'$ *is the natural map).*

Remark 7.9. — Note that $w_i \in W/W_P$. Also, even if $w_i^{-1}\nu_i$ is not well defined, its restriction to S_P is.

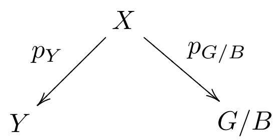
8. GIT-cone and moment polytope

We denote by $\mathcal{TC}^G_\Lambda(X)$, and call the *total G-cone*, the cone generated in $\Lambda_\mathbb{Q}$ by the line bundles (non-necessarily ample) $\mathcal{L} \in \Lambda$ which have nonzero G-invariant sections. Since the tensor product of two nonzero G-invariant sections is a nonzero G-invariant section, $\mathcal{TC}^G_\Lambda(X)$ is convex.

Let Y be a projective G-variety. Let us now explain the relation mentioned in the introduction between the moment polytopes of Y and some total G-cones of $X = G/B \times Y$.

Let $\mathcal{L}$ be an ample G-linearized line bundle on Y. We consider the set $P_G(Y, \mathcal{L})$ of points $p \in X(T)_\mathbb{Q}$ such that for some positive integer n, np is a dominant character of T and the dual V^*_{np} of V_{np} is a submodule of $\mathrm{H}^0(Y, \mathcal{L}^{\otimes n})$. In fact, $P_G(Y, \mathcal{L})$ is a polytope, called the *moment polytope*. Notice that "the dual" is not usual in the definition; but it will be practical for us.

Consider the two projections:



In this section, Λ will always denote the subgroup of $\mathrm{Pic}^G(X)$ generated by $p^*_{G/B}(\mathrm{Pic}^G(G/B))$ and $p^*_Y(\mathcal{L})$. Consider the affine subspace $\Lambda^{\mathrm{aff}}_\mathbb{Q}$ of $\Lambda_\mathbb{Q}$ generated by $p^*_Y(\mathcal{L}) \otimes p_{G/B}(\mathrm{Pic}^G(G/B))$. Note that, $\Lambda^{\mathrm{aff}}_\mathbb{Q}$ is an affine hyperplane of $\Lambda_\mathbb{Q}$ which does not contain 0.

Proposition 8.1. — *With the above notation,* $P_G(Y, \mathcal{L})$ *is the intersection of* $\mathcal{TC}^G_\Lambda(X)$ *with the hyperplane* $\Lambda^{\mathrm{aff}}_\mathbb{Q}$. *More explicitly, for all positive* *rational number m and for all $\nu \in X(T)$, we have:*

$$mp_Y^*(\mathcal{L}) \otimes p_{G/B}^*(\mathcal{L}_\nu) \in \mathcal{TC}_\Lambda^G(X) \iff \frac{\nu}{m} \in P_G(Y, \mathcal{L}).$$

Proof. — The proof which is a direct application of the Borel-Weil theorem is left to the reader. □

It is possible that $P_G(Y, \mathcal{L})$ does not intersect the interior of the Weyl chamber. In this case, $\mathcal{AC}_\Lambda^G(X)$ is empty and our results cannot be applied directly. But, the argument of [4, Section 5] shows that there exists a subgroup H (namely, the centralizer of the derived subgroup of a Levi subgroup of G) and a subvariety Y' of Y such that $P_G(Y, \mathcal{L}) = P_H(Y', \mathcal{L}_{Y'})$ and $P_H(Y', \mathcal{L}_{Y'})$ intersects the interior of the Weyl chamber of H. >From now on, we assume that $P_G(Y, \mathcal{L})$ contains regular points.

Let λ be a one-parameter subgroup of T. Set $B(\lambda) = B \cap P(\lambda)$. Let C be an irreducible component of Y^λ and

$$C^+ := \{x \in X \ : \ \lim_{t \to 0} \lambda(t)x \in C\}$$

the associated Białynicki-Birula cell.

Definition 8.2. — *The pair (C, λ) is said to be B-covering if the natural map $\eta_B : B \times_{B(\lambda)} C^+ \longrightarrow Y$ is birational. It is said to be well-B-covering if η induces an isomorphism over an open subset of Y intersecting C.*

The following lemma is obvious.

Lemma 8.3. — *With the above notation, the pair (C, λ) is B-covering (resp. well-B-covering) if and only if $(G^\lambda B/B \times C, \lambda)$ is covering (resp. well-covering).*

Let us recall that the subtori of T correspond bijectively to the linear subspaces of $X(T)_\mathbb{Q}$. If V is a linear subspace of $X(T)_\mathbb{Q}$, the associated torus is the neutral component of the intersection of kernels of elements in $X(T) \cap V$. If F is a convex subset of $X(T)_\mathbb{Q}$, the direction $\mathrm{dir}(F)$ of F is the subspace spanned by the differences of two elements of F.

We will denote by $\mathcal{C}^+$ the convex cone in $X(T)_\mathbb{Q}$ generated by the dominant weights. The next proposition is an improvement of [4, Theorem 1]:

Proposition 8.4. — *We keep the above notation and assume that Y is smooth and $P_G(Y, \mathcal{L})$ intersects the interior of the dominant chamber. Let $\mathcal{F}$ be a face of $P_G(Y, \mathcal{L})$ which intersects the interior of the dominant chamber. Let S the subtorus of T associated to $\mathrm{dir}(\mathcal{F})$.*

There exist a unique irreducible component C of Y^S and a one-parameter subgroup λ of S such that $G^\lambda = G^S$, (C, λ) is a well-B-covering pair, and $\mathcal{F} = P_{G^S}(C, \mathcal{L}_{|C}) \cap \mathcal{C}^+$.

Proof. — Let $\widetilde{\mathcal{F}}$ be the face of $\mathcal{AC}^G_\Lambda(X)$ corresponding to $\mathcal{F}$ and r denote its codimension. By Proposition 6.2, there exists an admissible well-covering pair (C_X, S') such that $\widetilde{\mathcal{F}} = \mathcal{F}(C_X)$ and S' is a r-dimensional torus. Up to conjugacy, we may assume that C_X intersects $Y \times B/B$, and S' is contained in T. Let λ be a one-parameter subgroup of S' such that (C_X, λ) is well-covering. Then, $C_X = G^\lambda B/B \times C$ for some irreducible component C of $Y^{S'}$.

The fact that $\widetilde{\mathcal{F}} = \mathcal{F}(C_X)$ readily implies that $\mathcal{F} = P_{G^{S'}}(C, \mathcal{L}_{|C}) \cap \mathcal{C}^+$. Since the direction of $P_{G^{S'}}(C, \mathcal{L}_{|C})$ is contained in $X(T)^{S'}$, this implies that $X(T)^S$ is contained in $X(T)^{S'}$. But S and S' have the same rank, it follows that $S = S'$.

The unicity part is a direct consequence of Proposition 6.7. □

Remark 8.5. — The improvement over [4] is the assertion that (C, λ) is a well-covering pair.

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