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Hussein MOURTADA

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JET SCHEMES OF COMPLEX PLANE BRANCHES AND EQUISINGULARITY

by Hussein MOURTADA (*)

ABSTRACT. — For $m \in \mathbb{N}$, we determine the irreducible components of the m -th Jet Scheme of a complex branch C and we give formulas for their number $N(m)$ and for their codimensions, in terms of m and the generators of the semigroup of C . This structure of the Jet Schemes determines and is determined by the topological type of C .

RÉSUMÉ. — Pour $m \in \mathbb{N}$, nous déterminons les composantes irréductibles des m -èmes espaces des jets d'une branche plane complexe C et nous donnons des formules pour leur nombre $N(m)$ et leurs codimensions, en fonction de m et des générateurs du semigroupe de C . Cette structure des espaces des jets détermine et elle est déterminée par le type topologique de C .

1. Introduction

Let \mathbb{K} be an algebraically closed field. The space of arcs X_∞ of an algebraic \mathbb{K} -variety X is a non-noetherian scheme in general. It has been introduced by Nash in [10]. Nash has initiated its study by looking at its image by the truncation maps $X_\infty \rightarrow X_m$ in the jet schemes of X . The m^{th} -jet scheme X_m of X is a \mathbb{K} -scheme of finite type which parametrizes morphisms $\text{Spec } \mathbb{K}[t]/(t)^{m+1} \rightarrow X$. From now on, we assume $\text{char } \mathbb{K} = 0$. In [10], Nash has derived from the existence of a resolution of singularities of X , that the number of irreducible components of the Zariski closure of the set of the m -truncations of arcs on X that send 0 into the singular locus of X is constant for m large enough. Besides a theorem of Kolchin asserts that if X is irreducible, then X_∞ is also irreducible. More recently,

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the jet schemes have attracted attention from various viewpoints. In [9], Mustata has characterized the locally complete intersection varieties having irreducible X_m for $m \geq 0$. In [2], a formula comparing the codimensions of Y_m in X_m with the log canonical threshold of a pair (X, Y) is given. In this work, we consider a curve C in the complex plane \mathbb{C}^2 with a singularity at 0 at which it is analytically irreducible (i.e. the formal neighborhood $(C, 0)$ of C at 0 is a branch). We determine the irreducible components of the space $C_m^0 := \pi_m^{-1}(0)$ where $\pi_m : C_m \rightarrow C$ is the canonical projection, and we show that their number is not bounded as m grows. More precisely, let x be a transversal parameter in the local ring $O_{\mathbb{C}^2, 0}$, i.e. the line $x = 0$ is transversal to C at 0 and following [2], for $e \in \mathbb{N}$, let

$$Cont^e(x)_m (\text{resp. } Cont^{>e}(x)_m) := \{\gamma \in C_m \mid ord_t x \circ \gamma = e (\text{resp. } > e)\},$$

where $Cont$ stands for contact locus. Let $\Gamma(C) = \langle \bar{\beta}_0, \dots, \bar{\beta}_g \rangle$ be the semigroup of the branch $(C, 0)$ and let $e_i = gcd(\bar{\beta}_0, \dots, \bar{\beta}_i)$, $0 \leq i \leq g$. Recall that $\Gamma(C)$ and the topological type of C near 0 are equivalent data and characterize the equisingularity class of $(C, 0)$ as defined by Zariski in [13]. We show in theorem 4.9 that the irreducible components of C_m^0 are

$$C_{m\kappa I} = \overline{Cont^{\kappa\bar{\beta}_0}(x)_m},$$

for $1 \leq \kappa$ and $\kappa\bar{\beta}_0\bar{\beta}_1 + e_1 \leq m$,

$$C_{m\kappa v}^j = \overline{Cont^{\frac{\kappa\bar{\beta}_0}{e_j-1}}(x)_m}$$

for $2 \leq j \leq g, 1 \leq \kappa, \kappa \not\equiv 0 \pmod{e_j}$ and $\kappa\frac{\bar{\beta}_0\bar{\beta}_1}{e_j-1} + e_1 \leq m < \kappa\bar{\beta}_j$,

$$B_m = \overline{Cont^{>\frac{\bar{\beta}_0}{e_1}q}(x)_m},$$

if $q\frac{\bar{\beta}_0}{e_1}\bar{\beta}_1 + e_1 \leq m < (q+1)n_1\bar{\beta}_1 + e_1$.

These irreducible components give rise to infinite and finite inverse systems represented by a tree. We recover $\langle \bar{\beta}_0, \dots, \bar{\beta}_g \rangle$ from the tree and the multiplicity $\bar{\beta}_0$ in corollary 4.13, and we give formulas for the number of irreducible components of C_m^0 and their codimensions in terms of m and $(\bar{\beta}_0, \dots, \bar{\beta}_g)$ in proposition 4.7 and corollary 4.10. We recover the fact coming from [2] and [6] that

$$\min_m \frac{codim(C_m^0, \mathbb{C}^2)}{m+1} = \frac{1}{\bar{\beta}_0} + \frac{1}{\bar{\beta}_1}.$$

The structure of the paper is as follows: The basics about Jet schemes and the results that we will need are presented in section 2. In section 3

we present the definitions and the results we will need about branches. The last section is devoted to the proof of the main result and corollaries.

2. Jet schemes

Let \mathbb{K} be an algebraically closed field of arbitrary characteristic. Let X be a \mathbb{K} -scheme of finite type over k and let $m \in \mathbb{N}$. The functor $F_m : \mathbb{K}\text{-Schemes} \rightarrow \text{Sets}$ which to an affine scheme defined by a \mathbb{K} -algebra A associates

$$F_m(\text{Spec}(A)) = \overline{\text{Hom}}_{\mathbb{K}}(\text{Spec}A[t]/(t^{m+1}), X)$$

is representable by a \mathbb{K} -scheme X_m [12]. X_m is the m -th jet scheme of X , and F_m is isomorphic to its functor of points. In particular the closed points of X_m are in bijection with the $\mathbb{K}[t]/(t^{m+1})$ points of X .

For $m, p \in \mathbb{N}, m > p$, the truncation homomorphism $A[t]/(t^{m+1}) \rightarrow A[t]/(t^{p+1})$ induces a canonical projection $\pi_{m,p} : X_m \rightarrow X_p$. These morphisms clearly verify $\pi_{m,p} \circ \pi_{q,m} = \pi_{q,p}$ for $p < m < q$.

Note that $X_0 = X$. We denote the canonical projection $\pi_{m,0} : X_m \rightarrow X_0$ by π_m .

Example 2.1. — Let $X = \text{Spec} \frac{\mathbb{K}[x_0, \dots, x_n]}{(f_1, \dots, f_r)}$ be an affine \mathbb{K} -scheme. For a \mathbb{K} -algebra A , to give a A -point of X_m is equivalent to give a \mathbb{K} -algebra homomorphism

$$\varphi : \frac{\mathbb{K}[x_0, \dots, x_n]}{(f_1, \dots, f_r)} \rightarrow A[t]/(t^{m+1}).$$

The map φ is completely determined by the image of $x_i, i = 0, \dots, n$

$$x_i \mapsto \varphi(x_i) = x_i^{(0)} + x_i^{(1)}t + \dots + x_i^{(m)}t^m$$

such that $f_l(\phi(x_0), \dots, \phi(x_n)) \in (t^{m+1}), l = 1, \dots, r$.

If we write

$$f_l(\phi(x_0), \dots, \phi(x_n)) = \sum_{j=0}^m F_l^{(j)}(\underline{x}^{(0)}, \dots, \underline{x}^{(j)}) t^j \text{ mod } (t^{m+1})$$

where $\underline{x}^{(j)} = (x_0^{(j)}, \dots, x_n^{(j)})$, then

$$X_m = \text{Spec} \frac{\mathbb{K}[\underline{x}^{(0)}, \dots, \underline{x}^{(m)}]}{(F_l^{(j)})_{l=1, \dots, r}^{j=0, \dots, m}}$$

Example 2.2. — From the above example, we see that the m -th jet scheme of the affine space $\mathbb{A}_{\mathbb{K}}^n$ is isomorphic to $\mathbb{A}_{\mathbb{K}}^{(m+1)n}$ and that the projection $\pi_{m,m-1} : (\mathbb{A}_{\mathbb{K}}^n)_m \rightarrow (\mathbb{A}_{\mathbb{K}}^n)_{m-1}$ is the map that forgets the last n coordinates.

Let $\text{char}(\mathbb{K}) = 0$, $S = \mathbb{K}[x_0, \dots, x_n]$ and $S_m = \mathbb{K}[\underline{x}^{(0)}, \dots, \underline{x}^{(m)}]$. Let D be the \mathbb{K} -derivation on S_m defined by $D(x_i^{(j)}) = x_i^{(j+1)}$ if $0 \leq j < m$, and $D(x_i^{(m)}) = 0$. For $f \in S$ let $f^{(1)} := D(f)$ and we recursively define $f^{(m)} = D(f^{(m-1)})$.

PROPOSITION 2.3. — *Let $X = \text{Spec}(S/(f_1, \dots, f_r)) = \text{Spec}(R)$ and $R_m = \Gamma(X_m)$. Then*

$$R_m = \text{Spec}\left(\frac{\mathbb{K}[\underline{x}^{(0)}, \dots, \underline{x}^{(m)}]}{(f_i^{(j)})_{i=1, \dots, r}^{j=0, \dots, m}}\right).$$

Proof. — For a \mathbb{K} -algebra A , to give an A -point of X_m is equivalent to give an homomorphism

$$\phi : \mathbb{K}[x_0, \dots, x_n] \rightarrow A[t]/(t^{m+1})$$

which can be given by

$$x_i \rightarrow \frac{x_i^{(0)}}{0!} + \frac{x_i^{(1)}}{1!}t + \dots + \frac{x_i^{(m)}}{m!}t^m.$$

Then for a polynomial $f \in S$, we have

$$\phi(f) = \sum_{j=0}^m \frac{f^{(j)}(\underline{x}^{(0)}, \dots, \underline{x}^{(j)})}{j!} t^j.$$

To see this, it is sufficient to remark that it is true for $f = x_i$, and that both sides of the equality are additive and multiplicative in f , and the proposition follows. □

Remark 2.4. — Note that the proposition shows the linearity of the equations $F_i^j(\underline{x}^{(0)}, \dots, \underline{x}^{(j)})$ defining X_m with respect to the new variables i.e $\underline{x}^{(j)}$. We can deduce from this that if X is a nonsingular \mathbb{K} -variety of dimension n , then the projections $\pi_{m,m-1} : X_m \rightarrow X_{m-1}$ are locally trivial fibrations with fiber $\mathbb{A}_{\mathbb{K}}^n$. In particular, X_m is a non singular variety of dimension $(m + 1)n$.

3. Semigroup of complex branches

The main references for this section are [14],[8],[1],[11],[5],[4],[7]. Let $f \in \mathbb{C}[[x, y]]$ be an irreducible power series, which is y -regular (i.e $f(0, y) =$

$y^{\beta_0}u(y)$ where u is invertible in $\mathbb{C}[[y]]$ and such that $mult_0 f = \beta_0$ and let C be the analytically irreducible plane curve(branch for short) defined by f in $Spec \mathbb{C}[[x, y]]$. By the Newton-Puiseux theorem, the roots of f are

$$y = \sum_{i=0}^{\infty} a_i w^i x^{\frac{i}{\beta_0}} \tag{1}$$

where w runs over the $\beta_0 - th$ -roots of unity in \mathbb{C} . This is equivalent to the existence of a parametrization of C of the form

$$\begin{aligned} x(t) &= t^{\beta_0} \\ y(t) &= \sum_{i \geq \beta_0} a_i t^i. \end{aligned}$$

We recursively define $\beta_i = \min\{i, a_i \neq 0, \gcd(\beta_0, \dots, \beta_{i-1}) \text{ is not a divisor of } i\}$.

Let $e_0 = \beta_0$ and $e_i = \gcd(e_{i-1}, \beta_i), i \geq 1$. Since the sequence of positive integers

$$e_0 > e_1 > \dots > e_i > \dots$$

is strictly decreasing, there exists $g \in \mathbb{N}$, such that $e_g = 1$. The sequence $(\beta_1, \dots, \beta_g)$ is the sequence of Puiseux exponents of C . We set

$$n_i := \frac{e_{i-1}}{e_i}, m_i := \frac{\beta_i}{e_i}, i = 1, \dots, g$$

and by convention, we set $\beta_{g+1} = +\infty$ and $n_{g+1} = 1$.

On the other hand, for $h \in \mathbb{C}[[x, y]]$, we define the intersection number

$$(f, h)_0 = (C, C_h)_0 := \dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{(f, h)} = ord_t h(x(t), y(t))$$

where C_h is the Cartier divisor defined by h and $\{x(t), y(t)\}$ is as above.

The mapping $v_f : \frac{\mathbb{C}[[x, y]]}{(f)} \rightarrow \mathbb{N}, h \mapsto (f, h)_0$ defines a divisorial valuation.

We define the semigroup of C to be the semigroup of v_f i.e $\Gamma(C) = \Gamma(v_f) = \{(f, h)_0 \in \mathbb{N}, h \neq 0 \text{ mod}(f)\}$.

The following propositions and theorem from [14] characterize the structure of $\Gamma(C)$.

PROPOSITION 3.1. — *There exists a unique sequence of $g + 1$ positive integers $(\bar{\beta}_0, \dots, \bar{\beta}_g)$ such that:*

- i) $\bar{\beta}_0 = \beta_0$,
- ii) $\bar{\beta}_i = \min\{\Gamma(C) \setminus \langle \bar{\beta}_0, \dots, \bar{\beta}_{i-1} \rangle\}, 1 \leq i \leq g$,
- iii) $\Gamma(C) = \langle \bar{\beta}_0, \dots, \bar{\beta}_g \rangle$,

where for $i = 1, \dots, g + 1, \langle \bar{\beta}_0, \dots, \bar{\beta}_{i-1} \rangle$ is the semigroup generated by $\bar{\beta}_0, \dots, \bar{\beta}_{i-1}$. By convention, we set $\bar{\beta}_{g+1} = +\infty$.

PROPOSITION 3.2. — The sequence $(\bar{\beta}_0, \dots, \bar{\beta}_g)$ verifies:
 i) $e_i = \gcd(\bar{\beta}_0, \dots, \bar{\beta}_i), 0 \leq i \leq g,$
 ii) $\bar{\beta}_0 = \beta_0, \bar{\beta}_1 = \beta_1$ and $\bar{\beta}_i = n_{i-1}\bar{\beta}_{i-1} + \beta_i - \beta_{i-1}$. In particular $n_i\bar{\beta}_i < \bar{\beta}_{i+1}$, for $i = 2, \dots, g$.

THEOREM 3.3. — The sequence $(\bar{\beta}_0, \dots, \bar{\beta}_g)$ and the sequence $(\beta_0, \dots, \beta_g)$ are equivalent data. They determine and are determined by the topological type of C .

Then from the appendix of [14], [1] or [11], we can choose a system of approximate roots (or a minimal generating sequence) $\{x_0, \dots, x_{g+1}\}$ of the divisorial valuation v_f . We set $x = x_0, y = x_1$; for $i = 2, \dots, g + 1, x_i \in \mathbb{C}[[x, y]]$ is irreducible; for $1 \leq i \leq g$, the analytically irreducible curve $C_i = \{x_i = 0\}$ has $i - 1$ Puiseux exponents and $C_{g+1} = C$. This sequence also verifies

- i) $v_f(x_i) = \bar{\beta}_i, 0 \leq i \leq g,$
- ii) $\Gamma(C_i) = \langle \frac{\bar{\beta}_0}{e_{i-1}}, \dots, \frac{\bar{\beta}_{i-1}}{e_{i-1}} \rangle$ and the Puiseux sequence of C_i is $(\frac{\bar{\beta}_1}{e_{i-1}}, \dots, \frac{\bar{\beta}_{i-1}}{e_{i-1}}), 2 \leq i \leq g + 1.$
- iii) for $1 \leq i \leq g$, there exists a unique system of nonnegative integers $b_{ij}, 0 \leq j < i$ such that for $1 \leq j < i, b_{ij} < n_j$ and $n_i\bar{\beta}_i = \sum_{0 \leq j < i} b_{ij}\bar{\beta}_j$. Furthermore, for $1 \leq i \leq g$, one can choose x_i such that they satisfy identities of the form

$$x_{i+1} = x_i^{n_i} - c_i x_0^{b_{i0}} \dots x_{i-1}^{b_{i(i-1)}} - \sum_{\gamma=(\gamma_0, \dots, \gamma_i)} c_{i,\gamma} x_0^{\gamma_0} \dots x_i^{\gamma_i}, (\star)$$

with, $0 \leq \gamma_j < n_j$, for $1 \leq j \leq i$, and $\sum_j \gamma_j \bar{\beta}_j > n_i \bar{\beta}_i$ and with $c_{i,\gamma}, c_i \in \mathbb{C}$ and $c_i \neq 0$. These last equations (\star) let us realize C as a complete intersection in $\mathbb{C}^{g+1} = \text{Spec } \mathbb{C}[[x_0, \dots, x_g]]$ defined by the equations

$$f_i = x_{i+1} - (x_i^{n_i} - c_i x_0^{b_{i0}} \dots x_{i-1}^{b_{i(i-1)}} - \sum_{\gamma=(\gamma_0, \dots, \gamma_i)} c_{i,\gamma} x_0^{\gamma_0} \dots x_i^{\gamma_i})$$

for $1 \leq i \leq g$, with $x_{g+1} = 0$ by convention.

Let $h \in \mathbb{C}[[x, y]]$ be a y -regular irreducible power series with multiplicity $p = \text{ord}_y h(0, y)$. Let $y(x^{\frac{1}{p}})$ and $z(x^{\frac{1}{p}})$ be respectively roots of f and h as in (1). We call contact order of f and h in their Puiseux series the following rational number

$$\begin{aligned} o_f(h) := \max\{\text{ord}_x(y(wx^{\frac{1}{\beta_0}}) - z(\lambda x^{\frac{1}{p}})); w^{\beta_0} = 1, \lambda^p = 1\} = \\ \max\{\text{ord}_x(y(wx^{\frac{1}{\beta_0}}) - z(x^{\frac{1}{p}})); w^{\beta_0} = 1\} = \\ \max\{\text{ord}_x(y(x^{\frac{1}{\beta_0}}) - z(\lambda x^{\frac{1}{p}})); \lambda^p = 1\} = o_h(f). \end{aligned}$$

The following formula is from [8], see also [5].

PROPOSITION 3.4. — Assume that f and h are as above; let $(\beta_1, \dots, \beta_g)$ the sequence of Puiseux exponents of f and let $i \leq g + 1$ be the smallest strictly positive integer such that $o_f(h) \leq \frac{\beta_i}{\beta_0}$. Then

$$\begin{aligned} \frac{(f, h)_0}{p} &= \sum_{k=1}^{i-1} \frac{e_{k-1} - e_k}{\beta_0} \beta_k + e_{i-1} o_f(h) \\ &= (\bar{\beta}_{i-1} e_{i-2} + (\beta_0 o_f(h) - \beta_{i-1}) e_{i-1}) \frac{1}{\beta_0}. \end{aligned}$$

COROLLARY 3.5. — [1][5] Let $i > 0$ be an integer. Then $o_f(h) \leq \frac{\beta_i}{\beta_0}$ iff $\frac{(f, h)_0}{p} \leq e_{i-1} \frac{\bar{\beta}_i}{\beta_0}$. Moreover $o_f(h) = \frac{\beta_i}{\beta_0}$ iff $\frac{(f, h)_0}{p} = e_{i-1} \frac{\bar{\beta}_i}{\beta_0}$. In particular $o_f(x_i) = \frac{\beta_i}{\beta_0}, 1 \leq i \leq g$. We say that $C_i x_i = 0$ has maximal contact with C .

4. Jet schemes of complex branches

We keep the notations of sections 2 and 3. We consider a curve $C \subset \mathbb{C}^2$ with a branch of multiplicity $\beta_0 > 1$ at 0, defined by f . Note that in suitable coordinates we can write

$$f(x_0, x_1) = (x_1^{n_1} - c x_0^{m_1})^{e_1} + \sum_{a\beta_0 + b\beta_1 > \beta_0\beta_1} c_{ab} x_0^a x_1^b; c \in \mathbb{C}^* \text{ and } c_{ab} \in \mathbb{C}. \quad (\diamond)$$

We look for the irreducible components of $C_m^0 := (\pi_m^{-1}(0))$ for every $m \in \mathbb{N}$, where $\pi_m : C_m \rightarrow C$ is the canonical projection. Let J_m^0 be the radical of the ideal defining $(\pi_m^{-1}(0))$ in \mathbb{C}^2_m .

In the sequel, we will denote the integral part of a rational number r by $[r]$.

PROPOSITION 4.1. — For $0 < m < n_1 \bar{\beta}_1$, we have that

$$(C_m^0)_{red} = (\pi_m^{-1}(0))_{red} = Spec \frac{\mathbb{C}[x_0^{(0)}, \dots, x_0^{(m)}, x_1^{(0)}, \dots, x_1^{(m)}]}{(x_0^{(0)}, \dots, x_0^{(\lfloor \frac{m}{\beta_1} \rfloor)}, x_1^{(0)}, \dots, x_1^{(\lfloor \frac{m}{\beta_0} \rfloor)})}$$

and

$$\begin{aligned} (C_{n_1 \bar{\beta}_1}^0)_{red} &= (\pi_{n_1 \bar{\beta}_1}^{-1}(0))_{red} \\ &= Spec \frac{\mathbb{C}[x_0^{(0)}, \dots, x_0^{(n_1 \bar{\beta}_1)}, x_1^{(0)}, \dots, x_1^{(n_1 \bar{\beta}_1)}]}{(x_0^{(0)}, \dots, x_0^{(n_1-1)}, x_1^{(0)}, \dots, x_1^{(m_1-1)}, x_1^{(m_1)^{n_1}} - c x_0^{(n_1)^{m_1}})}. \end{aligned}$$

Proof. — We write $f = \sum_{(a,b)} c_{ab} f_{ab}$ where $(a, b) \in \mathbb{N}^2, f_{ab} = x_0^a x_1^b, c_{ab} \in \mathbb{C}$ and $a\beta_0 + b\beta_1 \geq \beta_0\bar{\beta}_1$ (the segment $[(0, \beta_0)(\bar{\beta}_1, 0)]$ is the Newton Polygon of f). Let $supp(f) = \{(a, b) \in \mathbb{N}^2; c_{ab} \neq 0\}$.

For $0 < m < n_1\bar{\beta}_1$, the proof is by induction on m . For $m = 1$, we have that

$$F^{(1)} = \sum_{(a,b) \in \text{supp}(f)} c_{ab} F_{ab}^{(1)}$$

where $(F^{(0)}, \dots, F^{(i)})$ (resp. $(F_{ab}^{(0)}, \dots, F_{ab}^{(i)})$) is the ideal defining the i -th jet scheme C_i of C (resp. C_i^{ab} the i -th jet scheme of $C^{ab} = \{f_{ab} = 0\}$) in \mathbb{C}_i^2 . Then we have

$$F_{ab}^{(1)} = \sum_{\sum i_k=1} x_0^{(i_1)} \dots x_0^{(i_a)} x_1^{(i_{a+1})} \dots x_1^{(i_{a+b})}$$

where $\bar{\beta}_1(a+b) \geq a\beta_0 + b\bar{\beta}_1 \geq \beta_0\bar{\beta}_1$ so $a+b \geq \beta_0 > 1$. Then for every $(a,b) \in \text{supp}(f)$ and every $(i_1, \dots, i_a, \dots, i_{a+b}) \in \mathbb{N}^{a+b}$ such that $\sum_{k=1}^{a+b} i_k = 1$ there exists $1 \leq k \leq a+b$ such that $i_k \neq 0$, this means that $F_{ab}^{(1)} \in (x_0^{(0)}, x_1^{(0)})$ and since we are looking over the origin, we have that $(x_0^{(0)}, x_1^{(0)}) \subseteq J_1^0$ therefore $(\pi_1^{-1}(0))_{red} = \text{Spec} \frac{\mathbb{C}[x_0^{(0)}, x_0^{(1)}, x_1^{(0)}, x_1^{(1)}]}{(x_0^{(0)}, x_1^{(0)})}$ (In fact this is nothing but the Zariski tangent space of C at 0).

Suppose that the lemma holds until $m - 1$ i.e.

$$(\pi_{m-1}^{-1}(0))_{red} = \text{Spec} \frac{\mathbb{C}[x_0^{(0)}, \dots, x_0^{(m-1)}, x_1^{(0)}, \dots, x_1^{(m-1)}]}{(x_0^{(0)}, \dots, x_0^{(\lfloor \frac{m-1}{\beta_1} \rfloor)}, x_1^{(0)}, \dots, x_1^{(\lfloor \frac{m-1}{\beta_0} \rfloor)})}$$

First case: If $\lfloor \frac{m-1}{\beta_1} \rfloor = \lfloor \frac{m}{\beta_1} \rfloor$ and $\lfloor \frac{m-1}{\beta_0} \rfloor = \lfloor \frac{m}{\beta_0} \rfloor$. We have

$$F^{(m)} = \sum_{(a,b) \in \text{supp}(f)} c_{ab} \sum_{\sum i_k=m} x_0^{(i_1)} \dots x_0^{(i_a)} x_1^{(i_{a+1})} \dots x_1^{(i_{a+b})}$$

Let $(a,b) \in \text{supp}(f)$; if for every $k = 1, \dots, a$, we had $i_k \geq \lfloor \frac{m}{\beta_1} \rfloor + 1$, and for every $k = a+1, \dots, a+b$, we had $i_k \geq \lfloor \frac{m}{\beta_0} \rfloor + 1$, then

$$m \geq a(\lfloor \frac{m}{\beta_1} \rfloor + 1) + b(\lfloor \frac{m}{\beta_0} \rfloor + 1) > \frac{m}{\beta_1}a + \frac{m}{\beta_0}b = m \frac{a\beta_0 + b\bar{\beta}_1}{\beta_0\bar{\beta}_1} \geq m.$$

The contradiction means that there exists $1 \leq k \leq a$ such that $i_k \leq \lfloor \frac{m}{\beta_1} \rfloor$ or there exists $a+1 \leq k \leq a+b$ such that $i_k \leq \lfloor \frac{m}{\beta_0} \rfloor$. So $F^{(m)}$ lies in the ideal generated by J_{m-1}^0 in $\mathbb{C}[x_0^{(0)}, \dots, x_0^{(m)}, x_1^{(0)}, \dots, x_1^{(m)}]$ and $J_m^0 = J_{m-1}^0 \cdot \mathbb{C}[x_0^{(0)}, \dots, x_0^{(m)}, x_1^{(0)}, \dots, x_1^{(m)}]$.

Second case: If $\lfloor \frac{m-1}{\beta_1} \rfloor = \lfloor \frac{m}{\beta_1} \rfloor$ and $\lfloor \frac{m-1}{\beta_0} \rfloor + 1 = \lfloor \frac{m}{\beta_0} \rfloor$ (i.e. β_0 divides m). We have that

$$F^{(m)} = F_{0\beta_0}^{(m)} + \sum_{(a,b) \in \text{supp}(f); (a,b) \neq (0,\beta_0)} F_{ab}^{(m)}, \tag{**}$$

where

$$\begin{aligned}
 F_{0\beta_0}^{(m)} &= \sum_{\sum i_k=m} x_1^{(i_1)} \cdots x_1^{(i_{\beta_0})} \\
 &= x_1^{\binom{m}{\beta_0}} + \sum_{\sum i_k=m; (i_1, \dots, i_{\beta_0}) \neq (\frac{m}{\beta_0}, \dots, \frac{m}{\beta_0})} x_1^{(i_1)} \cdots x_1^{(i_{\beta_0})};
 \end{aligned}$$

but $\sum i_k = m$ and $(i_1, \dots, i_{\beta_0}) \neq (\frac{m}{\beta_0}, \dots, \frac{m}{\beta_0})$ implies that there exists $1 \leq k \leq \beta_0$ such that $i_k < \frac{m}{\beta_0}$, so

$$\begin{aligned}
 &\sum_{\sum i_k=m; (i_1, \dots, i_{\beta_0}) \neq (\frac{m}{\beta_0}, \dots, \frac{m}{\beta_0})} x_1^{(i_1)} \cdots x_1^{(i_{\beta_0})} \\
 &\qquad \qquad \qquad \in J_{m-1}^0 \cdot \mathbb{C}[x_0^{(0)}, \dots, x_0^{(m)}, x_1^{(0)}, \dots, x_1^{(m)}].
 \end{aligned}$$

For the same reason as above, we have that

$$\sum_{(a,b) \in \text{supp}(f); (a,b) \neq (0,\beta_0)} F_{ab}^{(m)} \in J_{m-1}^0 \cdot \mathbb{C}[x_0^{(0)}, \dots, x_0^{(m)}, x_1^{(0)}, \dots, x_1^{(m)}].$$

From $(\star\star)$ we deduce that $x_1^{\binom{m}{\beta_0}} \in J_m^0$ and

$$F^{(m)} \in (x_0^{(0)}, \dots, x_0^{\lfloor \frac{m}{\beta_1} \rfloor}, x_1^{(0)}, \dots, x_1^{\binom{m}{\beta_0}}).$$

Then $J_m^0 = (x_0^{(0)}, \dots, x_0^{\lfloor \frac{m}{\beta_1} \rfloor}, x_1^{(0)}, \dots, x_1^{\binom{m}{\beta_0}})$.

The third case i.e. if $\lfloor \frac{m-1}{\beta_1} \rfloor + 1 = \lfloor \frac{m}{\beta_1} \rfloor$ and $\lfloor \frac{m-1}{\beta_0} \rfloor = \lfloor \frac{m}{\beta_0} \rfloor$ is discussed as the second one. Note that these are the only three possible cases since $m < n_1 \bar{\beta}_1 = \text{lcm}(\beta_0, \bar{\beta}_1)$ (here *lcm* stands for the least common multiple).

For $m = n_1 \bar{\beta}_1$, we have that $F^{(m)}$ is the coefficient of t^m in the expansion of

$$f(x_0^{(0)} + x_0^{(1)}t + \cdots + x_0^{(m)}t^m, x_1^{(0)} + x_1^{(1)}t + \cdots + x_1^{(m)}t^m).$$

But since we are interested in the radical of the ideal defining the m -th jet scheme, and we have found that $x_0^{(0)}, \dots, x_0^{(n_1-1)}, x_1^{(0)}, \dots, x_1^{(m_1-1)} \in J_{m-1}^0 \subseteq J_m^0$, we can annihilate $x_0^{(0)}, \dots, x_0^{(n_1-1)}, x_1^{(0)}, \dots, x_1^{(m_1-1)}$ in the above expansion. Using (\diamond) , we see that the coefficient of t^m is $(x_1^{(m_1)})^{n_1} - cx_0^{(n_1 m_1)} \in \mathfrak{e}_1$. □

In the sequel if A is a ring, $I \subseteq A$ an ideal and $f \in A$, we denote by $V(I)$ the subvariety of *Spec* A defined by I and by $D(f)$ the open set in *Spec* A , $D(f) := \text{Spec} A_f$.

The proof of the following corollary is analogous to that of proposition 4.1.

COROLLARY 4.2. — *Let $m \in \mathbb{N}$; let $k \geq 1$ be such that $m = kn_1\bar{\beta}_1 + i$; $1 \leq i \leq n_1\bar{\beta}_1$. Then if $i < n_1\bar{\beta}_1$, we have that*

$$Cont^{>kn_1}(x_0)_m = (\pi_{m, kn_1\bar{\beta}_1}^{-1}(V(x_0^{(0)}, \dots, x_0^{(kn_1)})))_{red} =$$

$$Spec \frac{\mathbb{C}[x_0^{(0)}, \dots, x_0^{(m)}, x_1^{(0)}, \dots, x_1^{(m)}]}{(x_0^{(0)}, \dots, x_0^{(kn_1)}, \dots, x_0^{(kn_1 + \lceil \frac{i}{\bar{\beta}_1} \rceil)}, x_1^{(0)}, \dots, x_1^{(km_1)}, \dots, x_1^{(km_1 + \lceil \frac{i}{\bar{\beta}_1} \rceil)})}$$

and if $i = n_1\bar{\beta}_1$

$$(\pi_{m, kn_1\bar{\beta}_1}^{-1}(V(x_0^{(0)}, \dots, x_0^{(kn_1)})))_{red} =$$

$$Spec \frac{\mathbb{C}[x_0^{(0)}, \dots, x_0^{(m)}, x_1^{(0)}, \dots, x_1^{(m)}]}{(x_0^{(0)}, \dots, x_0^{((k+1)n_1-1)}, x_1^{(0)}, \dots, x_1^{((k+1)m_1-1)}, x_1^{((k+1)m_1)^{n_1}} - cx_0^{((k+1)n_1)^{m_1}})}$$

We now consider the case of a plane branch with one Puiseux exponent.

LEMMA 4.3. — *Let C be a plane branch with one Puiseux exponent. Let $m, k \in \mathbb{N}$, such that $k \neq 0$ and $m \geq kn_1\bar{\beta}_1 + 1$, and let $\pi_{m, kn_1\bar{\beta}_1} : C_m \rightarrow C_{kn_1\bar{\beta}_1}$ be the canonical projection. Then*

$$C_m^k := \pi_{m, kn_1\bar{\beta}_1}^{-1}(V(x_0^{(0)}, \dots, x_0^{(kn_1-1)}) \cap D(x_0^{(kn_1)}))_{red}$$

is irreducible of codimension $k(m_1 + n_1) + 1 + (m - kn_1\bar{\beta}_1)$ in \mathbb{C}_m^2 .

Proof. — First note that since $e_1 = 1$, we have $m_1 = \frac{\bar{\beta}_1}{e_1} = \bar{\beta}_1$. Let I_m^{0k} be the ideal defining C_m^k in $\mathbb{C}_m^2 \cap D(x_0^{(kn_1)})$. Since $m \geq kn_1\bar{\beta}_1$, by corollary 4.2, $x_1^{(0)}, \dots, x_1^{(km_1-1)} \in I_m^{0k}$. So I_m^{0k} is the radical of the ideal $I_m^{*0k} := (x_0^{(0)}, \dots, x_0^{(kn_1-1)}, x_1^{(0)}, \dots, x_1^{(km_1-1)}, F^{(0)}, \dots, F^{(m)})$. Now it follows from \diamond and proposition 2.3 that

$$F^{(l)} \in (x_0^{(0)}, \dots, x_0^{(kn_1-1)}, x_1^{(0)}, \dots, x_1^{(km_1-1)}) \text{ for } 0 \leq l < kn_1m_1,$$

$$F^{(kn_1m_1)} \equiv x_1^{(km_1)^{n_1}} - cx_0^{(kn_1)^{m_1}} \text{ mod } (x_0^{(0)}, \dots, x_0^{(kn_1-1)}, x_1^{(0)}, \dots, x_1^{(km_1-1)}),$$

$$F^{(kn_1m_1+l)} \equiv n_1x_1^{(km_1)^{n_1-1}}x_1^{(km_1+l)} - m_1cx_0^{(kn_1)^{m_1-1}}x_0^{(kn_1+l)} + H_l(x_0^{(0)}, \dots, x_0^{(kn_1+l-1)}, x_1^{(0)}, \dots, x_1^{(km_1+l-1)}) \text{ mod } (x_0^{(0)}, \dots, x_0^{(kn_1-1)}, x_1^{(0)}, \dots, x_1^{(km_1-1)}),$$

for $1 \leq l \leq m - kn_1m_1$.

This implies that

$$I_m^{*0k} = (x_0^{(0)}, \dots, x_0^{(kn_1-1)}, x_1^{(0)}, \dots, x_1^{(km_1-1)}, F^{(kn_1m_1)}, \dots, F^{(m)}).$$

Moreover the subscheme of $\mathbb{C}_m^2 \cap D(x_0^{(kn_1)})$ defined by I_m^{*0k} is isomorphic to the product of \mathbb{C}^* (\mathbb{C}^* is isomorphic to the regular locus of $x_1^{(km_1)n_1} - cx_0^{(kn_1)m_1}$) by an affine space and its codimension is $k(m_1 + n_1) + 1 + (m - kn_1m_1)$; so it is reduced and irreducible, and it is nothing but C_m^k , or equivalently $I_m^{0k} = I_m^{*0k}$. \square

COROLLARY 4.4. — *Let C be a plane branch with one Puiseux exponent. Let $m \in \mathbb{N}, m \neq 0$. let $q \in \mathbb{N}$ be such that $m = qn_1\bar{\beta}_1 + i; 0 < i \leq n_1\bar{\beta}_1$. Then $C_m^0 = \pi_m^{-1}(0)$ has $q + 1$ irreducible components which are:*

$$C_{mkI} = \overline{C_m^k}, 1 \leq k \leq q,$$

$$\text{and } B_m = \text{Cont}^{>qn_1}(x)_m = \pi_{m, qn_1\bar{\beta}_1}^{-1}(V(x_0^{(0)}, \dots, x_0^{(qn_1)})).$$

We have that

$$\text{codim}(C_{mkI}, \mathbb{C}_m^2) = k(m_1 + n_1) + 1 + (m - kn_1m_1)$$

and

$$\text{codim}(B_m, \mathbb{C}_m^2) = q(m_1 + n_1) + \left[\frac{i}{\beta_0}\right] + \left[\frac{i}{\beta_1}\right] + 2 = \left[\frac{m}{\beta_0}\right] + \left[\frac{m}{\beta_1}\right] + 2 \text{ if } i < n_1\bar{\beta}_1$$

$$\text{codim}(B_m, \mathbb{C}_m^2) = (q + 1)(m_1 + n_1) + 1 \text{ if } i = n_1\bar{\beta}_1.$$

Proof. — The codimensions and the irreducibility of B_m and C_{mkI} follow from corollary 4.2 and lemma 4.3. This shows that if $1 \leq k < k' \leq q$, we have $\text{codim}(C_{mk'I}, \mathbb{C}_m^2) < \text{codim}(C_{mkI}, \mathbb{C}_m^2)$, then $C_{mk'I} \not\subseteq C_{mkI}$. On the other hand, since $C_{mk'I} \subseteq V(x_0^{(kn_1)})$ and $C_{mkI} \not\subseteq V(x_0^{(kn_1)})$, we have that $C_{mkI} \not\subseteq C_{mk'I}$. This also shows that $\dim B_m \geq \dim C_{mkI}$ for $1 \leq k \leq q$, therefore $B_m \not\subseteq C_{mkI}, 1 \leq k \leq q$. But $C_{mkI} \not\subseteq B_m$ because $B_m \subseteq V(x_0^{(qn_1)})$ and $C_{mkI} \not\subseteq V(x_0^{(qn_1)})$ for $1 \leq k \leq q$. We thus have that $C_{mkI} \not\subseteq B_m$ and $B_m \not\subseteq C_{mkI}$. We conclude the corollary from the fact that by construction $C_m^0 = \cup_{k=1}^q C_{mkI} \cup B_m$. \square

To understand the general case, i.e. to find the irreducible components of C_m^0 where C has a branch with g Puiseux exponents at 0, since for $kn_1\bar{\beta}_1 < m \leq (k + 1)n_1\bar{\beta}_1, m, k \in \mathbb{N}$ we know by corollary 4.2 the structure of the m -jets that project to $V(x_0^{(0)}, \dots, x_0^{(kn_1)}) \cap C_{kn_1\bar{\beta}_1}^0$, we have to understand for $m > kn_1\bar{\beta}_1$ the m -jets that projects to $V(x_0^{(0)}, \dots, x_0^{(kn_1-1)}) \cap D(x_0^{(kn_1)})$, i.e. $C_m^k := \pi_{m, kn_1\bar{\beta}_1}^{-1}(V(x_0^{(0)}, \dots, x_0^{(kn_1-1)}) \cap D(x_0^{(kn_1)}))_{red}$.

Let $m, k \in \mathbb{N}$ be such that $m \geq kn_1\bar{\beta}_1$. Let $j = \max\{l, n_2 \cdots n_{l-1} \text{ divides } k\}$ (we set $j = 2$ if the greatest common divisor $(k, n_2) = 1$ or if $g = 1$). Set κ such that $k = \kappa n_2 \cdots n_{j-1}$, then we have $kn_1 = \kappa \frac{\beta_0}{n_j \cdots n_g}$.

PROPOSITION 4.5. — Let $2 \leq j \leq g + 1$; for $i = 2, \dots, g$, and $kn_1\bar{\beta}_1 < m < \kappa e_{i-1} \frac{\bar{\beta}_i}{e_{j-1}}$, we have that

$$C_m^k = \bar{\pi}_{m, [\frac{m}{n_i \cdots n_g}]^{-1}}(C_{i, [\frac{m}{n_i \cdots n_g}]}^k),$$

where $\bar{\pi}_{m, [\frac{m}{n_i \cdots n_g}]} : \mathbb{C}_m^2 \rightarrow \mathbb{C}_{[\frac{m}{n_i \cdots n_g}]}^2$ is the canonical map. For $j < g + 1$ and $m \geq \kappa \bar{\beta}_j$, we have that

$$C_m^k = \emptyset$$

Proof. — Let $\phi \in C_m^k$. Let $\tilde{\phi} : \text{Spec } \mathbb{C}[[t]] \rightarrow (\mathbb{C}^2, 0)$ be such that $\phi = \tilde{\phi} \bmod t^{m+1}$. Let $\tilde{f} \in \mathbb{C}[[x, y]]$ be a function that defines the branch \tilde{C} image of $\tilde{\phi}$. we may assume that the map $\text{Spec } \mathbb{C}[[t]] \rightarrow \tilde{C}$ induced by $\tilde{\phi}$ is the normalization of \tilde{C} . Since $\text{ord}_t x_0 \circ \tilde{\phi} = kn_1, \text{ord}_t x_1 \circ \tilde{\phi} = km_1$ the multiplicity $m(\tilde{f})$ of \tilde{C} at the origin is $\text{ord}_{x_1} \tilde{f}(0, x_1) = kn_1 = \kappa \frac{\beta_0}{n_j \cdots n_g}$.

Claim: If $(f, \tilde{f})_0 < \kappa e_{i-1} \frac{\bar{\beta}_i}{e_{j-1}}$ then $(f, \tilde{f})_0 = n_i \cdots n_g(x_i, \tilde{f})_0$.

Indeed, we have that $\frac{(f, \tilde{f})_0}{\text{ord}_y \tilde{f}(0, y)} < e_{i-1} \frac{\bar{\beta}_i}{\beta_0}$, therefore by corollary 3.5 we have that

$$o_f(\tilde{f}) < \frac{\beta_i}{\beta_0} = o_f(x_i).$$

We will prove that $o_f(\tilde{f}) = o_{x_i}(\tilde{f})$. (It was pointed by the referee that this follows from [1]. For the convenience of the reader we give a detailed proof below.)

Let $y(x^{\frac{1}{\beta_0}}), z(x^{\frac{1}{n_1 \cdots n_{i-1}}})$ and $u(x^{\frac{1}{m(\tilde{f})}})$ be respectively Puiseux-roots of f, x_i and \tilde{f} . There exist $w, \lambda \in \mathbb{C}$ such that $w^{\frac{\beta_0}{n_i \cdots n_g}} = 1, \lambda^{m(\tilde{f})} = 1$ and

$$o_f(\tilde{f}) = \text{ord}_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - y(x^{\frac{1}{\beta_0}}))$$

and

$$o_f(x_i) = \text{ord}_x(y(x^{\frac{1}{\beta_0}}) - z(wx^{\frac{1}{n_1 \cdots n_{i-1}}}))$$

Since $o_f(\tilde{f}) < o_f(x_i)$, we have that

$$\begin{aligned} o_f(\tilde{f}) &= \text{ord}_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - y(x^{\frac{1}{\beta_0}}) + y(x^{\frac{1}{\beta_0}}) - z(wx^{\frac{1}{n_1 \cdots n_{i-1}}})) \\ &= \text{ord}_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - z(wx^{\frac{1}{n_1 \cdots n_{i-1}}})) \leq o_{x_i}(\tilde{f}). \end{aligned}$$

On the other hand, there exist λ and $\delta \in \mathbb{C}$, such that $\lambda^{m(\tilde{f})} = 1, \delta^{\beta_0} = 1$ and such that

$$o_{x_i}(\tilde{f}) = \text{ord}_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - z(x^{\frac{1}{n_1 \cdots n_{i-1}}}))$$

and

$$o_f(x_i) = \text{ord}_x(y(\delta x^{\frac{1}{\beta_0}}) - z(x^{\frac{1}{n_1 \cdots n_{i-1}}}))$$

We have then that

$$o_{x_i}(\tilde{f}) = ord_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - y(\delta x^{\frac{1}{\beta_0}}) + y(\delta x^{\frac{1}{\beta_0}}) - z(wx^{\frac{1}{n_1 \cdots n_{i-1}}}))$$

Now

$$\begin{aligned} ord_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - y(\delta x^{\frac{1}{\beta_0}})) &\leq o_f(\tilde{f}) \\ &< o_f(x_i) = ord_x(y(\delta x^{\frac{1}{\beta_0}}) - z(wx^{\frac{1}{n_1 \cdots n_{i-1}}})) \end{aligned}$$

So

$$o_{x_i}(\tilde{f}) = ord_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - y(\delta x^{\frac{1}{\beta_0}})) \leq o_f(\tilde{f}).$$

We conclude that $o_f(\tilde{f}) = o_{x_i}(\tilde{f})$, and since the sequence of Puiseux exponents of C_i is $(\frac{\beta_0}{n_i \cdots n_g}, \dots, \frac{\beta_{i-1}}{n_i \cdots n_g})$, applying proposition 3.4 to C and C_i , we find that $(f, \tilde{f})_0 = n_i \cdots n_g(x_i, \tilde{f})_0$ and claim follows.

On the other hand by the corollary 3.5 applied to f and $\tilde{f}, (f, \tilde{f})_0 \geq \kappa e_{i-1} \frac{\bar{\beta}_i}{e_{j-1}}$ if and only if $o_f(\tilde{f}) \geq \frac{\beta_i}{\beta_0} = o_{x_i}(f) = o_f(x_i)$ so $o_f(\tilde{f}) \geq \frac{\beta_i}{\beta_0}$ if and only if $o_{x_i}(\tilde{f}) \geq \frac{\beta_i}{\beta_0}$, therefore $(x_i, \tilde{f})_0 \geq \kappa \frac{\bar{\beta}_i}{e_{j-1}}$. This proves the first assertion.

The second assertion is a direct consequence of lemma 5.1 in [5]. □

To further analyse the C_m^k 's, we realize, as in section 3, C as a complete intersection in $\mathbb{C}^{g+1} = Spec \mathbb{C}[x_0, \dots, x_g]$ defined by the ideal (f_1, \dots, f_g) where

$$f_i = x_{i+1} - (x_i^{n_i} - c_i x_0^{b_{i0}} \cdots x_{i-1}^{b_{i(i-1)}}) - \sum_{\gamma=(\gamma_0, \dots, \gamma_i)} c_{i,\gamma} x_0^{\gamma_0} \cdots x_i^{\gamma_i}$$

for $1 \leq i \leq g$ and $x_{g+1} = 0$. This will let us see the C_m^k 's as fibrations over some reduced scheme that we understand well.

We keep the notations above and let I_m^0 be the radical of the ideal defining C_m^0 in \mathbb{C}_m^{g+1} and let I_m^{0k} be the ideal defining

$$C_m^k = (V(I_m^0, x_0^{(0)}, \dots, x_0^{(kn_1-1)}) \cap D(x_0^{(kn_1)}))_{red} \text{ in } D(x_0^{(kn_1)}).$$

LEMMA 4.6. — *Let $k \neq 0, j$ and κ as above. For $1 \leq i < j \leq g$ (resp. $1 \leq i < j - 1 = g$) and for $\kappa n_i \cdots n_{j-1} \bar{\beta}_i \leq m < \kappa n_{i+1} \cdots n_{j-1} \bar{\beta}_{i+1}$, we have*

$$\begin{aligned} I_m^{0k} &= (x_0^{(0)}, \dots, x_0^{(\frac{\kappa \bar{\beta}_0}{n_j \cdots n_g} - 1)}, \\ x_l^{(0)}, \dots, x_l^{(\frac{\kappa \bar{\beta}_l}{n_j \cdots n_g} - 1)}, F_l^{(\frac{\kappa n_l \bar{\beta}_l}{n_j \cdots n_g})}, \dots, F_l^{(m)}, 1 \leq l \leq i, \\ x_{i+1}^{(0)}, \dots, x_{i+1}^{(\lfloor \frac{m}{n_{i+1} \cdots n_g} \rfloor)}, \\ F_l^{(0)}, \dots, F_l^{(m)}, i + 1 \leq l \leq g - 1). \end{aligned}$$

Moreover for $1 \leq l \leq i$,

$$F_l^{(\kappa \frac{n_l \bar{\beta}_l}{n_j \dots n_g})} \equiv - (x_l^{(\kappa \frac{\bar{\beta}_l}{n_j \dots n_g})^{n_l}} - c_l x_0^{(\kappa \frac{\bar{\beta}_0}{n_j \dots n_g})^{b_{l0}}} \dots x_{l-1}^{(\kappa \frac{\bar{\beta}_{l-1}}{n_j \dots n_g})^{b_{l(l-1)}}}) \pmod{((x_l^{(0)}, \dots, x_l^{(\kappa \frac{\bar{\beta}_l}{n_j \dots n_g} - 1)})_{0 \leq l \leq i}, x_{i+1}^{(0)}, \dots, x_{i+1}^{(\lfloor \frac{m}{n_{i+1} \dots n_g} \rfloor)})}$$

for $1 \leq l < i$ and $\kappa \frac{n_l \bar{\beta}_l}{n_j \dots n_g} < n < \kappa \frac{\bar{\beta}_{l+1}}{n_j \dots n_g}$ (resp. $l = i$ and $\kappa \frac{n_l \bar{\beta}_l}{n_j \dots n_g} < n \leq \lfloor \frac{m}{n_{i+1} \dots n_g} \rfloor$)

$$F_l^{(n)} \equiv - (n_l x_l^{(\kappa \frac{\bar{\beta}_l}{n_j \dots n_g})^{n_l-1}} x_l^{(\kappa \frac{\bar{\beta}_l}{n_j \dots n_g} + n - \kappa \frac{n_l \bar{\beta}_l}{n_j \dots n_g})} - c_l \sum_{0 \leq h \leq l-1} b_{lh} x_0^{(\kappa \frac{\bar{\beta}_0}{n_j \dots n_g})^{b_{l0}}} \dots x_h^{(\kappa \frac{\bar{\beta}_h}{n_j \dots n_g})^{b_{lh}-1}} x_h^{(\kappa \frac{\bar{\beta}_h}{n_j \dots n_g} + n - \kappa \frac{n_l \bar{\beta}_l}{n_j \dots n_g})} \dots x_{l-1}^{(\kappa \frac{\bar{\beta}_{l-1}}{n_j \dots n_g})^{b_{l(l-1)}}} + H_l(\dots, x_h^{(\kappa \frac{\bar{\beta}_h}{n_j \dots n_g} + n - \kappa \frac{n_l \bar{\beta}_l}{n_j \dots n_g} - 1)}, \dots)) \pmod{((x_l^{(0)}, \dots, x_l^{(\kappa \frac{\bar{\beta}_l}{n_j \dots n_g} - 1)})_{0 \leq l \leq i}, x_{i+1}^{(0)}, \dots, x_{i+1}^{(\lfloor \frac{m}{n_{i+1} \dots n_g} \rfloor)})}$$

for $1 \leq l < i$ and $\kappa \frac{\bar{\beta}_{l+1}}{n_j \dots n_g} \leq n \leq m$ (resp. $l = i$ and $\lfloor \frac{m}{n_{i+1} \dots n_g} \rfloor < n \leq m$), or $i + 1 \leq l \leq g - 1$ and $0 \leq n \leq m$,

$$F_l^{(n)} = x_{l+1}^{(n)} + H_l(x_0^{(0)}, \dots, x_0^{(n)}, \dots, x_l^{(0)}, \dots, x_l^{(n)}).$$

For $i = j - 1 = g$ and $m \geq \kappa n_g \bar{\beta}_g$,

$$I_m^{0k} = (x_0^{(0)}, \dots, x_0^{(\kappa \bar{\beta}_0 - 1)}, x_l^{(0)}, \dots, x_l^{(\kappa \bar{\beta}_l - 1)}, F_l^{(\kappa n_l \bar{\beta}_l)}, \dots, F_l^{(m)}), 1 \leq l \leq g,$$

where for $1 \leq l < g$ and $\kappa n_l \bar{\beta}_l \leq n \leq m$, the above formula for $F_l^{(n)}$ remains valid,

$$F_g^{(\kappa n_g \bar{\beta}_g)} \equiv - (x_g^{(\kappa \bar{\beta}_g)^{n_g}} - c_g x_0^{(\kappa \bar{\beta}_0)^{b_{g0}}} \dots x_{g-1}^{(\kappa \bar{\beta}_{g-1})^{b_{g(g-1)}}}) \pmod{((x_l^{(0)}, \dots, x_l^{(\kappa \bar{\beta}_l - 1)}))_{0 \leq l \leq g}}$$

and for $\kappa n_g \bar{\beta}_g < n \leq m$,

$$F_g^{(n)} \equiv - (n_g x_g^{(\kappa \bar{\beta}_g)^{n_g-1}} x_g^{(\kappa \bar{\beta}_g + n - \kappa n_g \bar{\beta}_g)} - c_g \sum_{0 \leq h \leq g-1} b_{g0} x_0^{(\kappa \bar{\beta}_0)^{b_{g0}}} \dots x_h^{(\kappa \bar{\beta}_h)^{b_{gh}-1}} x_h^{(\kappa \bar{\beta}_h + n - \kappa n_h \bar{\beta}_h)} \dots x_{g-1}^{(\kappa \bar{\beta}_{g-1})^{b_{g(g-1)}}} + H_g(\dots, x_h^{(\kappa \bar{\beta}_h + n - \kappa n_h \bar{\beta}_h)}, \dots)) \pmod{((x_l^{(0)}, \dots, x_l^{(\kappa \bar{\beta}_l - 1)}))_{0 \leq l \leq g}}$$

Proof. — First assume that $\kappa n_i \cdots n_{j-1} \bar{\beta}_i \leq m < \kappa n_{i+1} \cdots n_{j-1} \bar{\beta}_{i+1}$ for $1 \leq i < j \leq g$ (resp. $1 \leq i < j - 1 = g$). By proposition 4.5, we have that $C_m^k = \bar{\pi}_m^{-1} \left[\frac{m}{n_{i+1} \cdots n_g} \right] (C_{i+1, [\frac{m}{n_{i+1} \cdots n_g}]}^k)$ where $\bar{\pi}_m, [\frac{m}{n_{i+1} \cdots n_g}] : \mathbb{C}_m^2 \rightarrow \mathbb{C}_{[\frac{m}{n_{i+1} \cdots n_g}]}^2$ is the canonical map. Now $\mathbb{C}^2 = \text{Spec } \mathbb{C}[x_0, x_1]$ (resp. $C_{i+1} = V(x_{i+1})$) is realized as the complete intersection in $\mathbb{C}^{g+1} = \text{Spec } \mathbb{C}[x_0, \dots, x_g]$ defined by the ideal (f_1, \dots, f_{g-1}) (resp. $(f_1, \dots, f_{g-1}, x_{i+1})$). So since $m \geq \kappa n_1 \bar{\beta}_1, I_m^{0k}$ is the radical of the ideal $I_m^{*0k} =$

$$(x_0^{(0)}, \dots, x_0^{(\kappa n_1 - 1)}, x_1^{(0)}, \dots, x_1^{(\kappa m_1 - 1)}, F_1^{(0)}, \dots, F_1^{(m)}, \dots, F_{g-1}^{(0)}, \dots, F_{g-1}^{(m)}, x_{i+1}^{(0)}, \dots, x_{i+1}^{([\frac{m}{n_{i+1} \cdots n_g}]]}).$$

We first observe that $F_1^{(n)} \equiv x_2^{(n)} \text{ mod } (x_0^{(0)}, \dots, x_0^{(\kappa n_1 - 1)}, x_1^{(0)}, \dots, x_1^{(\kappa m_1 - 1)})$ for $0 \leq n < \kappa n_1 \bar{\beta}_1$. Now since $\frac{m}{n_2 \cdots n_g} \geq [\frac{m}{n_2 \cdots n_g}] \geq \kappa n_1 m_1$, we have

$$F_1^{(\kappa n_1 m_1)} \equiv -(x_1^{(\kappa m_1)^{n_1}} - c_1 x_0^{(\kappa n_1)^{m_1}})$$

$$\text{mod } (x_0^{(0)}, \dots, x_0^{(\kappa n_1 - 1)}, x_1^{(0)}, \dots, x_1^{(\kappa m_1 - 1)}, x_2^{(0)}, \dots, x_2^{([\frac{m}{n_2 \cdots n_g}]]}).$$

and

$$F_1^{(n)} \equiv -(n_1 x_1^{(\kappa m_1)^{n_1 - 1}} x_1^{(\kappa m_1 + n - \kappa n_1 m_1)} - m_1 c_1 x_0^{(\kappa n_1)^{m_1 - 1}} x_0^{(\kappa n_1 + n - \kappa n_1 m_1)}) + H_1(x_0^{(0)}, \dots, x_0^{(\kappa n_1 + n - \kappa n_1 m_1 - 1)}, x_1^{(0)}, \dots, x_1^{(\kappa m_1 + n - \kappa n_1 m_1 - 1)}) \text{ mod } (x_0^{(0)}, \dots, x_0^{(\kappa n_1 - 1)}, x_1^{(0)}, \dots, x_1^{(\kappa m_1 - 1)}, x_2^{(0)}, \dots, x_2^{([\frac{m}{n_2 \cdots n_g}]]}).$$

for $\kappa n_1 \bar{\beta}_1 < n \leq [\frac{m}{n_2 \cdots n_g}]$. Finally, for $l = 1$ and $[\frac{m}{n_2 \cdots n_g}] < n \leq m$, or $2 \leq l \leq g - 1$ and $0 \leq n \leq m$, we have

$$F_l^{(n)} = x_{l+1}^{(n)} + H_l(x_0^{(0)}, \dots, x_0^{(n)}, \dots, x_l^{(0)}, \dots, x_l^{(n)}).$$

As a consequence for $i = 1$, the subscheme of $\mathbb{C}^{g+1} \cap D(x_0^{(\kappa n_1)})$ defined by I_m^{*0k} is isomorphic to the product of \mathbb{C}^* by an affine space, so it is reduced and irreducible and $I_m^{*0k} = I_m^{0k}$ is a prime ideal in $\mathbb{C}[x_0^{(0)}, \dots, x_0^{(m)}, \dots, x_g^{(0)}, \dots, x_g^{(m)}]_{x_0^{(\kappa n_1)}}$, generated by a regular sequence, i.e the proposition holds for $i = 1$.

Assume that it holds for $i < j - 1 < g$ (resp. $i < j - 2 = g - 1$). For $\kappa n_{i+1} \cdots n_{j-1} \bar{\beta}_{i+1} \leq m < \kappa n_{i+2} \cdots n_{j-1} \bar{\beta}_{i+2}$, the ideal in $\mathbb{C}[x_0^{(0)}, \dots, x_0^{(m)}, \dots, x_g^{(0)}, \dots, x_g^{(m)}]_{x_0^{(\kappa n_1)}}$ generated by $I_{\kappa n_{i+1} \cdots n_{j-1} \bar{\beta}_{i+1}}^{0k}$ is contained in I_m^{0k} . By the inductive hypothesis, $x_l^{(0)}, \dots, x_l^{(\frac{\kappa \bar{\beta}_l}{n_j \cdots n_g} - 1)} \in$

I_m^{0k} for $l = 1, \dots, i + 1$. So I_m^{0k} is the radical of

$$\begin{aligned}
 I_m^{*0k} = & (x_0^{(0)}, \dots, x_0^{(\frac{\kappa\bar{\beta}_0}{n_j \cdots n_g} - 1)}, \\
 & x_l^{(0)}, \dots, x_l^{(\frac{\kappa\bar{\beta}_l}{n_j \cdots n_g} - 1)}, F_l^{(0)}, \dots, F_l^{(m)}, 1 \leq l \leq i + 1, \\
 & x_{i+2}^{(0)}, \dots, x_{i+2}^{(\lfloor \frac{m}{n_{i+2} \cdots n_g} \rfloor)}, \\
 & F_l^{(0)}, \dots, F_l^{(m)}, i + 2 \leq l \leq g - 1).
 \end{aligned}$$

Now for $0 \leq n < \frac{\kappa n_l \bar{\beta}_l}{n_j \cdots n_g}$, we have

$$\begin{aligned}
 F_l^{(n)} \equiv x_{l+1}^{(n)} \text{ mod } (x_0^{(0)}, \dots, x_l^{(\frac{\kappa\bar{\beta}_0}{n_j \cdots n_g} - 1)}, x_l^{(0)}, \dots, x_l^{(\frac{\kappa\bar{\beta}_l}{n_j \cdots n_g} - 1)}), \\
 1 \leq l \leq i + 1).
 \end{aligned}$$

Here since $\bar{\beta}_{l+1} > n_l \bar{\beta}_l$, for $1 \leq l \leq i$ and $\frac{m}{n_{i+2} \cdots n_g} \geq \lfloor \frac{m}{n_{i+2} \cdots n_g} \rfloor \geq \frac{\kappa n_{i+1} \bar{\beta}_{i+1}}{n_j \cdots n_g}$, we can delete $F_l^{(n)}$, $1 \leq l \leq i + 1$, $0 \leq n < \frac{\kappa n_l \bar{\beta}_l}{n_j \cdots n_g}$ from the above generators of I_m^{*0k} . The identities relative to the $F_l^{(n)}$ for $1 \leq l \leq i + 1$, $\frac{\kappa n_l \bar{\beta}_l}{n_j \cdots n_g} \leq n \leq m$ or $i + 2 \leq l \leq g - 1$ and $0 \leq n \leq m$ follow immediately from (\diamond) . Hence the subscheme of $\mathbb{C}^{g+1} \cap D(x_0^{(kn_1)})$ defined by I_m^{*0k} is isomorphic to the product of \mathbb{C}^* by an affine space, so it is reduced and irreducible and $I_m^{*0k} = I_m^{0k}$ is a prime ideal in $\mathbb{C}[x_0^{(0)}, \dots, x_0^{(m)}, \dots, x_g^{(0)}, \dots, x_g^{(m)}]_{x_0^{(kn_1)}}$, generated by a regular sequence, i.e the proposition holds for $i + 1$.

The case $i = j - 1 = g$ and $m \geq \kappa n_g \bar{\beta}_g$ follows by similar arguments. \square

As an immediate consequence we get

PROPOSITION 4.7. — *Let C be a plane branch with g Puiseux exponents. Let $k \neq 0, j$ and κ as above. For $m \geq \kappa n_1 \bar{\beta}_1$, let $\pi_{m, \kappa n_1 \bar{\beta}_1} : C_m \rightarrow C_{\kappa n_1 \bar{\beta}_1}$ be the canonical projection and let $C_m^k := \pi_{m, \kappa n_1 \bar{\beta}_1}^{-1}(D(x_0^{(kn_1)}) \cap V(x_0^{(0)}, \dots, x_0^{(kn_1-1)}))_{red}$. Then for $1 \leq i < j \leq g$ (resp. $1 \leq i < j - 1 = g$) and $\kappa n_i \cdots n_{j-1} \bar{\beta}_i \leq m < \kappa n_{i+1} \cdots n_{j-1} \bar{\beta}_{i+1}$, C_m^k is irreducible of codimension*

$$\frac{\kappa}{n_j \cdots n_g} (\bar{\beta}_0 + \bar{\beta}_1 + \sum_{l=1}^{i-1} (\bar{\beta}_{l+1} - n_l \bar{\beta}_l)) + (\lfloor \frac{m}{n_{i+1} \cdots n_g} \rfloor - \frac{\kappa n_i \bar{\beta}_i}{n_j \cdots n_g}) + 1$$

in \mathbb{C}_m^2 . (We suppose that the sum in the formula is equal to 0 when $i = 1$). For $j \leq g$ and $m \geq \kappa \bar{\beta}_j$ (resp. $j = g + 1$ and $m \geq \kappa n_g \bar{\beta}_g$),

$$C_m^k = \emptyset$$

(resp. C_m^k is of codimension

$$\kappa(\bar{\beta}_0 + \bar{\beta}_1 + \sum_{l=1}^{g-1} (\bar{\beta}_{l+1} - n_l \bar{\beta}_l)) + m - \kappa n_g \bar{\beta}_g + 1)$$

in \mathbb{C}_m^2 .

The referee kindly pointed out that for $m \in \mathbb{N}$ such that $\kappa n_i \cdots n_{j-1} \bar{\beta}_i \leq m < \kappa n_{i+1} \cdots n_{j-1} \bar{\beta}_{i+1}$, the codimension of C_m^k can also be written as :

$$\frac{\kappa}{e_{j-1}} (\bar{\beta}_0 + \beta_{i+1} - \bar{\beta}_{i+1}) + \left[\frac{m}{e_i} \right] + 1.$$

For $k' \geq k$ and $m \geq k' n_1 \bar{\beta}_1$, we now compare $\text{codim}(C_m^k, \mathbb{C}_m^2)$ and $\text{codim}(C_m^{k'}, \mathbb{C}_m^2)$.

COROLLARY 4.8. — For $k' \geq k \geq 1$ and $m \geq k' n_1 \bar{\beta}_1$, if C_m^k and $C_m^{k'}$ are nonempty, we have

$$\text{codim}(C_m^{k'}, \mathbb{C}_m^2) \leq \text{codim}(C_m^k, \mathbb{C}_m^2).$$

Proof. — Let $\gamma^k : [k n_1 \bar{\beta}_1, \infty[\rightarrow [k(n_1 + m_1), \infty[$ be the piecewise linear function given by

$$\gamma^k(m) = \frac{k}{e_1} (\bar{\beta}_0 + \bar{\beta}_1 + \sum_{l=1}^{i-1} (\bar{\beta}_{l+1} - n_l \bar{\beta}_l)) + \left(\frac{m}{e_i} - \frac{k n_i \bar{\beta}_i}{e_1} \right) + 1$$

for $1 \leq i \leq g$ and $\frac{k \bar{\beta}_i}{n_2 \cdots n_{i-1}} \leq m < \frac{k \bar{\beta}_{i+1}}{n_2 \cdots n_i}$. (Recall that by convention $\bar{\beta}_{g+1} = \infty$)

In view of proposition 4.7, we have that $\text{codim}(C_m^k, \mathbb{C}_m^2) = [\gamma^k(m)]$ for $k \equiv 0 \pmod{n_2 \cdots n_{j-1}}$ and $k \not\equiv 0 \pmod{n_2 \cdots n_j}$ with $2 \leq j \leq g$ and any integer $m \in [k n_1 \bar{\beta}_1, \frac{k \bar{\beta}_j}{n_2 \cdots n_{j-1}}[$ or for $k \equiv 0 \pmod{n_2 \cdots n_g}$ and any integer $m \geq k n_1 \bar{\beta}_1$. Similarly we define $\gamma^{k'} : [k' n_1 \bar{\beta}_1, \infty[\rightarrow [k'(n_1 + m_1), \infty[$ by changing k to k' .

Let Γ^k (resp. $\Gamma^{k'}$) be the graph of γ^k (resp. $\gamma^{k'}$) in \mathbb{R}^2 . Now let $\tau : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $\tau(a, b) = (a, b - 1)$ and let $\lambda^{k'/k} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $\lambda^{k'/k}(a, b) = \frac{k'}{k}(a, b)$. We note that $\tau(\Gamma^{k'}) = \lambda^{k'/k}(\tau(\Gamma^k))$; we also note that the endpoints of $\tau(\Gamma^k)$ and $\tau(\Gamma^{k'})$ lie on the line through 0 with slope $\frac{\beta_0 + \beta_1}{e_1 n_1 \beta_1} = \frac{1}{e_1} \frac{n_1 + m_1}{n_1 m_1} < \frac{1}{e_1}$. Since $\frac{k'}{k} \geq 1$, the image of $\tau(\Gamma^k)$ by $\lambda^{k'/k}$ lies in the interior subset of $\mathbb{R}_{\geq 0}^2$ with boundary the union of $\tau(\Gamma^k)$, of the segment joining its endpoint $(k n_1 \bar{\beta}_1, \frac{k}{e_1} (\beta_0 + \bar{\beta}_1))$ to $(k n_1 \bar{\beta}_1, 0)$ and of $[k n_1 \bar{\beta}_1, \infty[\times 0$. This implies that $\gamma^{k'}(m) \leq \gamma^k(m)$ for $m \geq k' n_1 \bar{\beta}_1$, hence $[\gamma^{k'}(m)] \leq [\gamma^k(m)]$ and the claim. \square

THEOREM 4.9. — Let C be a plane branch with $g \geq 2$ Puiseux exponents. Let $m \in \mathbb{N}$. For $1 \leq m < n_1\bar{\beta}_1 + e_1, C_m^0 = \text{Cont}^{>0}(x_0)_m$ is irreducible. For $qn_1\bar{\beta}_1 + e_1 \leq m < (q + 1)n_1\bar{\beta}_1 + e_1$, with $q \geq 1$ in \mathbb{N} , the irreducible components of C_m^0 are :

$$C_{m\kappa I} = \overline{\text{Cont}^{\kappa\bar{\beta}_0}(x_0)_m}$$

for $1 \leq \kappa$ and $\kappa\bar{\beta}_0\bar{\beta}_1 + e_1 \leq m$,

$$C_{m\kappa v}^j = \overline{\text{Cont}^{\frac{\kappa\bar{\beta}_0}{n_j \cdots n_g}}(x_0)_m}$$

for $j = 2, \dots, g, 1 \leq \kappa$ and $\kappa \not\equiv 0 \pmod{n_j}$ and such that $\kappa n_1 \cdots n_{j-1}\bar{\beta}_1 + e_1 \leq m < \kappa\bar{\beta}_j$,

$$B_m = \text{Cont}^{>n_1q}(x_0)_m.$$

Proof. — We first observe that for any integer $k \neq 0$ and any $m \geq kn_1\bar{\beta}_1$,

$$(C_m^0)_{red} = \cup_{1 \leq h \leq k} C_m^h \cup \text{Cont}^{>kn_1}(x_0)_m$$

where $C_m^h := \text{Cont}^{hn_1}(x_0)_m$. Indeed, for $k = 1$, we have that $(C_m^0)_{red} \subset V(x_0^{(0)}, \dots, x_0^{(n_1-1)})$ by proposition 4.1. Arguing by induction on k , we may assume that the claim holds for $m \geq (k - 1)n_1\bar{\beta}_1$. Now by corollary 4.2, we know that for $m \geq kn_1\bar{\beta}_1, \text{Cont}^{>(k-1)n_1}(x_0)_m \subset V(x_0^{(0)}, \dots, x_0^{(kn_1-1)})$, hence the claim for $m \geq kn_1\bar{\beta}_1$.

We thus get that for $qn_1\bar{\beta}_1 + e_1 \leq m < (q + 1)n_1\bar{\beta}_1 + e_1$,

$$(C_m^0)_{red} = \cup_{1 \leq k \leq q} C_m^k \cup \text{Cont}^{>qn_1}(x_0)_m.$$

By proposition 4.7, for $1 \leq k \leq q, C_m^k$ is either irreducible or empty. We first note that if $C_m^k \neq \emptyset$, then $\overline{C_m^k} \not\subset \text{Cont}^{>qn_1}(x_0)_m$. Similarly, if $1 \leq k < k' \leq q$ and if C_m^k and $C_m^{k'}$ are nonempty, then $\overline{C_m^k} \not\subset \overline{C_m^{k'}}$. On the other hand by corollary 4.8, we have that $\text{codim}(C_m^{k'}, \mathbb{C}_m^2) \leq \text{codim}(C_m^k, \mathbb{C}_m^2)$. So $\overline{C_m^{k'}} \not\subset \overline{C_m^k}$. Finally we will show that $\text{Cont}^{>qn_1}(x_0)_m \not\subset \overline{C_m^k}$ if $C_m^k \neq \emptyset$ for $1 \leq k \leq q$. To do so, it is enough to check that $\text{codim}(C_m^k, \mathbb{C}_m^2) \geq \text{codim}(\text{Cont}^{>qn_1}(x_0)_m, \mathbb{C}_m^2)$. For $m \in [qn_1\bar{\beta}_1 + e_1, (q + 1)n_1\bar{\beta}_1[$, we have

$$\begin{aligned} \delta^q(m) &:= \text{codim}(\text{Cont}^{>qn_1}(x_0)_m, \mathbb{C}_m^2) \\ &= 2 + q(n_1 + m_1) + \left\lceil \frac{m - qn_1\bar{\beta}_1}{\beta_0} \right\rceil + \left\lceil \frac{m - qn_1\bar{\beta}_1}{\bar{\beta}_1} \right\rceil \end{aligned}$$

by corollary 4.2. Let $\lambda^q : [qn_1\bar{\beta}_1 + e_1[\rightarrow [q(n_1 + m_1), \infty[$ be the function given by $\lambda^q(m) = q(n_1 + m_1) + \frac{m - qn_1\bar{\beta}_1}{e_1} + 1$. For simplicity, set $i = m - qn_1\bar{\beta}_1$. For any integer i such that $e_1 \leq i < n_1\bar{\beta}_1 = n_1m_1e_1$, we have $1 + [\frac{i}{n_1e_1}] + [\frac{i}{m_1e_1}] \leq [\frac{i}{e_1}]$. Indeed this is true for $i = e_1$ and it follows by induction on i from the fact that for any pair of integers (b, a) , we have $[\frac{b+1}{a}] = [\frac{b}{a}]$ if and only if $b + 1 \not\equiv 0 \pmod a$ and $[\frac{b+1}{a}] = [\frac{b}{a}] + 1$ otherwise, since $i < n_1m_1e_1$. So $\delta^q(m) \leq [\lambda^q(m)]$.

But in the proof of corollary 4.8, we have checked that if $C_m^k \neq \emptyset$, then $\text{codim}(C_m^k, \mathbb{C}_m^2) = [\gamma^k(m)]$. We have also checked that for $q \geq k$ and $m \geq qn_1\bar{\beta}_1$, $\gamma^k(m) \geq \gamma^q(m)$. Finally in view of the definitions of γ^q and λ^q , we have $\gamma^q(m) \geq \lambda^q(m)$, so $[\gamma^q(m)] \geq [\lambda^q(m)] \geq \delta^q(m)$.

For $m = (q + 1)n_1\bar{\beta}_1$, we have $\delta^q(m) = (q + 1)(n_1 + m_1) + 1$ by corollary 4.2. For $m \in [(q + 1)n_1\bar{\beta}_1, (q + 1)n_1\bar{\beta}_1 + e_1[$, we have

$$\text{Cont}^{>qn_1}(x_0)_m = C_m^{q+1} \cup \text{Cont}^{>(q+1)n_1}(x_0)_m$$

and

$$\text{Cont}^{>(q+1)n_1}(x_0)_m = V(x_0^{(0)}, \dots, x_0^{((q+1)n_1)}, x_1^{(0)}, \dots, x_1^{((q+1)m_1)})$$

again by corollary 4.2. If in addition we have $m < (q + 1)\bar{\beta}_2$, then by proposition 4.5 $C_m^{q+1} = V(x_0^{(0)}, \dots, x_0^{((q+1)n_1-1)}, x_1^{(0)}, \dots, x_1^{((q+1)m_1-1)}, x_1^{((q+1)m_1)^{n_1}} - c_1x_0^{((q+1)n_1)^{m_1}}) \cap D(x_0^{((q+1)n_1)})$, thus we have $\text{Cont}^{>qn_1}(x_0)_m = C_m^{q+1}$ and $\delta^q(m) = (q + 1)(n_1 + m_1) + 1$. We have $(q + 1)n_1\bar{\beta}_1 + e_1 \leq (q + 1)\bar{\beta}_2$ if $q + 1 \geq n_2$, because $\bar{\beta}_2 - n_1\bar{\beta}_1 \equiv 0 \pmod{e_2}$. If not, we may have $(q + 1)\bar{\beta}_2 < (q + 1)n_1\bar{\beta}_1 + e_1$, so for $(q + 1)\bar{\beta}_2 \leq m < (q + 1)n_1\bar{\beta}_1 + e_1$, we have $C_m^{q+1} = \emptyset, \text{Cont}^{>qn_1}(x_0)_m = \text{Cont}^{>(q+1)n_1}(x_0)_m$ and $\delta^q(m) = (q + 1)(n_1 + m_1) + 2$.

In both cases, for $m \in [(q + 1)n_1\bar{\beta}_1, (q + 1)n_1\bar{\beta}_1 + e_1[$, we have $\delta^q(m) \leq (q + 1)(n_1 + m_1) + 2$. Since $[\lambda^q(m)] = q(n_1 + m_1) + n_1m_1 + 1$, we conclude that $[\lambda^q(m)] \geq \delta^q(m)$, so for $1 \leq k \leq q$, if $C_m^k \neq \emptyset$, we have $[\gamma^k(m)] \geq \delta^q(m)$. This proves that the irreducible components of C_m^0 are the \bar{C}_m^k for $1 \leq k \leq q$ and $C_m^k \neq \emptyset$, and $\text{Cont}^{>qn_1}(x_0)_m$, hence the claim in view of the characterization of the nonempty $C_m^{k's}$'s given in proposition 4.5. \square

COROLLARY 4.10. — *Under the assumption of theorem 4.9, let $q_0 + 1 = \min\{\alpha \in \mathbb{N}; \alpha(\bar{\beta}_2 - n_1\bar{\beta}_1) \geq e_1\}$. Then $0 \leq q_0 < n_2$. For $1 \leq m < (q_0 + 1)n_1\bar{\beta}_1 + e_1, C_m^0$ is irreducible and we have $\text{codim}(C_m^0, \mathbb{C}_m^2) =$*

$$2 + [\frac{m}{\beta_0}] + [\frac{m}{\beta_1}] \text{ for } 0 \leq q \leq q_0 \text{ and } qn_1\bar{\beta}_1 + e_1 \leq m < (q + 1)n_1\bar{\beta}_1$$

$$\text{or } 0 \leq q \leq q_0 \text{ and } (q + 1)\bar{\beta}_2 \leq m < (q + 1)n_1\bar{\beta}_1 + e_1.$$

$$1 + \left\lfloor \frac{m}{\beta_0} \right\rfloor + \left\lfloor \frac{m}{\beta_1} \right\rfloor \text{ for } 0 \leq q < q_0 \text{ and } (q + 1)n_1\bar{\beta}_1 \leq m < (q + 1)\bar{\beta}_2$$

$$\text{or } (q_0 + 1)n_1\bar{\beta}_1 \leq m < (q_0 + 1)n_1\bar{\beta}_1 + e_1.$$

For $q \geq q_0 + 1$ in \mathbb{N} and $qn_1\bar{\beta}_1 + e_1 \leq m < (q + 1)n_1\bar{\beta}_1 + e_1$, the number of irreducible components of C_m^0 is:

$$N(m) = q + 1 - \sum_{j=2}^g \left(\left\lfloor \frac{m}{\beta_j} \right\rfloor - \left\lfloor \frac{m}{n_j\beta_j} \right\rfloor \right)$$

and $\text{codim}(C_m^0, \mathbb{C}_m^2) =$

$$2 + \left\lfloor \frac{m}{\beta_0} \right\rfloor + \left\lfloor \frac{m}{\beta_1} \right\rfloor \text{ for } qn_1\bar{\beta}_1 + e_1 \leq m < (q + 1)n_1\bar{\beta}_1.$$

$$1 + \left\lfloor \frac{m}{\beta_0} \right\rfloor + \left\lfloor \frac{m}{\beta_1} \right\rfloor \text{ for } (q + 1)n_1\bar{\beta}_1 \leq m < (q + 1)n_1\bar{\beta}_1 + e_1.$$

Proof. — We have already observed that $n_2(\bar{\beta}_2 - n_1\bar{\beta}_1) \geq e_1$ because $\bar{\beta}_2 - n_1\bar{\beta}_1 \equiv 0 \pmod{e_2}$, so $1 \leq q_0 + 1 \leq n_2$.

For $qn_1\bar{\beta}_1 + e_1 \leq m < (q + 1)n_1\bar{\beta}_1 + e_1$, with $q \geq 1$, we have seen in the proof of theorem 4.9 that the irreducible components of C_m^0 are the \overline{C}_m^k for $1 \leq k \leq q$ and $C_m^k \neq \emptyset$, and $\text{Cont}^{qn_1}(x_0)_m$. We thus have to enumerate the empty C_m^k for $1 \leq k \leq q$. By proposition 4.5, $C_m^k = \emptyset$ if and only if $j := \max\{l; l \geq 2 \text{ and } k \equiv 0 \pmod{n_2 \cdots n_{l-1}}\} \leq q$ and $m \geq \frac{k}{n_2 \cdots n_{j-1}}\bar{\beta}_j$.

Now recall that $\bar{\beta}_{i+1} > n_i\bar{\beta}_i$ for $1 \leq i \leq g - 1$ and that $\bar{\beta}_2 - n_1\bar{\beta}_1 \geq e_2$. This implies that for $3 \leq j \leq g$, we have $\bar{\beta}_j - n_1 \cdots n_{j-1}\bar{\beta}_1 > n_2 \cdots n_{j-1}(\bar{\beta}_2 - n_1\bar{\beta}_1) \geq n_2 \cdots n_{j-1}e_2 \geq e_1$. So if $j \geq 3$ and κ is a positive integer such that $m \geq \kappa\bar{\beta}_j$, we have $\frac{m - e_1}{n_1\bar{\beta}_1} > \kappa n_2 \cdots n_{j-1}$, hence $q = \left\lfloor \frac{m - e_1}{n_1\bar{\beta}_1} \right\rfloor \geq \kappa n_2 \cdots n_{j-1}$.

Therefore for $j \geq 3$, there are exactly $\left\lfloor \frac{m}{\beta_j} \right\rfloor$ integers $\kappa \geq 1$ such that $m \geq \kappa\bar{\beta}_j$ and $\kappa n_2 \cdots n_{j-1} \leq q$, among them $\left\lfloor \frac{m}{n_j\beta_j} \right\rfloor$ are $\equiv 0 \pmod{n_j}$.

Similarly if $(q + 1)n_1\bar{\beta}_1 + e_1 \leq (q + 1)\bar{\beta}_2$, or equivalently $q \geq q_0$, and if κ is a positive integer such that $m \geq \kappa\bar{\beta}_2$, we have $\kappa \leq \frac{m}{\bar{\beta}_2} < q + 1$. Therefore if $q \geq q_0 + 1$, we conclude that there are $\sum_{j=2}^g \left(\left\lfloor \frac{m}{\beta_j} \right\rfloor - \left\lfloor \frac{m}{n_j\beta_j} \right\rfloor \right)$ empty C_m^k 's with $1 \leq k \leq q$. Moreover we have shown in the proof of theorem 4.9 that $\text{codim}(C_m^0, \mathbb{C}_m^2) = \text{codim}(\text{Cont}^{>qn_1}(x_0)_m, \mathbb{C}_m^2) = 2 + \left\lfloor \frac{m}{\beta_0} \right\rfloor + \left\lfloor \frac{m}{\beta_1} \right\rfloor$ if $m < (q + 1)n_1\bar{\beta}_1$ (resp. $1 + (q + 1)(n_1 + m_1) = 1 + \left\lfloor \frac{m}{\beta_0} \right\rfloor + \left\lfloor \frac{m}{\beta_1} \right\rfloor$ for $m \geq (q + 1)n_1\bar{\beta}_1$). Also note that $q_0\bar{\beta}_2 < q_0n_1\bar{\beta}_1 + e_1 < (q_0 + 1)n_1\bar{\beta}_1 + e_1 \leq (q_0 + 1)\bar{\beta}_2 \leq n_2\bar{\beta}_2 < \bar{\beta}_3 \cdots$. Therefore for $q_0n_1\bar{\beta}_1 + e_1 \leq m < (q_0 + 1)n_1\bar{\beta}_1 + e_1$, we have $\left\lfloor \frac{m}{\beta_2} \right\rfloor = q_0, \left\lfloor \frac{m}{n_2\beta_2} \right\rfloor = \left\lfloor \frac{m}{\beta_3} \right\rfloor = \cdots = 0$, so $N(m) = 1$, i.e. C_m^0 is irreducible.

Finally, assume that $qn_1\bar{\beta}_1 + e_1 \leq m < (q + 1)n_1\bar{\beta}_1 + e_1$ with $q \geq 1$ and $q \leq q_0$. Since $q_0 < n_2$, for $1 \leq k \leq q$ we have $k \not\equiv 0 \pmod{n_2}$ and $m \geq qn_1\bar{\beta}_1 + e_1 > q\bar{\beta}_2$, hence for $1 \leq k \leq q$, $C_m^k = \emptyset$ and $C_m^0 = \text{Cont}^{qn_1}(x_0)_m$ is irreducible. (The case $q = q_0$ was already known). So for $n_1\bar{\beta}_1 \leq m < (q_0 + 1)n_1\bar{\beta}_1 + e_1$, C_m^0 is irreducible. (Recall that for $1 \leq m < q_0n_1\bar{\beta}_1 + e_1$, the irreducibility of C_m^0 is already known). It only remains to check the codimensions of C_m^0 for $1 \leq m \leq q_0n_1\bar{\beta}_1 + e_1$. Here again we have seen in the proof of Theorem 4.9 that $\text{codim}(C_m^0, \mathbb{C}_m^2) = \text{codim}(\text{Cont}^{>qn_1}(x_0)_m, \mathbb{C}_m^2) =: \delta^q(m)$ for any $q \geq 1$ and $qn_1\bar{\beta}_1 + e_1 \leq m < (q + 1)n_1\bar{\beta}_1 + e_1$ and that

$$\delta^q(m) = 2 + \left\lfloor \frac{m}{\beta_0} \right\rfloor + \left\lfloor \frac{m}{\beta_1} \right\rfloor \text{ for any } q \geq 1 \text{ and } qn_1\bar{\beta}_1 + e_1 \leq m < (q + 1)n_1\bar{\beta}_1$$

$$(q + 1)(n_1 + m_1) + 1 = 1 + \left\lfloor \frac{m}{\beta_0} \right\rfloor + \left\lfloor \frac{m}{\beta_1} \right\rfloor \text{ for } q < q_0 \text{ and } (q + 1)n_1\bar{\beta}_1 \leq m < (q + 1)\bar{\beta}_2$$

$$(q + 1)(n_1 + m_1) + 2 = 2 + \left\lfloor \frac{m}{\beta_0} \right\rfloor + \left\lfloor \frac{m}{\beta_1} \right\rfloor \text{ for } q < q_0 \text{ and } (q + 1)\bar{\beta}_2 \leq m < (q + 1)n_1\bar{\beta}_1 + e_1.$$

This completes the proof. □

In [6], Igusa has shown that the log-canonical threshold of the pair $((\mathbb{C}^2, 0), (C, 0))$ is $\frac{1}{\beta_0} + \frac{1}{\beta_1}$. Here $(\mathbb{C}^2, 0)$ (resp. $(C, 0)$) is the formal neighborhood of \mathbb{C}^2 (resp. C) at 0. Corollary 4.10 allows to recover corollary B of [2] in this special case.

COROLLARY 4.11. — *If the plane curve C has a branch at 0, with multiplicity β_0 , and first Puiseux exponent $\bar{\beta}_1$, then*

$$\min_m \frac{\text{codim}(C_m^0, \mathbb{C}_m^2)}{m + 1} = \frac{1}{\beta_0} + \frac{1}{\bar{\beta}_1}.$$

Proof. — For any $m, p \neq 0$ in \mathbb{N} , we have $m - p\left\lfloor \frac{m}{p} \right\rfloor \leq p - 1$ and $m - p\left\lfloor \frac{m}{p} \right\rfloor = p - 1$ if and only if $m + 1 \equiv 0 \pmod{p}$; so for any $m \in \mathbb{N}$, $2 + \left\lfloor \frac{m}{\beta_0} \right\rfloor + \left\lfloor \frac{m}{\beta_1} \right\rfloor \geq (m + 1)\left(\frac{1}{\beta_0} + \frac{1}{\beta_1}\right)$ and we have equality if and only if $m + 1 \equiv 0 \pmod{\beta_0}$ and $\pmod{\beta_1}$ or equivalently $m + 1 \equiv 0 \pmod{n_1\bar{\beta}_1}$ since $n_1\bar{\beta}_1$ is the least common multiple of β_0 and $\bar{\beta}_1$. If not we have $1 + \left\lfloor \frac{m}{\beta_0} \right\rfloor + \left\lfloor \frac{m}{\beta_1} \right\rfloor \geq (m + 1)\left(\frac{1}{\beta_0} + \frac{1}{\beta_1}\right)$. Now if $(q + 1)n_1\bar{\beta}_1 \leq m < (q + 1)n_1\bar{\beta}_1 + e_1$ with $q \in \mathbb{N}$, we have $(q + 1)n_1\bar{\beta}_1 < m + 1 \leq (q + 1)n_1\bar{\beta}_1 + e_1 < (q + 2)n_1\bar{\beta}_1$, so $m + 1 \not\equiv 0 \pmod{n_1\bar{\beta}_1}$. If $(q + 1)n_1\bar{\beta}_1 \leq m < (q + 1)\bar{\beta}_2$ with $q \in \mathbb{N}$ and $q < q_0$, then $(q + 1)n_1\bar{\beta}_1 < m + 1 \leq (q + 1)n_1\bar{\beta}_1 + e_1 < (q + 2)n_1\bar{\beta}_1$, so $m + 1 \not\equiv 0 \pmod{n_1\bar{\beta}_1}$.

$(n_1\bar{\beta}_1)$. So in both cases, we have $1 + [\frac{m}{\bar{\beta}_0}] + [\frac{m}{\bar{\beta}_1}] \geq (m + 1)(\frac{1}{\bar{\beta}_0} + \frac{1}{\bar{\beta}_1})$. The claim follows from corollary 4.10. □

It also follows immediately from corollary 4.10 .

COROLLARY 4.12. — *Let $q_0 \in \mathbb{N}$ as in corollary 4.10. There exists $n_1\bar{\beta}_1$ linear functions, $L_0, \dots, L_{n_1\bar{\beta}_1-1}$ such that $\dim(C_m^0) = L_i(m)$ for any $m \equiv i \pmod{(n_1\bar{\beta}_1)}$ such that $m \geq q_0n_1\bar{\beta}_1 + e_1$.*

The canonical projections $\pi_{m+1,m} : C_{m+1}^0 \longrightarrow C_m^0, m \geq 1$, induce infinite inverse systems

$$\cdots B_{m+1} \longrightarrow B_m \cdots \longrightarrow B_1$$

$$\cdots C_{(m+1)\kappa I} \longrightarrow C_{m\kappa I} \cdots \longrightarrow C_{(\kappa\beta_0\bar{\beta}_1+e_1)\kappa I} \longrightarrow B_{\kappa\beta_0\bar{\beta}_1+e_1-1}$$

and finite inverse systems

$$C_{(\kappa\bar{\beta}_j-1)\kappa v}^j \longrightarrow C_{m\kappa v}^j \cdots \longrightarrow C_{(\kappa n_1 \cdots n_{j-1}\bar{\beta}_1+e_1)\kappa v}^j \longrightarrow B_{\kappa n_1 \cdots n_{j-1}\bar{\beta}_1+e_1-1}$$

for $2 \leq j \leq g$, and $\kappa \not\equiv 0 \pmod{(n_j)}$.

We get a tree $T_{C,0}$ by representing each irreducible component of $C_m^0, m \geq 1$, by a vertex $v_{i,m}, 1 \leq i \leq N(m)$, and by joining the vertices $v_{i_1,m+1}$ and $v_{i_0,m}$ if $\pi_{m+1,m}$ induces one of the above maps between the corresponding irreducible components.

This tree only depends on the semigroup Γ .

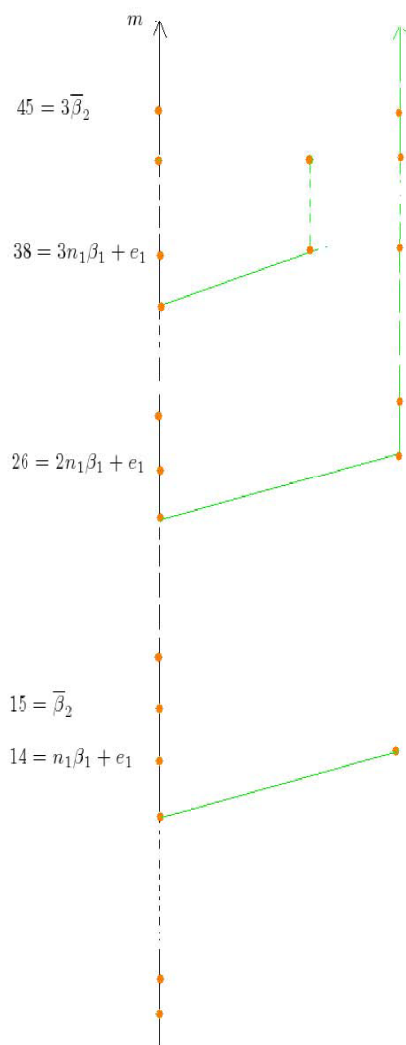
Conversely, we recover $\bar{\beta}_0, \dots, \bar{\beta}_g$ from this tree and $\max\{m, \text{codim}(B_m, \mathbb{C}_m^2) = 2\} = \bar{\beta}_0 - 1$. Indeed the number of edges joining two vertices from which an infinite branch of the tree starts is $\beta_0\bar{\beta}_1$. We thus recover $\bar{\beta}_1$ and e_1 . We recover $\bar{\beta}_2 - n_1\bar{\beta}_1, \dots, \bar{\beta}_j - n_1 \cdots n_{j-1}\bar{\beta}_1, \dots, \bar{\beta}_g - n_1 \cdots n_{g-1}\bar{\beta}_1$, hence $\bar{\beta}_2, \dots, \bar{\beta}_g$ from the number of edges in the finite branches.

COROLLARY 4.13. — *Let C be a plane branch with $g \geq 1$ Puiseux exponents. The tree $T_{C,0}$ described above and $\max\{m, \dim C_m^0 = 2m\}$ determines the sequence $\bar{\beta}_0, \dots, \bar{\beta}_g$ or equivalently the equisingularity class of C and conversely.*

We represent below the tree for the branch defined by

$$f(x, y) = (y^2 - x^3)^2 - 4x^6y - x^9 = 0,$$

whose semigroup is $\langle \bar{\beta}_0 = 4, \bar{\beta}_1 = 6, \bar{\beta}_2 = 15 \rangle$, and for which we have $e_1 = 2, e_2 = 1$ and $n_1 = n_2 = 2$.



BIBLIOGRAPHY

- [1] S. S. ABHYANKAR, “Irreducibility criterion for germs of analytic functions of two complex variables”, *Adv. Math.* **74** (1989), p. 190-257.
- [2] L. EIN, R. LAZARSFELD & M. MUSTĂŢĂ, “Contact loci in arc spaces”, *Compos. Math.* **140** (2004), p. 1229-1244.
- [3] E. GARCIA BARROSO, “Invariants des singularités des courbes planes et courbure de fibre de Milnor”, 2004, Servicio de Publicaciones de la ULL, 216 pages.

- [4] R. GOLDIN & B. TEISSIER, “Resolving singularities of plane analytic branches with one toric morphism”, in *Resolution of singularities (Obergrugl, 1997)*, Progr. Math., vol. 181, Birkhäuser, Basel, 2000, p. 315-340.
- [5] J. GWOŹDZIEWICZ & A. PŁOSKI, “On the approximate roots of polynomials”, *Ann. Polon. Math.*, LX.3, 1995.
- [6] J. IGUSA, “On the first terms of certain asymptotic expansions”, in *Complex analysis and algebraic geometry*, Iwanami Shoten, Tokyo, 1977, p. 357-368.
- [7] M. LEJEUNE-JALABERT & A. J. REGUERA-LÓPEZ, “Arcs and wedges on sandwiched surface singularities”, *Amer. J. Math.* **121** (1999), no. 6, p. 1191-1213.
- [8] M. MERLE, “Invariants polaires des courbes planes”, *Invent. Math.* **41** (1977), p. 103-111.
- [9] M. MUSTAŢĂ, “Jet schemes of locally complete intersection canonical singularities”, *Invent. Math.* **145** (2001), p. 397-424, With an appendix by David Eisenbud and Edward Frenkel.
- [10] J. F. NASH, JR., “Arc structure of singularities”, *Duke Math. J.* **81** (1995), p. 31-38.
- [11] M. SPIVAKOVSKY, “Valuations in function fields of surfaces”, *Amer. J. Math.* **112** (1990), no. 1, p. 107-156.
- [12] P. VOJTA, “Jets via Hasse-Schmidt derivations”, in *Diophantine geometry*, CRM Series, vol. 4, Ed. Norm., Pisa, 2007, p. 335-361.
- [13] O. ZARISKI, “Studies in equisingularity. I. Equivalent singularities of plane algebroid curves”, *Amer. J. Math.* **87** (1965), p. 507-536.
- [14] ———, *Le problème des modules pour les branches planes*, École Polytechnique, 1973, with a appendix by B. Teissier.

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Hussein MOURTADA
Université de Versailles Saint-Quentin
Laboratoire de Mathématiques de Versailles
45 avenue des États-Unis
78035 Versailles CEDEX (France)
mourtada@math.uvsq.fr