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SECOND COHOMOLOGY CLASSES OF THE GROUP OF C^1-FLAT DIFFEOMORPHISMS

by **Tomohiko ISHIDA**

Abstract. — We study the cohomology of the group consisting of all C^∞-diffeomorphisms of the line, which are C^1-flat to the identity at the origin. We construct non-trivial two second real cohomology classes and uncountably many second integral homology classes of this group.

Résumé. — On étudie la cohomologie du groupe des C^∞-difféomorphismes de la droite, qui sout C^1-tangents à l'identité à l'origine. On construit deux classes non-triviales de cohomologie réelle de degré deux et un nombre non-dénombrable de classes d'homologie de dimension deux de ce groupe.

1. Notations and main results

We denote by $\mathfrak{a}_1$ the Lie algebra of all formal vector fields on $\mathbb{R}$ with the Krull topology. For $k \geqslant 0$, we denote by $\mathfrak{a}_1^k$ the Lie subalgebra of $\mathfrak{a}_1$ consisting of formal vector fields which are C^k-flat at the origin. Let $\mathrm{Diff}_0^\infty(\mathbb{R})$ be the group of orientation-preserving C^∞-diffeomorphisms of $\mathbb{R}$ which fix the origin. Let $\mathcal{G}(1)$ be the group of germs of local C^∞-diffeomorphisms at the origin of $\mathbb{R}$. Let $G^\infty(1)$ be the group of ∞-jets of local C^∞-diffeomorphisms at the origin of $\mathbb{R}$. For $k \geqslant 1$, we denote by $\mathrm{Diff}_k^\infty(\mathbb{R})$, $\mathcal{G}_k(1)$ and $G_k^\infty(1)$ the subgroup of $\mathrm{Diff}_0^\infty(\mathbb{R})$, $\mathcal{G}(1)$ and $G^\infty(1)$ respectively, consisting of elements which are C^k-flat to the identity at the origin. The groups $G^\infty(1)$ and $G_k^\infty(1)$ can be considered as infinite-dimensional Lie groups, whose Lie algebras are $\mathfrak{a}_1^0$ and $\mathfrak{a}_1^k$, respectively.

We define the Gel'fand-Fuks cohomology [2] of $\mathfrak{a}_1^1$ in § 2. It is known to be 2-dimensional for each degree [3][7]. Moreover, Millionschikov proved its generators in degree greater than 1 can be described by the Massey

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products [5]. We carried out the calculation of the Massey products on $\mathrm{Diff}_1^\infty(\mathbb{R})$, and we give two 2-cocycles of $\mathrm{Diff}_1^\infty(\mathbb{R})$ in § 3.

For $l \geqslant k$, let α_l and $\alpha_{l_1}\dots\alpha_{l_i}$ be the 1-cochains of $\mathrm{Diff}_k^\infty(\mathbb{R})$ defined by

$$\alpha_l(f) = \frac{d^l}{dx^l} f(0) \quad \text{for } f \in \mathrm{Diff}_k^\infty(\mathbb{R}),$$

and

$$\alpha_{l_1}\dots\alpha_{l_i}(f) = \alpha_{l_1}(f)\dots\alpha_{l_i}(f) \quad \text{for } f \in \mathrm{Diff}_k^\infty(\mathbb{R}),$$

respectively. Then the following proposition holds.

Proposition 1.1. — *The following γ_-^2 and γ_+^2 are 2-cocycles of the group* $\mathrm{Diff}_1^\infty(\mathbb{R})$.

$$\begin{aligned}
\gamma_-^2 &= \Big(\frac{1}{2}\alpha_4 - 3\alpha_2\alpha_3 + 3\alpha_2^3\Big)\smile\Big(\alpha_3 - \frac{3}{2}\alpha_2^2\Big) - \frac{1}{2}\alpha_2\smile\Big(\alpha_3 - \frac{3}{2}\alpha_2^2\Big)^2,\\
\gamma_+^2 &= -\alpha_2\smile\Big(\frac{1}{10}\alpha_3\alpha_5 - \frac{1}{8}\alpha_4^2 - \frac{3}{20}\alpha_2^2\alpha_5\\
&\qquad + \frac{1}{2}\alpha_2\alpha_3\alpha_4 - \frac{4}{9}\alpha_3^3 + \frac{1}{2}\alpha_2^2\alpha_3^2 - \frac{3}{4}\alpha_2^4\alpha_3 + \frac{3}{8}\alpha_2^6\Big)\\
&\quad + \Big(\frac{1}{2}\alpha_4 - 3\alpha_2\alpha_3 + 3\alpha_2^3\Big)\smile\Big(\frac{1}{10}\alpha_5 - \alpha_2\alpha_4 - \frac{1}{3}\alpha_3^2 + 4\alpha_2^2\alpha_3 - 3\alpha_2^4\Big)\\
&\quad + \Big(\frac{1}{5}\alpha_5 - \frac{3}{2}\alpha_2\alpha_4 - \alpha_3^2 + 6\alpha_2^2\alpha_3 - \frac{15}{4}\alpha_2^4\Big)\smile\Big(-\frac{1}{2}\alpha_4 + 3\alpha_2\alpha_3 - 3\alpha_2^3\Big)\\
&\quad + \Big(\frac{1}{30}\alpha_6 - \frac{3}{10}\alpha_2\alpha_5 - \frac{1}{2}\alpha_3\alpha_4 + \frac{3}{2}\alpha_2^2\alpha_4 + 2\alpha_2\alpha_3^2 - 5\alpha_2^3\alpha_3 + \frac{9}{4}\alpha_2^5\Big)\\
&\qquad \smile\Big(\alpha_3 - \frac{3}{2}\alpha_2^2\Big).
\end{aligned}$$

Our main theorem is the following.

Theorem 1.2. — *Let $\gamma^2 : H_2(\mathrm{Diff}_1^\infty(\mathbb{R});\mathbb{Z}) \to \mathbb{R}^2$ be the homomorphism defined by*

$$\gamma^2(\xi) = (\gamma_-^2(\xi), \gamma_+^2(\xi)).$$

Then γ^2 is surjective.

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2. First cohomology of $\mathrm{Diff}_k^\infty(\mathbb{R})$

In this section, we review a result of Fukui [1] and compute 1-cocycles of $H^1(\mathrm{Diff}_k^\infty(\mathbb{R});\mathbb{R})$.

Definition 2.1. — *For a topological Lie algebra $\mathfrak{g}$, we denote by $A_C^*(\mathfrak{g})$ the differential graded algebra of all continuous alternating forms on $\mathfrak{g}$. The Gel'fand-Fuks cohomology of $\mathfrak{g}$ is defined to be the cohomology of the complex $(A_C^*(\mathfrak{g}), d)$. Here, d is the ordinary differential mapping of cochain complexes of Lie algebras. We denote the Gel'fand-Fuks cohomology of $\mathfrak{g}$ by $H_{GF}^*(\mathfrak{g})$.*

In the case $\mathfrak{g} = \mathfrak{a}_1^k$, the complex $A_C^*(\mathfrak{a}_1^k)$ is an exterior algebra generated by $\delta^{(k+1)}, \delta^{(k+2)}, \ldots$, where $\delta^{(l)}$'s are the 1-forms on $\mathfrak{a}_1^k$ defined by

$$\delta^{(l)}\left(f(x)\frac{d}{dx}\right) = (-1)^l f^{(l)}(0) \quad \text{for } f(x) \in \mathbb{R}[[x]].$$

Since it is easily seen that $d\delta^{(l)} = 0$ if and only if $k+1 \leqslant l \leqslant 2k+1$, we obtain the following proposition.

Proposition 2.2. — *For $k \geqslant 1$,*

$$H_{GF}^1(\mathfrak{a}_1^k) \cong \mathbb{R}^{k+1}.$$

Moreover, $\delta^{(k+1)}, \delta^{(k+2)}, \ldots, \delta^{(2k+1)}$ generate $H_{GF}^1(\mathfrak{a}_1^k)$.

In particular, $H_{GF}^1(\mathfrak{a}_1^1)$ is generated by δ'' and δ'''.

On the other hand, Fukui proved a proposition about the homology of groups corresponding to $\mathfrak{a}_1^k$.

Theorem 2.3 (Fukui[1]). — *For $k \geqslant 1$,*

$$H_1(\mathrm{Diff}_k^\infty(\mathbb{R});\mathbb{Z}) \cong \mathbb{R}^{k+1}.$$

Theorem 2.3 is obtained from the fact that the group homomorphism

$$\Psi_k : \mathrm{Diff}_k^\infty(\mathbb{R}) \to \dot{\mathbb{R}}^{k+1}$$

defined by

$$\Psi_k(f) := \left(\frac{1}{(k+1)!}f^{(k+1)}(0), \frac{1}{(k+2)!}f^{(k+2)}(0), \ldots, \frac{1}{(2k+1)!}f^{(2k+1)}(0)\right)$$

induces an isomorphism in the first homology. Here, $\dot{\mathbb{R}}^{k+1}$ means the group which is $\mathbb{R}^{k+1}$ as a set, where the addition is defined by

$$\begin{aligned}&(a_1, a_2, \ldots, a_{k+1}) + (b_1, b_2, \ldots, b_{k+1}) \\ &\qquad = (a_1 + b_1, a_2 + b_2, \ldots, a_k + b_k, a_{k+1} + b_{k+1} + (k+1)a_1 b_1).\end{aligned}$$

Since Ψ_k is a group homomorphism, α_{k+1}, α_{k+2}, $\ldots$, α_{2k} are 1-cocycles of $\mathrm{Diff}_k^\infty(\mathbb{R})$ with real coefficients. Moreover, if we denote the cochain $\tilde{\alpha}_{2k+1}$ by

$$\tilde{\alpha}_{2k+1} = \alpha_{2k+1} - \frac{1}{2}\binom{2k+1}{k}\alpha_{k+1}^2,$$

then it is also a 1-cocycle. In particular, α_2 and $\tilde{\alpha}_3 = \alpha_3 - \dfrac{3}{2}\alpha_2^2$ are 1-cocycles of $\mathrm{Diff}_1^\infty(\mathbb{R})$.

Remark 2.4. — The same argument can be applied to the groups $\mathcal{G}_1(1)$ and $G_1^\infty(1)$ instead of $\mathrm{Diff}_1^\infty(\mathbb{R})$. Hence Theorem 2.3 also holds for $\mathcal{G}_1(1)$ and $G_1^\infty(1)$. By regarding α_l's as the 1-cochains of $\mathcal{G}_1(1)$ and $G_1^\infty(1)$, the 1-cocycles α_{k+1}, α_{k+2}, $\ldots$, α_{2k} and $\tilde{\alpha}_{2k+1}$ of $\mathrm{Diff}_1^\infty(\mathbb{R})$ can be considered as 1-cocycles of the groups $\mathcal{G}_1(1)$ and $G_1^\infty(1)$, respectively.

3. Construction of the 2-cocycles of $\mathrm{Diff}_1^\infty(\mathbb{R})$

In this section, we recall the definition of the Massey products following [4], and construct the 2-cocycles $\gamma_\pm^2$ of the group $\mathrm{Diff}_1^\infty(\mathbb{R})$.

Definition 3.1 ([4]). — *Let $\mathcal{A} = (\mathcal{A}^n, d)$ be a differential graded algebra. For $u_i \in H^{p_i}(\mathcal{A})$, we set a_i a cocycle representative of u_i. We define $p(i,j)$ to be $\sum_{r=i}^{j}(p_r - 1)$. A collection of cochains $A = (a(i,j))$ for $1 \leqslant i \leqslant j \leqslant k$ and $(i,j) \neq (1,k)$ is a defining system of $\{a_1, \ldots, a_k\}$ if*

(i) $a(i,i) = a_i \in \mathcal{A}^{p_i}$,
(ii) $a(i,j) \in \mathcal{A}^{p(i,j)+1}$, *and*
(iii) $da(i,j) = \sum_{r=i}^{j-1}(-1)^{\deg a(i,r)}a(i,r)a(r+1,j)$.

Definition 3.2 ([4]). — *When a defining system A of $\{a_1, \ldots, a_k\}$ exists, we define $c(A) \in \mathcal{A}^{p(1,k)+2}$ by setting*

$$c(A) = \sum_{r=1}^{k-1}(-1)^{\deg a(i,r)}a(1,r)a(r+1,k).$$

Then $c(A)$ is a cocycle and the set

$$\{\text{a cohomology class of } c(A); A \text{ is a defining system of } \{a_1, \ldots, a_k\}\}$$

depends only on the cohomology classes $u_1, \ldots, u_k$. We call the elements of the set the Massey products of $\{u_1, \ldots, u_k\}$.

By Goncharova's theorem [3][7], which gives $\dim H^p_{GF}(\mathfrak{a}_1^k)$ for any $p, k \geqslant 1$, we know

$$H^p_{GF}(\mathfrak{a}_1^1) \cong \mathbb{R}^2 \quad \text{for any } p.$$

Furthermore, the following theorem is known.

THEOREM 3.3 (Millionschikov[5]). — *For any $p \geqslant 2$, there exist generators $g^p_-, g^p_+ \in H^p_{GF}(\mathfrak{a}_1^1)$ of $H^p_{GF}(\mathfrak{a}_1^1) \cong \mathbb{R}^2$, which are described by the Massey products. In particular, both the triple Massey product of $\{\delta'', \delta''', \delta'''\}$ and the 5-fold Massey product $\{\delta'', \delta''', \delta'', \delta'', \delta'''\}$ determine non-trivial cohomology classes in $H^2_{GF}(\mathfrak{a}_1^1)$, which are linearly independent.*

In fact, the defining systems of $\{\delta'', \delta''', \delta'''\}$ and $\{\delta'', \delta''', \delta'', \delta'', \delta'''\}$ can be written as

$$\begin{pmatrix} \delta'' & -\frac{1}{2}\delta^{(4)} & * \\ & \delta''' & 0 \\ & & \delta''' \end{pmatrix} \text{ and } \begin{pmatrix} \delta'' & -\frac{1}{2}\delta^{(4)} & -\frac{1}{5}\delta^{(5)} & -\frac{1}{30}\delta^{(6)} & * \\ & \delta''' & \frac{1}{2}\delta^{(4)} & \frac{1}{10}\delta^{(5)} & 0 \\ & & \delta'' & \frac{1}{3}\delta''' & \frac{1}{10}\delta^{(5)} \\ & & & \delta'' & -\frac{1}{2}\delta^{(4)} \\ & & & & \delta''' \end{pmatrix},$$

respectively.

Proof of Proposition 1.1. — For $\mathrm{Diff}^\infty_1(\mathbb{R})$ we checked that the defining systems of both of $\{\alpha_2, \tilde{\alpha}_3, \tilde{\alpha}_3\}$ and $\{\alpha_2, \tilde{\alpha}_3, \alpha_2, \alpha_2, \tilde{\alpha}_3\}$ also exist. In fact, they can be written as

$$\begin{pmatrix} \alpha_2 & \beta_1 & * \\ & \tilde{\alpha}_3 & \beta_2 \\ & & \tilde{\alpha}_3 \end{pmatrix} \text{ and } \begin{pmatrix} \alpha_2 & \beta_1 & \beta_5 & \beta_8 & * \\ & \tilde{\alpha}_3 & \beta_3 & \beta_6 & \beta_9 \\ & & \alpha_2 & \beta_4 & \beta_7 \\ & & & \alpha_2 & \beta_1 \\ & & & & \tilde{\alpha}_3 \end{pmatrix},$$

respectively. Here,

$$\beta_1 = -\frac{1}{2}\alpha_4 + 3\alpha_2\alpha_3 - 3\alpha_2^3, \qquad \beta_2 = \frac{1}{2}\tilde{\alpha}_3^2,$$
$$\beta_3 = \frac{1}{2}\alpha_4 - 2\alpha_2\alpha_3 + \frac{3}{2}\alpha_2^3, \qquad \beta_4 = \frac{1}{3}\alpha_3,$$
$$\beta_5 = -\frac{1}{5}\alpha_5 + \frac{3}{2}\alpha_2\alpha_4 + \alpha_3^2 - 6\alpha_2^2\alpha_3 + \frac{15}{4}\alpha_2^4,$$
$$\beta_6 = \frac{1}{10}\alpha_5 - \frac{1}{2}\alpha_2\alpha_4 - \frac{1}{3}\alpha_3^2 + \frac{3}{2}\alpha_2^2\alpha_3 - \frac{3}{4}\alpha_2^4,$$
$$\beta_7 = \frac{1}{10}\alpha_5 - \alpha_2\alpha_4 - \frac{1}{3}\alpha_3^2 + 4\alpha_2^2\alpha_3 - 3\alpha_2^4,$$
$$\beta_8 = -\frac{1}{30}\alpha_6 + \frac{3}{10}\alpha_2\alpha_5 + \frac{1}{2}\alpha_3\alpha_4 - \frac{3}{2}\alpha_2^2\alpha_4 - 2\alpha_2\alpha_3^2 + 5\alpha_2^3\alpha_3 - \frac{9}{4}\alpha_2^5,$$
$$\beta_9 = \frac{1}{10}\alpha_3\alpha_5 - \frac{1}{8}\alpha_4^2 - \frac{3}{20}\alpha_2^2\alpha_5 + \frac{1}{2}\alpha_2\alpha_3\alpha_4 - \frac{4}{9}\alpha_3^3 + \frac{1}{2}\alpha_2^2\alpha_3^2 - \frac{3}{4}\alpha_2^4\alpha_3 + \frac{3}{8}\alpha_2^6.$$

Following to the definition of the Massey products, we obtain cocycles

$$\gamma_-^2 = -\alpha_2 \smile \beta_2 - \beta_1 \smile \tilde{\alpha}_3,$$

and

$$\gamma_+^2 = -\alpha_2 \smile \beta_9 - \beta_1 \smile \beta_7 - \beta_5 \smile \beta_1 - \beta_8 \smile \tilde{\alpha}_3,$$

of Proposition 1.1. □

4. Proof of the main theorem

Throughout this section, for any two diffeomorphisms f and g, the multiplication fg means that g is applied first.

In this section, we prove the non-triviality of $\gamma_\pm^2$ by constructing uncountably many 2-cycles $\xi_2^\pm \in \mathbb{Z}[\mathrm{Diff}_1^\infty(\mathbb{R})^2]$ such that $\gamma_-^2(\xi_2^-) \neq 0$ and $\gamma_+^2(\xi_2^+) \neq 0$. Then this proves Theorem 1.2. To construct $\xi_2^\pm$, we use the following lemma.

Lemma 4.1. — *For any $k, l \geqslant 1 (k \neq l)$ and any $f \in \mathrm{Diff}_{k+l}^\infty(\mathbb{R})$, there exist $g \in \mathrm{Diff}_k^\infty(\mathbb{R})$ and $h \in \mathrm{Diff}_l^\infty(\mathbb{R})$ such that $f = [g, h]$. Here, $[g, h]$ means $ghg^{-1}h^{-1}$.*

Moreover, for any $k, l \geqslant 1 (k \neq l)$ it is true that $[\mathrm{Diff}_k^\infty(\mathbb{R}), \mathrm{Diff}_l^\infty(\mathbb{R})] = \mathrm{Diff}_{k+l}^\infty(\mathbb{R})$. In the case $k = l$, Fukui proved that $[\mathrm{Diff}_k^\infty(\mathbb{R}), \mathrm{Diff}_k^\infty(\mathbb{R})] = \mathrm{Diff}_{2k+1}^\infty(\mathbb{R})$ for $k \geqslant 1$ in [1].

Proof. — We may assume that $k > l$ and the ∞-jet of f is written as

$$f(x) = x + \sum_{n=k+l+1}^{\infty} a_n x^n.$$

If we take $h \in \mathrm{Diff}_l^\infty(\mathbb{R})$ so that h can be described as

$$h(x) = x + x^{l+1}$$

in a some neighborhood of 0, then the ∞-jet of fh at 0 is written as

$$fh(x) = x + x^{l+1} + a_{k+l+1}x^{k+l+1} + \dots.$$

Here, we apply the following theorem of the normal forms of diffeomorphisms of $(\mathbb{R}, 0)$.

THEOREM 4.2 (Takens[6]). — *For any $l \geqslant 1$ and $\psi \in \mathrm{Diff}_l^\infty(\mathbb{R})$, there exists $\varphi \in \mathrm{Diff}_0^\infty(\mathbb{R})$ such that*

$$\varphi\psi\varphi^{-1}(x) = x + \delta x^{l+1} + \alpha x^{2l+1},$$

in a some neighborhood of 0 for some $\delta = \pm 1$ and $\alpha \in \mathbb{R}$. Here δ and α are uniquely determined by the $(2l + 1)$-jet of ψ.

Because of the uniqueness of δ and α, there exists φ such that $\varphi^{-1}fh\varphi = h$ in some neighborhood U of 0. By Takens' construction of φ in Theorem 4.2, it is seen that one can choose φ to be C^l-flat to the identity at 0. We denote the composition $\varphi^{-1}fh\varphi$ by Φ. If we take h so that both of h and Φ have no fixed points except for 0, then Φ is conjugate to h. In the case l is odd and $x < 0$, there exists an integer $n_x \geqslant 0$ such that $\Phi^n(x)$ is in U for any $n \geqslant n_x$ and we define $\tilde{\varphi}(x) = \Phi^{-n_x}h^{n_x}(x)$. Otherwise, for any x there exists an integer $n_x \geqslant 0$ such that $\Phi^{-n}(x)$ is in U for any $n \geqslant n_x$ and we define $\tilde{\varphi}(x) = \Phi^{n_x}h^{-n_x}(x)$. Then $\tilde{\varphi}^{-1}\Phi\tilde{\varphi}$ coincides with h. If we set $g = \varphi\tilde{\varphi}$, then g is contained in $\mathrm{Diff}_k^\infty(\mathbb{R})$ and Lemma 4.1 is proved. □

Proof of Theorem 1.2. — If $f, g \in \mathrm{Diff}_1^\infty(\mathbb{R})$ and the ∞-jet of them are written as

$$f(x) = x + \sum_{n=2}^{\infty} a_n x^n, \quad g(x) = x + \sum_{n=2}^{\infty} b_n x^n,$$

then

$$\gamma_-^2(f, g) = 36(b_3 - b_2^2)(2a_4 - 6a_2a_3 + 4a_2^3 - a_2b_3 + a_2b_2^2).$$

Thus if $f_i \in \mathrm{Diff}_i^\infty(\mathbb{R})$ for $i = 1, 2, 3, 4$, then $\gamma_-^2(f_2, f_3) = \gamma_-^2(f_1, f_4) = \gamma_-^2(f_4, f_1) = 0$. On the other hand, if both of the coefficient of x^4 in the jet of f_3 and the coefficient of x^3 in the jet of f_2 are non-zero, then $\gamma_-^2(f_3, f_2) \neq 0$. Therefore, we assume $f_3 \in \mathrm{Diff}_3^\infty(\mathbb{R}) \setminus \mathrm{Diff}_4^\infty(\mathbb{R})$ and $f_2 \in \mathrm{Diff}_2^\infty(\mathbb{R}) \setminus$

$\mathrm{Diff}_3^\infty(\mathbb{R})$. By Lemma 4.1, we can choose $f_1 \in \mathrm{Diff}_1^\infty(\mathbb{R})$ and $f_4 \in \mathrm{Diff}_4^\infty(\mathbb{R})$ such that $[f_2, f_3] = [f_1, f_4]$. If we set

$$\xi_2^- = (f_3, f_2) - (f_2, f_3) + ([f_2, f_3], f_3 f_2) \\ - (f_4, f_1) + (f_1, f_4) - ([f_1, f_4], f_4 f_1) \in \mathbb{Z}[\mathrm{Diff}_1^\infty(\mathbb{R})^2],$$

then ξ_2^- is a cycle and

$$\gamma_-^2(\xi_2^-) = \gamma_2^-((f_3, f_2)) = \frac{1}{2}\alpha_4(f_3)\alpha_3(f_2) \neq 0.$$

Therefore, the non-triviality of γ_-^2 is proved.

The non-triviality of γ_+^2 can be proved similarly. Let $g_3 \in \mathrm{Diff}_3^\infty(\mathbb{R}) \setminus \mathrm{Diff}_4^\infty(\mathbb{R})$ and $g_4 \in \mathrm{Diff}_4^\infty(\mathbb{R}) \setminus \mathrm{Diff}_5^\infty(\mathbb{R})$. If we choose $g_1 \in \mathrm{Diff}_1^\infty(\mathbb{R})$ and $g_6 \in \mathrm{Diff}_6^\infty(\mathbb{R})$ such that $[g_3, g_4] = [g_1, g_6]$ and set

$$\xi_2^+ = (g_4, g_3) - (g_3, g_4) + ([g_3, g_4], g_4 g_3) \\ - (g_6, g_1) + (g_1, g_6) - ([g_1, g_6], g_6 g_1) \in \mathbb{Z}[\mathrm{Diff}_1^\infty(\mathbb{R})^2],$$

then ξ_2^+ is a cycle and

$$\gamma_+^2(\xi_2^+) = \gamma_2^+((g_4, g_3) - (g_3, g_4)) = -\frac{1}{20}\,\alpha_5(g_4)\alpha_4(g_3) \neq 0.$$

Consequently, $\gamma_\pm^2$ are non-trivial cohomology classes in $H^2(\mathrm{Diff}_1^\infty(\mathbb{R}); \mathbb{R})$. Furthermore, clearly $\gamma_-^2(\xi_2^+) = 0$ and $\gamma_-^2(\xi_2^-)$ can take any real value by changing f_2 or f_3. Similarly, $\gamma_+^2(\xi_2^+)$ also can take any value. This concludes the proof of Theorem 1.2. □

Moreover, the following corollary holds.

Corollary 4.3. — *For any $g \geqslant 2$, there exist uncountably many isomorphism classes of flat $\mathbb{R}$-bundles on genus g surface Σ_g, such that the images of their holonomy homomorphisms*

$$\pi_1(\Sigma_g) \to \mathrm{Diff}^\infty(\mathbb{R})$$

are contained in $\mathrm{Diff}_1^\infty(\mathbb{R})$.

Remark 4.4. — The same argument in §3, and §4 can be applied to the groups $\mathcal{G}_1(1)$ and $G_1^\infty(1)$ instead of $\mathrm{Diff}_1^\infty(\mathbb{R})$. Therefore, we can regard $\gamma_\pm^2$ as the 2-cochains of $\mathcal{G}_1(1)$ and $G_1^\infty(1)$, and Theorem 1.2 also holds for $\mathcal{G}_1(1)$ and $G_1^\infty(1)$, respectively.

On the other hand, for any group G and commuting $f, g \in G$, the chain $(f, g) - (g, f)$ is the simplest 2-cycle of G. However, if we regard $\gamma_\pm^2$ as the 2-cocycles of $G_1^\infty(1)$, then it is seen that $\gamma_\pm^2((f, g) - (g, f)) = 0$ for any commuting $f, g \in G_1^\infty(1)$. Hence the following is true.

PROPOSITION 4.5. — *For any group homomorphism* $\rho : \pi_1(T^2) \to G_1^\infty(1)$,

$$\rho^* \gamma_\pm^2 = 0.$$

BIBLIOGRAPHY

[1] K. FUKUI, "Homologies of the group $\mathrm{Diff}^\infty(\mathbf{R}^n, 0)$ and its subgroups", *J. Math. Kyoto Univ.* **20** (1980), no. 3, p. 475-487.

[2] I. M. GEL′FAND & D. B. FUKS, "Cohomologies of the Lie algebra of formal vector fields", *Izv. Akad. Nauk SSSR Ser. Mat.* **34** (1970), p. 322-337.

[3] L. V. GONCHAROVA, "The cohomologies of Lie algebras of formal vector fields on the line", *Funct. Anal. and Appl.* **7** (1973), p. 91-97, 194-203.

[4] D. KRAINES, "Massey higher products", *Trans. Amer. Math. Soc.* **124** (1966), p. 431-449.

[5] D. MILLIONSCHIKOV, "Algebra of formal vector fields on the line and Buchstaber's conjecture", *Funct. Anal. Appl.* **43** (2009), p. 264-278.

[6] F. TAKENS, "Normal forms for certain singularities of vectorfields", *Ann. Inst. Fourier (Grenoble)* **23** (1973), no. 2, p. 163-195, Colloque International sur l'Analyse et la Topologie Différentielle (Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1972).

[7] F. V. WEINSTEIN, "Filtering bases: a tool to compute cohomologies of abstract subalgebras of the Witt algebra", in *Unconventional Lie algebras*, Adv. Soviet Math., vol. 17, Amer. Math. Soc., Providence, RI, 1993, p. 155-216.

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