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# VECTOR BUNDLES ON NON-KAEHLER ELLIPTIC PRINCIPAL BUNDLES

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ABSTRACT. — We study relatively semi-stable vector bundles and their moduli on non-Kähler principal elliptic bundles over compact complex manifolds of arbitrary dimension. The main technical tools used are the twisted Fourier-Mukai transform and a spectral cover construction. For the important example of such principal bundles, the numerical invariants of a 3-dimensional non-Kähler elliptic principal bundle over a primary Kodaira surface are computed.

RÉSUMÉ. — Nous étudions les fibrés vectoriels relativement semi-stables sur des variétés non-kähleriennes qui sont des fibrés principaux elliptiques. Les principaux outils techniques utilisés sont la transformée de Fourier-Mukai tordue et une construction de revêtement spectral. Pour un exemple important de ces fibrés principaux, nous calculons les invariants numériques des fibrés elliptiques sur une surface de Kodaira primaire.

## 1. Introduction

The study of vector bundles over elliptic fibrations has been a very active area of research in both mathematics and physics over the past twenty years; in fact, there is by now a well understood theory for projective elliptic fibrations (see for example [25], [26], [29], [30], [14],[43], [12], *etc.* ). However, not very much is known about the non-Kähler case; the study of rank 2 vector bundles on non-Kähler elliptic surfaces is done in [15],[16]. In this article we study relatively semi-stable vector bundles on non-Kähler principal elliptic bundles over complex manifolds of arbitrary dimension with the invariant  $\delta \neq 0$ . One of the motivations for the study of vector bundles on

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non-Kähler elliptic  $n$ -folds comes from recent developments in superstring theory, where six-dimensional non-Kähler manifolds occur in the context of  $\mathcal{N} = 1$  supersymmetric heterotic and type II string compactifications with non-vanishing background  $H$ -field; in particular most of the non-Kähler examples appearing in the physics literature so far are non-Kähler principal elliptic fibrations (see [8], [22], [31]). There are also two classes of non-Kähler Calabi-Yau type threefolds appearing in the mathematical and physical literature: one is due to M. Gross (privately communicated to us by A. Căldăraru). Other examples appear in [2], [1], [38] and [20]. The main technical tools used are the twisted Fourier-Mukai transform, introduced by A. Căldăraru (see [21]) and the spectral cover construction, see [30], [24], [7], [43].

The paper is organized as follows. In the second section we determine the structure of the relative Jacobian of a principal elliptic bundle as a moduli space and find out that it is the product of the fiber with the basis. In the third and fourth sections, using the relative Jacobian, we adapt the construction of Căldăraru, [19], to our case, obtaining a twisted Fourier-Mukai transform. Similar results were obtained in different settings by O. Ben-Bassat [9] and I. Burban and B. Kreussler [18]. In the fifth section using this transform and the associated spectral cover we prove that the moduli space of rank  $n$ , relatively semi-stable vector bundles is corepresented by the relative Douady space of length  $n$  and relative dimension 0 subspaces of the relative Jacobian, see Theorem 5.

After reviewing some background results on torus bundles from [34] in the sixth section, we compute the numerical invariants (Hodge and Betti numbers) of a principal elliptic bundle over a primary Kodaira surface and use them to distinguish the non-trivial elliptic bundles. These invariants are also of interest for physicists working on heterotic string-theory models with non-Kähler Calabi-Yau type threefolds as backgrounds.

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## 2. Line bundles on elliptic principal bundles

In this section we shall be concerned with the study of the (coarse) moduli space of line bundles over a principal elliptic bundle  $\pi : X \rightarrow S$ , where  $S$  is a compact complex manifold, with fiber  $E := E_\tau := \mathbb{C}/\Lambda$  ( $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ ). Among the invariants of such bundles is the homomorphism  $\delta : H^1(E, \mathbb{C}) \rightarrow H^2(S, \mathbb{C})$  which is the  $d_2$ -differential  $E_2^{0,1} \rightarrow E_2^{2,0}$  of the Leray spectral sequence of the sheaf  $\mathbb{C}_X$  with terms  $E_2^{p,q} = H^p(S, R^q\pi_*\mathbb{C}_X) \cong H^p(S, \mathbb{C}) \otimes H^q(E, \mathbb{C})$ , see also Section 6 for more invariants.

We make the assumption that  $\delta \neq 0$ . In particular,  $X \rightarrow S$  does not have the topology of a product. We should note here that if  $S$  is Kähler, then  $X$  is non-Kähler if and only if  $\delta \neq 0$ , see [34].

We shall need in the sequel the following result of Deligne, [23], in the formulation of [34, Prop.5.2].

**THEOREM 2.1.** — *Let  $X \rightarrow S$  be a principal elliptic bundle. Then the following statements are equivalent:*

- a) *the Leray spectral sequence for  $\mathbb{C}_X$  degenerates at the  $E_2$ -level;*
- b)  *$\delta : H^1(E, \mathbb{C}) \rightarrow H^2(X, \mathbb{C})$  is the zero map;*
- c) *the restriction map  $H^2(X, \mathbb{C}) \rightarrow H^2(E, \mathbb{C})$  to a fibre takes a non-zero value in  $H_E^{1,1}$ .*

In our case the preceding theorem has a very important consequence.

**COROLLARY 2.2.** — *Let  $X \rightarrow S$  be a principal elliptic bundle with  $S$  a compact complex manifold and  $\delta \neq 0$ . Then for any vector bundle  $\mathcal{F}$  over  $X$  and any  $s \in S$  the bundle  $\mathcal{F}|_{X_s}$  has degree 0.*

*Proof.* — Indeed let  $r : H^2(X, \mathbb{C}) \rightarrow H^2(E, \mathbb{C})$  be the restriction map. We have that  $c_1(\mathcal{F}|_{X_s}) = r(c_1(\mathcal{F})) = 0$  by the theorem.  $\square$

Let us recall now the definition of the Jacobian variety  $J$  of a smooth curve  $C$ , see for instance [32, IV.4]. Let  $\text{Pic}^0(C/T)$  for any analytic space  $T$  denote the group

$$\{\mathcal{L} \in \text{Pic}(T \times C) \mid \deg(\mathcal{L}|_{\{t\} \times C}) = 0 \text{ for any } t \in T\} / p^* \text{Pic}(T),$$

where  $p : T \times C \rightarrow T$  is the first projection. The Jacobian variety of  $C$  will be a variety  $J$ , together with an element  $\mathcal{P} \in \text{Pic}^0(C/J)$  such that for any analytic space  $T$  and any  $\mathcal{M} \in \text{Pic}^0(C/T)$  there is a unique morphism  $f : T \rightarrow J$  such that  $(f \times \text{id}_C)^*\mathcal{P} \sim \mathcal{M}$  in  $\text{Pic}^0(T \times C)$ , i.e.,  $J$  represents the functor  $T \mapsto \text{Pic}^0(C/T)$ . It is well known that  $J$  exists for any smooth curve  $C$ . When  $C$  is an elliptic curve  $E$  then  $J = E^*$ , the dual torus, and  $\mathcal{P}$  is called a Poincaré bundle. In this case  $\mathcal{P}$  is a line bundle over  $E^* \times E$

such that  $\mathcal{P}|_{\{\mathcal{L}\} \times E} \simeq \mathcal{L}$ , and  $E^* \simeq \text{Pic}^0(E)$ . We pass now to the relative case for elliptic principal bundles.

DEFINITION 2.3. — *Let  $X \xrightarrow{\pi} S$  be an elliptic principal bundle with typical fibre an elliptic curve  $E_\tau$  and base  $S$  a smooth manifold. Let  $F : (An/S)^{op} \rightarrow (Sets)$  be the functor from the category of analytic spaces over  $S$  to the category of sets, given, for any commutative diagram*

$$(2.1) \quad \begin{array}{ccc} X_T & \longrightarrow & X \\ \downarrow \pi & & \downarrow \pi \\ T & \longrightarrow & S, \end{array}$$

where  $X_T := X \times_S T$ , by

$$F(T) := \{ \mathcal{L} \text{ invertible on } X_T \mid \text{deg}(\mathcal{L}|_{X_{T,t}}) = 0, \text{ for all } t \in T \} / \sim,$$

where  $\mathcal{L}_1 \sim \mathcal{L}_2$  if there is a line bundle  $L$  on  $T$  such that  $\mathcal{L}_1 \simeq \mathcal{L}_2 \otimes \pi^* L$ .

A variety  $J$  over  $S$  will be called the relative Jacobian of  $X$  if

- (i) it corepresents the functor  $F$ , see [36, Def. 2.2.1], i.e., there is a natural transformation  $F \xrightarrow{\sigma} \text{Hom}_S(-, J)$  and for any other variety  $N/S$  with a natural transformation  $F \xrightarrow{\sigma'} \text{Hom}_S(-, N)$  there is a unique  $S$ -morphism  $J \xrightarrow{\nu} N$  such that  $\nu_* \circ \sigma = \sigma'$ ;
- (ii) for any point  $s \in S$  the map  $F(\{s\}) \rightarrow \text{Hom}_S(\{s\}, J) \simeq J_s$  is bijective. Then each fibre  $J_s$  is the Jacobian of the fibre  $X_s \simeq E$ .

If  $X$  is projective, the existence of the relative Jacobian is well known, because it can be identified with the coarse relative moduli space of stable locally free sheaves of rank 1 and degree 0 on the fibres of  $X$ , see [36], [21]. The relative Jacobian exists also in our non-Kähler case. It is just the product  $S \times E^*$  and has the following special properties.

THEOREM 2.4.

- (i) The functor  $F$  is corepresented by  $J := S \times E^*$ .
- (ii) For any point  $s \in S$  the map  $F(\{s\}) \rightarrow \text{Hom}_S(\{s\}, J) \simeq J_s \simeq E^*$  is bijective.
- (iii) The map  $\sigma(T)$  is injective for any complex space  $T$ .
- (iv) The functor  $F$  is locally representable by  $J = S \times E^*$ , i.e., if  $U \subset S$  is a trivializing open subset,  $\sigma(U)$  is bijective.

For the proof we use the following Seesaw lemma and its Corollary.

LEMMA 2.5. — *Let  $Y \xrightarrow{q} T$  be a principal elliptic bundle over a complex analytic space  $T$  with fibre  $E$  and let  $\mathcal{M}$  be an invertible sheaf on  $Y$  such*

that  $\mathcal{M}_t = \mathcal{M}|_{Y_t}$  is trivial on any fibre of  $q$ . Then  $L = q_*\mathcal{M}$  is locally free of rank 1 and  $\mathcal{M} = q^*L$ .

*Proof.* — Because the statements are local over  $T$  and  $Y$  is locally trivial, we may assume that  $Y = T \times E$  and that  $q$  is the first projection  $p_1$ . We first note that the canonical homomorphism  $p_1^*p_{1*}\mathcal{M} \rightarrow \mathcal{M}$  is an isomorphism because each  $\mathcal{M}_t$  is trivial so that it is an isomorphism when restricted to a fibre of  $p_1$ . In order to show that  $p_{1*}\mathcal{M}$  is invertible we consider the sheaf  $\mathcal{M}(a) := \mathcal{M} \otimes p_2^*\mathcal{O}_E(a)$  for some  $a \in E$ . Then  $h^0(\mathcal{M}_t(a)) = 1$  and  $h^1(\mathcal{M}_t(a)) = 0$ , whereas  $h^1(\mathcal{M}_t) = 1$ . Let  $\phi^i(t) : (R^i p_{1*}\mathcal{M}(a))(t) \rightarrow H^i(X_t, \mathcal{M}_t(a))$  denote the base change homomorphism. Because  $\phi^1(t) = 0$  it follows that  $R^1 p_{1*}\mathcal{M}(a) = 0$ , and by that  $L := p_{1*}\mathcal{M}(a)$  is invertible, see e.g. [32, Th.12.11].

Let  $\mathbb{C}(a)$  be the sky-scraper sheaf with stalk  $\mathbb{C}$  at  $a \in E$ . Then  $\mathcal{M} \otimes p_2^*\mathbb{C}(a) = \mathcal{M}|_{T \times \{a\}}$  and we have the exact sequence

$$0 \rightarrow \mathcal{M} \rightarrow \mathcal{M}(a) \rightarrow \mathcal{M} \otimes p_2^*\mathbb{C}(a) \rightarrow 0,$$

giving rise to the long exact sequence

$$0 \rightarrow p_{1*}\mathcal{M} \rightarrow p_{1*}\mathcal{M}(a) \xrightarrow{\alpha} p_{1*}(\mathcal{M}|_{S \times \{a\}}) \rightarrow R^1 p_{1*}(\mathcal{M}) \rightarrow 0.$$

Denoting by  $\mathcal{A}$  the image of  $\alpha$  and pulling back to  $Y$ , we obtain the exact sequence

$$p_1^*p_{1*}\mathcal{M} \xrightarrow{\gamma} p_1^*p_{1*}\mathcal{M}(a) \rightarrow p_1^*\mathcal{A} \rightarrow 0.$$

The restriction of  $\gamma$  to a fibre becomes the canonical map

$$H^0(X_t, \mathcal{M}_t) \otimes \mathcal{O}_{Y_t} \rightarrow H^0(Y_t, \mathcal{M}_t(a)) \otimes \mathcal{O}_{Y_t},$$

which is an isomorphism. Hence  $(p_1^*\mathcal{A})_t = 0$  for any  $t \in T$ , and then also  $p_1^*\mathcal{A} = 0$  and finally  $\mathcal{A} = 0$ . This proves that  $p_{1*}\mathcal{M} \cong p_{1*}\mathcal{M}(a) = L$ , and we have  $\mathcal{M} \cong p_1^*p_{1*}\mathcal{M} \cong p_1^*L$ .

One should note here that  $p_1^*p_{1*}\mathcal{M}(a) \rightarrow \mathcal{M}(a)$  is not an isomorphism, having  $\mathcal{M}|_{T \times \{a\}}$  as its cokernel, because  $H^0(X_t, \mathcal{M}_t(a)) \otimes \mathcal{O}_{Y_t} \rightarrow \mathcal{M}_t(a)$  is not an isomorphism. □

**COROLLARY 2.6.** — *Let  $T$  be a complex space and  $T \times E \xrightarrow{\theta} T \times E$  an isomorphism of the form  $\theta(t, \alpha) = (t, \alpha + \lambda(t))$  with  $\lambda : T \rightarrow E$  a holomorphic map. Then for any invertible sheaf  $\mathcal{L}$  on  $T \times E$  with  $\deg(\mathcal{L}_t) = 0$  for any  $t \in T$ , there is an invertible sheaf  $L$  on  $T$  such that  $\theta^*\mathcal{L} \cong \mathcal{L} \otimes p_1^*L$ .*

*Proof.* — Because  $\mathcal{L}_s$  has degree 0,  $(\theta_s^*\mathcal{L}_s) \otimes \mathcal{L}_s^{-1}$  is trivial on any fiber (recall that a line bundle of degree 0 is isomorphic with its pull-back via a translation, see [10], [40]). Now apply the Lemma to  $\theta^*\mathcal{L} \otimes \mathcal{L}^{-1}$ . □

*Proof of Theorem 2.4.* — Let  $f : T \rightarrow S$  be a morphism of analytic spaces and let  $\mathcal{L}$  be a line bundle on  $X_T$ . Let  $\{S_i\}$  be an open cover of  $S$  that gives a trivialization of the bundle  $X$  (that is  $X_i := X|_{S_i} \simeq S_i \times E$ ). Taking the inverse image of this cover we get an open cover  $\{T_i = f^{-1}S_i\}$  of  $T$  with the same property. Let us denote  $X_{T,i} := X_T|_{T_i}$ . There are trivializing maps

$$(2.2) \quad \begin{array}{ccc} X_{T,i} & \xrightarrow{\theta_i} & T_i \times E \\ & \searrow & \swarrow \\ & T_i & \end{array}$$

Let  $\mathcal{L}_i$  be the sheaf on  $T_i \times E$  defined by  $\theta_i^* \mathcal{L}_i = \mathcal{L}|_{X_{T,i}}$  such that  $\mathcal{L}_j \simeq (\theta_i \circ \theta_j^{-1})^* \mathcal{L}_i$  over  $T_{ij}$ .

Let now  $\mathcal{P}$  be a Poincaré bundle on  $E^* \times E$ . Then for any  $i$  we'll have a **unique** morphism  $\phi_i : T_i \rightarrow E^*$  such that  $\mathcal{L}_i \sim (\phi_i \times \text{id})^*(\mathcal{P})$ . Taking into account that  $\theta_i \circ \theta_j^{-1} = \text{id} \times \theta_{ij}$  and  $\theta_{ij}$  acts by translations, the preceding corollary implies that  $\mathcal{L}_i \sim \mathcal{L}_j$  on  $T_{ij}$ . Therefore  $\phi_i = \phi_j$  on  $T_{ij}$ . So we are given a global morphism  $\phi : T \rightarrow E^*$ .

Let now

$$(2.3) \quad \begin{array}{ccc} T & \xrightarrow{\tilde{\phi}} & S \times E^* = J \\ & \searrow & \swarrow \\ & S & \end{array}$$

be the corresponding map  $\tilde{\phi} := (f, \phi)$ . This provides us with a map  $F(T) \xrightarrow{\sigma(T)} \text{Hom}_S(T, J)$ . It is straightforward to check that  $\sigma : F \rightarrow \text{Hom}_S(-, J)$  is a morphism of functors. The minimality of  $J$  will be proved after the proof of (iv).

ii) Property (ii) follows directly from the definition of the maps  $\sigma(\{s\})$ .

iii) We show next that each map  $\sigma(T) : F(T) \rightarrow \text{Hom}_S(T, J)$  is injective. For that let  $\mathcal{L}$  and  $\mathcal{L}'$  be two line bundles such that their respective maps  $\phi$  and  $\phi'$  are equal. We need to show that  $\pi_*(\mathcal{L}' \otimes \mathcal{L}^{-1}) =: L$  is locally-free.

We have that  $\phi_i = \phi'_i$  for any  $i$ . This implies that over  $X_{T,i}$  we have

$$(2.4) \quad \mathcal{L}_i \sim \mathcal{L}'_i.$$

Because of the relation (2.4)  $\mathcal{L}'_t \otimes \mathcal{L}_t^{-1}$  is trivial for any  $t \in T$ . By the above corollary we obtain that  $\pi_*(\mathcal{L}' \otimes \mathcal{L}^{-1})$  is locally free.

iv) Let  $U \subset S$  be trivializing for the bundle  $X$  such that  $X_U \simeq U \times E$ . By the universal property of the dual torus  $E^*$ , the map  $\sigma(U)$  as well as all the maps  $\sigma(T)$  for  $T \rightarrow U$  are bijective.

Let now  $F \xrightarrow{\sigma'} \text{Hom}_S(-, N)$  be any natural transformation. For any  $s \in S$  there is the map  $\nu_s := \sigma'(\{s\}) \circ \sigma(\{s\})^{-1} : J_s \rightarrow N_s$ , thus defining a map  $\nu : J \rightarrow N$  over  $S$ . In order to show that  $\nu$  is a morphism, we just remark that  $\nu_s$  is the restriction of the map  $\sigma'(U \times E^*) \circ \sigma(U \times E^*)^{-1}(\text{id}) : J_U \rightarrow N_U$  for a trivializing open subset  $U \subset S$  and any  $s \in U$ .

Finally  $\nu_* \circ \sigma = \sigma'$  follows from (iv) of the theorem and the fact that the functor  $\text{Hom}_S(-, N)$  is a sheaf, using a trivializing covering for  $X$  of  $S$ . This completes the proof of (i). □

*Remark 2.7.* — There is a very convenient description of  $\sigma(S)$  as follows. Let  $\mathcal{L}$  be an arbitrary line bundle over  $X$ . We know that  $[\mathcal{L}] \in F(S)$  by Corollary 2.2. By the above proof  $\phi := \sigma(S)([\mathcal{L}]) : S \rightarrow S \times E^*$  is given by  $\phi(s) = (s, x)$  with  $x = [\mathcal{L}|_{X_s}]$ .

It will follow from Theorem 3.2 that the relative Jacobian  $J = S \times E^*$  is only a coarse moduli space under our assumption on  $X$ . However, by property (iv) of the theorem one can find a system of local universal sheaves which will form a twisted sheaf in the next section as in [21], chapter 4.

### 3. The twisted universal sheaf

In the following we replace the relative Jacobian  $J$  by  $S \times E$  via the canonical isomorphism between  $E$  and  $E^*$ . Then the local trivializations  $X_i \xrightarrow{\theta_i} S_i \times E$  are at the same time isomorphisms between  $X_i$  and  $J_i := S_i \times E$ . The local universal sheaves  $\mathcal{U}_i$  on  $X_{i,J} = J \times_S X_i = J_i \times_{S_i} X_i$  are then given as pull backs of the universal sheaf  $\mathcal{O}_{E \times E}(\Delta) \otimes p_2^* \mathcal{O}_E(-p_0)$  for the classical Jacobian of the elliptic curve  $E$ , after fixing an origin  $p_0 \in E$  and where  $\Delta$  is the diagonal.

Denoting by  $\rho_i$  the composition of maps

$$X_{i,J} \xrightarrow{id \times \theta_i} J \times_S (S_i \times E) \simeq S_i \times E \times E \rightarrow E \times E,$$

and by  $p_X$  the projection from  $X_{i,J}$  to  $X_i$ , the local universal sheaf becomes

$$\mathcal{U}_i = \rho_i^*(\mathcal{O}_{E \times E}(\Delta) \otimes p_2^* \mathcal{O}_E(-p_0)) \simeq \mathcal{O}_{X_{i,J}}(\Gamma_i) \otimes p_X^* \mathcal{O}_{X_i}(-s_i),$$

where  $\Gamma_i$  is the inverse of the diagonal (or the graph of the map  $\theta_i$ ) and  $s_i$  is the section of  $X_i$  corresponding to the reference point  $p_0$  under the isomorphism  $\theta_i$ , see [21], prop. 4.2.3.



To measure the failure of these bundles to glue to a global universal one let us consider the line bundles  $\mathcal{M}_{ij} := \mathcal{U}_i \otimes \mathcal{U}_j^{-1}$  over  $J \times_S X_{ij}$ . Then the restriction of  $\mathcal{M}_{ij}$  to a fibre  $X_s$  of the projection  $J \times_s X_i \xrightarrow{q_i} J$  is trivial because both  $\mathcal{U}_j$  and  $\mathcal{U}_i$  restrict to isomorphic sheaves. By Lemma 2.5 there are invertible sheaves  $\mathcal{F}_{ij}$  on  $J_{ij} = S_{ij} \times E$  such that  $\mathcal{M}_{ij} = q_i^* \mathcal{F}_{ij}$ .

This collection of line bundles satisfies the following properties:

1.  $\mathcal{F}_{ii} = \mathcal{O}_{J_i}$ ;
2.  $\mathcal{F}_{ji} = \mathcal{F}_{ij}^{-1}$ ;
3.  $\mathcal{F}_{ij} \otimes \mathcal{F}_{jk} \otimes \mathcal{F}_{ki} =: \mathcal{F}_{ijk}$  is trivial, with trivialization induced by the canonical one of  $\mathcal{M}_{ij} \otimes \mathcal{M}_{jk} \otimes \mathcal{M}_{ki}$ ;
4.  $\mathcal{F}_{ijk} \otimes \mathcal{F}_{jkl}^{-1} \otimes \mathcal{F}_{kli} \otimes \mathcal{F}_{lij}^{-1}$  is canonically trivial.

These conditions tell us that the collection  $\{\mathcal{F}_{ij}\}$  represents a gerbe (see [26]) and gives rise to an element  $\alpha \in H^2(J, \mathcal{O}_J^*)$ . More explicitly,  $\alpha$  is defined as follows. We may assume that the sheaves  $\mathcal{F}_{ij}$  are already trivial with trivializations  $a_{ij} : \mathcal{O}_J \simeq \mathcal{F}_{ij}$  over  $J_{ij}$ .

If  $c_{ijk} : \mathcal{O}_J \simeq \mathcal{F}_{ijk}$  is the isomorphism which is induced by the canonical trivialization of  $\mathcal{M}_{ij} \otimes \mathcal{M}_{jk} \otimes \mathcal{M}_{ki}$ , then

$$(3.1) \quad a_{ij} \otimes a_{jk} \otimes a_{ki} = \alpha_{ijk} c_{ijk}$$

with scalar functions  $\alpha_{ijk}$  which then define a cocycle for the sheaf  $\mathcal{O}_J^*$ , thus defining the class  $\alpha \in H^2(J, \mathcal{O}_J^*)$ , see [21], section 4.3. It is straightforward to prove:

LEMMA 3.1. — *The sheaves  $\mathcal{U}_i$  can be glued to a global universal sheaf if and only if the class  $\alpha = 0$ .*

The element  $\alpha$  is related to the element  $\xi \in H^1(S, \mathcal{O}_S(E))$  which is defined by the cocycle of the elliptic bundle  $X \rightarrow S$ , using the Ogg-Shafarevich group  $\text{III}_S(J)$  of  $J$ , see [21], section 4.4. There is an exact sequence  $0 \rightarrow \text{Br}(S) \rightarrow \text{Br}(J) \xrightarrow{\pi} \text{III}_S(J) \rightarrow 0$ , where  $\text{Br}(S) \simeq H^2(J, \mathcal{O}_J^*)$  is the analytic Brauer group of  $S$  and  $\text{III}_S(J)$  is isomorphic to  $H^1(S, \mathcal{O}_S(E))$  in our setting. We have the

THEOREM 3.2 ([21], Th 4.4.1).

$$\xi = \pi(\alpha).$$

Because  $\xi \neq 0$  in our case,  $\alpha \neq 0$ , and thus the local universal sheaves cannot be glued to a global universal sheaf by preserving the bundle structure on the elliptic fibres.

### 4. The twisted Fourier-Mukai transform

The collection of local universal sheaves above can be considered as an  $\alpha$ -twisted sheaf with which one can define a Fourier-Mukai transform. Recall the definition of an  $\alpha$ -twisted sheaf on a complex space or on an appropriate scheme  $X$ .

DEFINITION 4.1. — Let  $\alpha \in C^2(\mathfrak{U}, \mathcal{O}_X^*)$  be a Čech 2-cocycle, given by an open cover  $\mathfrak{U} = \{U_i\}_{i \in I}$  and sections  $\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)$ . An  $\alpha$ -twisted sheaf on  $X$  will be a pair of families  $(\{\mathcal{F}_i\}_{i \in I}, \{\varphi_{ij}\}_{i,j \in I})$  with  $\mathcal{F}_i$  a sheaf of  $\mathcal{O}_X$ -modules on  $U_i$  and  $\varphi_{ij} : \mathcal{F}_j|_{U_i \cap U_j} \rightarrow \mathcal{F}_i|_{U_i \cap U_j}$  isomorphisms such that

- $\varphi_{ii}$  is the identity for all  $i \in I$ .
- $\varphi_{ij} = \varphi_{ji}^{-1}$ , for all  $i, j \in I$ .
- $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{kl}$  is multiplication by  $\alpha_{ijk}$  on  $\mathcal{F}_i|_{U_i \cap U_j \cap U_k}$  for all  $i, j, k \in I$ .

It is easy to see that the coherent  $\alpha$ -twisted sheaves on  $X$  make up an abelian category and thus give rise to a derived category  $\mathcal{D}^b(X, \alpha)$ . For further properties of  $\alpha$ -twisted sheaves, see [21].

With the notation above, the family  $(\mathcal{U}_i)$  becomes a twisted sheaf  $\mathcal{U}$  w.r.t. the cocycle  $p_J^* \alpha$  of the sheaf  $\mathcal{O}_{J \times_S X}$  as follows. The trivializations  $a_{ij}$  of the  $\mathcal{F}_{ij}$  induce isomorphisms  $\phi_{ij} : \mathcal{U}_j \simeq \mathcal{U}_i$  which satisfy the definition of a twisted sheaf because of identity 3.1. We also need the dual  $\mathcal{V}$  of  $\mathcal{U}$  on  $J \times_S X$  which locally over  $S_i$  is given by

$$\mathcal{V}_i = \rho_i^*(\mathcal{O}_{E \times E}(-\Delta) \otimes p_2^* \mathcal{O}_E(p_0)) \simeq \mathcal{O}_{X_{iJ}}(-\Gamma_i) \otimes p_X^* \mathcal{O}_{X_i}(s_i).$$

It follows that  $\mathcal{V}_i$  is  $\alpha^{-1}$ -twisted. We let  $\mathcal{V}^0$  and  $\mathcal{U}^0$  denote the extensions of  $\mathcal{V}$  and  $\mathcal{U}$  to  $J \times X$  by zero.

The following theorem supplies us with the main tool for the treatment of the moduli spaces  $M_X(n, 0)$  of relatively semistable vector bundles on  $X$  of rank  $n$  and degree 0 on the fibres  $X_s$  in Section 5. It is an analog of theorem [21, Th.6.5.4](and also [19]):

THEOREM 4.2. — Let  $X \xrightarrow{\pi} S$  be an elliptic principal fiber bundle, where  $S$  has trivial canonical bundle. Let  $\alpha \in Br(J)$  be the obstruction to the existence of the universal sheaf on  $J \times_S X$  and let  $\mathcal{U}$  be the associated  $p_J^*(\alpha)$ -twisted universal sheaf on  $J \times_S X$  with its dual  $\mathcal{V}$  as above.

Then the twisted Fourier-Mukai transform  $\Psi : \mathcal{D}^b(J, \alpha) \rightarrow \mathcal{D}^b(X)$ , given by  $\Psi(\mathcal{F}) := Rp_{X*}(\mathcal{V}^0 \otimes^L Lp_J^* \mathcal{F})$  is an equivalence of categories, where  $p_J$  and  $p_X$  are the product projections

$$(4.1) \quad J \xleftarrow{p_J} J \times X \xrightarrow{p_X} X$$

Note here that  $\mathcal{V}^0 \otimes^L Lp_{J*} \mathcal{F}$ ) is a complex in the category of sheaves on  $J \times X$ .

*Proof.* — The theorem follows from the Căldăraru’s version of the Bridgeland (Orlov, Mukai, etc.)-criterion ([41],[13], [21, Th.3.2.1]), applied to our case. Due to a private communication this criterion works also in the case when  $\alpha$  is not torsion<sup>(1)</sup>. It follows that the functor  $\Psi$  is fully faithful if and only if for each point  $y \in J$  and its skyscraper sheaf  $\mathbb{C}(y)$ ,  $\text{Hom}(\Psi(\mathbb{C}(y)), \Psi(\mathbb{C}(y))) = \mathbb{C}$  and for any  $y_1, y_2 \in J$ ,  $\text{Ext}^i(\Psi(\mathbb{C}(y_1)), \Psi(\mathbb{C}(y_2))) = 0$ , unless  $y_1 = y_2$  and  $0 \leq i \leq \dim(J)$ . Moreover  $\Psi$  is an equivalence of categories if and only if for any  $y \in J$  we have  $\Psi(\mathbb{C}(y)) \otimes \omega_X \simeq \Psi(\mathbb{C}(y))$ . Note that in our case the canonical bundle is trivial, see formula (6.1), so that the last condition is automatically satisfied.

In order to compute  $\Psi(\mathbb{C}(y))$  for a point  $y \in J$ , let  $s \in S_i$  be the image in  $S$  and consider  $\mathcal{V}^0 \otimes p_J^* \mathbb{C}(y)$ . Its support is  $(J \times_S X) \cap (\{y\} \times X) = \{y\} \times X_s$ . We may therefore identify  $\mathcal{V}$  with  $\mathcal{V}_i$  and obtain  $\mathcal{V}^0|_{\{y\} \times X} \simeq \mathcal{O}_{X_s}(-x + s_i(s))$ , where  $\theta_i(x) = y$  and  $s_i$  denotes the local section of  $X$  corresponding to  $p_0$ . Because  $p_J^* \mathbb{C}(y) = \mathcal{O}_{y \times X_s}$ , we obtain  $\Psi(\mathbb{C}(y)) \simeq p_{X*} \mathcal{O}_{X_s}(-x + s_i(s)) \simeq i_{s*} \mathcal{O}_{X_s}(-x + s_i(s))$  by the base change isomorphism for the inclusion  $i_s$  which holds in this case because  $X_s$  is smooth as is the projection  $p_X$ , see [11, Lemma 1.3] Using this, we conclude that  $\text{Hom}(\Psi(\mathbb{C}(y)), \Psi(\mathbb{C}(y))) = \mathbb{C}$  and we proceed in the same way for  $\text{Ext}$ .  $\square$

*Remark 4.3.* — One can see this result in connection with Section 6 of [37], since the element  $\alpha \in \text{Br}(J)$  is not torsion.

In the sequel we shall work with the adjoint transform

$$\Phi(-) = Rp_{J*}(\mathcal{U}^0 \otimes^L Lp_X^*(-))$$

of  $\Psi$ , with kernel  $\mathcal{U}^0$ . It is the reverse equivalence, see [14, 8.4], [35], [6] for the untwisted situation.

We need the following special cases of base change properties.

PROPOSITION 4.4. — *For any  $s \in S$  let  $i_s : X_s \rightarrow X$  and  $j_s : \{s\} \times E \rightarrow J$  be the natural inclusions. Then the canonical morphism of functors*

$$(4.2) \quad \mathbf{L}j_s^* \circ \Phi \simeq \Phi_s \circ \mathbf{L}i_s^*,$$

*is an isomorphism, where  $\Phi_s$  is the classical Fourier-Mukai transform associated with the Poincaré bundle over  $E^* \times E$ .*

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<sup>(1)</sup>The authors are indebted to Andrei Căldăraru for this information.

*Proof.* — Let  $\tilde{j}_s$  denote the inclusion of  $J_s \times X_s$  into  $J \times_S X$  and let  $p_{J_s}$  be the first projection of  $J_s \times X_s$ . By [11, Lemma 1.3]  $Lj_s^* p_{J_s} = Rp_{J_s} \tilde{j}_s^*$ . Then

$$Lj_s^* p_{J_s}(\mathcal{V}^0 \otimes p_X^* \mathcal{F}) \simeq Rp_{J_s} \tilde{j}_s^* \mathcal{V}^0 \otimes p_X^* \mathcal{F} \simeq (\mathcal{V}|_{J_s \times X_s}) \otimes p_{X_s}^* i_s^* \mathcal{F},$$

which implies the formula. □

The following definition is very useful for dealing with the spectral covers in the next section.

DEFINITION 4.5 ([39]). — We denote by  $\Phi^i(\mathcal{F})$  the  $i$ -th term of the complex  $\Phi(\mathcal{F})$ . We say that the sheaf  $\mathcal{F}$  is  $\Phi$ -WIT $_i$  (the weak index theorem holds) if  $\Phi^i(\mathcal{F}) \neq 0$  and  $\Phi^j(\mathcal{F}) = 0$  for any  $j \neq i$ . Moreover if  $\mathcal{F}$  is WIT $_i$  and  $\Phi^i(\mathcal{F})$  is locally free we say that  $\mathcal{F}$  is IT $_i$ .

Consider now a rank  $n$  vector-bundle  $\mathcal{F}$  over the principal elliptic bundle  $X$  and denote its restriction to a fibre  $X_s$  by  $\mathcal{F}_s$ . From Proposition 4.4 it follows that if  $\mathcal{F}_s$  is  $\Phi_s$ -WIT $_i$  for any  $s$  then  $\mathcal{F}$  is  $\Phi$ -WIT $_i$ .

### 5. A spectral cover and vector bundles on $X$

In this section we shall apply the twisted Fourier-Mukai transform to the moduli problem for rank- $n$  relatively semi-stable vector bundles on the principal elliptic bundle  $X$ . By Deligne’s theorem (Theorem 2.1), the degree of the restriction  $\mathcal{F}_s$  of any vector bundle  $\mathcal{F}$  on  $X$  is 0 for any  $s \in S$ . Therefore we consider the set  $MS_X(n, 0)$  of rank- $n$  vector bundles on  $X$  which are fibrewise semistable and of degree zero, together with its quotient

$$M_X(n, 0) := MS_X(n, 0) / \sim$$

of equivalence classes, where two bundles are defined to be equivalent if they are fibrewise S-equivalent.

Let us recall that a vector bundle  $\mathcal{E}$  on a smooth projective curve is called semistable if for any proper subbundle  $\mathcal{E}'$ ,

$$\text{deg}(\mathcal{E}') / \text{rank}(\mathcal{E}') \leq \text{deg}(\mathcal{E}) / \text{rank}(\mathcal{E}).$$

For such bundles there is the standard notion of S-equivalence, see e.g. [36].

It is well-known that the semistable vector bundles of degree zero on the elliptic curve  $E$  are direct sums

$$(5.1) \quad \mathcal{E} = \bigoplus_i \mathcal{A}_{n_i} \otimes \mathcal{O}_E(x_i - p_0),$$

where the  $\mathcal{A}_n$  denote the indecomposable Atiyah bundles of degree zero which are inductively defined by nontrivial extensions  $0 \rightarrow \mathcal{O}_E \rightarrow \mathcal{A}_n \rightarrow \mathcal{A}_{n-1} \rightarrow 0$  with  $\mathcal{A}_1 = \mathcal{O}_E$ , see [3], [30, Def. 1.12] or [44]. It follows that each such  $\mathcal{E}$  is S-equivalent to a direct sum  $gr(\mathcal{E}) = \bigoplus_j \mathcal{O}_E(y_j - p_0)^{\oplus m_j}$  with pairwise distinct points  $y_j$ .

PROPOSITION 5.1 ([43], [7]). — *Let  $\mathcal{F}$  be a member of  $MS_X(n, 0)$ . Then*

- (i)  $\mathcal{F}$  is  $\Phi$ -WIT<sub>1</sub>.
- (ii) For any  $s \in S$  with  $\mathcal{F}_s$  as in 5.1, the sheaf  $\Phi^1(\mathcal{F}_s)$  is a skyscraper sheaf  $\bigoplus_j \mathcal{C}_j$  with  $\text{Supp}(\mathcal{C}_j) = \{-y_j\}$ , (the point of the dual bundle  $\mathcal{O}_E(-y_j + p_0) \simeq \mathcal{O}_E([-y_j] - p_0)$ ,  $[-y_j]$  denoting the divisor of  $-y_j \in E$ ) and  $\text{length}(\mathcal{C}_j) = m_j$ .

*Proof.* — The first part follows from [7, Prop. 2.7 and Coro. 2.12]. The second part follows by direct computation and the base change property Proposition 4.4. □

Remark 5.2. — The proof shows that the sheaves  $\Phi^1(\mathcal{F}_s)$  and  $\Phi^1(\text{gr}(\mathcal{F}_s))$  are the same.

Remark 5.3. — The condition for  $\mathcal{F}_s$  to be  $\Phi_s$ -WIT<sub>1</sub> is even equivalent for  $\mathcal{F}_s$  to be of degree 0 and semi-stable, see [43].

For a  $\Phi$ -WIT<sub>1</sub>-sheaf  $\mathcal{F}$  on  $X$ , we define the **spectral cover** of  $\mathcal{F}$  as follows.

DEFINITION 5.4. — *Let  $\mathcal{F}$  be a WIT<sub>1</sub> sheaf on  $X$ . The spectral cover  $C(\mathcal{F})$  of  $\mathcal{F}$  is the 0–th Fitting subscheme of  $J$  given by the Fitting ideal sheaf  $\text{Fitt}_0(\Phi^1(\mathcal{F}))$  of  $\Phi^1(\mathcal{F})$ .*

Because we work over a non-algebraic manifold and because the image of  $\Phi$  is not in the derived category of coherent sheaves, but in that of twisted sheaves, we need to prove that the Fitting scheme is well-defined in our case. But this follows from the fact that the Fitting ideals are independent of the finite presentation of the local sheaves  $\mathcal{F}_i$  of an  $\alpha$ -sheaf, see [28, 20.4]. Thus we have well-defined sheaves of ideals  $\text{Fitt}_l(\mathcal{F})$  given locally by the ideal sheaves  $I_{p_i-l}(\mathcal{F}_i)$  of minors of size  $p_i - l$  of the matrix  $F_i$  of a local presentation  $\mathcal{O}_J^{q_i} \xrightarrow{F_i} \mathcal{O}_J^{p_i} \rightarrow \mathcal{F}_i \rightarrow 0$  over the open set  $U_i$ . This sheaf gives us an analytic subspace  $V(\text{Fitt}_l(\mathcal{F}))$  called the  $l$ -th Fitting scheme by abuse of notation in the analytic category.

By Proposition 5.1 (ii), for a single fiber  $X_s$ , we are given a map  $M_{X_s}(n, 0) \rightarrow \text{Sym}^n J_s$  from the moduli space of semistable vector bundles of rank  $n$

and degree 0 to the  $n$ -th symmetric power of the torus  $J_s = \{s\} \times E$ , defined by

$$\mathcal{F}_s \mapsto \Sigma_j m_j(s, -y_j).$$

In this way we obtain a map from  $M_X(n, 0)$  to  $S \times \text{Sym}^n E$ , where  $\text{Sym}^n E := E^n / \mathfrak{S}_n$  is the  $n$ -th symmetric power of  $E$  as the quotient of  $E^n$  by the symmetric group  $\mathfrak{S}_n$ . Then  $S \times \text{Sym}^n E$  is a complex manifold of dimension  $2 + \dim S$  and can be thought of as the relative space of cycles of degree  $n$  in  $E$ . We will show that this map is part of a transformation of functors with target  $\text{Hom}_S(-, S \times \text{Sym}^n E)$  and that  $S \times \text{Sym}^n E$  corepresents the moduli functor  $\mathcal{M}_X(n, 0)$  for  $M_X(n, 0)$  defined as follows.

For any complex space  $T$  over  $S$  let the set  $\mathcal{M}_X(n, 0)(T)$  be defined by

$$\mathcal{M}_X(n, 0)(T) := \mathcal{MS}_X(n, 0)(T) / \sim,$$

where  $\mathcal{MS}_X(n, 0)(T)$  is the set of vector bundles on  $X_T$  of rank  $n$  and fibre degree 0, and where the equivalence relation  $\mathcal{F} \sim \mathcal{G}$  is defined by  $S$ -equivalence of the restricted sheaves  $\mathcal{F}_t$  and  $\mathcal{G}_t$  on the fibres  $X_{T_t}$ . The functor property is then defined via pull backs.

We are going to describe the spectral cover as a functor below. For that let  $T \rightarrow S$  be a complex space over  $S$  and let  $\Phi_T$  be the Fourier-Mukai transform for the product  $J_T \times X_T$  with the pull back  $\mathcal{U}_T$  of  $\mathcal{U}$  as kernel. By [7, Prop. 2.7 and Coro. 2.12], any bundle  $\mathcal{F}_T$  in  $\mathcal{MS}_X(n, 0)(T)$  is also  $\Phi_T - WIT_1$  and admits a spectral cover  $C(\mathcal{F}_T) \subset T \times E$  defined by the Fitting ideal  $\text{Fitt}_0 \Phi_T^1(\mathcal{F}_T)$ .

LEMMA 5.5. — *If  $T$  is reduced, then  $C(\mathcal{F}_T)$  is flat over  $T$ .*

*Proof.* — The fibres of  $C(\mathcal{F}_T)$  are finite of constant length  $n$  as in the case of  $S$  above. Because the projection to  $T$  is surjective, flatness follows from Douady’s criterion in [27]. □

LEMMA 5.6. — *The spectral cover is compatible with base change: For any morphism*

$$h : T' \rightarrow T \text{ over } S \text{ and any bundle } \mathcal{F}_T \text{ in } \mathcal{MS}_X(n, 0)(T),$$

$$h^*C(\mathcal{F}_T) \simeq C(h^*\mathcal{F}_T).$$

*Proof.* — Because the fibres of the morphisms  $p_J : J_T \times X_T \rightarrow J_T$  are 1-dimensional and the sheaves  $\mathcal{F}_T$  are locally free, base change holds for  $R^1 p_{J*}$ , see [7, Prop. 2.6] When the induced map  $J_{T'} \rightarrow J_T$  is denoted by  $h_J$ , then

$$h_J^* \Phi_T^1(\mathcal{F}_T) \simeq \Phi_{T'}^1(h^*\mathcal{F}_T)$$

for any  $\mathcal{F}_T \in \mathcal{MS}_X(n, 0)(T)$ . Since the Fitting ideals are also compatible with base change, the claim follows. □

The spectral covers  $C(\mathcal{F}_T)$  lead us to consider the relative Douady functors

$$\mathcal{D}^n : (An/S)^{op} \rightarrow (Sets),$$

where  $(An/S)$  denotes the category of complex analytic spaces over  $S$  and where a set  $\mathcal{D}^n(T)$  for a morphism  $T \rightarrow S$  is defined as the set of analytic subspaces  $Z \subset T \times E$  which are flat over  $T$  and have 0-dimensional fibres of constant length  $n$ . The Douady functor  $\mathcal{D}^n$  is represented by a complex space  $D^n(S \times E/S)$  over  $S$ , see [42]. For a point  $s \in S$ ,  $\mathcal{D}^n(\{s\})$  is the set of 0-dimensional subspaces of length  $n$  and can be identified with the symmetric product  $\text{Sym}^n(E)$  because it is well known that the Hilbert-Chow morphism, in our case the Douady-Barlet morphism,  $\mathcal{D}^n(\{s\}) \rightarrow \{s\} \times \text{Sym}^n(E)$  is an isomorphism for the smooth curve  $E$ , see [4, Ch. V]. It is then easy to show that also the relative Douady-Barlet morphism  $D^n(S \times E/S) \rightarrow S \times \text{Sym}^n(E)$  is an isomorphism. This implies that for any complex space  $T$  over  $S$  there is bijection

$$(5.2) \quad \mathcal{D}^n(T) \xrightarrow{\sim} \text{Hom}_S(T, S \times \text{Sym}^n(E)).$$

One should note here that the behavior of families of cycles is more difficult to describe than of those for the Douady space.

Let now  $\mathcal{D}_r^n$  respectively  $\mathcal{M}_X(n, 0)_r$  be the restriction of the functors  $\mathcal{D}^n$  and  $\mathcal{M}_X(n, 0)$  to the category  $(Anr/S)$  of reduced complex analytic spaces. By the Lemmas 5.5 and 5.6 the spectral covers give rise to a transformation of functors

$$(5.3) \quad \mathcal{M}_X(n, 0)_r \xrightarrow{\gamma} \mathcal{D}_r^n \simeq \text{Hom}_S(-, S \times \text{Sym}^n(E)),$$

where for a reduced space  $T$  over  $S$  and for a class  $[\mathcal{F}_T]$  in  $\mathcal{M}_X(n, 0)(T)$  we have  $\gamma(T)(\mathcal{F}_T) = C(\mathcal{F}_T)$ . Note that by flatness  $C(\mathcal{F}_T)$  depends only on the equivalence class of  $\mathcal{F}_T$ . We are now able to present the main theorem which generalises Theorem 2.4.

**THEOREM 5.7.** — *Let  $X \rightarrow S$  be an elliptic principal bundle over a compact complex manifold  $S$  of arbitrary dimension with invariant  $\delta \neq 0$ . Then the spectral cover induces a transformation of functors  $\gamma : \mathcal{M}_X(n, 0)_r \rightarrow \text{Hom}_S(-, S \times \text{Sym}^n(E))$  with the following properties:*

- (i) *The functor  $\mathcal{M}_X(n, 0)_r$  is corepresented by  $S \times \text{Sym}^n(E)$  via the transformation  $\gamma$ .*
- (ii) *For any point  $s \in S$  the induced map  $\mathcal{M}_{X_s}(n, 0) \rightarrow \text{Sym}^n(E)$  is bijective.*
- (iii) *The map  $\gamma(T)$  is injective for any reduced complex space  $T$  over  $S$ .*

(iv)  $\mathcal{M}_X(n, 0)_r$  is locally representable by  $S \times \text{Sym}^n(E)$ , i.e., if  $U \subset S$  is a trivializing open subset for  $X$  and  $T$  is a complex space over  $U$ , then  $\gamma(T)$  is bijective.

*Proof.* — Property (ii) is clear by the construction of the functors.

We begin proving the injectivity in (iii). Let  $[\mathcal{F}_1], [\mathcal{F}_2] \in \mathcal{M}_X(n, 0)(T)$  such that  $\gamma(T)([\mathcal{F}_1]) = \gamma(T)([\mathcal{F}_2])$ . This implies that for every  $t \in T$  the spectral covers of  $(\mathcal{F}_1)_t$  and  $(\mathcal{F}_2)_t$  are the same. If  $(\mathcal{F}_1)_t$  is S-equivalent to  $\oplus_j \mathcal{O}_E(y_j - p_0)^{\oplus m_j}$ , then  $C((\mathcal{F}_1)_t) = \Sigma_j m_j(-y_j)$  and vice versa by Atiyah’s classification. Hence  $(\mathcal{F}_1)_t$  and  $(\mathcal{F}_2)_t$  are S-equivalent. But this is precisely the equivalence relation for the classes  $[\mathcal{F}_1]$  and  $[\mathcal{F}_2]$ .

To prove (iv), let  $U \subset S$  be an open subset over which  $X$  is trivial and let  $T \rightarrow U$  be a reduced complex space over  $U$ . Then we can assume that  $X_T = T \times E$ . First we define a map

$$\text{Hom}_S(T, S \times E^n) \xrightarrow{b(T)} \mathcal{M}_X(n, 0)(T)$$

as follows. Given a morphism  $(p, f) : T \rightarrow U \times E^n$  over  $S$ , let  $f_\nu : T \rightarrow E$  be the  $\nu$ -th component of  $f$ . Let then

$$\mathcal{L}_\nu := (f_\nu \times id)^* \mathcal{O}_{E \times E}(-\Delta) \otimes p_2^* \mathcal{O}_E(p_0)$$

on  $T \times E$  be the pull back of the dual Poincaré bundle. Then the spectral cover of  $\mathcal{L}_{\nu,t}$  for any point  $t \in T$  consists of the point  $f_\nu(t) \in E$ . The map  $b(T)$  can now be defined by  $(p, f) \mapsto [\mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_n]$ . This map is obviously  $\mathfrak{S}_n$ -equivariant and thus can be factorized through  $\text{Hom}_S(T, S \times \text{Sym}^n(E))$ , giving a map

$$\text{Hom}_S(T, S \times \text{Sym}^n(E)) \xrightarrow{\beta(T)} \mathcal{M}_X(n, 0)(T).$$

By construction,  $\beta(T)$  is an inverse of  $\gamma(T)$ .

The proof of (i) is now analogous to that of (i) for Theorem 2.4, using (iv). □

### 6. Invariants of torus bundles

Let  $M$  be an  $n$ -dimensional compact complex manifold,  $T = V/\Lambda$  an  $m$ -dimensional complex torus and  $X \xrightarrow{\pi} M$  a principal bundle with fiber  $T$ . The theory of principal torus bundles is developed in great detail in [34]; see also [17]. It is well known that such bundles are described by elements of  $H^1(M, \mathcal{O}_M(T))$ , where  $\mathcal{O}_M(T)$  denotes the sheaf of local holomorphic maps from  $M$  to  $T$ . Considering the exact sequence of groups

$$0 \rightarrow \Lambda \rightarrow V \rightarrow T \rightarrow 0$$



and taking local sections we obtain the following exact sequence

$$0 \rightarrow \Lambda \rightarrow \mathcal{O}_M \otimes V \rightarrow \mathcal{O}_M(T) \rightarrow 0.$$

Passing to the cohomology we have the long exact sequence

$$\begin{aligned} \dots \rightarrow H^1(M, \Lambda) \rightarrow H_M^{0,1} \otimes V \rightarrow H^1(M, \mathcal{O}_M(T)) \xrightarrow{c^{\mathbb{Z}}} \\ \xrightarrow{c^{\mathbb{Z}}} H^2(M, \Lambda) \rightarrow H_M^{0,2} \otimes V \rightarrow \dots \end{aligned}$$

By taking the image of the co-cycle defining the bundle via the map  $c^{\mathbb{Z}}$  we obtain a characteristic class  $c^{\mathbb{Z}}(X) \in H^2(M, \Lambda) = H^2(M, \mathbb{Z}) \otimes \Lambda$  and also a characteristic class  $c(X) \in H^2(M, \mathbb{C}) \otimes V$ .

Concerning some important sheaves on  $X$  we have (see [34]):

$$(6.1) \quad \mathcal{K}_X = \pi^* \mathcal{K}_M, \quad R^i \pi_* \mathcal{O}_X = \mathcal{O}_M \otimes_{\mathbb{C}} H^{0,i}(T)$$

and the exact sequence

$$(6.2) \quad 0 \rightarrow \Omega_M^1 \rightarrow \pi_* \Omega_X^1 \rightarrow \mathcal{O}_M \otimes_{\mathbb{C}} H^{1,0}(T) \rightarrow 0.$$

All the informations concerning the topology of the bundle  $X \rightarrow M$  are given by the following invariants:

- a) The exact sequence (6.2) gives rise to an element  $\gamma \in \text{Ext}^1(\mathcal{O}_M \otimes H^{1,0}(T), \Omega_M^1) = H^1(\Omega_M^1) \otimes H^{1,0}(T)^*$ . Thus  $\gamma$  is a map  $H^{1,0}(T) \rightarrow H^{1,1}(M)$ .
- b) The first non-trivial  $d_2$ -differential in the Leray spectral sequence  $(d_2 : E_2^{0,1} \rightarrow E_2^{2,0})$  of the sheaf  $\mathbb{C}_X$ . We obtain in this way a map  $\delta : H^1(T, \mathbb{C}) \rightarrow H^2(M, \mathbb{C})$ . In the same way we may define the maps  $\delta^{\mathbb{Z}} : H^1(T, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z})$ .
- c) The first non-trivial  $d_2$ -differential in the Leray spectral sequence of  $\mathcal{O}_X$ , where  $d_2 : H^0(R^1 \pi_* \mathcal{O}_X) \rightarrow H^2(\pi_* \mathcal{O}_X)$ . Via the identifications (6.1) we get a map  $\epsilon : H^{0,1}(T) \rightarrow H^{0,2}(M)$ .
- d) The characteristic classes  $c^{\mathbb{Z}}(X)$  and  $c(X)$  defined above.

These invariants are related by the following theorem of Höfer:

**THEOREM 6.1.** — *Let  $X \xrightarrow{\pi} M$  be a holomorphic principal  $T$ -bundle. Then:*

- (1) *The Borel spectral sequence ([33, Appendix Two by A. Borel])  ${}_{p,q} E_2^{s,t} = \sum H^{i,s-i}(M) \otimes H^{p-i,t-p+i}(T)$  degenerates on  $E_3$ -level and the  $d_2$ -differential is given by  $\epsilon$  and  $\gamma$ .*
- (2) *The Leray spectral sequence  $E_2^{s,t} = H^s(M, \mathbb{C}) \otimes H^t(T, \mathbb{C})$  degenerates on  $E_3$ -level and the  $d_2$ -differential is given by  $\delta$ .*
- (3) *Via the identification  $H^1(T, \mathbb{Z}) = \text{Hom}(\Lambda, \mathbb{Z})$  the characteristic class  $c^{\mathbb{Z}}$  and the map  $\delta^{\mathbb{Z}}$  coincide.*

- (4)  $\delta$  is determined by  $\delta^{\mathbb{Z}}$  via scalar extension.
- (5) If  $H^2(M)$  has Hodge decomposition then  $\delta$  determines  $\epsilon$  and  $\gamma$  and conversely.

In order to compute the Dolbeault cohomology of  $X$  we need to use the Borel spectral sequence because the direct-image sheaves  $R^j \pi_* \Omega_X^p$  are non-trivial for  $p > 0$  and the Leray spectral sequence is more difficult to use.

### 7. Invariants of elliptic principal bundles over surfaces

In what follows we shall consider fiber bundles with basis  $M$  a smooth complex surface and fiber  $T$  an elliptic curve. In this case a more detailed description is possible. Let  $(1, \tau)$  be a basis of  $\Lambda$ , let  $(dt, d\bar{t})$  be a basis of  $H^1(T, \mathbb{C})$  given by the decomposition  $H^1(T, \mathbb{C}) = H^{1,0}(T) \oplus H^{0,1}(T)$  (another basis on  $H^1(T, \mathbb{C})$  is given by the canonical coordinates  $(dx_1, dx_2)$ ). Assume that  $c^{\mathbb{Z}} = a \otimes 1 + b \otimes \tau$ , then  $c = (a + b \cdot \tau) \otimes 1 = \eta \otimes 1$ , with  $\eta^{02} = 0$ . Then we have

$$\begin{aligned} \delta : dt &\mapsto a + \tau \cdot b = \eta \\ d\bar{t} &\mapsto a + \bar{\tau} \cdot b = \bar{\eta} \\ \epsilon : d\bar{t} &\mapsto (a + \bar{\tau} \cdot b)^{02} = \bar{\eta}^{02} \\ \gamma : dt &\mapsto (a + \tau \cdot b)^{11} = \eta^{11}. \end{aligned}$$

The only non-zero terms in the Borel spectral sequence are (7.1)

$$\begin{array}{ccc} & H^{p-1, q-2}(M) \otimes H^{1,1}(T) & \\ \swarrow \gamma & & \searrow -\epsilon \\ H^{p, q-1}(M) \otimes H^{0,1}(T) & \oplus & H^{p-1, q}(M) \otimes H^{1,0}(T) \\ \searrow \epsilon & & \swarrow \gamma \\ & H^{p, q+1}(M) \otimes H^{0,0}(T). & \end{array}$$

From now on we shall be concerned with the case when  $M$  has trivial canonical bundle. By (6.1) this implies that  $X$  also has trivial canonical bundle. The case when  $M$  is Kähler was considered by Höfer. This leaves us with the case when  $M$  is a primary Kodaira surface. We shall use the preceding diagram to compute the Hodge numbers for  $X$  in this case. Recall

that the Hodge diamond for a primary Kodaira together with Betti numbers is, see [5, V.5.]:

$$\begin{array}{cccc}
 & & 1 & & 1 \\
 & & 2 & 1 & 3 \\
 & 1 & 2 & 1 & 4 \\
 & & 1 & 2 & 3 \\
 & & & 1 & 1.
 \end{array}$$

Taking into account the Hodge numbers from above and the fact that all the Dolbeault groups of the elliptic curve that appear in (7.1) are 1-dimensional we obtain the following Hodge diamond for  $X$

$$\begin{array}{cccccc}
 & & & 1 & & \\
 & & & 2 - e & & 3 - g \\
 & & 3 - e & 6 - g - h & 3 - g & 2 - g \\
 (7.2) & 1 & 5 - g - h & 5 - g - h & 1 & \\
 & & 2 - g & 6 - g - h & & 3 - e \\
 & & & 2 - g & & 1 - e \\
 & & & & 1, & 
 \end{array}$$

where  $e = \text{Rank}(\epsilon); g = \text{Rank}(\gamma)$  and  $h$  is the rank of the map given by the multiplication with  $\gamma(dt) - \epsilon(d\bar{t})$ .

The first Betti number is given by  $b_1(X) = b_1(M) + \dim \text{Ker}(\delta) = 3 + 2 - d = 5 - d$ , where  $d = \text{Rank}(\delta)$ . To compute the Betti number  $b_2(X)$  we shall use the Leray spectral sequence for the constant sheaf  $\mathbb{C}_X$ . We have  $E_2^{p,q} = H^p(M, R^q\pi_*\mathbb{C}_X) = H^p(M, \mathbb{C}) \otimes H^q(T, \mathbb{C})$ , the  $d_2$ -differential is determined by  $\delta : E_2^{0,1} = H^1(T, \mathbb{C}) \rightarrow E_2^{2,0} = H^2(M, \mathbb{C})$  and the sequence degenerates at the  $E_3$  level. In this case  $E_\infty^{0,2} = E_3^{0,2} = \text{Ker}(E_2^{0,2} \rightarrow E_2^{2,1}) = \text{Ker}(H^2(T, \mathbb{C}) \rightarrow H^2(M, \mathbb{C}) \otimes H^1(T, \mathbb{C}))$ , and so  $E_3^{0,2} \simeq 0$  (we assumed that  $\delta \neq 0$ ). Moreover,  $E_3^{1,1} = \text{Ker}(E_2^{1,1} \rightarrow E_2^{3,0}) = \text{Ker}(H^1(M, \mathbb{C}) \otimes H^1(T, \mathbb{C}) \rightarrow H^3(M, \mathbb{C}))$ . It follows that  $\dim(E_3^{1,1}) = 6 - d'$ , where  $d'$  is the rank of the map obtained by composing  $\delta$  and the cup-product. Similarly,  $\dim(E_3^{2,0}) = 4 - d$ . We have the filtration  $0 \subset F_2 \subset F_1 \subset F_0 = H^2(X, \mathbb{C})$  associated with the spectral sequence (that is  $F_2 \simeq E_\infty^{2,0}, F_1/F_2 \simeq E_\infty^{1,1}$  and  $F_0/F_1 \simeq E_\infty^{0,2} = 0$ ). So we obtain an exact sequence  $0 \rightarrow F_2 \rightarrow F_1 \rightarrow F_1/F_2 \rightarrow 0$ . From the above computations it follows that  $b_2(X) = \dim(E_\infty^{2,0}) + \dim(E_\infty^{1,1}) = 10 - d - d'$ .

For  $b_3(X)$  we remark that in the Leray filtration  $0 \subset F_3 \subset F_2 \subset F_1 \subset F_0 = H^3(X, \mathbb{C})$  we have  $F_1 = F_0$ . This makes the things easier and by a 2-step computation we obtain that  $b_3(X) = 12 - 2d'$  so we can complete



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