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STABLE COMMUTATOR LENGTH ON MAPPING CLASS GROUPS

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Abstract. — Let G be a finite index subgroup of the mapping class group $MCG(\Sigma)$ of a closed orientable surface Σ, possibly with punctures. We give a precise condition (in terms of the Nielsen-Thurston decomposition) when an element $g \in G$ has positive stable commutator length. In addition, we show that in these situations the stable commutator length, if nonzero, is uniformly bounded away from 0. The method works for certain subgroups of infinite index as well and we show *scl* is uniformly positive on the nontrivial elements of the Torelli group. The proofs use our previous construction of group actions on quasi-trees.

Résumé. — Soit G un sous-groupe d'indice fini du groupe modulaire $MCG(\Sigma)$ d'une surface fermée orientable, possiblement épointée. Nous donnons une condition précise (en termes de la décomposition de Nielsen-Thurston) pour qu'un élément $g \in G$ ait une longueur stable des commutateurs strictement positive. Nous montrons de plus que dans ces situations, la longueur stable des commutateurs est soit nulle, soit uniformément minorée par un réel strictement positif. Notre méthode permet aussi de traiter le cas de certains sous-groupes d'indice infini, et nous montrons l'existence d'un minorant strictement positif pour la longueur stable des commutateurs des éléments non triviaux du groupe de Torelli. Les démonstrations utilisent notre prééédente construction d'actions de groupes sur des quasi-arbres.

1. Introduction and statement of results

Let G be a group, and $[G, G]$ its commutator subgroup. For an element $g \in [G, G]$, let $cl(g) = cl_G(g)$ denote the *commutator length* of g, the least

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number of commutators whose product is equal to g. We define $cl(g) = \infty$ for an element g not in $[G, G]$. For $g \in G$, the *stable commutator length*, $\mathrm{scl}(g) = \mathrm{scl}_G(g)$, is defined by

$$\mathrm{scl}(g) = \liminf_{n \to \infty} \frac{cl(g^n)}{n} \leqslant \infty.$$

See for example [15] for a thorough account on scl.

A function $H : G \to \mathbb{R}$ is a *quasi-morphism* if

$$\Delta(H) := \sup_{x,y \in G} |H(xy) - H(x) - H(y)| < \infty.$$

$\Delta(H)$ is called the *defect* of H. Recall that a quasi-morphism $H : G \to \mathbb{R}$ is *homogeneous* if $H(g^m) = mH(g)$ for all $g \in G$ and $m \in \mathbb{Z}$.

A theme in the subject is to classify elements g in a given group for which $scl(g) > 0$. Note that in the following situations $cl(g^n)$ is bounded and therefore $scl(g) = 0$:

(a) g has finite order,

(b) g is *achiral*, i.e. g^k is conjugate to g^{-k} for some $k \neq 0$,[(1)]

(c) (Endo-Kotschick [22]) $g = g_1 g_2 = g_2 g_1$ and g_1 is conjugate to g_2^{-1},

(d) more generally, g is expressed as a commuting product

$$g = g_1 \cdots g_p$$

and $g_i^{n_i}$ are all conjugate for some $n_i \neq 0$, and

$$\sum_i \frac{1}{n_i} = 0,$$

(e) $g = g_1 \cdots g_p$ is a commuting product and $cl(g_i^n) = 0$ are bounded for all i.

Roughly speaking, our main theorem states that for mapping classes g we have $scl(g) > 0$ unless writing an arotational power of g as a commuting product as in the Nielsen-Thurston decomposition implies $scl(g) = 0$ by the above properties.

Brooks [13] showed that in free groups $scl(g) > 0$ for every nontrivial element g. Then Epstein-Fujiwara [23], generalizing the Brooks construction, proved that in hyperbolic groups the above obstructions (a) and (b) are the only ones, namely, if g has infinite order and g is *chiral* (i.e. not achiral) then $scl(g) > 0$.

For $G = MCG(\Sigma)$ questions due to Geoff Mess about commutator length of powers of Dehn twists appear on Kirby's list [27]. Using 4-manifold

(1) Of course, $scl(g) = 0$ if g^k is conjugate to g^l for some $k \neq l$, but in mapping class groups this is possible only when $k = \pm l$ or g has finite order.

invariants, Endo-Kotschick [21] and Korkmaz [28] prove that $scl(g) > 0$ if g is a Dehn twist and Baykur [2] gave an argument based on the Milnor-Wood inequality that Dehn twist in the boundary curve has $scl = \frac{1}{2}$. Endo-Kotschick [22] also note that in $MCG(\Sigma)$ there are additional obstructions to $scl > 0$: if g, h commute and h is conjugate to g then $scl(gh^{-1}) = 0$; for example this occurs if g, h are Dehn twists in disjoint curves in the same orbit. By contrast, Calegari-Fujiwara [16] prove that if g is pseudo-Anosov and chiral then $scl(g) > 0$. The argument is based on the action of $MCG(\Sigma)$ on the curve graph, which is hyperbolic by [32], with pseudo-Anosov classes acting as hyperbolic isometries.

In this paper, for a subgroup $G < MCG(\Sigma)$ of finite index of the mapping class group of a closed orientable surface Σ (possibly with punctures) we characterize those elements $g \in G$ for which $scl(g) > 0$, or equivalently there exists a quasi-morphism $G \to \mathbb{R}$ which is unbounded on the powers of g. The new feature is that we consider actions of a certain subgroup $\mathcal{S} < MCG(\Sigma)$ of finite index on hyperbolic spaces constructed in [5]. The subgroup $\mathcal{S}$ is called *the color preserving subgroup*. In this way, for *any* nontrivial element of $\mathcal{S}$ there is an action where this element is a hyperbolic isometry.

The novelty of the paper is that it gives the first example of a group for which we exactly know which element g has $scl(g) > 0$ and the answer is interesting and complicated. For example, although it was known that $scl > 0$ for all Dehn-twists, but among the multitwists, it turns out that $scl > 0$ only for certain elements. The result is in sharp contract to the case where $scl = 0$ for higher rank lattices, [14], and the case where $scl > 0$ for all non-trivial elements for torsion-free word-hyperbolic groups, [16]. As we said a few important but special cases have been handled by various methods, and it was not easy to predict the general result for elements in MCG, but our result and method covers all cases in a unified way.

Here is our main result. The definition of *chiral essential classes* will be given later, but we describe them informally below.

Main Theorem (Theorem 4.2). — *Let $G < MCG(\Sigma)$ be a subgroup of finite index and $g \in G$. Then $scl(g) > 0$ if and only if some chiral equivalence class of pure components of g is essential.*

Since $scl(g) > 0$ if and only if $scl(g^k) > 0$ for $k \neq 0$ we are free to pass to powers, and we may assume that g has the Nielsen-Thurston decomposition in which there are no rotations, so g is expressed as a commuting product of Dehn twists and pseudo-Anosov pure components. Now consider the chirality of the pure components, e.g. every Dehn twist is chiral. Two chiral components of g are *equivalent* if they have nontrivial powers that are conjugate. A chiral equivalence class is *essential* if after conjugating all to the same supporting subsurface with the same pair of (un)stable foliations, the product has infinite order.

We now state some corollaries and extensions of the Main Theorem.

COROLLARY. — *Let G be a subgroup of $MCG(\Sigma)$ of finite index.*

- *There is $\epsilon = \epsilon(G) > 0$ so that if $g \in G$ with $scl_G(g) > 0$ then $scl_G(g) > \epsilon$ (Proposition 4.6).*
- *If $scl_G(g) = 0$ then the sequence $cl_G(g^n)$ is bounded (Proposition 4.8).*
- *There is a specific finite index subgroup $\mathcal{S} < MCG(\Sigma)$ so that $scl_{\mathcal{S}}(g) > 0$ for every nontrivial $g \in \mathcal{S}$ (the remark after Theorem 4.2).*
- *When $G = Ker[MCG(\Sigma) \to GL(H_1(\Sigma); \mathbb{Z}_3)]$, every exponentially growing element of G has $scl_G > 0$ (Corollary 5.3).*
- *If $G = \mathcal{T}$ is the Torelli subgroup and $g \neq 1 \in G$ then $scl_G(g) > 0$ (Theorem 5.6).*

Convention. — All our constants will depend linearly on the previous constants, so for example when we say “there is $C = C(\delta, \xi)$” we mean that C is bounded by a fixed multiple of $\delta + \xi + 1$. All (L, A)-quasi-geodesics and (L, A)-quasi-isometries will have L uniformly bounded (e.g. by 2 or 4) and A will depend linearly on the previous constants.

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2. Review

Here we review some background.

2.1. scl

The following facts will be used (see for example [15]).

Proposition 2.1. — *Let $H : G \to \mathbb{R}$ be a quasi-morphism and $\Delta(H)$ its defect.*

(i) *every quasi-morphism $H : G \to \mathbb{R}$ differs from a unique homogeneous quasi-morphism $\hat{H}$ by a bounded function; in fact*

$$\hat{H}(g) = \lim_{n\to\infty} \frac{H(g^n)}{n}$$

(ii) *Suppose $g \in [G, G]$. Then $\mathrm{scl}(g) \geqslant \frac{|\hat{H}(g)|}{4\Delta(H)}$, and if H is homogeneous then $\mathrm{scl}(g) \geqslant \frac{|H(g)|}{2\Delta(H)}$.*

Also note that any homogeneous quasi-morphism is constant on conjugacy classes and it is a homomorphism when restricted to an abelian subgroup.

Remark 2.2. — The Bavard duality asserts that for a given g with $0 < \mathrm{scl}(g)(< \infty)$, second inequality in (ii) is actually an equality for a suitable homogeneous H with $0 < \Delta(H)$. See [1] and [15].

2.2. Quasi-trees and the bottleneck criterion

All graphs will be connected and endowed with the path metric in which edge lengths are 1. A *quasi-tree* is a graph quasi-isometric to a tree. It is a theorem of Manning [31] that a graph Q is a quasi-tree if and only if it satisfies the *bottleneck property*: there is a number $\Delta \geqslant 0$ (referred to as a *bottleneck constant*) such that any two vertices $x, y \in Q$ are connected by a path α with the property that any other path connecting x to y necessarily contains α in its Δ-neighborhood. A quasi-tree is δ-hyperbolic with δ depending linearly only on Δ. i.e. it is bounded by a fixed multiple of $\Delta + 1$

Theorem 2.3 ([31]). — *Let Q be a graph satisfying the bottleneck property with constant Δ. Then there is a tree T and a $(4, A)$-quasi-isometry $\alpha : Q \to T$ with A depending linearly on Δ.*

Proof. — This is not exactly the way it is stated in [31], so we give an outline. We will assume that $\Delta \geqslant 1$ is an integer; in general we replace it by the next larger integer and this doesn't change the conclusion. The idea is to fix $R = 20\Delta$ and a base vertex $* \in Q$, and then define the vertices of T as the path components of $B(*, R), B(*, 2R) - B(*, R), B(*, 3R) - B(*, 2R), \cdots$. Manning produces an explicit quasi-isometry $\beta : T \to Q$ satisfying

$$8\Delta d(x, y) - 16\Delta \leqslant d(\beta(x), \beta(y)) \leqslant 26\Delta d(x, y)$$

and which is 20Δ-almost onto. If the metric on T is rescaled by the factor 8Δ, β becomes a $(4, 16\Delta)$-quasi-isometric embedding and the standard inverse is a $(4, A)$-quasi-isometry for A a linear function of Δ. □

In particular, if $[a, b]$ is a $(2, 10\delta + 10)$-quasi-geodesic in Q, the image in T is contained in the ϵ-neighborhood of the segment spanned by the images of a and b, and ϵ is a linear function of Δ.

2.3. Quasi-axes

Unfortunately, in general, hyperbolic isometries do not have axes. The notion of quasi-axes (with uniform constants) will serve as a surrogate. Assume $g : X \to X$ is a hyperbolic isometry of a δ-hyperbolic graph. Let $x_0 \in X$ be a vertex with $D = d(x_0, g(x_0))$ minimal possible. This implies that $d(x_0, g^2(x_0)) \geqslant 2D - 4\delta - 4$ for otherwise a vertex near the middle of $[x_0, g(x_0)]$ would be displaced less than D. In other words, the piecewise geodesic $[x_0, g(x_0)] \cup [g(x_0), g^2(x_0)]$ is a $(1, 4\delta + 4)$-quasi-geodesic of length $2D$. Now recall that local quasi-geodesics are global quasi-geodesics: a $(100\delta+100)$-local $(1, 4\delta+4)$-quasi-geodesic is a global $(2, 10\delta+10)$-quasi-geodesic (see e.g. [12, Theorem III.1.13] and [19, Theorem 4.1](2)). We can arrange that $\cup_i[g^i(x_0), g^{i+1}(x_0)]$ is g-invariant, and thus if $D \geqslant 100\delta + 100$ it is a $(2, 10\delta + 10)$-quasi-geodesic. We refer to it as a *quasi-axis* of g; Any two quasi-axes of g are in uniformly bounded Hausdorff neighborhoods of each other, and the bound is a linear function of δ. A *virtual quasi-axis* of g is a quasi-axis of a power of g. Thus every hyperbolic isometry has a virtual quasi-axis. In the situations we consider in this paper the power of g can be taken to be uniformly bounded.

It is convenient to introduce the following notation: for hyperbolic isometries g, h of a δ-hyperbolic space X we write

$$\Pi_g(h) = \Pi_g^X(h) \leqslant \eta$$

if the (nearest point) projection of any virtual quasi-axis of h to any virtual quasi-axis of g has diameter $\leqslant \eta$. Note that for any two pairs of choices of virtual quasi-axes the diameters of projections differ by a number bounded by a linear function of δ.

We will also write

$$\tilde\Pi_g(h) = \tilde\Pi_g^X(h) \leqslant \eta$$

if $\Pi_g(h') \leqslant \eta$ for every conjugate h' of h.

(2) [12] proves this for local geodesics and [19] proves that local (L, A)-quasi-geodesics are (L', A')-quasi-geodesics; the arguments prove our assertion.

2.4. WWPD

Throughout the paper we will be concerned with the following setting:

- G is a group acting on a δ-hyperbolic graph X,
- $g \in G$ is a hyperbolic element,
- $C = C(g) < G$ is a subgroup that fixes the points $g^{\pm\infty}$ at infinity fixed by g; equivalently, for every virtual quasi-axis ℓ the orbit $C\ell$ is contained in a Hausdorff neighborhood of ℓ and no element of C flips the ends, and
- there is $\xi = \xi_g > 0$ such that for every $\gamma \in G - C$ we have $\Pi_g(g^\gamma) \leqslant \xi$.

When these properties hold we will say that (G, X, g, C) satisfy $WWPD$. This is a weakening of WPD [7] which requires in addition that C be virtually cyclic.(3) As we shall see below, every mapping class group has a torsion-free subgroup $\mathcal{S}$ of finite index such that *every* nontrivial element g admits a WWPD action. For example, when g is a pseudo-Anosov mapping class, we may take the curve graph for X (and then the action is WPD), but for a general element we will use the construction in [5] and obtain only WWPD actions.

It is tempting to think that in this situation there is always a homomorphism $C \to \mathbb{R}$ given by the (signed) translation length and then modify X so that C acts by translations on a fixed line (axis of g). However, in general this is not possible since $C \to \mathbb{R}$ is only a quasi-morphism. This leads to the perhaps surprising fact that (many) mapping class groups can act by isometries on a δ-hyperbolic graph with a Dehn twist acting hyperbolically, while in general this is impossible on complete $CAT(0)$ spaces, as observed by Martin Bridson [11].

2.5. Mapping class group and the curve graph

Let Σ be a closed orientable surface, possibly with punctures. The mapping class group $MCG(\Sigma)$ of Σ is the group of orientation preserving homeomorphisms preserving the set of punctures, modulo isotopy rel punctures. The curve graph $\mathcal{C}(\Sigma)$ has a vertex for every isotopy class of essential simple closed curves in Σ, and an edge corresponding to pairs of simple closed curves that intersect minimally.

(3) Strictly speaking WPD allows flips; one could talk about $WWPD^\pm$ vs. $WWPD$ but we will keep it simple.

It is a fundamental theorem of Masur and Minsky [32] that the curve graph is hyperbolic. Moreover, they show that an element g acts hyperbolically if and only if g is pseudo-Anosov, and that the translation length

$$\tau_g = \lim \frac{d(x_0, g^n(x_0))}{n}$$

of g is uniformly bounded below by a positive constant that depends only on Σ.

Recall also that to an annulus A one associates the "curve graph" quasi-isometric to a line, whose vertices are represented by isotopy classes of spanning arcs and edges by disjointness.

It was proved in [7] that the action of $MCG(\Sigma)$ on the curve graph $\mathcal{C}(\Sigma)$ satisfies WPD. This means that every hyperbolic element (i.e. a pseudo-Anosov class) g is a WPD element, that is, for every g there exists ξ_g such that $\Pi_g(g') \leqslant \xi_g$ for every conjugate g' of g whose virtual quasi-axes aren't parallel to those of g and the stabilizer of $g^{\pm\infty}$ is virtually cyclic. Bowditch [9] improved this result and showed that the action of $MCG(\Sigma)$ on $\mathcal{C}(\Sigma)$ is *acylindrical*. Denoting by δ the hyperbolicity constant of $\mathcal{C}(\Sigma)$ this means that if x, y are two vertices sufficiently far apart, say distance at least M, then the set

$$\{h \in MCG(\Sigma) \mid d(x, h(x)) \leqslant 10\delta, d(y, h(y)) \leqslant 10\delta\}$$

is finite and has cardinality bounded by some $N = N(\Sigma) < \infty$. This allows us to estimate ξ_g as follows.

Lemma 2.4. — *There are constants A, B that depend only on the surface Σ so that we may take $\xi_g = A + B\tau_g$, where τ_g is the translation length of g. More generally, if f is another hyperbolic element with $\tau_f \leqslant \tau_g$ then $\Pi_g(f) \leqslant \xi_g$ or else f and g have parallel virtual quasi-axes.*

Proof. — First note that after replacing g with a bounded power we may assume that g has a quasi-axis; this is because the translation length is bounded below. We will prove the lemma assuming g has an axis ℓ; the case of a quasi-axis requires straightforward changes. If for $h \in G$ the projection of $h(\ell)$ to ℓ has diameter $D > 10\delta + 10$, then there are segments $I \subset \ell$ and $J \subset h(\ell)$ of length $D - 4\delta$ that are in each other's 2δ-neighborhood. Denote by $\phi = hgh^{-1}$, the conjugate of g with axis $h(\ell)$. We will assume that I and J are oriented in the same direction; otherwise replace ϕ by ϕ^{-1}. If $D > N \cdot \tau(g) + M + 10\delta + 10$, the elements $1, g\phi^{-1}, g^2\phi^{-2}, \cdots, g^N\phi^{-N}$ move each point of a segment of length M a distance $\leqslant 10\delta$, so from acylindricity we deduce $g^i\phi^{-i} = g^j\phi^{-j}$ for some $i < j$, i.e. $g^{i-j} = \phi^{i-j}$, so in particular ℓ and $h(\ell)$ are parallel.

For the second part, we increase B by 1 and assume that f violates the conclusion. Then $f(\ell)$ has projection to ℓ of diameter $> \xi_g$, so we must have that $f(\ell)$ is parallel to ℓ and the conclusion follows. □

The following lemma was proved in [5] in the case of closed surfaces; when Σ has punctures the statement is easily reduced to the closed case by doubling. (In [5] we find a subgroup which acts trivially in $\mathbb{Z}/2$-homology, but we can further take a finite index subgroup which acts trivially in $\mathbb{Z}/3$-homology.)

Proposition 2.5 ([5, Lemma 4.7]). — *There is a finite index normal subgroup $\mathcal{S} \subset MCG(\Sigma)$ which is torsion-free, fixes all punctures, acts trivially in $\mathbb{Z}/3$-homology of Σ, and for every $h \in \mathcal{S}$ and every simple closed curve α on Σ, $i(\alpha, h(\alpha)) = 0$ implies $h(\alpha) = \alpha$.*

When $S \subset \Sigma$ is a π_1-injective subsurface we denote by $\hat{S}$ the surface obtained from S by collapsing each boundary component to a puncture. Note that every mapping class $f : \Sigma \to \Sigma$ that preserves S induces a mapping class $\hat{f} : \hat{S} \to \hat{S}$. The following is immediate from a theorem of Ivanov [26] (see Section 5).

Corollary 2.6. — *Let $f \in \mathcal{S}$ preserve a subsurface $S \subset \Sigma$. If $\hat{f}$ has finite order in $MCG(\hat{S})$ then $\hat{f} = id$.*

2.6. The projection complex

Recall [33] that when S, S' are π_1-injective subsurfaces of Σ with $\partial S \cap \partial S' \neq \emptyset$ there is a coarse *subsurface projection* $\pi_S(S') \subset \mathcal{C}(S)$, a uniformly bounded subset of the curve complex $\mathcal{C}(S)$ obtained by closing up each component of $\partial S' \cap S$ along ∂S. Let $\mathbf{Y}$ be an $\mathcal{S}$-orbit of subsurfaces of Σ, where $\mathcal{S} \subset MCG(\Sigma)$ is the subgroup as in Proposition 2.5. Then distinct subsurfaces in $\mathbf{Y}$ have intersecting boundaries (see [5]) and the following two properties below hold. For the first see [3] (for a simple proof due to Leininger see [29, 30]), and for the second see [33] (for a simpler proof see [5]). When $A, B, C \in \mathbf{Y}$ define $d^\pi_A(B, C) = \operatorname{diam}\{\pi_A(B) \cup \pi_A(C)\}$. Then there is $\eta > 0$ such that

- of the three numbers $d^\pi_A(B, C), d^\pi_B(A, C), d^\pi_C(A, B)$ at most one is larger than η, and
- for every $A, B \in \mathbf{Y}$ the set $\{C \in \mathbf{Y} \mid d^\pi_C(A, B) > \eta\}$ is finite.

Section 3 of [5] proves the following theorem.

Proposition 2.7. — *Let $\mathbf{Y}$ be an $\mathcal{S}$-orbit of subsurfaces of Σ. Then $\mathcal{S}$ acts by isometries on a hyperbolic graph $\mathcal{C}(\mathbf{Y})$ with the following properties:*

(i) *For every surface $S \in \mathbf{Y}$ the curve graph $\mathcal{C}(S)$ is embedded isometrically as a convex subgraph in $\mathcal{C}(\mathbf{Y})$, and when $S \neq S'$ then $\mathcal{C}(S)$ and $\mathcal{C}(S')$ are disjoint.*

(ii) *The inclusion*
$$\bigsqcup_{S\in\mathbf{Y}} \mathcal{C}(S) \hookrightarrow \mathcal{C}(\mathbf{Y})$$
is $\mathcal{S}$-equivariant, where on the left $\phi \in \mathcal{S}$ sends a curve $\alpha \in \mathcal{C}(S)$ to the curve $\phi(\alpha) \in \mathcal{C}(\phi(S))$.

(iii) *For $S \neq S'$ the nearest point projection to $\mathcal{C}(S')$ sends $\mathcal{C}(S)$ to a uniformly bounded set, and this set is within uniformly bounded distance from $\pi_{S'}(S)$.*

(iv) *Assume $g \in \mathcal{S}$ is pure, i.e. supported on $S \in \mathbf{Y}$ and the restriction is pseudo-Anosov or, in case S is an annulus, a power of a Dehn twist. Denote by C the subgroup of $\mathcal{S}$ consisting of elements f that leave S invariant and, if S is not an annulus, $\hat{f} : \hat{S} \to \hat{S}$ preserves the stable and unstable foliations of $\hat{g}$. Then $(\mathcal{S}, \mathcal{C}(\mathbf{Y}), g, C)$ satisfies WWPD.*

Proof. — The graph $\mathcal{C}(\mathbf{Y})$ is constructed in Section 3.1 of [5] from which it is clear that (ii) holds. Hyperbolicity is Theorem 3.15, convexity is Lemma 3.1 and (iii) is Lemma 3.11. For (iv) use Corollary 2.6 to see that there are no flips. If g' is conjugate to g and its virtual quasi-axis is contained in $\mathcal{C}(S)$, then the projection to a quasi-axis ℓ of g is uniformly bounded (or the two are parallel) by Lemma 2.4. If the virtual quasi-axis of g' is contained in some other $\mathcal{C}(S')$ the projection to ℓ is bounded by the third bullet. □

Lemma 2.8. — *Let $\phi \in \mathcal{S}$ be supported on a subsurface F so that $\phi|F$ is pseudo-Anosov (or a Dehn twist if F is an annulus). Also suppose that F does not contain any $S \in \mathbf{Y}$. Then for each $S \in \mathbf{Y}$, there exists a vertex in $\mathcal{C}(\mathbf{Y})$ such that the nearest point projection to $\mathcal{C}(S)$ of its ϕ-orbit is uniformly bounded (independently of F, ϕ and S; the bound depends only on Σ).*

Thus, if ϕ is hyperbolic in $\mathcal{C}(\mathbf{Y})$, its virtual quasi-axis can intersect $\mathcal{C}(S)$ only in a uniformly bounded length segment.

Proof. — If $S \cap F = \emptyset$ then ϕ fixes $\mathcal{C}(S)$ pointwise (ϕ is elliptic) and the claim is clear. Otherwise $\partial F \cap S \neq \emptyset$ and all $\phi^i(\partial F)$, $i \in \mathbb{Z}$, have the same projection to S in Σ. The first part of the lemma follows from Proposition 2.7(iii).

Now, by an elementary argument in δ-hyperbolic geometry, if ϕ is hyperbolic and has a virtual quasi-axis, then the ϕ-orbit of a point on it has the smallest projection (in diameter) to $\mathcal{C}(S)$ among the ϕ-orbits of points (up to a constant depending on δ). Therefore the last part of the claim follows. □

2.7. Promoting hyperbolic spaces to quasi-trees

In this section we promote a $WWPD$ action (G, X, g, C) with X a δ-hyperbolic graph to a $WWPD$ action (G, Q, g, C) where Q is a quasi-tree.

Proposition 2.9. — *Let X be a δ-hyperbolic graph and assume (G, X, g, C) satisfies $WWPD$ with the constant $\xi = \xi_g^X$. Then there is an action of G on a quasi-tree Q such that:*

(i) *The bottleneck constant $\Delta = \Delta(\delta, \xi)$ for Q depends only on δ and $\xi = \xi_g^X$, and it is bounded by a multiple of $\delta + \xi + 1$,*
(ii) *(G, Q, g, C) satisfies $WWPD$ with ξ_g^Q bounded by a multiple of $\delta + \xi + 1$,*
(iii) *if $h \in G$ is elliptic on X then h is elliptic on Q,*
(iv) *if $h \in G$ is hyperbolic on X and if $\tilde{\Pi}_g^X(h) \leqslant \eta$ then either h is elliptic on Q, or h is hyperbolic on Q and $\tilde{\Pi}_g^Q(h) \leqslant \eta + P$ for some constant $P = P(\delta, \xi)$ which is a fixed multiple of $\delta + \xi + 1$.*

Proof. — This is also a special case of the construction in [5]. Consider the conjugates of g, and say two are equivalent if they have parallel quasi-axes. For each equivalence class take the union of all quasi-axes of all of its members with the subspace metric – this is a quasi-line. The collection $\mathbf{Y}$ of all these quasi-lines satisfies the axioms in Section 3 of [5] since by assumption the projections are uniformly bounded. The space $\mathcal{C}(\mathbf{Y})$ constructed there is a quasi-tree by Theorem 3.10 of [5], and we name it Q. $\mathcal{C}(\mathbf{Y})$ contains the quasi-line Y for each Y. The main observation for the proof of (i) is that the constant K used in the definition of the projection complex depends only on δ and ξ and the dependence is linear. Then (ii) follows from Lemma 3.11 of [5] and (iii) is clear from the construction.

Suppose h is hyperbolic in Q and has long overlap with one of the quasi-lines Y. Then $h^{-N}(Y)$ and $h^N(Y)$ have large projection in Y measured in Q, hence also in X (this again uses Lemma 3.11 in [5]). But then a virtual quasi-axis of h has large projection to Y in X. □

3. Construction of quasi-morphisms

In this section we show how to construct quasi-morphisms $G \to \mathbb{R}$ if (G, Q, g, C) satisfies $WWPD$ and Q is a quasi-tree, generalizing the Brooks construction.

PROPOSITION 3.1. — *For every Δ there is $M = M(\Delta)$, a fixed multiple of $\Delta + 1$, such that the following holds. Let (G, Q, g, C) satisfy $WWPD$ where Q is a quasi-tree with bottleneck constant Δ and assume $\tau_g \geqslant \xi_g + M$. Then there is a quasi-morphism $F : G \to \mathbb{R}$ such that*

(a) *the defect of F is $\leqslant 12$,*

(b) *F is unbounded on the powers of g; more precisely,*

$$\hat{F}(g) \geqslant \frac{1}{2}$$

where $\hat{F}$ is the homogeneous quasi-morphism equivalent to F, and moreover if h is hyperbolic with virtual quasi-axes parallel to those of g and both g, h translating in the same direction then

$$\frac{\hat{F}(h)}{\tau_h} = \frac{\hat{F}(g)}{\tau_g}$$

and in particular $\hat{F}(h) \geqslant \frac{\tau_h}{2\tau_g}$,

(c) *F is bounded on the powers of any elliptic element of G, and*

(d) *F is bounded on the powers of any hyperbolic element α such that $\tilde{\Pi}_g(\alpha) \leqslant \tau_g - M$.*

Proof. — The proof is a modification of the classical Brooks construction for free groups [13]. There are two variants, one counts the number of *all* subwords of a given word isomorphic to a fixed word w, and the other counts the maximal number of *non-overlapping* subwords isomorphic to w. The first version is more convenient when working with coefficients (see [4]). The second version is more convenient when control on the defect is important, and this is the version we pursue here.

We start by fixing an $(4, A)$-quasi-isometry $\phi : Q \to T$ to a tree T and a constant $\epsilon \geqslant 0$ so that the ϕ-image of a $(2, 10\delta{+}10)$-quasi-geodesic $[a, b]$ is in the ϵ-neighborhood of $[\phi(a), \phi(b)]$. Note that A, ϵ depend only on Δ and can be arranged to be fixed multiples of $\Delta{+}1$, see Secton 2.2. Let x_0 be a vertex with $D = d(x_0, g(x_0))$ minimal possible and let $w = [x_0, g(x_0)]$, viewed as an oriented segment. By taking M sufficiently large we may assume that $D \gg \delta, A, \epsilon$ and the union of $\langle g \rangle$-translates of w forms a quasi-axis ℓ of g. From this data we will construct a quasi-morphism $F = F_{\phi,\epsilon,w} : G \to \mathbb{R}$.

A *copy* of w is a translate γw, also viewed as an oriented segment. For $q, q' \in Q$ we write $\gamma w \overset{\circ}{\subset} [q, q']$ if there exists $\beta \in G$ such that $\phi(\beta\gamma w)$ is contained in the ϵ-neighborhood of the segment $[\phi(\beta(q)), \phi(\beta(q'))]$ and $\phi(\beta(q))$ is closer to the ϕ-image of the initial endpoint of $\beta\gamma w$ than the terminal endpoint. Since w is long compared to ϵ and the quasi-isometry constants of ϕ, the condition says that the copy γw is nearly contained in $[q, q']$ in the oriented sense. Note that the notion is equivariant, i.e. if $\gamma w \overset{\circ}{\subset} [q, q']$ then $\beta\gamma w \overset{\circ}{\subset} [\beta(q), \beta(q')]$ for any $\beta \in G$. Also, if $\gamma w \overset{\circ}{\subset} [q, q']$ and $\beta' \in G$ is arbitrary, then $\phi(\beta'\gamma w)$ is contained in the ϵ'-neighborhood of the segment $[\phi(\beta'(q)), \phi(\beta'(q'))]$, where ϵ' also depends linearly on δ.

We say that two copies γw and $\gamma' w$ are *non-overlapping* if for some $\beta \in G$ the images $\phi(\beta\gamma w)$ and $\phi(\beta\gamma' w)$ are disjoint. This notion is also equivariant, and if $\beta' \in G$ is arbitrary the intersection $\phi(\beta'\gamma w) \cap \phi(\beta'\gamma' w)$ has uniformly bounded diameter, say by ϵ'', which also depends linearly on δ. The constant M and hence the length of w will be large compared to all these constants.

Now define the *non-overlapping count* $N_w(q, q')$ as the maximal number of pairwise non-overlapping copies $\gamma w \overset{\circ}{\subset} [q, q']$. To see that this number is finite, note that the projection to T of any $\gamma w \overset{\circ}{\subset} [q, q']$ has a long overlap with $[\phi(q), \phi(q')]$ while the pairwise overlaps are bounded.

Claim. — *Let δ be the hyperbolicity constant for Q and assume $|w| \gg \delta$. If $r \in Q$ is 2δ-close to a geodesic from q to q' then*

$$|N_w(q, q') - N_w(q, r) - N_w(r, q')| \leqslant 2$$

Indeed, the union of maximal collections for (q, r) and (r, q') gives a non-overlapping collection for (q, q'), perhaps after removing the two copies closest to r, and conversely, and maximal collection for (q, q') breaks up into two non-overlapping collections for (q, r) and (r, q'), perhaps after removing two copies closest to r.

Now define $F : G \to \mathbb{R}$ by

$$F(\alpha) = N_w(x_0, \alpha(x_0)) - N_{w^{-1}}(x_0, \alpha(x_0)) = N_w(x_0, \alpha(x_0)) - N_w(\alpha(x_0), x_0)$$

It is straightforward to check $(a) - (d)$.

Proof of (a). — This is the standard Brooks tripod argument. Let $\alpha, \beta \in G$ and let $r \in Q$ be within 2δ of each of 3 geodesics joining $x_0, \alpha(x_0), \beta\alpha(x_0)$.

Now we have $N_w(x_0, \alpha(x_0)) \sim N_w(x_0, r) + N_w(r, \alpha(x_0))$ by the Claim. Write 5 more such approximate equalities, for each oriented side of the triangle $x_0, \alpha(x_0), \beta\alpha(x_0)$ and note that e.g.

$$N_w(\alpha(x_0), \alpha\beta(x_0)) = N_w(x_0, \beta(x_0)).$$

Adding these (approximate) equalities, we find that

$$|F(\alpha\beta) - F(\alpha) - F(\beta)| \leqslant 12$$

Proof of (b). — Note that $\{g^{2k}(w)\}_{k\in\mathbb{Z}}$ are non-overlapping, $k \in \mathbb{Z}$. Thus $N_w(x_0, g^{2k}(x_0)) \geqslant k$ for every $k = 1, 2, \cdots$. It remains to observe that $N_w(g^k(x_0), x_0) = 0$ by the $WWPD$ assumption. Thus $F(g^{2k}) \geqslant k$ and so $\hat{F}(g) \geqslant \frac{1}{2}$ where $\hat{F}$ is the homogenous quasi-morphism equivalent to F. For the second assertion, note that for a fixed x_0 and every n the points $g^{[n/\tau_g]}(x_0)$ and $h^{[n/\tau_h]}(x_0)$ are uniformly close, so the statement follows from the above Claim.

Proof of (c). — If the orbit $\{\alpha^i(x_0)\}$ is bounded, then so are the translates $\{\beta\alpha^i(x_0)\}$ and their ϕ-images, and hence $N_{w^{\pm 1}}(x_0, \alpha^i(x_0))$ are uniformly bounded, and so are $F(\alpha^i)$.

Proof of (d). — If $F(\alpha^N) \neq 0$ for large N, then there must be a copy of w near a virtual quasi-axis of α and we see that the projection of this quasi-axis to the corresponding copy of ℓ will contain the copy of w except for segments near the endpoints bounded by a fixed multiple of $\Delta + 1$. □

Corollary 3.2. — *There is a constant $R = R(\delta, \xi)$, a fixed multiple of $\delta + \xi + 1$, so that the following holds. Let (G, X, g, C) satisfy $WWPD$ where X is a δ-hyperbolic graph and $\tau_g \geqslant R$. Then there is a quasi-morphism $F : G \to \mathbb{R}$ such that*

(a) *the defect of F is $\leqslant 12$,*
(b) *F is unbounded on the powers of g; in fact $\hat{F}(g) \geqslant \frac{1}{2}$,*
(c) *F is bounded on the powers of any elliptic element of G, and*
(d) *F is bounded on the powers of any hyperbolic element α with $\tilde{\Pi}_g(\alpha) \leqslant \tau_g - R$.*

Proof. — This is immediate from Propositions 2.9 and 3.1. □

Remark 3.3. — In applications we will not necessarily have $\tau_g \geqslant R$, but will have to pass to a power g^N of g to achieve this. For our uniform estimates it will be important that N is uniformly bounded. In the setting of the curve graph $\mathcal{C}(\Sigma)$ of a fixed surface Σ this follows from two facts:

- $\tau_g \geqslant \epsilon_\Sigma > 0$ for every hyperbolic g [32, Proposition 4.6], and
- ξ_g is bounded by a fixed multiple of $\tau_g + 1$ (see Lemma 2.4).

Similarly, uniformity on powers holds in hyperbolic spaces $\mathcal{C}(\mathbf{Y})$ constructed in Proposition 2.7. More precisely, if g is supported on a subsurface $S \in \mathbf{Y}$ and the restriction is pseudo-Anosov, then its translation length in $\mathbf{Y}$ is equal to its translation length in $\mathcal{C}(S)$ (this follows from Proposition 2.7(i) and (ii)) and the projections of quasi-axes of conjugates are bounded by a fixed multiple of $\tau_g + 1$ (this follows from Proposition 2.7(iii) and Lemma 2.4).

Regarding the uniformity in the second bullet in the above, Delzant [20] calls an action of G on X *weakly acylilndrical* if there is a constant D such that for every hyperbolic element $g \in G$, (G, X, g, C) satisfy WWPD and $\xi_g \leqslant D(\tau_g + 1)$.

4. Stable commutator length on mapping class groups

Now assume that $G < MCG(\Sigma)$ is a finite index subgroup and $g \in G$. By the Nielsen-Thurston theory (see e.g. [17]) there is a unique minimal g-invariant collection $\mathcal{C}$ (possibly empty) of pairwise disjoint simple closed curves which are non-parallel and no curve bounds a disk or a punctured disk and so that after replacing g by a power:

- each puncture of Σ is fixed,
- each curve in $\mathcal{C}$ is g-invariant,
- each component of $\Sigma - \cup_{c \in \mathcal{C}} \mathcal{C}$ is g-invariant,
- the restriction of g to each complementary component is homotopic to the identity or a pseudo-Anosov homeomorphism.

Let S_i be a complementary component on which g is pseudo-Anosov. Collapsing all boundary components to punctures produces a closed surface $\hat{S}_i$ with punctures and g induces a pseudo-Anosov homeomorphism $\hat{g}_i : \hat{S}_i \to \hat{S}_i$. There is a (projectively) $\hat{g}_i$-invariant measured (singular) foliation $\hat{\mathcal{F}}_i$ on $\hat{S}_i$ without saddle connections. Each puncture is a k-prong singularity for some $k = 1, 2, \cdots$ (when $k = 2$ it is a regular point). After passing to a higher power of g we may assume that

- each such $\hat{g}_i$ is *arotational* i.e. all prongs (directions of leaves) out of any puncture are fixed.

We may reverse the collapsing process and blow up the newly created punctures back to boundary components. A point in the boundary circle is a tangent direction out of the puncture. The foliation $\hat{\mathcal{F}}_i$ lifts to a foliation $\mathcal{F}_i$ on S_i with k leaves transverse to the boundary circle. The homeomorphism $\hat{g}_i$ naturally lifts to a homeomorphism $g_i : S_i \to S_i$, and it has $2k$ fixed points on the boundary circle (k from the stable and k from the unstable foliation). In any case, there is a canonical way to isotope g_i to a homeomorphism that fixes the boundary pointwise, keeping the $2k$ points fixed throughout the isotopy.

We can now glue the surfaces S_i together to form S, but we will also insert an annulus between any two boundary components to be glued. The purpose of this is that otherwise the homeomorphism of the glued surface that agrees with g_i on S_i may not be g; it may differ from g by a product of Dehn twists in the curves in $\mathcal{C}$. We realize any such Dehn twists on the inserted annuli. Extend each $g_i : S_i \to S_i$ by the identity in the complement of S_i to obtain a homeomorphism of S, also denoted g_i. We summarize the discussion as follows.

Theorem 4.1. — *For every $g \in G$ there is $N > 0$ such that*

$$g^N = g_1 \cdots g_k \delta_1^{n_1} \cdots \delta_l^{n_l}$$

where δ_j are (left) Dehn twists supported on annuli around the curves in a subset of the reducing multicurve $\mathcal{C}$, $n_i \neq 0$ and each g_i is a pseudo-Anosov supported on a complementary subsurface S_i. Any two homeomorphisms above commute.

We now make some further definitions. First, after possibly taking a further power of g, we may assume:

- each g_i and $\delta_j^{n_j}$ is in G (this is where we are using that G has finite index in $MCG(S)$) and also in $\mathcal{S}$ (see Proposition 2.5),
- for each i, either g_i is conjugate (in G) to g_i^{-1} or g_i^m is not conjugate to g_i^{-m} for any $m > 0$ (the latter is equivalent to saying that no $\gamma \in G$ interchanges the stable and unstable foliation of g_i).

If g_i is conjugate to g_i^{-1} in G we say g_i is *achiral*, and otherwise it is *chiral*. If g_i and g_j are both chiral, we say they are *equivalent* if a nontrivial power of g_i is conjugate in G to a power of g_j. In other words, g_i and g_j are equivalent if there is some element $\gamma \in G$ that takes S_i to S_j and takes the stable foliation of g_i to either the stable or the unstable foliation of g_j. We make the same definition for Dehn twists (recall that a power of a Dehn twist is not conjugate to its inverse, so we may view them as chiral): $\delta_i^{n_i}$ and $\delta_j^{n_j}$ are *equivalent* if some of their nontrivial powers are conjugate (equivalently, the corresponding annuli are in the same G-orbit).

Let $\{g_{i_1}, g_{i_2}, \cdots, g_{i_p}\}$ be an equivalence class. Thus $g_{i_1}^{m_1}, g_{i_2}^{m_2}, \cdots, g_{i_p}^{m_p}$ are all conjugate for certain $m_j \neq 0$. We will say this equivalence class is *essential* if

$$\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_p} \neq 0$$

and *inessential* otherwise (an example of an inessential class has appeared in [22]). Since g_i^m conjugate to g_i^n implies $m = \pm n$ for any g_i, and implies $m = n$ for chiral g_i, the exponents m_i above are unique up to a common multiple. We make the same definition for equivalence classes of powers of

Dehn twists $\{\delta_{i_1}^{n_1}, \cdots, \delta_{i_p}^{n_p}\}$ with m_i chosen so that all $\delta_{i_j}^{n_j m_j}$ are pairwise conjugate.

Theorem 4.2. — *Let $G < MCG(\Sigma)$ be a subgroup of finite index and $g \in G$. Then $scl(g) > 0$ if and only if some chiral equivalence class is essential.*

Note that if G is a subgroup of $\mathcal{S}$ then every class is chiral (by Corollary 2.6) and has one element, so every nontrivial $g \in G$ has $scl_G(g) > 0$.

Proof of Theorem 4.2. — We first prove that $scl(g) = 0$ if every chiral class is inessential. Let $H : G \to \mathbb{R}$ be a homogeneous quasi-morphism. We will argue that $H(g) = 0$. If g_i is achiral then $H(g_i) = H(g_i^{-1})$ so $H(g_i) = 0$. Let $\{g_{i_1}, g_{i_2}, \cdots, g_{i_p}\}$ be a chiral equivalence class with $g_{i_1}^{m_1}, \cdots, g_{i_p}^{m_p}$ all conjugate. Then $H(g_{i_1}^{m_1}) = \cdots = H(g_{i_p}^{m_p})$; call the common value A. Thus $H(g_{i_j}) = \frac{A}{m_j}$ and $H(g_{i_1} g_{i_2} \cdots g_{i_p}) = A(\frac{1}{m_1} + \cdots + \frac{1}{m_p})$, which is 0 for an inessential class. A similar argument applies to inessential classes of powers of Dehn twists. It now follows that $H(g) = 0$ since H is additive on commuting elements.

Now assume that, after reindexing, $\{g_1, \cdots, g_p\}$ is an essential chiral equivalence class. In case there are several such classes we choose one with highest complexity ($= -\chi(S_1)$ where S_1 is the surface supporting g_1). E.g. annuli have the smallest complexity, so powers of Dehn twists would be chosen only if nothing else is available. Further, in case there are several essential chiral classes with maximal complexity we choose one whose primitive root has largest translation length. That is, if $\{g_{i_1}, \cdots, g_{i_s}\}$ is another essential chiral class of maximal complexity, and we write $g_1 = h_1^{N_1}$, $g_{i_1} = h_{i_1}^{N_{i_1}}$ with $N_1, N_{i_1} > 0$ maximal possible and with h_1 $[h_{i_1}]$ supported on the same subsurface as g_1 $[g_{i_1}]$, then $\tau_{h_1} \geqslant \tau_{h_{i_1}}$ (translation lengths are with respect to the curve graph of the supporting subsurface).

We wish to construct a quasi-morphism $H : G \to \mathbb{R}$ which is unbounded on the powers of $g_1 \cdots g_p$ but bounded on the powers of all other chiral g_j and Dehn twists that belong to essential classes. It will then follow that H is unbounded on the powers of g and hence $scl(g) > 0$. For simplicity we assume g_1 is not a Dehn twist.

Let $G' = G \cap \mathcal{S}$, where $\mathcal{S}$ is the subgroup in Proposition 2.5, so G' is normal in G. We will now consider the action of G' on the graph $\mathcal{C}(\mathbf{Y})$ of Proposition 2.7, where $\mathbf{Y}$ is the $\mathcal{S}$-orbit of S_1. According to Proposition 2.7 $(G', \mathcal{C}(\mathbf{Y}), g_1, C)$ satisfies $WWPD$ where $C < G'$ preserves the stable and unstable foliations of g_1 on S_1.

Choose coset representatives $1 = \gamma_1, \gamma_2, \cdots, \gamma_s \in G$ of G/G'.

Let $H' : G' \to \mathbb{R}$ be the associated quasi-morphism as in Corollary 3.2. We are really replacing g_1 here with a bounded power when applying this corollary (see Remark 3.3). Finally, define $H : G' \to \mathbb{R}$ by

$$H(\gamma) = \sum_{i=1}^{s} H'(\gamma_i \gamma \gamma_i^{-1}).$$

It was verified in [8, Section 7] that H extends to a quasi-morphism on G. (If we first replace H by the homogeneous quasi-morphism $\hat{H}$, then $\hat{H}(f) = \frac{1}{n}\hat{H}(f^n)$ when $f^n \in G'$ extends $\hat{H}$ to G. Alternatively, we can first replace H' by $\hat{H}'$, then define H, which is automatically homogeneous). Note that rechoosing the coset representatives changes H by a uniformly bounded amount.

Claim 1. — *H is unbounded on the powers of g_1.*

Proof of Claim 1. — The summand $H'(g_1^N)$ corresponding to the trivial coset is unbounded on the powers of g_1 by construction (see Corollary 3.2) and we only need to see that summands $H'(\gamma_i g_1^N \gamma_i^{-1})$ are bounded or have the same sign as $H'(g_1^N)$. The support of $\gamma_i g_1^N \gamma_i^{-1}$ is the surface $\gamma_i(S_1)$ where S_1 is the support of g_1.

If $\gamma_i(S_1) \notin \mathbf{Y}$ then by Lemma 2.8 $\gamma_i g_1 \gamma_i^{-1}$ has virtual quasi-axes that intersect every $\mathcal{C}(S)$ in a uniformly bounded segment, so their projections to the translates of virtual quasi-axes ℓ of g_1 are uniformly bounded. It now follows that $H'(\gamma_i g_1^N \gamma_i^{-1})$ is bounded by Proposition 3.2(d).

If $\gamma_i(S_1) \in \mathbf{Y}$ then $\gamma_i g_1 \gamma_i^{-1}$ preserves $\mathcal{C}(\gamma_i(S_1))$. Its translation length on $\mathcal{C}(\gamma_i(S_1))$ is τ_{g_1} since γ_i conjugates the action of g_1 on $\mathcal{C}(S_1)$ and the action of $\gamma_i g_1 \gamma_i^{-1}$ on $\mathcal{C}(\gamma_i(S_1))$. Thus it follows from Lemma 2.4 that the projection of a quasi-axis of $\gamma_i g_1 \gamma_i^{-1}$ to a G'-translate of ℓ is either bounded by a linear function of τ_{g_1}, or the two lines are parallel. In the former case, after replacing g_1 by a definite power, we may assume the projections are bounded by $\tau_{g_1} - R$ and so H' is bounded on the powers of $\gamma_i g_1 \gamma_i^{-1}$ by Corollary 3.2. In the latter case $\gamma_i g_1 \gamma_i^{-1} \in C$ and by chirality $\gamma_i g_1 \gamma_i^{-1}$ does not translate the opposite way from g_1, so $H'(\gamma_i g_1^N \gamma_i^{-1})$ has the same sign as $H'(g_1^N)$.

Claim 2. — *H is unbounded on the powers of $g_1 \cdots g_p$.*

Proof of Claim 2. — Denote by $\hat{H}$ the homogeneous quasi-morphism bounded distance away from H. If $\hat{H}(g_1^{m_1}) = A \neq 0$ then $\hat{H}(g_i^{m_i}) = A$, $\hat{H}(g_i) = \frac{A}{m_i}$ and $\hat{H}(g_1 \cdots g_p) = A(\frac{1}{m_1} + \cdots + \frac{1}{m_p}) \neq 0$ since the class is essential.

Claim 3. — *Let $\{g_{i_1}, g_{i_2}, \cdots, g_{i_q}\}$ be an equivalence class distinct from $\{g_1, \cdots, g_p\}$. Then H is bounded on the powers of $g_{i_1} g_{i_2} \cdots g_{i_q}$.*

Proof of Claim 3. — If the class is achiral or inessential chiral then we showed that every quasi-morphism is bounded on the powers. Now assume the class is essential chiral. The argument is similar to Claim 1. Consider a conjugate $\gamma_i g_{i_1} \gamma_i^{-1}$. Let S_{i_1} be the support of g_{i_1}. By the maximality assumption the surface $\gamma_i(S_{i_1})$, which supports $\gamma_i g_{i_1} \gamma_i^{-1}$, does not properly contain any surface in $\mathbf{Y}$. If it is not equal to any surface in $\mathbf{Y}$ then H' is bounded on the powers of $\gamma_i g_{i_1} \gamma_i^{-1}$ as in Claim 1. If $\gamma_i(S_{i_1}) \in \mathbf{Y}$ then we can apply Lemma 2.4 again since $\tau_{\gamma_i h_{i_1} \gamma_i^{-1}} \leqslant \tau_{h_1}$ for the primitive roots h_1 of g_1 and h_{i_1} of h_1 to deduce that H' is bounded on the powers of $\gamma_i g_{i_1} \gamma_i^{-1}$ (it is not possible for the virtual quasi-axes to be parallel here since g_1 and g_{i_1} belong to distinct classes). It now follows that H is unbounded on the powers of g.

The argument when the essential class consists of powers of Dehn twists is similar. We then use the collection $\mathbf{Y}$ of annuli consisting of the G'-orbit of the annuli supporting Dehn twists in the collection to build a hyperbolic graph $X = \mathcal{C}(\mathbf{Y})$. The role of the curve graph $\mathcal{C}(S_1)$ is played by the curve graph of the annulus, which is quasi-isometric to $\mathbb{R}$. □

We now state a number of consequences of the above proof. Some of them require going back and checking a few things.

4.1. Separability

We say that $g, g' \in G$ are *inseparable* if for any two homogeneous quasi-morphisms $H, H' : G \to \mathbb{R}$ the vectors $(H(g), H'(g)) \in \mathbb{R}^2$ and $(H(g'), H'(g')) \in \mathbb{R}^2$ are linearly dependent. Otherwise, g, g' are *separable* (see [34], [22]).

- If g has $scl(g) = 0$ then g, g' are inseparable for every g'.
- g^n and g^m are inseparable for every m, n,
- if g, h are (in)separable, then g^n, h^m are (in)separable for any $m, n \neq 0$.

Suppose $g, g' \in G$ have $scl(g) > 0$, $scl(g') > 0$. After passing to powers we may assume each can be written as the product of powers of Dehn twists and pseudo-Anosov homeomorphisms on subsurfaces, as discussed above. Suppose there are essential chiral classes $\{g_{i_1}, \cdots, g_{i_p}\}$ for g and $\{g'_{j_1}, \cdots, g'_{j_q}\}$ for g' so that g_{i_1} and g'_{j_1} have conjugate powers (i.e. the two classes are equivalent to each other). Then for any homogeneous quasi-morphism $H : G \to \mathbb{R}$ the ratio

$$\frac{H(g_{i_1} \cdots g_{i_p})}{H(g'_{j_1} \cdots g'_{j_q})}$$

does not depend on H (as long as it is not $\frac{0}{0}$). We call it the *characteristic ratio* of the essential chiral class that occurs in both g and g'. We make a similar definition for conjugacy classes of powers of Dehn twists that occur in both g and g'. Note that the characteristic ratio is always rational, and it can be computed from knowing which powers of the mapping classes in it are conjugate.

Proposition 4.3. — *Let $g, g' \in G$ be two elements with $scl(g) > 0$, $scl(g') > 0$. Then g, g' are inseparable if and only if every essential chiral class of g also occurs in g' and vice-versa, and all characteristic ratios are equal.*

Proof. — If there is an essential chiral class that occurs in g but not g', the proof of Theorem 4.2 produces a homogeneous quasi-morphism with $H(g) \neq 0$ and $H(g') = 0$, so g, g' are separable. Otherwise, for each characteristic ratio r there is a homogeneous quasi-morphism H with $\frac{H(g)}{H(g')} = r$.

If all essential chiral classes for g occur in g' and vice-versa, and all characteristic ratios are equal to r, then $\frac{H(g)}{H(g')} = r$ (or $\frac{0}{0}$) for any homogeneous quasi-morphism H, so g, g' are inseparable. □

For example, Dehn twists in curves in different G-orbits are separable.

There is a more general statement along the same lines. Denote by $\mathcal{X}$ the real vector space whose basis consists of equivalence classes (over G) of pure mapping classes which are chiral. In each class $[\gamma]$ choose a representative γ. Thus if $g \in MCG(\Sigma)$ then a definite power g^N decomposes as a product of pure classes, and after ignoring achiral components we get an element $\chi(g) \in \mathcal{X}$ by setting

$$\chi(g) = \sum_\gamma n_\gamma [\gamma]$$

where n_γ is computed as follows. Say $g_1, \cdots, g_k$ are the components of g^N equivalent to γ, so that $g_i^{m_i}$ is conjugate to γ^{r_i}. Then let $n_\gamma = \frac{1}{N} \sum_i \frac{r_i}{m_i}$. The arguments above show:

Proposition 4.4. — *Let $h_1, \cdots, h_p \in G$. The dimension of the space of functions $\{h_1, \cdots, h_p\} \to \mathbb{R}$ which are restrictions of homogeneous quasi-morphisms $G \to \mathbb{R}$ is equal to the dimension of the subspace of $\mathcal{X}$ spanned by $\chi(h_1), \cdots, \chi(h_p)$.*

Even more generally, let $C_1(G)$ be the vector space of chains $r_1h_1 + \cdots + r_ph_p$ with $r_i \in \mathbb{R}$ and $h_i \in G$. Any homogeneous quasi-morphism on G extends by linearity to $C_1(G)$ and there is a linear map $\chi : C_1(G) \to \mathcal{X}$ defined on the basis by the discussion above.

Proposition 4.5. — *Let $c_1, \cdots, c_p \in C_1(G)$. The dimension of the space of functions $\{c_1, \cdots, c_p\} \to \mathbb{R}$ which are restrictions of homogeneous quasi-morphisms $C_1(G) \to \mathbb{R}$ is equal to the dimension of the subspace of $\mathcal{X}$ spanned by $\chi(c_1), \cdots, \chi(c_p)$.*

4.2. Lower bound to scl

Proposition 4.6. — *Let G be a finite index subgroup of $MCG(\Sigma)$. There is a constant $\epsilon = \epsilon(G, \Sigma) > 0$ so that the following holds. Let $g \in G$ be any element. If $scl_G(g) > 0$ there is a homogeneous quasi-morphism $H : G \to \mathbb{R}$ such that $\frac{H(g)}{2\Delta(H)} \geqslant \epsilon$.*

As a consequence, by Proposition 2.1(iii)), we have that $\mathrm{scl}_G(g) > 0$ implies $\mathrm{scl}_G(g) \geqslant \epsilon$.

Proof. — The argument is same as the one for Theorem 4.2, but we need to bound constants carefully. We only give an outline of the argument. In the special case $G = MCG(S)$ and $g \in G$ pseudo-Anosov class this statement was proved by Calegari-Fujiwara in [16]. As before, set $G' = G \cap \mathcal{S}$.

Step 1. – In the course of the proof of Theorem 4.2, we replaced a given element g by an arotational power g^N. We note that the power can be taken to be uniformly bounded. This follows from the fact that the number of curves in a g-invariant pairwise disjoint collection is uniformly bounded, the number of complementary components is uniformly bounded, the number of singularities of the stable foliation and the number of prongs is uniformly bounded at any point. Moreover, after taking a further bounded power, the component maps g_i in the Nielsen-Thurston decomposition are in G'. We will also rename g^N as g.

Step 2. – Since $scl(g) > 0$, g has an essential chiral class. Among all essential chiral classes we choose one, say $g_1, \cdots, g_p$, where the complexity (absolute value of the Euler characteristic) of the supporting subsurface of each map in the class is largest possible, and among these, we arrange that the primitive root of g_1 has maximal translation length. Let $\hat{\mathcal{F}}_1$ be the stable foliation of $\hat{g}_1$ on the associated punctured surface $\hat{S}_1$. We have a short exact sequence

$$1 \to K \to Stab(\hat{\mathcal{F}}_1) \to \mathbb{Z} \to 1$$

where the map to $\mathbb{Z} \cong \{\lambda^n \mid n \in \mathbb{Z}\} \subset \mathbb{R}^+$ is given by the stretch factor, and K is a finite group whose size is bounded in terms of Σ. We now use the following fact from group theory: if two elements $\phi, \psi \in Stab(\hat{\mathcal{F}}_1)$ have the same image in $\mathbb{Z}$ then their powers ϕ^r, ψ^r are equal for $r = |K|$. Let $\hat{h}$ denote an element of $Stab(\hat{\mathcal{F}}_1)$ that maps to $1 \in \mathbb{Z}$. Then a certain bounded power $\hat{h}^N$ is arotational and lifts to a homeomorphism h on S_1 and extends by the identity to Σ. By taking a uniformly bounded power if necessary, we also assume that $h \in G'$. Putting all this together, we deduce that, perhaps after replacing g with a bounded power, for some integers $n_1, \cdots, n_p$ each g_i is conjugate to h^{n_i}. That the class is essential implies that $n_1 + \cdots + n_p \neq 0$.

Step 3. – We now construct $H' : G' \to \mathbb{R}$ using the action of G' on the space $X = \mathcal{C}(\mathbf{Y})$, where $\mathbf{Y}$ is the G'-orbit of the subsurface S_1 supporting h. To do this, we apply Corollary 3.2 to a uniformly bounded power of h, thus:

- the defect $\Delta(H')$ is at most 12,
- $\hat{H}'(h) \geqslant \epsilon > 0$.

Also note that $\Delta(\hat{H}') \leqslant 4\Delta(H') \leqslant 48$ (from the definition $\hat{H}'(\gamma) = \lim_{n\to\infty} H'(\gamma^n)/n$ we see that $|H' - \hat{H}'| \leqslant \Delta(H')$ and the statement follows from the triangle inequality).

Step 4. – Now let $\gamma_i \in G$ be the coset representatives of G/G' and define $\hat{H}'' : G' \to \mathbb{R}$ by $\hat{H}''(\gamma) = \sum_i \hat{H}'(\gamma_i \gamma \gamma_i^{-1})$. $\hat{H}''$ is homogeneous. Then

- $\Delta(\hat{H}'')$ is bounded above (by $[G : G']\Delta(\hat{H}')$, and $[G : G'] \leqslant [MCG(S) : \mathcal{S}]$),
- $|\hat{H}''(h)|$ is bounded away from 0 (no two summands have opposite sign, by chirality).

Then $\hat{H}''$ extends to a homogeneous quasi-morphism on G (via $\hat{H}(\gamma) = \hat{H}(\gamma^n)/n$ for $n = [G : G']$) and $\Delta(\hat{H}) \leqslant [G : G']\Delta(\hat{H}'')$, see [8, Lemma 7.2]. It remains to show that $|\hat{H}(g)|$ is bounded away from 0. First,

$$\hat{H}(g_1 \cdots g_p) = \hat{H}(g_1) + \cdots + \hat{H}(g_p) = (n_1 + \cdots + n_p)\hat{H}(h)$$

and since $|n_1 + \cdots + n_p| \geqslant 1$ and $\hat{H}(h) = \hat{H}''(h)$ we see that $|\hat{H}(g_1 \cdots g_p)|$ is bounded away from 0. It now suffices to note that for $j > p$ $\hat{H}(g_j) = 0$ whenever g_j belongs to an essential chiral class □

Remark 4.7. — Let Σ_h be the closed surface of genus h. For $G = MCG(\Sigma_g)$ it would be interesting to know how $s_h := \inf\{scl(g) \mid g \in G, scl(g) > 0\}$ behaves when the genus $h \to \infty$. On the plus side, subsurface projection constants are uniform (see Leininger's proof in [29, 30]) and so is the hyperbolicity constant δ of curve graphs, see [10, 25, 18]. The acylindicity constants in Lemma 2.4 are known explicitly [35], translation lengths in curve complexes are not uniform, but the asymptotics is understood [24]. The main deficiency of our argument is that it passes to the subgroup $\mathcal{S}$, whose index goes to ∞. There is a case where this can be avoided, namely when the genus $h = 2m$ is even and g is the Dehn twist in a curve that separates Σ_h into two subsurfaces of genus m. Then all of $MCG(\Sigma_h)$ acts on $\mathcal{C}(\mathbf{Y})$, where $\mathbf{Y}$ is the $MCG(\Sigma_h)$-orbit of annuli containing the support of g. We conclude that $scl(g) > \epsilon > 0$ independently of $h = 2m$. This implies that scl of a boundary Dehn twist is uniformly bounded below (in fact it is $\frac{1}{2}$ by [2]).

4.3. A bound on the commutator length

Proposition 4.8. — *For a finite index subgroup $G < MCG(S)$ and $g \in [G, G]$, $\mathrm{scl}(g) = 0$ if and only if $cl(g^n)$ is bounded for $n \in \mathbb{Z}$. Moreover, there are numbers $B = B(G)$ and $N = N(G) > 0$ such that for every $g \in [G, G]$ with $scl(g) = 0$, g^N can be written as a product of B commutators.*

Proof. — First note that the second statement implies the first, for if g is a product of $K = K(g)$ commutators and g^{iN} is a product of B commutators for every i, then every power of g is a product of $B + (N-1)K$ commutators.

If g is achiral, i.e. if $\gamma g \gamma^{-1} = g^{-1}$ for some $\gamma \in G$, then $\gamma g^k \gamma^{-1} = g^{-k}$, so $g^{2k} = g^k h^{-1} g^{-k} h = [g^k, h^{-1}]$ is a single commutator.

If $g = g_1 \cdots g_p$ is a single inessential chiral class, write $g_i = h_i^{n_i}$ with all h_i conjugate, say $h_i = \gamma_i h_{i-1} \gamma_i^{-1}$ for $i = 2, 3, \cdots, p$. Then

$$g = h_1^{n_1} h_2^{n_2} \cdots h_p^{n_p}$$

with $\sum n_i = 0$. Thus we can write g as a product of $(p-1)$ commutators:

$$g = [h_1^{n_1}, \gamma_2][h_2^{n_1+n_2}, \gamma_3] \cdots [h_{p-1}^{n_1+n_2+\cdots+n_{p-1}}, \gamma_p]$$

and note that p is uniformly bounded by the topology of Σ.

Now $scl(g) = 0$ implies, according to Theorem 4.2, that some nontrivial power g^k can be written as a commuting product of a bounded number of achiral elements and inessential classes of chiral elements. The claim follows from the observation that the power k is bounded in terms of Σ and G (see Step 1 in the proof of Proposition 4.6). □

4.4. Restrictions to subsurfaces

Recall that for a group G the vector space of all quasi-morphisms modulo homomorphisms plus bounded functions is denoted by $\widetilde{QH}(G)$ and this vector space is naturally isomorphic to the kernel of the comparison map $H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R})$ from the bounded to the regular cohomology of G. Also recall that when Σ supports pseudo-Anosov homeomorphisms the space $\widetilde{QH}(MCG(\Sigma))$ is infinite dimensional [7].

Proposition 4.9. — *Let $S \subset \Sigma$ be a subsurface that supports pseudo-Anosov homeomorphisms. Then the restriction map*

$$\widetilde{QH}(MCG(\Sigma)) \to \widetilde{QH}(MCG(S))$$

has infinite dimensional image.

Proof. — Recall that [7] produces a sequence $f_1, f_2, \cdots$ of chiral, pairwise inequivalent pseudo-Anosov homeomorphisms on S. The proof of the main theorem gives, for every i, a quasi-morphism $H_i : MCG(\Sigma) \to \mathbb{R}$ which is unbounded on the powers of f_i, and 0 on all powers of f_j for $j < i$. The statement also follows from Proposition 4.4. □

5. Example: Level subgroups and the Torelli group

We start by looking at the level subgroup $G = G_p$ for $p \geqslant 3$ consisting of mapping classes that act trivially in $H_1(\Sigma;\mathbb{Z}_p)$. Recall the theorem of Ivanov [26, Theorem 1.7] that every $f \in G_p$ fixes the punctures, the Nielsen-Thurston reducing curves are each invariant, and the restriction of f to a complementary component (after collapsing boundary to punctures) is either identity or pseudo-Anosov. If at least one of them is pseudo-Anosov, we say f has *exponential growth*.

Recall that two simple closed curves in Σ are homologous over $\mathbb{Z}_p$ if and only if either both are separating, or cobound a subsurface (with compatible boundary orientation).

The following Lemma is left as an exercise (find a non-separating loop b such that $f(b)$ and b are not homologous if $i(a, f(a)) = 0$ and $f(a) \neq a$).

Lemma 5.1. — *If $f \in G_p$ and a is a separating curve then either $f(a) = a$ or $i(a, f(a)) > 0$, and in the former case f preserves the orientation of a.*

Lemma 5.2. — *If S is a subsurface of Σ which is not an annulus and $f \in G_p$ then $f(S) \cap S \neq \emptyset$.*

Proof. — Assume $f(S) \cap S = \emptyset$. First, by the previous lemma if the genus of S is > 0 then S contains a separating curve which cannot be moved off itself (and if fixed the orientation is preserved), so $f(S) \cap S \neq \emptyset$, impossible. Thus S is a planar surface. If any of the boundary components are separating, they are fixed with the same orientation (otherwise, there will be a non-separating curve, in one of the components of the complement of the separating curve, whose homology class is not preserved by f), so $f(S) \cap S \neq \emptyset$, impossible. Thus they are all nonseparating, and in fact the only relation among them in homology is that the sum is 0 (this is equivalent to $\Sigma - S$ being connected), or otherwise S contains a separating curve . Let a be a boundary component, so $f(a)$ is a boundary component of $f(S)$. Since a and $f(a)$ are homologous over $\mathbb{Z}_p$, there are two subsurfaces cobounded by a and $f(a)$, one contains S and the other contains $f(S)$. Denote by A the one that contains S (use $f(S) \cap S = \emptyset$). Similarly, choose another boundary component b and let B be the subsurface cobounded by b and $f(b)$ that contains S. Then $A \cap B$ is a subsurface with two boundary components a and b, showing that $a + b = 0$ in homology, i.e. S is an annulus. □

Corollary 5.3. — *Let $g = g_1 \cdots g_k \delta_1^{n_1} \cdots \delta_l^{n_l}$ be the decomposition of $g \in G_p$ into pure parts as in Section 4. Then every g_i is chiral and forms its own equivalence class.*

In particular, every element of G_p of exponential growth has positive scl_{G_p}.

Proof. — Suppose $\gamma \in G_p$ conjugates g_i to g_i^{-1}. Restricting to the supporting surface and collapsing boundary to punctures gives a mapping class $\hat{\gamma}$ of finite order[(4)] that conjugates $\hat{g}_i$ to $\hat{g}_i^{-1}$. This is impossible by Ivanov's Theorem (Corollary 2.6). The fact that g_i is not equivalent to g_j for $i \neq j$ follows from the lemma. □

[(4)] Any mapping class that conjugates a pseudo-Anosov homeomorphism to its inverse flips the axis in Teichmüller space and must have a fixed point, so it has finite order.

Lemma 5.4. — *Suppose a, b are distinct nonseparating homologous curves so that $a, b, f(a), f(b)$ have pairwise intersection number 0 and $f \in G_p$. Then $f(a) = a$ and $f(b) = b$.*

Proof. — The four curves are cyclically ordered along Σ and if the conclusion fails f takes a cobounding subsurface off of itself. (Notice that it does not happen that $f(a) = b$ and $f(b) = a$ by [26, Theorem 1.7].) ☐

Corollary 5.5. — *Let $f \in G_p$ be a multitwist in a multicurve M. If M contains a separating curve then $scl(f) > 0$ and otherwise $scl(f) > 0$ if and only if the sum of the powers of Dehn twists over some homology class of curves in M is nonzero.*

Proof. — After taking a power, the twists in all curves occur with power divisible by p. It does not happen that there are two distinct (up to homotopy) separating curves a, b and $f \in G$ with $f(a) = b$ since otherwise there will be a non-separating curve c (in a component of the complement of a) whose homology class is not fixed by f. Therefore if M contains a separating curve then $scl(f) > 0$. Homologous nonseprating curves are in the same G_p-orbit (in fact, the same Torelli-orbit), so the Dehn twists in these curves form a chiral class. ☐

5.1. The Torelli group $\mathcal{T}$

Now let $\mathcal{T}$ be the group of mapping classes acting trivially in the integral homology of Σ and let $f \in \mathcal{T}$.

Theorem 5.6. — *If $f \neq 1$ then $scl_{\mathcal{T}}(f) > 0$.*

Proof. — Since $\mathcal{T} < G_3$, if f grows exponentially we have $scl(f) > 0$ (Corollary 5.3). So suppose f is a multitwist supported on the multicurve M. If M contains a separating curve then $scl(f) > 0$ (Corollary 5.5), and otherwise f has infinite order in the abelianization of $\mathcal{T}$, see [6], therefore $scl(f) > 0$ by Corollary 5.5. ☐

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