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POLYNOMIAL BOUND ON THE LOCAL BETTI NUMBERS OF A REAL ANALYTIC GERM

by Lionel F. ALBERTI

Abstract. — This paper proves the existence of a bound on the sum of local Betti numbers of a real analytic germ by a polynomial function of the multiplicity of the germ. This result can be interpreted as a localization of the classical Oleinik–Petrovsky bound (aka. Thom–Milnor bound) on the sum of Betti numbers of a semi-algebraic set. The key elements of the proof are the tangent cone of the germ, the Thom–Mather topological trivialization theorem, the Oleinik–Petrovsky bound, and a result by D. Mumford and J. Heintz bounding the degrees of the generators of an ideal by a polynomial function of the geometric degree of its associated variety. Our result is then applied to yield bounds on known geometric invariants: the Lipschitz–Killing invariants, and the Vitushkin variations.

Résumé. — Cet article borne de manière explicite la somme des nombres de Betti locaux d'un germe analytique réel par un polynôme en la multiplicité du germe. Ce résultat peut être interprété comme une localisation de la borne classique d'Oleinik–Petrovsky (ou borne de Thom–Milnor) de la somme des nombres de Betti d'un ensemble semi-algébrique. Les éléments clefs de la preuve sont le cône tangent du germe, le théorème de trivialité topologique de Thom–Mather, la borne d'Oleinik–Petrovsky, et un résultat de D. Mumford et J. Heintz bornant le degré des générateurs d'un idéal par un polynôme en la multiplicité du degré géométrique de la variété qui lui est associée. Le résultat est ensuite utilisé pour borner des invariants géométriques connus : les invariants de Lipschitz–Killing et les variations de Vitushkin.

Introduction

Experience from the complex case shows that multiplicity is key to controlling the local Betti numbers. In the real case, there are also results localizing the Oleinik–Petrovsky bound. In [16], F. Loeser presents bounds in terms of the monodromy and homological invariants. In [7], R.N. Draper

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carries out a very thorough investigation of multiplicity from another angle, and characterizes it in several different manners, such as the Lelong number (see [7, Theorem 7.3]), and with a similar approach as ours, as the geometric degree of the tangent cone to the germ by looking at the local ring of the germ (see [7, Theorem 6.5]). It is thus natural to think that the multiplicity could be used to bound the sum of local Betti numbers in the real case too. However, we show it is in fact not always possible to do so.

We first give a set of sufficient conditions on the dimension of the singular locus of the tangent cone for a polynomial bound in the multiplicity to exist (Theorem 1.13). The proof is carried out in Section 2 by applying the Thom–Mather topological trivialization theorem ([11], [17]) to relate the topology of the germ to that of its tangent cone, followed by the Mumford–Heintz theorem ([13], [19]) and the Oleinik–Petrovsky bound to control the sum of local Betti numbers in the tangent cone and thus the original germ.

We then show that these conditions are actually optimal (Theorem 1.17) in Section 3 by exhibiting families of germs with constant multiplicity but arbitrarily large 0^{th} Betti number whenever the conditions of Theorem 1.13 are not met.

Finally in Section 4, we apply our bound to interesting geometric invariants from singularity theory such as the local density of the germ (see [4, 6]), and with greater generality, the Lipschitz–Killing curvature invariants ([2, 6, 23], and Definition 4.1) and the Vitushkin variations ([14, 23], and Definition 4.2). These quantities play an important role in defining notions of equisingularity for real varieties ([5, 6]). The local number of connected components in a generic affine section is also relevant to localizing results on entropy by means of the Vitushkin variations (see [23, Theorem 3.5]). These quantities are linked to the local Betti numbers by an integral known as the local multidimensional Cauchy–Crofton formula ([6] for the multidimensional case, and [4, 5] for the original result about the density only). It inherits its name from the classical Cauchy–Crofton formula (see [9] or [10]) which is a global quantity. The sum of Betti numbers bounds both the number of connected components which serves to define the Vitushkin variations and the Euler characteristic which serves to define the Lipschitz–Killing curvature invariants (the density being both a Vitushkin variation and a Lipschitz–Killing invariant). By the local multidimensional Cauchy–Crofton formula it is thus possible to derive sharp local bounds in terms of the multiplicity for the density, the Lipschitz–Killing invariants, and the Vitushkin variations.

1. Statement of Main Results

1.1. Local Betti Numbers

We set the following conventions:

- We will use the letter X for real varieties, Z for complex varieties, and Y for varieties that can be either real or complex.
- The symbol $\mathbb{G}_{k,n}$ will denote the Grassmanian, and $\overline{\mathbb{G}}_{k,n}$ the affine Grassmanian, which we will parametrize by $D \in \mathbb{G}_{k,n}$ and $t \in D^{\perp}/\{0\}$.
- A property will be generically true iff it is true outside a set of measure zero (algebraically generic properties are thus *a fortiori* generic).

Definition 1.1. — *Let $S \subset \mathbb{R}^n$ be a triangulable set. The i^{th} Betti number $\mathfrak{b}_i(S)$ of S is the rank of the i^{th} homology group of S.*

Proposition 1.2 (Local Betti Numbers). — *Let $X \subset \mathbb{R}^n$ be an analytic germ at 0, let $D \in \mathbb{G}_{k,n}$ and $t \in D^{\perp}/\{0\}$.*

Define the local directional Betti number $\mathfrak{b}_i(X, D, t)$ as

$$\mathfrak{b}_i(X, D, t) = \lim_{\epsilon \to 0} \lim_{\lambda \to 0} \mathfrak{b}_i(X \cap (\lambda t + D) \cap B_{0,\epsilon}),$$

where $\epsilon > 0$ and $\lambda \in \mathbb{R}_+^*$.

Define the local Betti number $\mathfrak{b}_i(X, k)$ as the infimum of the constants c such that $c \geqslant \mathfrak{b}_i(X, D, t)$ generically for $(D + \lambda t) \in \overline{\mathbb{G}}_{k,n}$.

Then the limits defining $\mathfrak{b}_i(X, D, t)$ always exist, and thus $\mathfrak{b}_i(X, D, T)$ and $\mathfrak{b}_i(X, k)$ are well-defined.

Proof. — The family $(X \cap (\lambda t + D) \cap B_{0,\epsilon})_{\lambda,\epsilon}$ is subanalytic. We can thus apply Hardt's theorem which shows that there are only finitely many topological types over λ for each fixed ϵ. The limit $\lim_{|\lambda| \to 0} \mathfrak{b}_i(X \cap (\lambda t + D) \cap B_{0,\epsilon})$ thus exists for every ϵ, and the function $\epsilon \mapsto \lim_{|\lambda| \to 0} \mathfrak{b}_i(X \cap (\lambda t + D) \cap B_{0,\epsilon})$ is well-defined. Since limits can be written as first order logical formulas the graph of this function is also subanalytic, and thus analytic in a neighborhood of 0. Since the function is discrete (values in $\mathbb{N}$) it is actually constant in this neighborhood, and the limit $\lim_{\epsilon \to 0} \lim_{|\lambda| \to 0} \mathfrak{b}_i(X \cap (\lambda t + D) \cap B_{0,\epsilon})$ exists. □

Definition 1.3 (Local Sum of Betti Numbers). — *Let the local sum of directional Betti numbers for D and t be*

$$\sum_{\mathfrak{b}}(X, D, t) = \sum_{i=0}^{n} \mathfrak{b}_i(X, D, t).$$

Let the local sum of Betti numbers $\sum_{\mathfrak{b}}(X,k)$ be the infimum of the constants $c \in \mathbb{R}$ such that $c \geqslant \sum_{\mathfrak{b}}(X,D,t)$ generically for $(D+\lambda t) \in \overline{\mathbb{G}}_{k,n}$.

Remark 1.4. — Let $\sum'_{\mathfrak{b}}(X,k) = \sum_{i=0}^{n} \mathfrak{b}_i(X,k)$, the sum of local Betti numbers. Note that $\sum_{\mathfrak{b}}(X,k) \geqslant \sum'_{\mathfrak{b}}(X,k)$ since $\sum'_{\mathfrak{b}}(X,k)$ is a sum of infima and is thus less than $\sum_{\mathfrak{b}}(X,k)$, the infimum of the sum. This is why we choose to control the local sum of Betti numbers rather than the sum of local Betti numbers.

1.2. Algebraic Tangent Cone

As we mentioned earlier, for a real analytic germ X, it is not always possible to bound $\mathfrak{b}_0(X,k)$ (hence $\sum_{\mathfrak{b}}$) in terms of the multiplicity. We introduce the notion of algebraic tangent cone that will serve to formulate the condition under which a bound exists, and define the multiplicity of a germ in relation to it.

In the following, $\mathbb{K}$ is a notation for $\mathbb{C}$ or $\mathbb{R}$.

Definition 1.5. — *For any analytic germ at the origin $f \in \mathbb{K}((X_1,\ldots,X_n))$, let $\mu(f)$ denote the lowest total degree of the monomials of f, and let the initial part of f, $\mathrm{I}nit(f)$, be the sum of the monomials of f with degree $\mu(f)$.*

Since $\mu(f)$ is the lowest degree of the monomials of f, $\epsilon^{\mu(f)}$ can be factored out from the power series expansion of f at 0, and thus $f(\epsilon X_1, \ldots, \epsilon X_n)/\epsilon^{\mu(f)}$ is also an analytic germ in $\mathbb{K}((X_1,\ldots,X_n,\epsilon))$ (hence well-defined at $\epsilon = 0$). Based on this observation, we can define the conic blow-up of f:

Definition 1.6 (Conic Blow-up). — *Let $Y \in \mathbb{K}^n$ be an analytic germ at 0. Let $\mathcal{J} \subset \mathbb{K}((X_1,\ldots,X_n))$ be the ideal of functions vanishing on Y. We define the conic blow-up ideal as*

$$\tilde{J} = \{f(\epsilon X_1,\ldots,\epsilon X^n)/\epsilon^{\mu(f)} \mid f \in \mathcal{J}\} \subset \mathbb{K}((X_1,\ldots,X_n,\epsilon)).$$

We call $V(\tilde{J})$ the conic blow-up of Y, and we conceive of it as a family of varieties in $\mathbb{K}^n$ over the dimension $\epsilon \in \mathbb{K}$, therefore we write

$$(\tilde{Y}_\epsilon)_{\epsilon\in\mathbb{K}} = V(\tilde{J}) \subset \mathbb{K}^n \times \mathbb{K},$$

and $\tilde{Y}_\epsilon$ denotes the fiber in $\mathbb{K}^n$ defined by $\tilde{J}$ with ϵ fixed.

One can note that $\tilde{Y}_0 = V(\{\mathrm{I}nit(f) \mid f \in \mathcal{J}\})$, and $\tilde{Y}_1 = Y$. We will also use the fact that for all non-zero ϵ the $\tilde{Y}_\epsilon$'s are homothetic to each other.

DEFINITION 1.7 (Algebraic tangent cone). — *Let $Y \in \mathbb{K}^n$ be an analytic germ at 0. The algebraic tangent cone $T(Y)$ of Y is defined as the scheme $T(Y) = \tilde{Y}_0$.*

Remark 1.8. — It is worth noting that although Y is analytic, $T(Y)$ is in fact algebraic because the initial parts of the defining analytic functions are polynomials. The geometric significance of this observation is given by a result by H. Whitney in [21, Theorem 5.8] which asserts that for a complex germ Z, the tangent cone $T(Z)$ is equal to the set of limits of secants of the germ at 0. However, for a real analytic germ, the algebraic tangent cone may contain additional points that do not come from limits of secants.

We will repeatedly interpret cones as projective varieties and we introduce this definition to remove ambiguity:

DEFINITION 1.9 (Cone as Projective Variety). — *Let $C \subset \mathbb{K}^n$ be a cone centered at 0, then $C^{\mathbb{P}}$ denotes its associated interpretation as a subset of $\mathbb{P}^{n-1}(\mathbb{K})$.*

We can now give a definition of the multiplicity based on the tangent cone. This approach stems from the work of R.N. Draper in [7].

PROPOSITION 1.10. — *Let $Z \subset \mathbb{C}^n$ be a complex analytic germ of dimension d at 0. Let T be the algebraic tangent cone to Z, and let T_d be the union of the components of T of dimension d.*

Then for all $D \in \mathbb{G}_{n-d+1,n}$ such that $T_d^{\mathbb{P}} \cap D \subset \mathbb{P}^{n-1}(\mathbb{C})$ is a zero-dimensional variety, the dimension over $\mathbb{C}$ of the function sheaf of $\left(T_d^{\mathbb{P}} \cap D\right)$ is always equal to the same integer $\mu(Z)$.

We call $\mu(Z)$ the multiplicity of the germ Z. By extension, for $X \subset \mathbb{R}^n$ a real analytic germ, $\mu(X)$ denotes the multiplicity of the complexification of X.

Proof. — This is a straightforward consequence of the results in [7]: [7, Theorem 6.4] shows that the degree of the projective variety $T_d^{\mathbb{P}}$ is equal to the degree of Z at 0, and then [7, Theorem 6.5] shows that the degree of Z at 0 is the multiplicity of the germ as defined classically by means of the Hilbert polynomial. □

At this point we emphasize the use of the word scheme in the definition of the tangent cone: the tangent cone is not necessarily defined by a radical ideal. The properties of this potentially non-radical ideal are key to proving our results. For instance, for $Z = V(x^2 - y^3)$ in $\mathbb{C}^2$, $T(Z) = x^2$, and the multiplicity according to the previous definition is 2, not 1.

With this distinction in mind, let us define the singular locus of the algebraic tangent cone:

DEFINITION 1.11 (Singularity of tangent cone). — *Let T be the tangent cone for an analytic variety. We define* $\operatorname{Sing} T$ *as the singular locus of T: the points p where the local function sheaf at p is not regular.*

Note that the previous definition is applicable whether T is considered a cone in $\mathbb{K}^n$ or a variety in $\mathbb{P}(\mathbb{K})^{n-1}$ and yields the same result. Also, because regularity of the local ring can be expressed as a condition on the rank of the Jacobian matrix, $\operatorname{Sing} T$ itself is an algebraic variety.

Remark 1.12. — Effective calculation of the tangent cone for algebraic germs can be carried out using Gröbner bases (also known as standard bases). If $g_1, \ldots, g_l$ is a set of generators of $\mathcal{I}$ such that $(\operatorname{Init}(g_1), \ldots, \operatorname{Init}(g_l)) = \operatorname{Init}(\mathcal{I})$, it is called a Gröbner basis of $\mathcal{I}$. Therefore the initial parts of the generators of the Gröbner basis are generators of the algebraic tangent cone. One of the most efficient algorithms to compute Gröbner bases is Faugère's F_5 algorithm [8].

1.3. Main Theorem and Optimality Thereof

We now state our main result about the local sum of Betti numbers in a generic affine section of a real analytic germ (Theorem 1.13) and the associated optimality Theorem 1.17.

THEOREM 1.13 (Main Theorem). — *Let $X \subset \mathbb{R}^n$ be a real analytic germ at 0 of dimension d. Let T be the algebraic tangent cone to X, and let $s = \dim(\operatorname{Sing} T)$. Then for any $k \in \mathbb{N}$ such that $2 \leqslant k \leqslant n-1$*

(1) *if $k < n-d$, for A generic in $\overline{\mathbb{G}}_{k,n}$, A avoids X, and thus $\sum_{\mathfrak{b}}(X,k) = 0$.*
(2) *if $k = n-d$, we have the inequality $\mathfrak{b}_0(X,k) \leqslant \mu(X)$.*
Since $k = n-d$, $\forall i > 0$, $\mathfrak{b}_i(X,k) = 0$ and the previous inequality can be rewritten $\sum_{\mathfrak{b}}(X,k) \leqslant \mu(X)$.
(3) *if $n-d < k < n-s$ and the complexification of X is pure dimensional, then*

$$\sum_{\mathfrak{b}}(X,k) \leqslant \mu(X)(2\mu(X)-1)^{k-1}.$$

Remark 1.14. — Point (2) in Theorem 1.13 can be seen as a generalization of [20, Lemma 1.4]. This lemma shows that $\mathfrak{b}_0(X,k) \leqslant \mu(X)$ for

hypersurfaces and the Main Theorem generalizes it to the case of an arbitrary germ X.

The following proposition shows that pure dimensionality of the complexification of a real germ is strictly weaker than pure dimensionality of the real germ itself:

PROPOSITION 1.15. — *Let $X \subset \mathbb{R}^n$ be a real analytic germ at 0.*

(1) *If X is pure dimensional of dimension d, then its complexification $X_{\mathbb{C}}$ is also pure dimensional of dimension d.*
(2) *The real germ defined by $f(x,y,z) = x^4 + 4x^3z + 4x^2z^2 - y^2z^2$ is not pure dimensional but its complexification is.*

Proof of (1). — For any point p and an analytic variety Y, we define the vector space of differentials $D(Y,p) = \{D_p(f) \mid f \in V(Y)\}$. Since the defining functions of $X_{\mathbb{C}}$ and X are the same, at any point $p \in X$, $D(X,p) \otimes \mathbb{C} = D(X_{\mathbb{C}},p)$ and $\dim_{\mathbb{R}}(D(X,p)) = \dim_{\mathbb{C}}(D(X_{\mathbb{C}},p))$. In particular at any smooth point p, the codimension of a variety is the dimension of the vector space of differentials, so the dimensions of $X_{\mathbb{C}}$ and X are the same at any smooth point.

Let Z be any irreducible component of $X_{\mathbb{C}}$. Since the complexification is the minimal complex variety containing X, Z must intersect $\mathbb{R}^n$. Since $Z \cap \mathbb{R}^n \subset X$ is itself an analytic variety it contains a smooth point p in X. By what precedes the dimensions of Z and X are the same. □

Proof of (2). — The graph of $f(x,y,1)$, has two humps: one emerges over the plane $f = 0$ and the other one is tangent to it, which creates an oval and a point. However, the complex variety is pure dimensional as f is irreducible (this can be shown elementarily by attempting a factorization in y, since $\deg_y f$ is only 2). □

The following theorem precisely states how our Main Theorem 1.13 is optimal:

DEFINITION 1.16 (Counter-example Family). — *For any $(n,k,s) \in \mathbb{N}^3$ we say that a family of germs $(X_l)_{l\in\mathbb{N}}$ is a counter-example family of type (n,k,s) if and only if $\forall l$, $X_l \subset \mathbb{R}^n$, $\dim(\operatorname{Sing} T(X_l)) = s$, $\mu(X_l)_{l\in\mathbb{N}}$ is bounded by a constant, and $\lim_{l\to\infty} \mathfrak{b}_0(X_l,k) = +\infty$*

As a short-hand we will say that a counter-example family is pure dimensional if the complexifications *of the germs in the family are all pure dimensional.*

Note that the definition imposes a condition on $\mathfrak{b}_0$ rather than a weaker one on $\sum_{\mathfrak{b}}$ only because our counter-examples only require $\mathfrak{b}_0$.

THEOREM 1.17 (Optimality of Main Theorem). — *For any $n, k \in \mathbb{N}$ such that $2 \leqslant k \leqslant n-1$,*

(1) *for any $s \in \mathbb{N}$ such that $s \leqslant n-1$, there exists a counter-example family of type (n, k, s) that is not pure dimensional.*
(2) *for any $s \in \mathbb{N}$ such that $n-k \leqslant s \leqslant n-1$, there exists a pure dimensional counter-example family of type (n, k, s).*

Remark 1.18. — The optimality Theorem 1.17 complements Theorem 1.13 exactly since for any $s \in \{0, \dots, n-1\}$, Theorem 1.13 covers pure dimensional germs for all $k < n - s$, and Theorem 1.17 provides counter-example families that are either not pure dimensional, or pure dimensional with $k \geqslant n - s$.

2. Proof of Main Theorem

We now prove the Main Theorem 1.13. Point (1) of the Main Theorem is self-evident since generic sections by affine space of dimension less than the codimension of the germ are generically empty. In Subsection 2.1, we begin by proving point (2), where the dimension of the intersecting affine spaces is equal to the codimension of the germ, by using our careful characterization of multiplicity based on the tangent cone. We then prove point (3), where the dimension of the intersecting affine spaces is greater than the codimension of the germ, by showing that for a generic affine space, the topology of the intersection with the tangent cone is the same as the topology of the intersection with the germ (Subsection 2.2), hence their Betti numbers are the same. The equivalence between both topologies is proved using the Thom–Mather topological trivialization theorem (see [17] or [11]). Then, we use Proposition 2.2 to prove that the multiplicity of the germ is equal to the geometric degree of the sections of the tangent cone. This finally allows us to use the Heintz–Mumford result (Theorem 2.11) to bound the degree of the generators of the sections of the tangent cone by the multiplicity, and thereby to bound the sum of the Betti numbers by a polynomial function of the multiplicity using the Oleinik–Petrovsky bound (Theorem 2.9).

2.1. Proof of point (2) of the Main Theorem

It is a basic fact that the multiplicity of a *complex* germ is equal to the number of points close enough to the origin that lie in the intersection of

the germ with a generic affine space whose dimension is the codimension of the germ. Therefore, point (2) of the Main Theorem 1.13 could seem straightforward. However, we are considering real germs, and real affine spaces are contained in a set of complex affine spaces of measure zero, so generic equality over $\mathbb{C}$ may not entail generic equality over $\mathbb{R}$. We thus need a slightly more refined characterization of the multiplicity than this to conclude.

We introduce such a characterization as Lemma 2.2, which is formulated as geometric conditions on the tangent cone so that we can apply it to prove point (2) of the Main Theorem as Corollary 2.3, as well as to prove point (3) of the Main Theorem later on in the proof of Theorem 2.12.

DEFINITION 2.1. — *Let $Z \in \mathbb{C}^n$ a Zariski zero-dimensional scheme, then $\dim_{\mathbb{C}} \mathcal{O}(Z)$ denotes the vector space dimension over $\mathbb{C}$ of $\mathcal{O}(Z)$, the function sheaf of Z. In other words, $\dim_{\mathbb{C}} \mathcal{O}(Z)$ counts points in Z with multiplicity. This is the same as the length of $\mathcal{O}(Z)$ localized at p over the maximal ideal at p, which is the classical definition of module multiplicity algebraically (see* [12, Chapter 7] *for instance).*

LEMMA 2.2. — *Let $Z \in \mathbb{C}^n$ be a complex germ at 0 of dimension d and multiplicity μ. Let $T = \tilde{Z}_0$ be the tangent cone to Z, and let $T_{<d}$ be the union of the components of the tangent cone with dimension less than d. Let $k = n - \dim(Z)$. Let $D \in \mathbb{G}_{k,n}$ and $t \in D^{\perp}/\{0\}$. Assume $D \cap T = \{0\}$, $(t+D) \cap T$ is 0-dimensional and $(t+D) \cap T_{<d} = \emptyset$ in $\mathbb{C}^n$. Then*

$$\exists \epsilon_0 > 0,\ s.t.\ \forall \epsilon \in]0, \epsilon_0[,\ \exists \lambda_0 > 0,\ s.t.\ \forall \lambda \in]0, \lambda_0[,$$
$$\dim_{\mathbb{C}} \mathcal{O}\big((\lambda t + D) \cap B_{0,\epsilon} \cap Z\big) = \dim_{\mathbb{C}} \mathcal{O}\big((t+D) \cap T\big) = \mu$$

Proof. — Consider $Z'_\epsilon = (\mathbb{C}t + D) \cap B_{0,1} \cap \tilde{Z}_\epsilon$. For $\epsilon \neq 0$, it is homothetic to $(\mathbb{C}t + D) \cap B_{0,\epsilon} \cap Z$ because $\epsilon \tilde{Z}_\epsilon = \tilde{Z}_1 = Z$, $\epsilon B_{0,1} = B_{0,\epsilon}$ and $(\mathbb{C}t + D)$ is a vector space hence is invariant by homothety. Since $(t+D) \cap T$ is 0-dimensional, for $\epsilon < \epsilon_0$ small enough Z'_ϵ consists of points and 1-dimensional branches only. By possibly reducing ϵ_0 further, we can ensure that Z'_ϵ only contains 1-dimensional branches that are all connected to the origin. Therefore, each of these branches will give rise to a point in the projective variety $(\mathbb{C}t+D) \cap T$. Since by hypothesis $D \cap T = \{0\}$, there are no points at infinity, and each branch is thus transverse to the $(\lambda t + D)$ for $\lambda < \lambda_1$ small enough. Therefore, branches of Z'_ϵ are in one to one correspondence with the points in $(\lambda t + D) \cap B_{0,1} \cap \tilde{Z}_\epsilon = (\lambda \epsilon t + D) \cap B_{0,\epsilon} \cap Z$, and each

branch gives rise to one point in $(t+D)\cap T$ in $\mathbb{C}^n$. Since $(t+D)\cap T_{<d}=\emptyset$ by hypothesis, all the points in $(t+D)\cap T$ come from a branch of Z'_ϵ, and there is also a one to one correspondence between branches of $\tilde{Z}_\epsilon$ and points in $(t+D)\cap T$. This proves the first equality for ϵ_0 and $\lambda_0=\epsilon_0\lambda_1$.

The equality with the multiplicity stems from the fact that $(t+D)$ avoids $T_{<d}$ by hypothesis, so $\dim_{\mathbb{C}}\mathcal{O}\big((t+D)\cap T\big)=\dim_{\mathbb{C}}\mathcal{O}\big((t+D)\cap T_d\big)$, and since $D\cap T=\{0\}$, $\dim_{\mathbb{C}}\mathcal{O}((t+D)\cap T_d)=\dim_{\mathbb{C}}\mathcal{O}((\mathbb{C}t+D)\cap T_d^{\mathbb{P}})$, which is equal to μ by Proposition 1.10. □

The previous lemma allows us to see that the multiplicity of the germ can be obtained using a limit of intersecting affine spaces chosen in a Zariski open set of the Grassmannian, and not only chosen in an unspecified generic subset of it. This added structure allows us to prove the first item of the Main Theorem:

COROLLARY 2.3. — *Let $X\subset\mathbb{R}^n$ be a real analytic germ at 0 of dimension d. Then $\sum_{\mathfrak{b}}(X,n-d)=\mathfrak{b}_0(X,n-d)\leqslant\mu$.*

Proof. — As the intersection of X and a generic affine space of dimension $(n-d)$ is generically zero-dimensional, $\sum_{\mathfrak{b}}(X,n-d)=\mathfrak{b}_0(X,n-d)$.

The conditions on D and t in Lemma 2.2 are all satisfied on a *Zariski* open set of the affine complex Grassmannian $\overline{\mathbb{G}}_{n-d,n}(\mathbb{C})$. Since the real affine Grassmannian is a subscheme of the complex one, the lemma applies to a Zariski open set of the real affine Grassmannian, which is always generic in the *real* Grassmannian $\overline{\mathbb{G}}_{n-d,n}(\mathbb{R})$. Therefore by Lemma 2.2, the complexification of a generic real affine section of the germ contains μ points, and thus this generic real affine section cannot contain more than μ points. □

2.2. Sections of tangent cone and germ have the same topology

We know that analytic sets admit Whitney stratifications ([22, Theorem 19.2]). In the following it is understood that we stratify the sets we consider with an arbitrary Whitney stratification the first time we encounter them and all the subsequent proofs are carried out with respect to these arbitrary stratifications.

This section shows that we can relate the topology of the tangent cone to that of the germ, and therefore equate their Betti numbers.

THEOREM 2.4. — *Let $X \in \mathbb{R}^n$ be a real analytic germ at 0 and $\tilde{X}_\epsilon$ its conic blow-up at the origin (def. 1.6). Let T be the algebraic tangent cone to X. For any element $D \in \mathbb{G}_{k,n}$ and $t \in D^\perp$, assume that*

(1) *T and D, as stratified projective varieties, are transverse (i.e. the sum of their tangent spaces at intersection points is of maximal dimension, that is $n-1$).*
(2) *T and $(\mathbb{R}t + D)$, as stratified projective varieties, are transverse.*
(3) *The projective variety $(\mathbb{R}t + D)$ does not intersect $\operatorname{Sing} T^{\mathbb{P}}$.*

Then we have

$$\exists \lambda_0 > 0, \exists \epsilon_0 > 0,\ \forall \epsilon \in]0, \epsilon_0],\ \forall \lambda \in]0, \lambda_0],\ \exists h \text{ homeomorphism},$$
$$(t + D) \cap \tilde{X}_0 \overset{h}{\approx} (\lambda t + D) \cap B_{0,1} \cap \tilde{X}_\epsilon$$

The proof hinges on the Thom–Mather topological trivialization theorem. We use a variant known as the Moving the Wall Theorem ([11, Theorem 1]), which we recall here

THEOREM 2.5 (Moving the Wall). — *Let $X \subset \mathbb{R}^n$ be a Whitney stratified set, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth map. Let $X_t := f^{-1}(t) \cap X$, and for any stratum σ of X and $t \in \mathbb{R}$, let $\sigma_t := f^{-1}(t) \cap \sigma$.*

We endow $X_0 \times \mathbb{R}$ with the stratification $\mathcal{S}_0 = \{\sigma_0 \times \mathbb{R} \mid \sigma \text{ stratum of } X\}$. If $f|_X$ is proper, and $\forall t \in \mathbb{R}$, f is a submersion on each stratum of X, then $\mathcal{S}_0$ is a Whitney stratification for $X_0 \times \mathbb{R}$ and there exists a homeomorphism $h : X \to X_0 \times \mathbb{R}$, which is smooth on the strata of X and such that $\Pi_2 \circ h = f$ with Π_2 the projection to $\mathbb{R}$, the second component. In addition h is stratum preserving, that is, for any stratum σ, we have $h(\sigma) \subset (\sigma_0 \times \mathbb{R})$ (note that $(\sigma_0 \times \mathbb{R}) \in \mathcal{S}_0$).

We introduce the following definitions to carry out the proof of Theorem 2.4

DEFINITION 2.6. — *For any $D \in \mathbb{G}_{k,n}(\mathbb{R})$, $t \in D^\perp$ and $\lambda, \kappa \in \mathbb{R}$, we define:*

$$C_\lambda = (\lambda t + D) \cap S_{0,1},$$
$$D_\lambda = (\lambda t + D) \cap B_{0,1},$$
$$H_\lambda = ((\lambda + \mathbb{R}_+^*)t + D) \cap S_{0,1} = \cup_{\kappa > \lambda} C_\kappa,$$
$$F_{\lambda,\kappa} = \left\{ p \in (\lambda + \mathbb{R}_+^*)t + D \mid (1-\kappa)\left(\|p\|_2^2 - 1\right) + \kappa\left((p|t) - \lambda \|t\|_2^2\right) = 0 \right\}$$

where $(.|.)$ is the scalar product, and $\|.\|_2$ is the Euclidian norm.

We also define the topological closures of these sets: $\overline{D_\lambda} = D_\lambda \cup C_\lambda$, $\overline{H_\lambda} = H_\lambda \cup C_\lambda$, and $\overline{F_{\lambda,\kappa}} = F_\lambda \cup C_\lambda$.

The formal definition of $F_{\lambda,\kappa}$ may appear a bit obscure, but $F_{\lambda,\kappa}$ is simply a family of spherical sectors deforming $F_{\lambda,0} = H_\lambda$ into $F_{\lambda,1} = D_\lambda$ while leaving C_λ fixed. The $F_{\lambda,0}, F_{\lambda,0.5}$, and $F_{\lambda,1}$ are depicted in Figure 2.1 as part of the homeomorphism h_3.

Proof of Theorem 2.4. — The complete homeomorphism is built in four steps:

$$(t+D)\cap\tilde{X}_0 \overset{h_1}{\approx} H_0(t,D)\cap\tilde{X}_0 \overset{h_2}{\approx} H_\lambda(t,D)\cap\tilde{X}_0 \overset{h_3}{\approx} D_\lambda(t,D)\cap\tilde{X}_0 \overset{h_4}{\approx} D_\lambda(t,D)\cap\tilde{X}_\epsilon$$

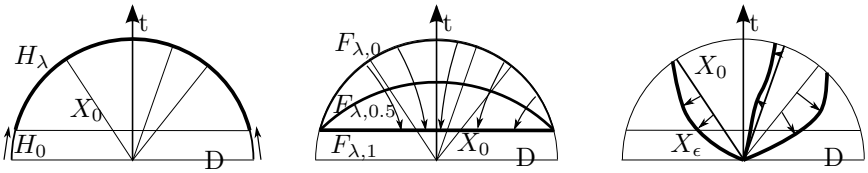


Figure 2.1. From left to right, h_2 retracts the hemisphere H_0 up to H_λ. Then h_3 flattens $H_\lambda (= F_{\lambda,0})$ to $D_\lambda (= F_{\lambda,1})$ (an intermediate transition $F_{\lambda,0.5}$ is also depicted). Finally h_4 perturbs the tangent cone $\tilde{X}_0$ into $\tilde{X}_\epsilon$, which is homothetic to the original germ X.

We define h_1 as

$$h_1 :\ p \in (t+D) \mapsto \frac{p}{\|p\|_2} \in H_0(t,D).$$

This is clearly a homeomorphism, and since $\tilde{X}_0$ is a cone, h_1 leaves $\tilde{X}_0$ stable, and is thus a homeomorphism between $(t+D)\cap\tilde{X}_0$ and $H_0(t,D)\cap\tilde{X}_0$.

We construct the homeomorphisms h_2, h_3 and h_4 depicted in Figure 2.1 by applying the Moving the Wall Theorem to:

$$f_2 = \left\{(\overline{H_\lambda}\cap\tilde{X}_0,\lambda) \subset (\mathbb{R}t+D)\times\mathbb{R} \mid \lambda\in\mathbb{R}\right\} \mapsto \lambda$$
$$f_3 = \left\{(\overline{F_{\lambda,\kappa}}\cap\tilde{X}_0,\kappa) \subset (\mathbb{R}t+D)\times\mathbb{R} \mid \kappa\in\mathbb{R}\right\} \mapsto \kappa$$
$$f_4 = \left\{(\overline{D_\lambda}\cap\tilde{X}_\epsilon,\epsilon) \subset \mathbb{R}^n\times\mathbb{R} \mid \epsilon\in\mathbb{R}\right\} \mapsto \epsilon$$

According to the Moving the Wall Theorem we need only stratify the source spaces of these functions and show they are proper submersions on each stratum to prove the existence of the required homeomorphisms. The possible obstructions that can prevent the functions f from being submersive are depicted in Figure 2.2 and correspond to conditions (1), (2), and (3) in

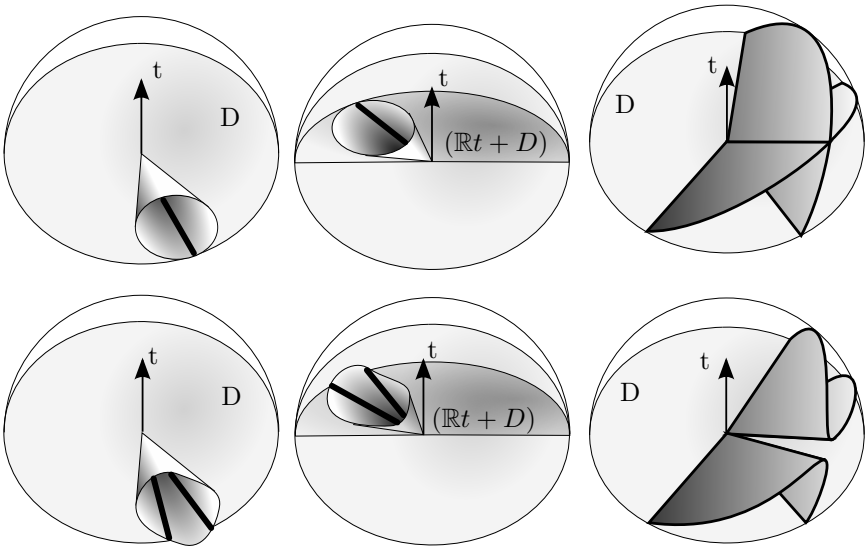


Figure 2.2. The obstructions are shown on top with the unwanted change in topology that may result below. From left to right, they correspond to conditions (1), (2), and (3) in Theorem 2.4: the tangent cone is tangent to D, tangent to $\mathbb{R}t + D$, or singular. In the left and middle column, a line splits into two, and in the right column the surface splits into two wings.

Theorem 2.4. It is key to notice that all the functions f are globally defined for parameters in $\mathbb{R}$, and not just for non-negative λ, κ and ϵ. This is how we prove that f_4 is a submersion at 0.

By condition (3) of Theorem 2.4, $T = \tilde{X}_0$ is smooth in a neighborhood of $(\mathbb{R}t+D)$. By condition (2), $\tilde{X}_0$ and $(\mathbb{R}t+D)$ are transverse and $\tilde{X}_0 \cap (\mathbb{R}t+D)$ is thus also smooth. Therefore the constant family $(\tilde{X}_0 \cap (\mathbb{R}t + D)) \times \mathbb{R} \subset (\mathbb{R}t + D) \times \mathbb{R}$ consists of a single smooth stratum.

On the other hand the families $\overline{H_\lambda}$, and $\overline{F_{\lambda,\kappa}}$ are Whitney stratified by their interiors (H_λ and $F_{\lambda,\kappa}$) and C_λ. To apply the Moving the Wall Theorem to f_2 and f_3, it thus suffices to show that the stratifications of these latter families are transerve to $(\tilde{X}_0 \cap (\mathbb{R}t + D)) \times \mathbb{R} \subset (\mathbb{R}t + D) \times \mathbb{R}$.

For f_2 the two strata to consider are C_λ and H_λ. For C_λ, condition (1) in Theorem 2.4 ensures that C_0 is transverse to $T = \tilde{X}_0$, and since transversality is an open condition and we work in a compact set C_0, there exists $\lambda_0 > 0$ such that C_λ is transverse to $\tilde{X}_0$ for all $\lambda \in [0, \lambda_0]$. Since $\tilde{X}_0$ is a cone, the tangent space at $p \in \tilde{X}_0$ contains the direction $(p - 0)$ which is

never contained in the tangent space at $p \in H_\lambda$. Since H_λ is of codimension 1 in $(\mathbb{R}t + D)$, the sum of the tangent spaces to $\tilde{X}_0 \cap (\mathbb{R}t + D)$ and to H_λ is of maximal dimension, that is $\tilde{X}_0 \cap (\mathbb{R}t + D)$ and H_λ are transverse for all $\lambda \geqslant 0$. Therefore $C_\lambda \cap \tilde{X}_0$ and $H_\lambda \cap \tilde{X}_0$ stratify the source space of f_2, and since projection to the parameter of a family (λ here) is automatically a submersion, f_2 is submersive on this stratification. The Moving the Wall Theorem thus asserts the existence of h_2.

For f_3, we consider the strata C_λ and $F_{\lambda,\kappa}$. As we just mentioned C_λ is transverse to $\tilde{X}_0$ for λ small enough, and since C_λ is fixed as the parameter κ varies this transversality is retained and $C_\lambda \cap \tilde{X}_0$ is a stratum. In a similar way as for H_λ above, the sets $F_{\lambda,\kappa}$ ($\kappa \in [0,1]$) are all transverse to $\tilde{X}_0$ in $(\mathbb{R}t + D)$ since $\tilde{X}_0$ is a cone. Therefore $F_{\lambda,\kappa} \cap \tilde{X}_0$ and $C_\lambda \cap \tilde{X}_0$ stratify the source space of f_3, and f_3 is a submersion because it projects to the parameter κ on these strata. This shows the existence of h_3 by the Moving the Wall Theorem.

For f_4 we need to consider the strata C_λ and D_λ that make up $\overline{D_\lambda}$. As λ was picked small enough earlier so that C_λ and D_λ are transverse to $\tilde{X}_0$, and since transversality is an open condition and $\overline{D_\lambda}$ is compact, $\tilde{X}_\epsilon$ will remain transverse to C_λ and D_λ in $(\mathbb{R}t + D)$ for all $\epsilon \in (-\epsilon_1, \epsilon_1)$ for ϵ_1 small enough. Furthermore, by condition (2), $(\mathbb{R}t + D)$ and $\tilde{X}_0$ are transverse, so by reducing ϵ_1 further to $\epsilon_0 > 0$, we can ensure that $\tilde{X}_\epsilon$ is transverse to C_λ and D_λ in $\mathbb{R}^n$ for $\epsilon \in (-\epsilon_0, \epsilon_0)$.

Therefore $C_\lambda \cap \tilde{X}_\epsilon$ and $D_\lambda \cap \tilde{X}_\epsilon$ ($\epsilon \in (-\epsilon_0, \epsilon_0)$) form a Whitney stratification of the source space of f_4, and h_4 exists by the Moving the Wall Theorem. □

COROLLARY 2.7. — *Let $X \in \mathbb{R}^n$ be a real analytic germ at 0 and T its tangent cone. Let $\mathfrak{b}_i$ be any local Betti number. For any elements $D \in \mathbb{G}_{k,n}$ and $t \in D^\perp$, if conditions (1), (2) and (3) in Theorem 2.4 are satisfied, then*

$$\mathfrak{b}_i(X, D, t) = \mathfrak{b}_i(T, D, t).$$

Proof. — We have:

$$\begin{aligned}
\mathfrak{b}_i(X, D, t) &= \lim_{\epsilon \to 0} \lim_{\lambda \to 0} \mathfrak{b}_i((\lambda t + D) \cap B_{0,\epsilon} \cap X) && \text{by definition} \\
&= \lim_{\epsilon \to 0} \lim_{\lambda \to 0} \mathfrak{b}_i((\lambda/\epsilon t + D) \cap B_{0,1} \cap \tilde{X}_\epsilon) && \text{by } 1/\epsilon\text{-homothety} \\
&= \mathfrak{b}_i((t + D) \cap \tilde{X}_0). && \text{by Theorem 2.4}
\end{aligned}$$

As T is already a cone, its conic blow-up is the constant family $\tilde{T}_\epsilon = \tilde{X}_0 = T$, and thus $\mathfrak{b}_i((t + D) \cap \tilde{X}_0) = \mathfrak{b}_i(T, D, t)$, which completes the proof. □

We use the previous corollary, which is dependent on a direction D, to yield the desired relation between the Betti numbers of the germ and of its tangent cone, which is direction-independent.

COROLLARY 2.8. — *For $X \subset \mathbb{R}^n$ be a real analytic germ with pure dimensional complexification. Let T be the tangent cone to X, and let s be the dimension of* $\operatorname{Sing} T \subset \mathbb{R}^n$. *Let $k \in \mathbb{N}$ such that $k+s < n$, and $\mathfrak{b}$ be any function of the local Betti numbers $(\mathfrak{b}_i(.,k))_{i\in\mathbb{N}}$. Then $\mathfrak{b}(X,k) = \mathfrak{b}(T,k)$.*

Proof. — It suffices to show that the three conditions of Theorem 2.4 are satisfied for $D \in \mathbb{G}_{k,n}$ generic and $t \in D^\perp$ generic, because then all the directional local Betti numbers will be equal at once.

We first consider conditions (1) and (2). It is true in general that for $L \in \mathbb{G}_{m,n}$ and for $W^{\mathbb{P}}$, a Whitney stratified projective analytic variety, $L^{\mathbb{P}}$ and $W^{\mathbb{P}}$ are generically transverse. Applying it to $L \in \mathbb{G}_{k+1,n}$ and $W^{\mathbb{P}} = T^{\mathbb{P}}$ we have that $L^{\mathbb{P}}$ and $T^{\mathbb{P}}$ are transverse for L in a set $G \subset \mathbb{G}_{k+1,n}$ of measure one. Therefore for $D \in \mathbb{G}_{k,n}$ in a subset $G_2 \subset \mathbb{G}_{k,n}$ of measure one, we have $(\mathbb{R}t+D) \in G$ for t generic in $D^\perp$. In other words, condition (2) of Theorem 2.4 is met for $D \in G_2$ and t generic in $D^\perp$.

Similarly, for $D \in \mathbb{G}_{k,n}$ and $W^{\mathbb{P}} = T^{\mathbb{P}}$ we have that D and $T^{\mathbb{P}}$ are transverse over a set $G_1 \subset \mathbb{G}_{k,n}$ of measure one.That is, condition 1 is met for $D \in G_1$. Since $G_1 \cap G_2 \subset \mathbb{G}_{k,n}$ is of measure one as the intersection of two measure one sets, conditions 1 and 2 are met for $D \in \mathbb{G}_{k,n}$ generic and $t \in D^\perp$ generic.

For condition (3), we have $\dim\left((\mathbb{R}t + D)^{\mathbb{P}}\right) = k$ and $\dim(\operatorname{Sing} T^{\mathbb{P}}) = s-1$ as projective varieties. Since $k + s < n$, $\dim\left((\mathbb{R}t + D)\right) + \dim\left(\operatorname{Sing} T^{\mathbb{P}}\right) = k+s-1 < n-1$, we know that $(\mathbb{R}t+D)$ generically avoids $\operatorname{Sing} T$ in $\mathbb{P}^{n-1}$. That is, condition (3) is generically true. □

2.3. Proof of point (3) of Main Theorem

In order to finish the proof of the Main Theorem we need two more auxiliary results. The first one is the well-known Oleinik–Petrovsky bound on the number of connected components of a real (affine or projective) algebraic variety (see [18, proof of Theorem 2] or [1, 3]).

THEOREM 2.9 (Oleinik-Petrovski/Thom–Milnor bound). — *If X is an algebraic variety defined by polynomials of degree at most d in $\mathbb{R}^n$ or $\mathbb{P}^n(\mathbb{R})$ then*

$$\sum_{\mathfrak{b}}(X) \leqslant d(2d-1)^{n-1}.$$

The second auxiliary theorem was proved by J. Heintz ([13, Proposition 3]). D. Mumford also obtained a similar result that appears in the proof of [19, Theorem 1] (the proof is less detailed than in J. Heintz's presentation). It states that if μ is the geometric degree of a pure dimensional affine variety, then its defining ideal is generated in degree μ.

DEFINITION 2.10 (Geometric Degree). — *Let $Z \subset \mathbb{C}^n$ be a variety of dimension d such that the intersection with a generic $(n-d)$ affine space contains $\delta(Z)$ points (counted with multiplicity). Then $\delta(Z)$ is the geometric degree of Z.*

We will use the following straightforward generalization of their result:

THEOREM 2.11 (Generators in bounded degree). — *Let $Z \subset \mathbb{C}^n$ be a pure dimensional variety with geometric degree δ. Then there exist finitely many generators $(g_i)_{i\in I} \in \mathcal{O}(\mathbb{C}^n)$ such that $\deg g_i \leqslant \delta$ $(\forall i \in I)$, and $Z = V((g_i)_{i\in I})$.*

Proof. — J. Heintz's result in [13, Proposition 3] states that for an irreducible variety $Y \subset \mathbb{C}^n$, there exist an ideal $\mathcal{I} = (g_1, \ldots, g_{n+1}) \subset \mathcal{O}(\mathbb{C}^n)$ such that $Y = V(\mathcal{I})$ and $\deg g_i \leqslant \delta(Y)$ where $\delta(Y)$ is the geometric degree of Y.

Since Z is equidimensional, it can be broken down into its irreducible components $V(\mathcal{I}_i)$ so that $Z = V(\prod_i \mathcal{I}_i)$, it is thus clear from the definition of the geometric degree that $\delta(Z) = \sum_i \delta(V(\mathcal{I}_i))$. And by applying J. Heintz's result to each irreducible component we obtain generators g_{ij} such that $\mathcal{I}_i = (g_{i1}, \ldots, g_{ik_i})$ with $\deg g_{ij} \leqslant \delta(V(\mathcal{I}_i))$. Consequently, $V(Z)$ is generated by products $\prod_i g_{ij_i}$, and each of these products is of degree $\sum_i \delta(V(\mathcal{I}_i)) = \delta(Z)$. □

Equipped with these last two theorems we can finally prove the third item of the Main Theorem 1.13.

THEOREM 2.12. — *For $X \subset \mathbb{R}^n$ an analytic germ of pure dimension d, let μ be the multiplicity $\mu(X)$ of X, and let s be the dimension of the singular locus of $T(X)$. If $k + s < n$, then $\sum_{\mathfrak{b}}(X, k) \leqslant \mu(2\mu - 1)^{k-1}$.*

Proof. — The conditions that $s + k < n$ and X is pure dimensional are the conditions required to apply Corollary 2.8. By applying it to the sum of Betti numbers, we obtain $\sum_{\mathfrak{b}}(X, k) = \sum_{\mathfrak{b}}(T, k)$. We now bound $\sum_{\mathfrak{b}}(T, k)$.

First we show that generic sections of T have geometric degree μ. Let $D \in \mathbb{G}_{k,n}(\mathbb{R})$ and $t \in D^{\perp}/\{0\}$. Let $T_{\mathbb{C}}$, $t_{\mathbb{C}}$ and $D_{\mathbb{C}}$ be the complexifications of T, t and D, and let $S = (t_{\mathbb{C}} + D_{\mathbb{C}}) \cap T_{\mathbb{C}}$. Let $T_{<d}$ be the union of components of $T_{\mathbb{C}}$ of dimension less than d. As X is pure d-dimensional, $T_{<d} \subset \operatorname{Sing} T$.

The condition $k+s<n$ thus implies that $\dim T_{<d}<n-k$, and thus for generic $D_{\mathbb{C}}$ and $t_{\mathbb{C}}$, $(t_{\mathbb{C}}+D_{\mathbb{C}})$ avoids $T_{<d}$. Furthermore, since $(t_{\mathbb{C}}+D_{\mathbb{C}})$ is generically transverse to $T_{\mathbb{C}}$, we have that S is a pure dimensional complex variety of dimension $k+d-n$ in $t_{\mathbb{C}}+D_{\mathbb{C}}\approx\mathbb{C}^k$.

Now that we have picked a suitable generic section S, we show its geometric degree is μ. Let $D'\in\mathbb{G}_{n-d,n}(\mathbb{C})$, and $t'\in D'^{\perp}$ generic such that $t'+D'\subset t_{\mathbb{C}}+D_{\mathbb{C}}$. As D is generic and $D'\subset D$ we can assume that $T\cap D'=\{0\}$, and $(t'+D')\cap T$ is Zariski 0-dimensional. Also $(t'+D')\cap T_{<d}=\emptyset$ as $(t_{\mathbb{C}}+D_{\mathbb{C}})$ avoids $T_{<d}$. Therefore we can apply Lemma 2.2 to T,t', and D', and we have $\mu=\dim_{\mathbb{C}}\mathcal{O}((t'+D')\cap T)$. Since $(t'+D')\cap T=(t'+D')\cap S$, we generically have $\mu=\dim_{\mathbb{C}}\mathcal{O}((t'+D')\cap S)$, that is the geometric degree of S is μ.

As we have made sure that S is pure dimensional, we can apply the Heintz–Mumford result (Theorem 2.11), and conclude that there exist generators of the defining ideal of S in degree μ. Since S is a complexification, $S\cap\mathbb{R}^n=(t+D)\cap T$, and the generators of S can be chosen with real coefficients.

Finally, by the Oleinik–Petrovsky bound (Theorem 2.9), $\sum_{\mathfrak{b}}(T\cap(t+D))\leqslant\mu(2\mu-1)^{k-1}$. Since this is true generically for $D\in\mathbb{G}_{k,n}$ and $t\in D^{\perp}$, we can conclude that $\sum_{\mathfrak{b}}(T,k)\leqslant\mu(2\mu-1)^{k-1}$. □

3. Proof of Optimality of Main Theorem

In this section we prove the optimality of our Main Theorem, which we have formalized in our optimality Theorem 1.17. It is proved by means of counter-examples whose 0^{th} Betti number is not controllable by the multiplicity of the germ (hence *a fortiori* neither is $\sum_{\mathfrak{b}}$). First, in Subsection 3.1 we give an example that shows why it is necessary to assume that the complexification of the germ is pure dimensional. This proves the first point in the optimality Theorem 1.17. Then, in Subsection 3.2 we build counter-example families of every type required by Theorem 1.17 by giving two explicit counter-example families, and introducing two simple transformations on germs to generate the remaining counter-example families.

3.1. Necessity of pure dimensionality

The first example shows a phenomenon that is not specific to the real case and that also occurs in the complex case: only the top dimensional components of a germ are taken into account by the multiplicity, but of course,

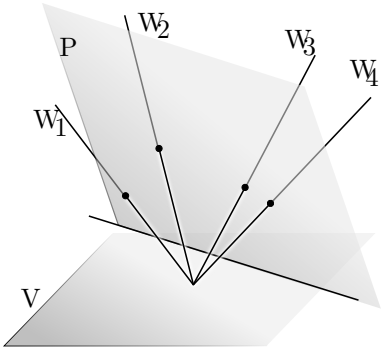


Figure 3.1. The germ consists of a plane V and an arbitrary number of lines W_i. Therefore a generic planar section by P has as many connected components as desired because the W_i do not count toward the multiplicity.

lower dimensional components of the germ can provide connected components when intersecting with an affine space whose dimension is greater than the codimension of the germ. Consequently, these lower dimensional components do not change the multiplicity but can make the number of connected components in a generic section grow arbitrarily large.

Here is one of many ways to construct a family of counter-examples. For any given $l \in \mathbb{N}$, let V be a d-dimensional vector space. Let $k \in \mathbb{N}$ such that $n - k < d$, and choose $W_1, \ldots, W_l$, $(n-k)$-dimensional vector spaces such that $\forall i \neq j,\ W_i \cap W_j = \{0\}$ and $W_i \cap V = \{0\}$. Let X be the union of V and all the $W_1, \ldots, W_l$. The situation for $n = 3,\ d = 2,\ k = 2,\ l = 4$ is shown in Figure 3.1.

The multiplicity of X is clearly $\mu = 1$ as an $(n-d)$-dimensional affine space will generically avoid all the $W_1, \ldots, W_l$ (as $(n-d) + (n-k) < (n-d) + d = n$), and the number of points in a generic intersection with V is 1. On the other hand, a generic k-dimensional affine space will intersect V and all the $W_1, \ldots, W_l$, giving rise to $l+1$ connected components in the intersection.

In addition, X and its tangent cone (which are here the same thing) have an isolated singularity at 0 and their intersection with a generic k-dimensional affine space is smooth, therefore the only condition of the Main Theorem that is not satisfied is that the complexification of the germ should be pure dimensional. This proves the first claim of the optimality Theorem 1.17.

3.2. Necessity of controlling the tangent cone

To find counter-example families of all types (n, k, s) as required by the optimality Theorem 1.17, we use one counter-example family of type $(3, 2, 1)$, a series of counter-example families of type $(n, n-1, n-1)$, and two transformations that we can apply to the previous germ families to cover all remaining counter-example family types.

3.2.1. Counter-example family of type (3,2,1)

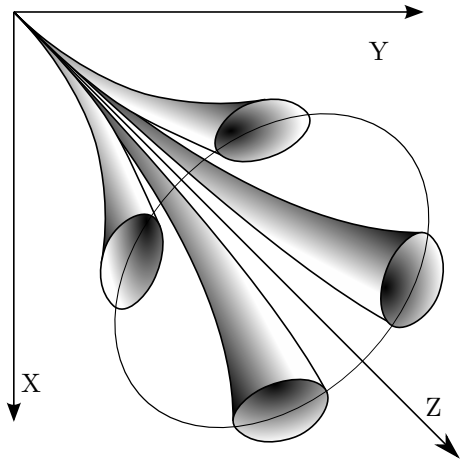


Figure 3.2. The tangent cone is a singular line, but it can arise from any number of higher-dimensional components that degenerate on this line.

The counter-example family of type (3,2,1) is constructed by making ovals degenerate onto a single tangent line as they approach the origin. For any $l \geqslant 2$, consider

$$g_l = (x^2 + y^2 - z^4)^2 + \prod_{i=0}^{2l-1} \left(y - \frac{i - l + 0.5}{l} z^2 \right).$$

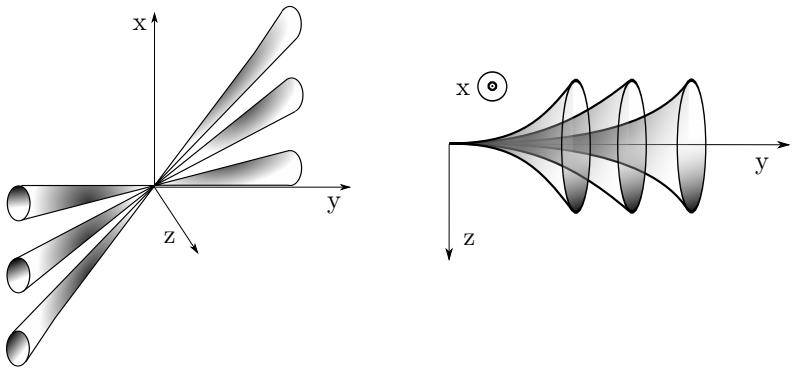
Let X_l be the associated germs. The germ X_3 is depicted in Figure 3.2.

Since we are considering hypersurfaces, the tangent cone is the initial part $(x^2 + y^2)^2$ of the generator, which defines a real singular line, and the multiplicity is the degree of that initial part $\mu(X_l) = 4$. The complexification of the hypersurface is also automatically pure dimensional.

On planar sections with constant z, the g_l can be seen as a circle with radius z^2 (left summand) that is perturbed by an oscillating polynomial along the y direction (right summand). Because the right summand is a product of an even number of factors, it is positive for $y < (-1 + 0.5/l)z^2$ and $y > (1-0.5/l)z^2$. In between its sign alternates on strips of width z^2/l. The positive strips separate ovals created in negative strips. The first and last negative strips ($i = 0$ to $i = 1$, and $i = 2l-2$ to $i = 2l-1$) contain the extremities ($x = 0, y = \pm z^2$) of the circle defined by the first summand. On all other strips it is elementary to check that the perturbation from the second summand stays smaller than the first summand at $x = 0$, which creates two ovals on the circle (since the first summand vanishes on the circle).

We thus have 1 oval for each negative strip at the two extremities of the circle and two ovals for each of the remaining $l - 2$ negative strips across the circle, and therefore $2(l - 1)$ ovals in total. This shows that $\mathfrak{b}_0(X_l, 2) \geqslant 2(l - 1)$.

3.2.2. Counter-example families of type (n,n-1,n-1)



(a) frontal view *(b)* side view

Figure 3.3. A generic plane cannot avoid the funnel-like branches of the germs, but the algebraic tangent cone is the singular plane $z = 0$.

For any $n \in \mathbb{N}$, $n \geqslant 3$, we define the families of germ $X_l = V(f_l) \subset \mathbb{R}^n$ by

$$f_l = \prod_{r=0}^{2l-1} (x - ry) + z^2 + \sum_{i=1}^{n-3} t_i^4,$$

where the first three components of $\mathbb{R}^n$ are associated to the variables x, y, z, the remaining $n-3$ components are associated to the $(t_i)_{i\in\{1,\ldots,n-3\}}$, and the last summand is taken as being zero for $n = 3$. Figure 3.3 shows what X_3 looks like for $n = 3$.

We have $T(X_l) = V(\mathrm{I}nit(f_l)) = V(z^2)$, and according to Lemma 2.2 $\mu(X_l)$ is the number of points counted with multiplicities in the intersection of $T(X_l)$ with a generic line. Since $T(X_l)$ is defined by z^2 alone, there are 2 points in a generic section and $\mu(X_l) = 2$. Also, as required for this counter-example, the Zariski dimension of $T(X_l)$ is $n-1$ and the complexification of the germ is also automatically pure dimensional since it is a hypersurface.

On any intersecting hyperplane H for y constant, it is easy to verify that the first summand of f_l is negative iff $x \in \cup_{r=0}^{r=l-1}(2ry, 2ry + y)$, zero iff $x \in \{0, \ldots, (2l-1)y\}$, and positive elsewhere. Since the sum of the second and third summands is 0 when $z = t_1 = \ldots = t_{n-1} = 0$, and positive elsewhere, f_l doesn't vanish where the first summand is positive, and is negative at $x = 2ry + y/2$, $z = t_1 = \ldots = t_{n-1} = 0$ ($r \in \{0, \ldots, l-1\}$). Therefore by continuity, f_l must vanish over each interval of the form $(2ry, 2ry + y)$, and the connected components over each interval are distinct from components over other intervals since there are intervening intervals of the form $(2ry + y, 2ry + 2y)$ where f_l is positive. Since there are l intervals of the form $(2ry, 2ry + y)$, this shows that $X_l \cap H$ has at least l connected components (there are in fact exactly l compact connected components, which we do not prove in detail for the sake of brevity).

One can verify elementarily that the number of connected components in the intersection doesn't change for any intersecting hyperplane in a neighborhood of any y-constant hyperplane in $\overline{\mathbb{G}}_{n-1,n}$. In the spirit of this article we can show this by invoking the fact that the germ can be Whitney stratified, and for a hyperplane H close enough to the origin, H is transverse to all strata. Since transversality is an open condition, by the Moving the Wall Theorem the topology is unchanged for all hyperplanes in a neighborhood of H, thereby showing that hyperplane sections of X_l do not generically contains less than l connected components. Hence $\lim_{l\to\infty} \mathfrak{b}_0(X_l, k) = \infty$.

3.2.3. Transformations generating the remaining counter-examples

We now use two simple transformations to generate counter-example families of all types (n, k, s) such that $2 \leqslant k \leqslant n-1$, and $n-k \leqslant s \leqslant n-1$ as specified in the optimality Theorem 1.17.

Definition 3.1 (Germ Transformations). — *If X_l is a family of real germs in $\mathbb{R}^n$, the product transformation of X_l is*

$$\mathfrak{P}(X_l) = X_l \times \mathbb{R} \ \subset \mathbb{R}^{n+1}.$$

The embedding transformation of X_l is

$$\mathfrak{E}(X_l) = X_l \times \{0\} \ \subset \mathbb{R}^{n+1}.$$

Proposition 3.2. — *If a counter-example family X_l in $\mathbb{R}^n$ is of type (n, k, s) then $\mathfrak{P}(X_l)$ is of type $(n+1, k, s+1)$ and $\mathfrak{E}(X_l)$ is of type $(n+1, k+1, s)$. Also, if X_l is pure dimensional, then the families $\mathfrak{P}(X_l)$ and $\mathfrak{E}(X_l)$ are also pure dimensional.*

Proof. — One can check that the multiplicity of the germs, and the pure dimensionality of the families is preserved by both transformations, as well as the fact that $\mathfrak{b}_0(X_l, k) = \mathfrak{b}_0(\mathfrak{P}(X_l), k) = \mathfrak{b}_0(\mathfrak{E}(X_l), k+1)$, and that the dimension of the singular locus of the tangent cone is increased by one by $\mathfrak{P}$ and preserved by $\mathfrak{E}$. These proofs are simple, and the details are left to the reader. □

We now use the two transformations to generate all the types of counter-example families we are missing, and prove the second point of the optimality Theorem 1.17.

Proposition 3.3. — *For any combination of $(n, k, s) \in \mathbb{N}^3$ that satisfies the inequalities $2 \leqslant k \leqslant n-1$ and $n-k \leqslant s \leqslant n-1$, there exists a pure dimensional counter-example family $(X_l)_{l\in\mathbb{N}}$ of type (n, k, s).*

Proof. — We work by induction on n. We start at $n = 3$ since $2 \leqslant k \leqslant n-1$ forces $n \geqslant 3$. For $n = 3$ the conditions give two possible types: $(3, 2, 1)$ which was given by our first counter-example family in 3.2.1 and $(3, 2, 2)$ which is given by our series of counter-example families in 3.2.2 of type $(n, n-1, n-1)$ for $n = 3$.

We now prove the induction step. Let $(n, k, s) \in \mathbb{N}^3$ be a triplet satifying the inequalities in the statement of the proposition.

- If $k < n-1$, then $(n-1, k, s-1)$ still satisfies the inequalities, and by induction we can find a pure dimensional counter-example family of that type. By applying the product transformation, according to Proposition 3.2, we obtain a counter-example family of type (n, k, s).
- If $k = n-1$ and $s < n-1$, then $(n-1, k-1, s)$ still satisfies the inequalities, and by induction we can find a pure dimensional counter-example family of that type. By applying the embedding

transformation, according to Proposition 3.2, we obtain a counter-example family of type (n, k, s).

- if $k = n - 1$ and $s = n - 1$, then the counter-example family series in 3.2.2 provides a suitable family of type $(n, n-1, n-1)$. □

4. Bound on Density and Lipschitz–Killing Invariants

In this section we present a polynomial bound on the local Lipschitz–Killing invariants of a real analytic germ, the first of which is the density (Definition 4.3). The bound we give on the density and the Lipschitz–Killing invariants of a real analytic germ (Proposition 4.9) is relevant to the work in [4, 5, 6] which all involve these quantities. The result is obtained from a localized version of the multidimensional Cauchy–Crofton formula ([6, Theorem 3.1]) and from our Main Theorem.

In the first place we define the Lipschitz–Killing invariants by a multidimensional Cauchy–Crofton formula. We define the *local* Lipschitz–Killing invariants by a process of limit from the original Lipschitz–Killing invariants. Then we quote [4, Theorem 3.1] which gives a direct formula for the local Lipschitz–Killing invariant. This latter formula can be seen as a local version of the multidimensional Cauchy–Crofton formula. Finally we apply the Main Theorem 1.13 to this expression of the local Lipschitz–Killing invariants to bound them. In order to illustrate the usefulness of these new bounds we conclude by using it to bound the density of an example.

DEFINITION 4.1 (k^{th} Lipschitz–Killing invariants). — *Let $\gamma_{k,n}$ be the unit measure $\mathcal{O}_n(\mathbb{R})$-invariant on $\mathbb{G}_{k,n}$, where $\mathcal{O}_n(\mathbb{R})$ is the orthogonal group of $\mathbb{R}^n$. Let $\mathcal{H}^k$ be the usual Lebesgue measure on $\mathbb{R}^k$. For any real sub-analytic set X in $\mathbb{R}^n$, the k^{th} Lipschitz–Killing invariant $\Lambda^k(X)$ is defined by:*

$$\Lambda_k(X) = \beta(k,n)^{-1} \int_{V \in \mathbb{G}_{k,n}} \int_{y \in V} \chi\left(X \cap \pi_V^{-1}(\{y\})\right) d\mathcal{H}^k(y) d\gamma_{k,n}(V),$$

where χ is the Euler characteristic, $\beta(k,n)$ only depends on k and n and π_V is the orthogonal projection to V.

These invariants are to be paralleled with the Vitushkin variations:

DEFINITION 4.2 (Vitushkin variations). — *For any set $S \subset \mathbb{R}^n$, let $V_0(S)$ be the number of connected components of S, and*

$$V_i(S) = c(i) \int_{L \in \mathcal{G}_{n-i}} V_0(S \cap L)\, \mathrm{dL},$$

where $\mathcal{G}_k$ is the Grassmannian of affine spaces of dimension k in $\mathbb{R}^n$, dL is the canonical measure on $\mathcal{G}_{n-i}$, and $c(i) = \left(\int_{L \in \mathcal{G}_{n-i}} V_0([0,1]^i \cap L) \, \mathrm{dL} \right)^{-1}$, *(so that $V_i([0,1]^i) = 1$ and $c(n) = 1$).*

The Vitushkin variations have a more straightforward geometric interpretation since their definition is based on the number of connected components, but the Lipschitz–Killing invariants have more interesting algebraic properties. This is due to the fact that the Lipschitz–Killing invariants enjoy the same additivity property as the Euler characteristic from which they are derived (i.e. $\Lambda_k(X \cup Y) = \Lambda_k(X) + \Lambda_k(Y) - \Lambda_k(X \cap Y)$).

Definition 4.3 (Local Lipschitz–Killing invariants and Density). — *For a real sub-analytic germ $X \subset \mathbb{R}^n$ at the origin, the k^{th} local Lipschitz–Killing invariant of X is defined by:*

$$\Lambda_k^{loc}(X) = \lim_{r \to 0} \frac{\Lambda_k\big(X \cap B^n(0,r)\big)}{\mathcal{H}^k\big(B^k(0,r)\big)},$$

where $B^k(0,r)$ stands for the k-dimensional ball centered at the origin of radius r. It is proved in [15, Theorem 2.2] *that this limit exists.*

When $k = \dim(X)$, the fibers $\pi_V^{-1}(\{y\})$ are generically 0-dimensional, and the Euler characteristic simply counts the number of points in those fibers. In this case the k^{th} local Lipschitz–Killing invariant of X is called the k-density of X and it is defined as:

$$\Theta_k(X) = \Lambda_k^{loc}(X) = \lim_{r \to 0} \frac{\Lambda_k\big(X \cap B^n(0,r)\big)}{\mathcal{H}^k\big(B^k(0,r)\big)}.$$

At this point, it would already be possible to use the Main Theorem 1.13 to derive a bound on the k-density and the local Lipschitz–Killing invariants. If we did so we would obtain a bound that is not sharp. Conceptually this would amount to approximating the balls by their bounding cubes. To avoid losing this sharpness we use an alternate characterization of the k-density and the local Lipschitz–Killing invariants. It comes from a local version of the multidimensional Cauchy–Crofton formula.

Definition 4.4 (Local polar profiles and local polar Euler characteristic). — *Let X be a real sub-analytic germ at the origin. Let $\mathrm{Crit}(\pi_V|_X)$ be the critical locus of π_V on X_{smooth}, where $X_{\text{smooth}} = X \setminus \operatorname{Sing} X$. Let $\mathcal{O}_X^k$ be the vector spaces $V \in \mathbb{G}_{k,n}$ such that $T(X) \cap V^\perp = \{0\}$, where $T(X)$ is the tangent cone to X.*

The local polar profiles of X for V are the connected components K_j^V ($j \in \{0, \dots, n_V\}$) of the open germ $\pi_V(X) \setminus \pi_V\left(\mathrm{Crit}(\pi_V|_X)\right)$. The definition

of $\mathrm{Crit}(\pi_V|_X)$ *entails that the topology of the fibers* $\big(X \cap \pi^{-1}(y)\big)$ $(y \in V)$ *is constant when* y *runs over a given* K_j^V*. Therefore, the Euler characteristic of* $\big(X \cap \pi^{-1}(y)\big)$ *is constant over the* K_j^V*.*

We call this constant the local polar Euler characteristic χ_j^V *associated to* K_j^V*. When* $\dim(V) = \dim(X)$*, the fiber contains finitely many points, and* χ_j^V *is thus the number of points in each fiber. In this case, we denote* χ_j^V *by* e_j^V*.*

Definition 4.5 (Local polar invariants). — *Let* X *be a real sub-analytic germ of* $\mathbb{R}^n$*. The* k^{th} *local polar invariant associated to* X *is:*

$$\sigma_k(X) = \int_{V \in \mathcal{O}_X^k} \left(\sum_{j=0}^{n_V} \chi_j^V \Theta_k(K_j^V) \right) d\gamma_{k,n}(V).$$

Theorem 4.6 (Local multidimensional Cauchy–Crofton formula). — *Let* X *be a real sub-analytic germ of* $\mathbb{R}^n$*. Theorem 3.1 in* [4] *states that there exists an upper-triangular matrix* $M \in \mathcal{M}_n(\mathbb{R})$ *such that*

$$\begin{pmatrix} \Lambda_1^{loc} \\ \vdots \\ \Lambda_n^{loc} \end{pmatrix} = \begin{pmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,n-1} & M_{1,n} \\ 0 & M_{2,2} & \dots & M_{2,n-1} & M_{2,n} \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & M_{n,n} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix}$$

where $M_{i,i} = 1$*, and for* $i < j \leqslant n$*:*

$$M_{i,j} = \frac{\alpha_j}{\alpha_{j-i}\ \alpha_i} C_j^i - \frac{\alpha_{j-1}}{\alpha_{j-1-i}\ \alpha_i} C_{j-1}^i,$$

with α_k *the* k*-dimensional volume of the unit ball in* $\mathbb{R}^k$ *and* C_j^i *the usual binomial coefficients.*

Remark 4.7. — The previous result extends the local Cauchy–Crofton formula for the density in [4] which asserts that for a k-dimensional real analytic germ X at 0 in $\mathbb{R}^n$:

$$\Theta_k(X) = \int_{V \in \mathcal{O}_X^k} \left(\sum_{j=0}^{n_V} e_j^V \Theta_k(K_j^V) \right) d\gamma_{k,n}(V).$$

This is simply because for $i > k$, the $(n-i)$-dimensional spaces avoid X generically and thus, $\sigma_i = 0$. As M is upper-triangular the only non-zero coefficient involved in the expression of $\Theta_k(X)$ $(= \Lambda_k^{loc}(X))$ is the diagonal coefficient $M_{k,k} = 1$, hence giving the above equality for the density.

The relation given by Theorem 4.6 between the σ_i and the local Lipschitz–Killing invariants enables us to derive the following bounds on the Λ_i^{loc}:

PROPOSITION 4.8 (Bound on the σ_i). — *Let $X \subset \mathbb{R}^n$ be a real analytic germ at the origin. Let T be the algebraic tangent cone to X. Then*

(1) *If $l = \dim(X)$, we have the inequality*

$$\sigma_l(X) = \Theta_l(X) \leqslant \mu(X).$$

(2) *For any $l \in \mathbb{N}$, if $\dim(\operatorname{Sing} T) < l$ and $X_{\mathbb{C}}$ is pure dimensional, then*

$$\sigma_l(X) \leqslant \mu(X)(2\mu(X) - 1)^{n-l-1}.$$

Proof. — We start from the expression of $\sigma_l(X)$. Let $k = n - l$. The bound stems from the simple fact that the sum of Betti numbers for a given set is a bound on the Euler characteristic of that set since this latter one is an alternating sum of Betti numbers. Under the hypothesis of the proposition, we can apply the Main Theorem 1.13, and we can conclude that for a generic $V \in \mathbb{G}_{l,n}$ and $y \in V$

$$\begin{aligned}
&\lim_{\lambda \to 0} \chi\left(X \cap \pi_V^{-1}(\lambda y)\right) \leqslant \mu(X) && \text{in case 1,}\\
&\lim_{\lambda \to 0} \chi\left(X \cap \pi_V^{-1}(\lambda y)\right) \leqslant \mu(X)(2\mu(X) - 1)^{n-l-1} && \text{in case 2.}
\end{aligned}$$

If $\Theta_l(K_j^V)$ is non-zero, for some y we have a whole segment $(0, y) \subset \pi_V(K_j^V)$. Therefore, for that y we have

$$\lim_{\lambda \to 0} \chi\left(X \cap \pi_V^{-1}(\lambda y)\right) = \chi\left(X \cap \pi_V^{-1}(y)\right) = \chi_j^V.$$

We then replace χ_j^V in the definition of $\sigma_l(X)$ with $\lim_{\lambda \to 0} \chi\left(X \cap \pi_V^{-1}(\lambda y)\right)$ and use the inequalities above to yield

$$\begin{aligned}
&\sigma_l(X) \leqslant \mu(X) \int_{V \in \mathcal{O}_X^l} \left(\sum_{j=0}^{n_V} \Theta_l(K_j^V) \right) d\gamma_{l,n}(V) && \text{in case 1,}\\
&\sigma_l(X) \leqslant \mu(X)(2\mu(X) - 1)^{l-1} \int_{V \in \mathcal{O}_X^l} \left(\sum_{j=0}^{n_V} \Theta_l(K_j^V) \right) d\gamma_{l,n}(V) && \text{in case 2.}
\end{aligned}$$

Notice that if $\Theta_l(K_j^V)$ was zero, the inequalities on χ_j^V might not hold (e.g. the counter-example families in Section 3), but this does not matter as they do not count in the Cauchy–Crofton expression of $\sigma_l(X)$. Finally by definition of the K_j^V we have:

$$\cup_{j=0}^{n_V} K_j^V = V_0 \setminus \pi_V\left(\operatorname{Crit}(\pi_V|_X)\right),$$

where V_0 denotes the germ at 0 associated to V. Furthermore the previous union is disjoint and $\pi_V\left(\mathrm{Crit}(\pi_V|_X)\right)$ is of measure 0. This shows that

$$\sum_{j=0}^{n_V} \Theta_l(K_j^V) = \Theta_l\Big(V_0 \setminus \pi_V\big(\mathrm{Crit}(\pi_V|_X)\big)\Big) = \Theta_l(V_0) = 1.$$

In the end we obtain the desired result, since $\mathcal{O}_X^l$ is dense in $\mathbb{G}_{l,n}$:

$$\begin{aligned}
\sigma_l(X) &\leqslant \mu(X) \int_{V\in\mathcal{O}_X^l} 1 \; d\gamma_{l,n}(V) \\
&\leqslant \mu(X) && \text{in case 1,} \\
\sigma_l(X) &\leqslant \mu(X)(2\mu(X)-1)^{l-1} \int_{V\in\mathcal{O}_X^l} 1 \; d\gamma_{l,n}(V) \\
&\leqslant \mu(X)(2\mu(X)-1)^{l-1} && \text{in case 2.} \qquad \square
\end{aligned}$$

Proposition 4.9 (Bound on local Lipschitz–Killing invariants). — *Let X be an analytic germ at 0 of dimension d such that $\dim(\mathrm{Sing}\, T) < k$. Then we have the following bound on the k^{th} local Lipschitz–Killing invariant of X:*

$$\Lambda_k^{loc}(X) \leqslant M_{k,d}\ \mu(X) + \sum_{l=k}^{d-1} M_{k,l}\ \mu(X)(2\mu(X)-1)^{l-1},$$

where M is defined as in Theorem 4.6.

Proof. — By Proposition 4.8, we have a bound in terms of the multiplicity for every $\sigma_l(X)$. We can store those bounds in a vector b. According to Theorem 4.6, the vector $(\Lambda_l(X))$ which is made up of the local Lipschitz–Killing invariants, is the image of the vector $(\sigma_l(X))$ through M. The coefficients of M are all non-negative, therefore after applying M to the vector b, we obtain a vector whose components bound the $\Lambda_l(X)$. By expanding the product Mb we obtain the bounds announced in the proposition. The sum ends at $M_{k,d}$ because the $\sigma_l(X)$ for $l > d$ all vanish (as $(n-l)$-dimensional spaces generically avoid X). $\square$

Remark 4.10. — For $k = \dim(X)$, the previous proposition gives:

$$\Lambda_k^{loc}(X) \leqslant M_{k,k}\mu(X) = \mu(X).$$

As $\Lambda_k^{loc}(X) = \Theta_k(X)$ by definition, the previous proposition yields the same result as the first item of Proposition 4.8 in this case.

So that the reader can appreciate the sharpness of our bound in view of existing results, we now recall the known Oleinik–Petrovsky/Thom–Milnor bound for the density:

THEOREM 4.11 (The Oleinik–Petrovsky/Thom–Milnor Bound on Density). — *For an analytic germ $X \subset \mathbb{R}^n$ of dimension l defined by functions $f_1, \dots, f_k$ of degree $d_1, \dots, d_k$,*

$$\Theta_l(X) \leqslant (d+1)(2d+1)^{n-l-1} \quad \text{where} \quad d = \sum_{i=1}^{k} d_i.$$

Proof. — This is a straightforward application of [23, Theorem 5.5] for semi-algebraic sets (using 2 inequalities there to encode 1 equality here). □

Our bound is sharper because it is a localization of the Oleinik–Petrovsky bound: the defining functions may store information on what happens away from the origin, but the multiplicity only accounts for what happens at the origin.

For further illustration, we can work out an example where the variety X is defined by

$$\big(x(x-z^3)(x-2z^2), y(y-z^3)(y-2z^2), (x+y)(x+y-z^3)\big).$$

Those generators have degrees $6, 6$, and 4. Therefore by the Oleinik–Petrovsky bound for the density 4.11 we have

$$\Theta_1(X) \leqslant (16+1)(2\times 16+1)^{3-1-1} = 17 \times 33 = 561.$$

The multiplicity can be shown to be 3 (see Remark 1.12 for a way to do this using Gröbner bases). Therefore, the bound in Proposition 4.8 states that

$$\Theta_1(X) \leqslant 3.$$

In fact, the intersection of X with a generic plane has exactly 3 points in it, therefore the density of X is equal to 3, and our bound was optimal.

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BIBLIOGRAPHY

[1] R. Benedetti, F. Loeser & J.-J. Risler, "Bounding the number of connected components of a real algebraic set", *Discrete Comput. Geom.* **6** (1991), no. 3, p. 191-209.

[2] A. Bernig & L. Bröcker, "Lipschitz-Killing invariants", *Math. Nachr.* **245** (2002), p. 5-25.

[3] J. Bochnak, M. Coste & M.-F. Roy, *Géométrie algébrique réelle*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 12, Springer-Verlag, Berlin, 1987, x+373 pages.

[4] G. Comte, "Formule de Cauchy-Crofton pour la densité des ensembles sous-analytiques", *C. R. Acad. Sci. Paris Sér. I Math.* **328** (1999), no. 6, p. 505-508.

[5] ———, "Équisingularité réelle : nombres de Lelong et images polaires", *Ann. Sci. École Norm. Sup.* **33** (2000), no. 6, p. 757-788.

[6] G. Comte, P. Graftiaux & M. Merle, "Équisingularité réelle II : invariants locaux et conditions de régularité", *Ann. Sci. École Norm. Sup.* **41** (2008), no. 2, p. 1-48.

[7] R. N. Draper, "Intersection theory in analytic geometry", *Math. Ann.* **180** (1969), p. 175-204.

[8] J.-C. Faugère, "A new efficient algorithm for computing Gröbner bases without reduction to zero (F_5)", in *Proceedings of the 2002 International Symposium on Symbolic and Algebraic Computation* (New York), ACM, 2002, p. 75-83 (electronic).

[9] H. Federer, "The (φ, k) rectifiable subsets of n-space", *Trans. Amer. Soc.* **62** (1947), p. 114-192.

[10] ———, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969, xiv+676 pages.

[11] M. Goresky & R. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 14, Springer-Verlag, Berlin, 1988, xiv+272 pages.

[12] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52, xvi+496 pages.

[13] J. Heintz, "Definability and fast quantifier elimination in algebraically closed fields", *Theoret. Comput. Sci.* **24** (1983), no. 3, p. 239-277.

[14] L. D. Ivanov, *Variatsii mnozhestv i funktsii [Variations of sets and functions]*, Izdat. "Nauka", Moscow, 1975, Edited by A. G. Vituškin, 352 pages.

[15] K. Kurdyka & G. Raby, "Densité des ensembles sous-analytiques", *Ann. Inst. Fourier (Grenoble)* **39** (1989), no. 3, p. 753-771.

[16] F. Loeser, "Un analogue local de l'inégalité de Petrowsky-Oleinik", in *Séminaire sur la géométrie algébrique réelle, Tome I, II*, Publ. Math. Univ. Paris VII, vol. 24, Univ. Paris VII, Paris, 1986, p. 73-83.

[17] J. Mather, "Notes on topological stability", *Bull. Amer. Math. Soc. (N.S.)* **49** (2012), no. 4, p. 475-506.

[18] J. Milnor, "On the Betti numbers of real varieties", *Proc. Amer. Math. Soc.* **15** (1964), p. 275-280.

[19] D. Mumford, "Varieties defined by quadratic equations", in *Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969)*, Edizioni Cremonese, Rome, 1970, p. 29-100.

[20] D. Trotman, "Multiplicity as a C^1 invariant", in *Real analytic and algebraic singularities (Nagoya/Sapporo/Hachioji, 1996)*, Pitman Res. Notes Math. Ser., vol. 381, Longman, Harlow, 1998, p. 215-221.

[21] H. Whitney, "Local properties of analytic varieties", in *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*, Princeton Univ. Press, Princeton, N. J., 1965, p. 205-244.

[22] ———, "Tangents to an analytic variety", *Ann. of Math.* **81** (1965), p. 496-549.

[23] Y. Yomdin & G. Comte, *Tame geometry with application in smooth analysis*, Lecture Notes in Mathematics, vol. 1834, Springer-Verlag, Berlin, 2004, viii+186 pages.

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